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Periodic solutions to a coupled two-dimensional lattice presented by Blaszak and Szum with Riemann–theta function

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Abstract
A coupled two-dimensional lattice presented by Blaszak and Szum is studied with the aid of Riemann–theta function and the bilinear method. By utilizing a bilinear form of the equation, we have obtained one-periodic and two-periodic solutions. In order to analyze the solution, we study asymptotic behavior and draw the solution plots.

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1 Introduction

The subject of the discrete system has been attracting interest [1–3]. Especially, Toda lattice equations have been discussed by many researchers. In [4], Dai was dedicated to the study of integrable variable-coefficient Toda lattice by using the dressing method. Nakamura in [5] discussed the 3 + 1-dimensional Toda equation and derived the solutions by using the Bessel functions. The authors in [6] obtained the solutions of 2 + 1-dimensional Toda lattice by using Darboux transformation. Tian and Hu discussed semi-discrete KP and BKP equations by utilizing nonlocal symmetries in [7]. By using the Hirota bilinear method with the help of Riemann–theta function, Nakamura [8, 9] studied some famous equations such as KdV, Boussinesq, Toda, etc., and Dai et al. demonstrated for KP equation and Toda lattice [10, 11]. Recently, a lot of researchers have been concerned with the method [12–15]. However, the coupled discrete system and high-dimensional equations have less been studied in the previous literature.

In this paper, we consider the two-dimensional lattice presented by Blaszak and Szum in [16]:

$$\begin{aligned} &u_t(n) = u(n)\big(w(n) - w(n-1)\big),\\ &v_t(n) = u(n+1) - u(n) + w_y(n),\\ &w_t(n+1) + w_t(n) = v(n+1) - v(n) - w^2(n+1) + w^2(n), \end{aligned} \tag{1.1}$$

which is a coupled discrete system. Tam and Hu in [17] discussed its bilinear forms and its solutions. Yu et al. [18] derived its pfaffianization and molecule solutions. We will obtain one-periodic solution and two-periodic solution by utilizing the bilinear method and the Riemann–theta function presented in [8–10].

The paper is organized as follows. In Sect. 2, we obtain one-periodic wave solution and study its asymptotic behavior. The solution is also studied graphically. In Sect. 3, we obtain two-periodic wave solutions whose asymptotic behaviors are studied and the plots are given.

2 One-periodic solution and its asymptotic behavior

In the section, we study one-periodic solution of (1.1). Through the transformation [17],

$$\begin{aligned} &u(n) = \frac{f(n+1)f(n-1)}{f^2(n)}, \qquad v(n) = \frac{D_t^2 f(n)\cdot f(n+1)}{f(n)f(n+1)}, \\ &w(n) = \left(\ln\frac{f(n+1)}{f(n)}\right)_t, \end{aligned} \tag{2.1}$$

then (1.1) can be written as

$$\left(D_z e^{\frac{1}{2}D_n} - D_t^2 e^{\frac{1}{2}D_n} + c_1\right) f(n)\cdot f(n) = 0, \tag{2.2}$$

$$(D_t D_z - D_t D_y - 2\cosh D_n + 2 + c_2) f(n)\cdot f(n) = 0, \tag{2.3}$$

where z is an auxiliary variable, c_1 and c_2 are integration constants. The Hirota bilinear differential operator is defined as [19]

$$D_x^m D_y^n a(x,y)\cdot b(x,y) \equiv (\partial_x - \partial_{x'})^m (\partial_y - \partial_{y'})^n a(x,y)\times b\left(x',y'\right)\big|x'=x, y'=y$$

and the difference operator is defined as

$$\begin{aligned} &e^{D_n} a_n\cdot b_n = a_{n+1}b_{n-1}, \qquad e^{-D_n} a_n\cdot b_n = a_{n-1}b_{n+1}, \\ &\cosh D_n a_n\cdot b_n = \frac{1}{2}(a_{n+1}b_{n-1} + a_{n-1}b_{n+1}). \end{aligned}$$

From the definition of Hirota bilinear operator, we have

$$D_x^m D_y^l e^{\zeta_1}\cdot e^{\zeta_2} = (l_1 - l_2)^m (\rho_1 - \rho_2)^l e^{\zeta_1+\zeta_2},$$

where $\zeta_j = l_j x + \rho_j y + \eta_j n + \zeta_{j0}$ $(j = 1,2)$. Moreover, it is easy to deduce

$$\cosh D_n e^{\zeta_1}\cdot e^{\zeta_2} = \cosh(\eta_1 - \eta_2) e^{\zeta_1+\zeta_2}, \tag{2.4}$$

$$G(D_x, D_y, \cosh D_n) e^{\zeta_1}\cdot e^{\zeta_2} = G(l_1 - l_2, \rho_1 - \rho_2, \eta_1 - \eta_2) e^{\zeta_1+\zeta_2}. \tag{2.5}$$

