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Dynamics of water eutrophication model with control

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Abstract

In this work, we consider a water eutrophication model with impulsive dredging. We prove that all solutions of the investigated system are uniformly bounded. There exists globally asymptotically stable periodic solution of microorganism-extinction when some condition is satisfied. The condition for permanence of the system is also obtained. It is concluded that the approach of impulsive dredging provides a reliable theoretical basis for the management of water eutrophication.

Keywords: Impulsive dredging; Water eutrophication; Extinction; Permanence

1 Introduction

Due to human activities, there are more and more nutrients in water, which causes excessive reproduction of algae and other aquatic organisms, decrease in water transparency, dissolved oxygen reduction, deterioration of water quality, and so on. This kind of pollution phenomenon, which is caused by the change in the ecological balance of the whole water body, is called the eutrophication of water body [1].

Eutrophication occurs in ponds, reservoirs, lakes, and so on and it has attracted the attention of many experts and scholars in the early twentieth century. Especially in recent years, the study on the eutrophication of water body has become quite active like fuzzy mathematics, stochastic model, gray system, artificial intelligence, and other theoretical methods combined with computer technology [2–4]. For instance, based on the analysis of the present situation and causes of pollution in Chaohu, Li et al. [5] proposed specific measures for eutrophication control of Chaohu. Yu et al. [6] used three methods—Carlson comprehensive index method, Grey clustering method, and principal component analysis—to analyze the main factors of eutrophication, and they provided theoretical basis for controlling water pollution in Poyang Lake. In addition, many scholars [7–9] have studied the dynamics model of a single population in the polluted environment.

Many scholars have done in-depth research on the problems of eutrophication. But most of them studied the reasons and the prevention and treatment of water eutrophication from the technical aspects. Few researchers studied the problem of eutrophication from the viewpoint of biological dynamics. In this paper, we discuss water eutrophication with impulsive dredging.

2 Model formulation

In this paper, we consider the following eutrophication model with control:

$$
\begin{cases}
\left.\begin{aligned}
\frac{ds(t)}{dt} &= \lambda - d_1 s(t) - \beta s(t) x(t) \\
\frac{dx(t)}{dt} &= \frac{\beta}{\delta} s(t) x(t) - d_2 x(t)
\end{aligned}\right\} & t \neq n\tau, t \neq (n+l)\tau, \\[2mm]
\left.\begin{aligned}
\Delta s(t) &= \mu \\
\Delta x(t) &= 0
\end{aligned}\right\} & t = n\tau, \\[2mm]
\left.\begin{aligned}
\Delta s(t) &= -\mu_1 s(t) \\
\Delta x(t) &= -\mu_2 x(t)
\end{aligned}\right\} & t = (n+l)\tau, 0 < l < 1, n = 1, 2, \ldots,
\end{cases}
\tag{2.1}
$$

where $s(t)$ denotes the concentration of nutrients in water. $x(t)$ denotes the concentration of microorganism in water. $\lambda > 0$ is the original concentration of nutrients $s(t)$ in water, $\beta > 0$ is called the transmission coefficient. $\frac{\beta}{\delta} > 0$ is called the transmission of $x(t)$ absorbed nutrients. $d_1 > 0$ is called the loss of $s(t)$. $d_2 > 0$ is called the death coefficient of $x(t)$. $\mu > 0$ is the increase amount of $s(t)$ at $t = n\tau$. $0 \leq \mu_1 < 1$, $0 \leq \mu_2 < 1$ respectively represent the dredging intensity of $s(t)$ and $x(t)$ at $t = (n+l)\tau$ ($0 < l < 1$), τ is the period of the impulsive effect.

3 Analyzing model

Let $R_+ = [0, \infty)$, The solution of system (2.1), denoted by $y(t) = (s(t), x(t))^T$, is a piecewise continuous function $y : R_+ \to R_+^2$, $y(t)$ is continuous on $(n\tau, (n+l)\tau]$, $((n+l)\tau, (n+1)\tau]$, $n \in Z_+$ and $y(n\tau^+) = \lim_{t \to n\tau^+} y(n\tau)$, $y((n+l)\tau^+) = \lim_{t \to (n+l)\tau^+} y((n+l)\tau)$ exist. Obviously the global existence and uniqueness of the solutions of (2.1) are guaranteed by the smoothness of function y, which denotes the mapping defined by the right-hand side of system (2.1) (see [10]). Before discussing the main results, we will give some definitions, notations, and lemmas. Since $s(t) = 0$ whenever $s'(t) = \lambda$, $x(t) = 0$ whenever $x'(t) = 0$, and $s(n\tau^+) = s(n\tau) + \mu$, $\mu \geq 0$, so we have the following.

Lemma 1 *Suppose that $y(t)$ is a solution of (2.1), with $y(0^+) \geq 0$, then $y(t) \geq 0$ for $t \geq 0$ and further $y(t) > 0$, $t \geq 0$ for $y(0^+) > 0$.*

Lemma 2 (see [10, p. 23, Lemma 2.2]) *Let the function $m \in PC'[R^+, R]$ satisfy the inequalities*

$$
\begin{cases}
m'(t) \leq p(t)m(t) + q(t), & t \geq t_0, t \neq t_k, k = 1, 2, \ldots, \\
m(t_k^+) \leq d_k m(t_k) + b_k, & t = t_k,
\end{cases}
$$

where $p, q \in PC[R^+, R]$, $d_k \geq 0$, b_k are constants, then

$$
m(t) \leq m(t_0) \prod_{t_0 < t_k < t} d_k \exp\left(\int_{t_0}^t p(s)\,ds\right) + \sum_{t_0 < t_k < t}\left(\prod_{t_0 < t_k < t} d_j \exp\left(\int_{t_0}^t p(s)\,ds\right)\right) b_k
$$

$$
+ \int_{t_0}^t \prod_{s < t_k < t} d_k \exp\left(\int_s^t p(\sigma)\,d\sigma\right) q(s)\,ds, \quad t \geq t_0.
$$

Now, we show that all solutions of (2.1) are uniformly ultimately bounded.

Lemma 3 *There exists a constant $M > 0$ such that $S(t) \le M$, $x(t) \le M$ for each solution $(s(t), x(t))$ of* (2.1) *with all t large enough.*

Proof Definition $V(t) = \frac{1}{\delta}s(t) + x(t)$, $d = \min\{d_1, d_2\}$. Then $t \ne n\tau$, $t \ne (n + l)\tau$. We have

$$D^+V(t) + dV(t) = \frac{1}{\delta}\left(\lambda - d_1 s(t) - \beta s(t)x(t)\right) + \frac{\beta}{\delta}s(t)x(t) - d_2 x(t) + \frac{d}{\delta}s(t) + dx(t)$$

$$= \frac{\lambda}{\delta} - \frac{d_1 - d}{\delta}s(t) - (d_2 - d)x(t)$$

$$\le \frac{\lambda}{\delta}.$$

When $t = n\tau$,

$$V\left(n\tau^+\right) = \frac{1}{\delta}s\left(n\tau^+\right) + x\left(n\tau^+\right) = \frac{1}{\delta}\left(s(n\tau) + \mu\right) + x(n\tau) = \frac{1}{\delta}s(n\tau) + x(n\tau) + \frac{\mu}{\delta} = V(n\tau) + \frac{\mu}{\delta}.$$

When $t = (n + l)\tau$,

$$V\left((n + l)\tau^+\right) = \frac{1}{\delta}s\left((n + l)\tau^+\right) + x\left((n + l)\tau^+\right)$$

$$= \frac{1}{\delta}(1 - \mu_1)s\left((n + l)\tau\right) + (1 - \mu_2)x\left((n + l)\tau\right) \le V\left((n + l)\tau\right).$$