2.1 One-periodic wave solution

In view of [8, 9], we consider the Riemann–theta function solution of the bilinear form (2.2) and (2.3)

$$f = \sum_{k\in Z^N} e^{\pi i\langle \tau k,k\rangle + 2\pi i\langle \zeta,k\rangle}, \tag{2.6}$$

where $\langle\cdot,\cdot\rangle$ is the inner product, $k = (k_1,\dots,k_N)^T$, $\zeta = (\zeta_1,\dots,\zeta_N)^T$ and τ is a symmetric matrix, $\zeta_j = p_jt + l_jy + \mu_jz + \eta_jn + \zeta_{0j}$ $(j = 1,\dots,N)$. In order to obtain one-periodic wave solution, we consider the case for $N = 1$, and we denote $k = k_1$, $\zeta = \zeta_1$, $\zeta_0 = \zeta_{01}$. The direct calculations show that $\pi i\langle\tau k,k\rangle = \pi ik^2\tau$, $2\pi i\langle\zeta,k\rangle = 2\pi ik\zeta$. Thus (2.6) becomes

$$f = \sum_{k=-\infty}^{\infty} e^{2\pi ik\zeta+\pi ik^2\tau}. \tag{2.7}$$

Inserting (2.7) into (2.2) and using the bilinear properties, we have

$$\begin{aligned}
&F_1\bigl(D_z,D_t,e^{\frac{1}{2}D_n}\bigr)f(n)\cdot f(n)\\
&\quad\equiv \bigl(D_ze^{\frac{1}{2}D_n} - D_t^2e^{\frac{1}{2}D_n} + c_1\bigr)f(n)\cdot f(n)\\
&\quad= \sum_{k,k'=-\infty}^{\infty} F_1\bigl(D_z,D_t,e^{\frac{1}{2}D_n}\bigr)\exp\bigl(2\pi ik\zeta + \pi ik^2\tau\bigr)\cdot\exp\bigl(2\pi ik'\zeta + \pi ik'^2\tau\bigr)\\
&\quad= \sum_{k,m=-\infty}^{\infty} F_1\bigl(D_z,D_t,e^{\frac{1}{2}D_n}\bigr)\exp\bigl(2\pi ik\zeta + \pi ik^2\tau\bigr)\cdot\exp\bigl(2\pi i(m-k)\zeta + \pi i(m-k)^2\tau\bigr)\\
&\quad= \sum_{k,m=-\infty}^{\infty} F_1\bigl(2\pi i(2k-m)\mu, 2\pi i(2k-m)p, e^{\pi i(2k-m)\eta}\bigr)\\
&\qquad\times\exp\bigl(2\pi im\zeta + \pi i\bigl[k^2 + (k-m)^2\bigr]\tau\bigr)\\
&\quad= \sum_{m=-\infty}^{\infty} \tilde{F}_1(m)\exp(2\pi im\zeta) = 0,
\end{aligned}$$

where the new summation index $m = k + k'$ has been introduced and $\tilde{F}_1(m)$ is defined by

$$\tilde{F}_1(m) = \sum_{k=-\infty}^{\infty} F_1\bigl(2\pi i(2k-m)\mu, 2\pi i(2k-m)p, e^{\pi i(2k-m)\eta}\bigr)e^{\pi i[k^2+(k-m)^2]\tau}. \tag{2.8}$$

Thus,

$$\tilde{F}_1(0) = \sum_{k=-\infty}^{\infty}\bigl(4\pi ik\mu e^{2\pi ik\eta} - 16\pi^2k^2p^2e^{2\pi ik\eta} + c_1\bigr)e^{2\pi ik^2\tau} = 0, \tag{2.9}$$

$$\begin{aligned}
\tilde{F}_1(1) &= \sum_{k=-\infty}^{\infty}\bigl[2\pi i(2k-1)\mu e^{\pi i(2k-1)\eta} - 4\pi^2(2k-1)^2p^2e^{\pi i(2k-1)\eta} + c_1\bigr]e^{\pi i[k^2+(k-1)^2]\tau}\\
&= 0.
\end{aligned} \tag{2.10}$$