By Lemma 2, for $t \in (n\tau, n + l)\tau]$, $t \in ((n + l)\tau, (n + 1)\tau]$, we have

$$\begin{cases} D^+V(t) \le -dV(t) + \frac{\lambda}{\delta}, & t \ne n\tau, t \ne (n + l)\tau, \\ V(t^+) = V(t) + \frac{\mu}{\delta}, & t = n\tau, \\ V(t^+) \le V(t), & t = (n + l)\tau, \end{cases}$$

then

$$V(t) \le V(0)\exp(-dt) + \int_0^t \frac{\lambda}{\delta}\exp\left(-d(t - s)\right)ds + \sum_{0 < n\tau < t} \frac{\mu}{\delta}\exp\left(-d(t - n\tau)\right)$$

$$\to \frac{\lambda}{\delta d} + \frac{\mu\exp(d\tau)}{\delta(\exp(d\tau) - 1)}, \quad \text{as } t \to \infty.$$

So $V(t)$ is uniformly ultimately bounded. Hence, by the definition of $V(t)$, there exists a constant $M > 0$ such that $S(t) \le M$, $x(t) \le M$ for t large enough. The proof is complete. □

If $x(t) = 0$, we have the following subsystem of (3.1):

$$\begin{cases} s'(t) = \lambda - d_1 s(t), & t \ne n\tau, t \ne (n + l)\tau, \\ \Delta s(t) = \mu, & t = n\tau, \\ \Delta s(t) = -\mu_1 s(t), & t = (n + l)\tau, \\ s(0^+) = s(0) > 0. \end{cases} \tag{3.1}$$

We can easily obtain the analytic solution of (3.1) between pulses as follows:

$$s(t) = \begin{cases} \frac{1}{d_1}[\lambda - (\lambda - d_1 s(n\tau^+))e^{-d_1(t - n\tau)}], & t \in (n\tau, (n + l)\tau], \\ \frac{1}{d_1}[\lambda - (\lambda - d_1 s((n + l)\tau^+))e^{-d_1(t - (n+l)\tau)}], & t \in ((n + l)\tau, (n + 1)\tau]. \end{cases} \tag{3.2}$$

Considering the third equation of (3.1), we have

$$s\big((n+l)\tau^+\big) = \frac{1-\mu_1}{d_1}\big[\lambda - \big(\lambda - d_1 s(n\tau^+)\big)e^{-d_1 l\tau}\big]. \tag{3.3}$$

Considering the fifth equation of (3.1), we also have

$$s\big((n+1)\tau^+\big) = \frac{1}{d_1}\big[\lambda - \big(\lambda - d_1 s((n+l)\tau^+)\big)e^{-d_1(1-l)\tau}\big] + \mu. \tag{3.4}$$

Substituting (3.3) into (3.4), we have

$$\begin{aligned} s\big((n+1)\tau^+\big) ={}& \frac{\lambda}{d_1} - \frac{\lambda e^{-d_1(1-l)\tau}}{d_1} + \frac{\lambda(1-\mu_1)e^{-d_1(1-l)\tau}}{d_1} \\ & - \frac{\lambda(1-\mu_1)e^{-d_1\tau}}{d_1} + (1-\mu_1)s(n\tau^+)e^{-d_1\tau} + \mu. \end{aligned} \tag{3.5}$$

Equation (3.1) has one fixed point as follows:

$$s^* = \frac{\lambda - \lambda\mu_1 e^{-d_1(1-l)\tau} - \lambda(1-\mu_1)e^{-d_1\tau} + \mu d_1}{d_1[1 - (1-\mu_1)e^{-d_1\tau}]}. \tag{3.6}$$

Then the periodic solution of (3.1) is

$$\tilde{s}(t) = \begin{cases} \frac{1}{d_1}[\lambda - (\lambda - d_1 s^*)e^{-d_1(t-n\tau)}], & t \in [n\tau, (n+l)\tau), \\ \frac{1}{d_1}[\lambda - (\lambda - d_1 s^{**})e^{-d_1(t-(n+l)\tau)}], & t \in [(n+l)\tau, (n+1)\tau). \end{cases} \tag{3.7}$$

Here s^* is determined as (3.6), s^{**} is defined as

$$s^{**} = \frac{1-\mu_1}{d_1}\big[\lambda - \big(\lambda - d_1 s^*\big)e^{-d_1 l\tau} + d_1(1-\mu_1)s^* e^{-d_1 l\tau}\big]. \tag{3.8}$$

It is similar to reference [8], we can obtain the following lemma.

Lemma 4 *System* (3.1) *has a positive periodic solution $\tilde{s}(t)$. For every solution $s(t)$ of system* (3.1), *we have $s(t) \to \tilde{s}(t)$ as $t \to \infty$.*

4 The dynamics

From the above discussion, we know that system (2.1) has a microorganism-extinction periodic solution $(\tilde{s}(t), 0)$. So we can have the following theorem.

Theorem 1 *If*

$$\ln\frac{1}{1-\mu_2} > \frac{\lambda\tau}{d_1} - \frac{\lambda - d_1 s^*}{d_1^2}\big(1 - e^{-d_1 l\tau}\big) - \frac{\lambda - d_1 s^{**}}{d_1^2}\big(1 - e^{-d_1(1-l)\tau}\big)e^{-d_1\tau} - d_2\tau \tag{4.1}$$

holds, then the microorganism-extinction periodic solution $(\tilde{s}(t), 0)$ of system (2.1) *is globally asymptotically stable.*

Proof First we prove the local stability. Defining $S(t) = s(t) - \tilde{s}(t)$, $X(t) = x(t)$, we have the following linearly similar system for (2.1) which concerns one periodic solution $(\tilde{s}(t), 0)$:

$$
\begin{pmatrix} S'(t) \\ X'(t) \end{pmatrix} = \begin{pmatrix} -d_1 & \beta\tilde{s}(t) \\ 0 & \frac{\beta}{\delta}\tilde{s}(t) - d_2 \end{pmatrix} \begin{pmatrix} S(t) \\ X(t) \end{pmatrix}.
$$

It is easy to obtain the fundamental solution matrix

$$
\Phi(t) = \begin{pmatrix} e^{-d_1 t} & * \\ 0 & \exp(\int_0^t (\frac{\beta}{\delta}\tilde{s}(s) - d_2)\, ds) \end{pmatrix}.
$$

There is no need to calculate the exact form of $*$ as it is not required in the analysis that follows. The linearization of the third and fourth equations of (2.1) is as follows:

$$
\begin{pmatrix} S(n\tau^+) \\ X(n\tau^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S(n\tau) \\ X(n\tau) \end{pmatrix}.
$$

The linearization of the fifth and sixth equations of (2.1) is as follows:

$$
\begin{pmatrix} S((n+l)\tau^+) \\ X((n+l)\tau^+) \end{pmatrix} = \begin{pmatrix} 1-\mu_1 & 0 \\ 0 & 1-\mu_2 \end{pmatrix} \begin{pmatrix} S((n+l)\tau) \\ X((n+l)\tau) \end{pmatrix}.
$$

The stability of the periodic solution $(\tilde{s}(t), 0)$ is determined by the eigenvalues of

$$
M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1-\mu_1 & 0 \\ 0 & 1-\mu_2 \end{pmatrix} \phi(\tau)
$$

which are

$$
\lambda_1 = (1-\mu_1)e^{-d_1 t} < 1, \qquad \lambda_2 = (1-\mu_2)\exp\left(\int_0^\tau \left(\frac{\beta}{\delta}\tilde{s}(s) - d_2\right) ds\right).
$$

According to the Floquet theory [11], if $|\lambda_2| < 1$, i.e.,

$$
\left| (1-\mu_2)\exp\left(\int_0^\tau \left(\frac{\beta}{\delta}\tilde{s}(t) - d_2\right) dt\right) \right| < 1,
$$

that is,

$$
\ln \frac{1}{1-\mu_2} > \frac{\lambda\tau}{d_1} - \frac{\lambda - d_1 s^*}{d_1^2}\left(1 - e^{-d_1 l\tau}\right) - \frac{\lambda - d_1 s^{**}}{d_1^2}\left(1 - e^{-d_1(1-l)\tau}\right)e^{-d_1\tau} - d_2\tau
$$

holds, then $(\tilde{s}(t), 0)$ is locally stable.

The following work is to prove the global attraction. Choose $\varepsilon > 0$ such that

$$
\rho = (1-\mu_2)\exp\left(\int_0^\tau \left[\frac{\beta}{\delta}(\tilde{s}(s) - \varepsilon) - d_2\right] ds\right) < 1.
$$

From the first equation of (2.1), we notice that $\frac{ds(t)}{dt} \leq \lambda - d_1 s(t)$. So we consider the following impulsive differential equations:

$$\begin{cases} \frac{dy(t)}{dt} = \lambda - d_1 y(t), & t \neq n\tau, t \neq (n+l)\tau, \\ \Delta y(t) = \mu, & t = n\tau, \\ \Delta y(t) = -\mu_2 y(t), & t = (n+l)\tau. \end{cases} \tag{4.2}$$

From Lemma 2 and the comparison theorem of impulsive differential equations (see Theorem 3.1.1 in [11]), we have $s(t) \leq y(t)$, and, $y(t) \to \tilde{s}(t)$, as $t \to \infty$, that is,

$$s(t) \leq y(t) \leq \tilde{s}(t) + \varepsilon \tag{4.3}$$

for all t large enough. For convenience, we may assume (4.3) holds for all $t > 0$. From (2.1) and (4.3), we get

$$\begin{cases} \frac{dx(t)}{dt} \leq \frac{\beta}{\delta}(\tilde{s}(t) + \varepsilon)x(t) - d_2 x(t), & t \neq n\tau, t \neq (n+l)\tau, \\ \Delta x(t) = 0, & t = n\tau, \\ \Delta x(t) = -\mu_2 x(t), & t = (n+l)\tau. \end{cases} \tag{4.4}$$

So

$$x\big((n+1)\tau\big) \leq x\big(n\tau^+\big)(1-\mu_2)\exp\left(\int_{n\tau}^{(n+1)\tau}\left[\frac{\beta}{\delta}(\tilde{s}(s)-\varepsilon)-d_2\right]ds\right).$$

Hence

$$x\big((n+l)\tau\big) \leq x\big(0^+\big)\rho^n$$

and $x((n+l)\tau) \to 0$, as $t \to \infty$, therefore $x(t) \to 0$, as $t \to \infty$.

In what follows, we prove that $s(t) \to \tilde{s}(t)$, as $t \to \infty$, let $0 < \varepsilon \leq d_1$, there must exist $t_0 > 0$ such that $t \geq t_0$, $0 < x(t) < \varepsilon$. Without loss of generality, we assume that $0 < x(t) < \varepsilon$ for all $t > 0$, then, for system (2.1), we have

$$\lambda - (d_1 + \varepsilon)s(t) \leq \frac{ds(t)}{dt} \leq \lambda - d_1 s(t). \tag{4.5}$$

Then we have $z_1(t) \leq s(t) \leq z_2(t)$ and $z_1(t) \to \tilde{s}(t)$, $z_2(t) \to \tilde{s}(t)$ as $t \to \infty$. While $z_1(t)$ and $z_2(t)$ are the solutions of

$$\begin{cases} \frac{dz_1(t)}{dt} = \lambda - (d_1 + \varepsilon)z_1(t), & t \neq n\tau, t \neq (n+l)\tau, \\ \Delta z_1(t) = \mu, & t = n\tau, \\ \Delta z_1(t) = -\mu_1 z_1(t), & t = (n+l)\tau \end{cases} \tag{4.6}$$

and

$$\begin{cases} \frac{dz_2(t)}{dt} = \lambda - d_1 z_2(t), & t \neq n\tau, t \neq (n+l)\tau, \\ \Delta z_2(t) = \mu, & t = n\tau, \\ \Delta z_2(t) = -\mu_1 z_2(t), & t = (n+l)\tau, \end{cases} \tag{4.7}$$

respectively

$$
z_1(t) = \begin{cases} \frac{1}{d_1+\varepsilon}[\lambda - (\lambda - (d_1 + \varepsilon)z_1^*)e^{-(d_1+\varepsilon)(t-n\tau)}], & t \in [n\tau, (n+l)\tau), \\ \frac{1}{d_1+\varepsilon}[\lambda - (\lambda - (d_1 + \varepsilon)z_1^{**})e^{-(d_1+\varepsilon)(t-(n+l)\tau)}], & t \in [(n+l)\tau, (n+1)\tau). \end{cases} \tag{4.8}
$$

Here z_1^* and z_1^{**} are defined as

$$
z_1^* = \frac{\lambda - \lambda\mu_1 e^{-(d_1+\varepsilon)(1-l)\tau} - \lambda(1-\mu_1)e^{-(d_1+\varepsilon)\tau} + (d_1 + \varepsilon)\mu}{(d_1 + \varepsilon)[1 - (1-\mu_1)e^{-(d_1+\varepsilon)\tau}]}
$$

and

$$
z_1^{**} = \frac{\lambda - \mu_1}{(d_1 + \varepsilon)}\left[\lambda - \left(\lambda - (d_1 + \varepsilon)z_1^*\right)e^{-(d_1+\varepsilon)l\tau}\right].
$$

Therefore, for any $\varepsilon_1 > 0$, there exists t_1, $t > t_1$, such that

$$
z_1(t) - \varepsilon_1 < s(t) < \tilde{s}(t) + \varepsilon_1.
$$

Let $\varepsilon \to 0$, so we have

$$
\tilde{s}(t) - \varepsilon_1 < s(t) < \tilde{s}(t) + \varepsilon_1
$$

for t large enough, which implies $s(t) \to \tilde{s}(t)$ as $t \to \infty$. This completes the proof. □

The following work is to investigate the permanence of system (2.1). Before starting this work, we should give the following definition.

Definition 1 System (2.1) is said to be permanent if there are constants $m, M > 0$ (independent of the initial value) and a finite time T_0 such that for all solutions $(s(t), x(t))$ with all initial values $s(0^+) > 0$, $x(0^+) > 0$, $m \le s(t) \le M$, $m \le x(t) \le M$ hold for all $t \ge T_0$. Here T_0 may depend on the initial values $(s(0^+), x(0^+))$.

Theorem 2 *Let $(s(t), x(t))$ be any solution of system* (2.1). *If*

$$
\ln\frac{1}{1-\mu_2} < \frac{\lambda\tau}{d_1} - \frac{\lambda - d_1 s^*}{d_1^2}\left(1 - e^{-d_1 l\tau}\right) - \frac{\lambda - d_1 s^{**}}{d_1^2}\left(1 - e^{-d_1(1-l)\tau}\right)e^{-d_1\tau} - d_2\tau
$$

holds, then system (2.1) *is permanent.*

Proof Let $(s(t), x(t))$ be a solution of (2.1) with $s(0) > 0$, $x(0) > 0$. By Lemma 3, we have proved there exists a constant of $M > 0$ such that $s(t) \le M$, $x(t) \le M$ for t large enough. We may assume $s(t) \le M$, $x(t) \le M$ for $t \ge 0$. Thus we only need to find $m_1 > 0$ and ε_3 such that $x(t) \ge m_1$ for t large enough. Otherwise, we can select $m_3 > 0$ small enough and prove $x(t) < m_3$ does not hold for $t \ge 0$. By the condition of Theorem 2, let

$$
\sigma = \int_{n\tau}^{(n+1)\tau}\left(\frac{\beta}{\delta}\left(\tilde{z}(t) - \varepsilon_1\right) - d_2\right)dt > 0.
$$

We will prove that $x(t) < m_3$ cannot hold for $t \geq 0$. Otherwise,

$$
\begin{cases}
\frac{ds(t)}{dt} \geq \lambda - (d_1 + \beta m_3)s(t), & t \neq n\tau, t \neq (n+l)\tau, \\
\Delta s(t) = \mu, & t = n\tau, \\
\Delta s(t) = -\mu_1 s(t), & t = (n+l)\tau.
\end{cases} \tag{4.9}
$$

By Lemma 3, we have $s(t) \geq z(t)$ and $z(t) \to \tilde{z}(t)$, $t \to \infty$, where $z(t)$ is the solution of

$$
\begin{cases}
\frac{dz(t)}{dt} = \lambda - (d_1 + \beta m_3)z(t), & t \neq n\tau, t \neq (n+l)\tau, \\
\Delta z(t) = \mu, & t = n\tau, \\
\Delta s(t) = -\mu_1 z(t), & t = (n+l)\tau
\end{cases} \tag{4.10}
$$

and

$$
\tilde{z}(t) =
\begin{cases}
\frac{1}{(d_1 + \beta m_3)}[\lambda - (\lambda - (d_1 + \beta m_3)z^*)e^{-(d_1 + \beta m_3)(t-n\tau)}], & t \in [n\tau, (n+l)\tau), \\
\frac{1}{(d_1 + \beta m_3)}[\lambda - (\lambda - (d_1 + \beta m_3)z^{**})e^{-(d_1 + \beta m_3)(t-(n+l)\tau)}], & t \in [(n+l)\tau, (n+1)\tau).
\end{cases} \tag{4.11}
$$

Here z^* and z^{**} are defined as

$$
z^* = \frac{\lambda(1 - \mu_1 e^{-(d_1 + \beta m_3)(1-l)\tau}) - \lambda(1 - \mu_1)e^{-(d_1 + \beta m_3)\tau} + (d_1 + \beta m_3)\mu}{(d_1 + \beta m_3)[1 - (1 - \mu_1)e^{-(d_1 + \beta m_3)\tau}]}
$$

and

$$
z^{**} = \frac{1 - \mu_1}{(d_1 + \beta m_3)}\left[\lambda - (\lambda - (d_1 + \beta m_3)z^*)e^{-(d_1 + \beta m_3)l\tau}\right].
$$

Therefore, there exists $T_1 > 0$ such that

$$
s(t) \geq z(t) \geq \tilde{z}(t) - \varepsilon_1,
$$

and

$$
\begin{cases}
\frac{dx(t)}{dt} \geq (\frac{\beta}{\delta}(\tilde{z}(t) - \varepsilon_1) - d_2)x(t), & t \neq n\tau, t \neq (n+l)\tau, \\
\Delta x(t) = 0, & t = n\tau, \\
\Delta x(t) = -\mu_2 x(t), & t = (n+l)\tau
\end{cases} \tag{4.12}
$$

for $t \geq T_1$. Let $N_1 \in N$ and $N_1\tau > T$. Integrating (4.12) on $(n\tau, (n+1)\tau]$ $(n \geq N_1)$, we have

$$
x((n+1)\tau) \geq (1 - \mu_2)x(n\tau^+)\exp\left(\int_{n\tau}^{(n+1)\tau} \left(\frac{\beta}{\delta}(\tilde{z}(t) - \varepsilon_1) - d_2\right)dt\right)
$$

$$
= (1 - \mu_2)x(n\tau^+)e^\sigma.
$$

Then $x((N_1 + K)\tau) \geq (1 - \mu_2)^K x(N_1\tau^+)e^{K\sigma} \to \infty$, as $K \to \infty$, which is a contradiction to the boundedness of $x(t)$. Hence there exists $t_1 > 0$ such that $x(t) \geq m_3$.

Thus, we can also obtain that $\frac{ds(t)}{dt} \geq \lambda - (d_1 + \beta M)s(t)$, then the following comparatively impulsive differential equation is

$$
\begin{cases}
\frac{dz_3(t)}{dt} = \lambda - (d_1 + \beta M)z_3(t), & t \neq n\tau, t \neq (n+l)\tau, \\
\Delta z_3(t) = \mu, & t = n\tau, \\
\Delta z_3(t) = -\mu_2 z_3(t), & t = (n+l)\tau.
\end{cases}
\tag{4.13}
$$

Similar to (3.7), we have

$$
\tilde{z}_3(t) =
\begin{cases}
\frac{1}{(d_1+\beta M)}[\lambda - (\lambda - (d_1 + \beta M)z_3^*)e^{-(d_1+\beta M)(t-n\tau)}], & t \in [n\tau, (n+l)\tau), \\
\frac{1}{(d_1+\beta M)}[\lambda - (\lambda - (d_1 + \beta M)z_3^{**})e^{-(d_1+\beta M)(t-(n+l)\tau)}], & t \in [(n+l)\tau, (n+1)\tau).
\end{cases}
\tag{4.14}
$$

Here z_3^* and z_3^{**} are defined as

$$
z_3^* = \frac{\lambda(1 - \mu_1 e^{-(d_1+\beta m_3)(1-l)\tau}) - \lambda(1 - \mu_1)e^{-(d_1+\beta m_3)\tau} + (d_1 + \beta m_3)\mu}{(d_1 + \beta m_3)[1 - (1 - \mu_1)e^{-(d_1+\beta m_3)\tau}]}
$$

and

$$
z_3^{**} = \frac{1 - \mu_1}{(d_1 + \beta m_3)}\big[\lambda - \big(\lambda - (d_1 + \beta m_3)z^*\big)e^{-(d_1+\beta m_3)l\tau}\big].
$$

For any ε_4 small enough, we obtain

$$
z_3(t) > \tilde{z}_3(t) - \varepsilon_4.
$$

From the comparison theorem of impulsive differential equations, we have

$$
s(t) > z_3(t) > \tilde{z}_3(t) - \varepsilon_4 > \big(z_3^*(t) + z_3^{**}(t)\big) - \varepsilon_4 = m_4,
$$

i.e., $s(t) > m_4$. This completes the proof. □

Remark Let

$$
F(\tau) = \ln\frac{1}{1 - \mu_2} - \frac{\lambda\tau}{d_1} - \frac{\lambda - d_1 s^*}{d_1^2}\big(1 - e^{-d_1 l\tau}\big) - \frac{\lambda - d_1 s^{**}}{d_1^2}\big(1 - e^{-d_1(1-l)\tau}\big)e^{-d_1\tau} - d_2\tau.
$$

We easily know $F(0) = \frac{1}{\ln(1-\mu_2)} > 0$, $F(\tau) \to -\infty$ as $\tau \to \infty$, and $F''(\tau) > 0$, so $F(\tau) = 0$ has a unique positive root, denoted by $\tau_{\max}$. From Theorems 1 and 2, we know $\tau_{\max}$ is a threshold. If $\tau < \tau_{\max}$, the microorganism-extinction periodic solution $(s(t), 0)$ of (2.1) is globally asymptotically stable. If $\tau < \tau_{\max}$, system (2.1) is permanent.

5 Discussion

In this paper, according to the actual situation, we discussed the impulsive dredging at a fixed time. The global asymptotic stability of the microorganism-extinction periodic solution $(s(t), 0)$ of (2.1) is discussed and the condition for the permanence of the system is also obtained. From Theorems 1 and 2, we can deduce that there must exist a threshold

μ_2^*. If $\mu_2 > \mu_2^*$, the microorganism-extinction periodic solution $(s(t), 0)$ of (2.1) is globally asymptotically stable. If $\mu_2 < \mu_2^*$, system (2.1) is permanent. Our results indicate that the impulsive dredging and the period of impulsive dredging play an important role in water eutrophication management.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JJ corrected the manuscript, PL carried out the main part of this article, HF gave some suggestions for this article. All authors have read and approved the final manuscript.

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