We denote

$$\begin{aligned}
&d_1 = \exp 2\pi ik^2\tau, \qquad d_2 = \exp\pi i\bigl[k^2 + (k-1)^2\bigr]\tau,\\
&\Delta_1 = \sum_{k=-\infty}^{\infty} d_1, \qquad \Delta_2 = \sum_{k=-\infty}^{\infty} d_2,\\
&a_{11} = \sum_{k=-\infty}^{\infty} 4\pi ikd_1\exp 2\pi ik\eta, \qquad a_{21} = \sum_{k=-\infty}^{\infty} 2\pi i(2k-1)d_2\exp\pi i(2k-1)\eta.
\end{aligned}$$

Then (2.9) and (2.10) are written as

$$\mu a_{11} + 2a_{11,\eta}p^2 + c_1\Delta_1 = 0, \qquad \mu a_{21} + 2a_{21,\eta}p^2 + c_1\Delta_2 = 0,$$

from which we have

$$\mu = 2p^2\frac{a_{21,\eta}\Delta_1 - a_{11,\eta}\Delta_2}{a_{11}\Delta_2 - a_{21}\Delta_1}, \qquad c_1 = 2p^2\frac{a_{11,\eta}a_{21} - a_{21,\eta}a_{11}}{a_{11}\Delta_2 - a_{21}\Delta_1}. \tag{2.11}$$

Similarly, substituting (2.7) into (2.3), we derive

$$\begin{aligned}
&F_2(D_z, D_t, D_y, \cosh D_n)f(n)\cdot f(n)\\
&\quad\equiv (D_tD_z - D_tD_y - 2\cosh D_n + 2 + c_2)f(n)\cdot f(n)\\
&\quad= \sum_{k,k'=-\infty}^{\infty} F_2(D_z, D_t, D_y, \cosh D_n)\exp\bigl(2\pi ik\zeta + \pi ik^2\tau\bigr)\cdot\exp\bigl(2\pi ik'\zeta + \pi ik'^2\tau\bigr)\\
&\quad= \sum_{k,m=-\infty}^{\infty} F_2\bigl(2\pi i(2k-m)\mu, 2\pi i(2k-m)p, 2\pi i(2k-m)l, \cosh 2\pi i(2k-m)\eta\bigr)\\
&\qquad\times\exp\bigl(2\pi im\zeta + \pi i\bigl[k^2 + (k-m)^2\bigr]\tau\bigr)\\
&\quad= \sum_{m=-\infty}^{\infty}\tilde{F}_2(m)\exp(2\pi im\zeta) = 0,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{F}_2(m) = {}&\sum_{k=-\infty}^{\infty} F_2\bigl[2\pi i(2k-m)\mu, 2\pi i(2k-m)p, 2\pi i(2k-m)l, \cosh 2\pi i(2k-m)\eta\bigr]\\
&\times\exp\pi i\bigl[k^2 + (k-m)^2\bigr]\tau.
\end{aligned}$$

It is easy to know that if $\tilde{F}_2(0) = 0$, $\tilde{F}_2(1) = 0$, then all $\tilde{F}_2(m) = 0$ are proved.

$$\begin{aligned}
\tilde{F}_2(0) &= \sum_{k=-\infty}^{\infty}\bigl(-16\pi^2k^2p\mu + 16\pi^2k^2pl - 2\cosh 4\pi ik\eta + 2 + c_2\bigr)e^{2\pi ik^2\tau} = 0,\\
\tilde{F}_2(1) &= \sum_{k=-\infty}^{\infty}\bigl(4\pi^2(2k-1)^2(-p\mu + pl) - 2\cosh 2\pi i(2k-1)\eta + 2 + c_2\bigr)\\
&\quad\times e^{\pi i[k^2+(k-1)^2]\tau}\\
&\quad= 0.
\end{aligned}\tag{2.12}$$

Letting $b_{11} = \sum_{k=-\infty}^{\infty}16\pi^2k^2d_1$, $b_{12} = \sum_{k=-\infty}^{\infty}\cosh(4\pi ik\eta)d_1$, $b_{21} = \sum_{k=-\infty}^{\infty}4\pi^2(2k-1)^2d_2$, $b_{22} = \sum_{k=-\infty}^{\infty}\cosh 2\pi i(2k-1)\eta d_2$, thus, (2.12) can be written as

$$\begin{aligned}
&p(l-\mu)b_{11} - 2b_{12} + (2 + c_2)\Delta_1 = 0,\\
&p(l-\mu)b_{21} - 2b_{22} + (2 + c_2)\Delta_2 = 0.
\end{aligned}$$

Solving the above system, we have

$$l = \mu + \frac{2}{p}\frac{b_{22}\Delta_1 - b_{12}\Delta_2}{b_{21}\Delta_1 - b_{11}\Delta_2}, \qquad 2 + c_2 = 2\frac{b_{12}b_{21} - b_{22}b_{11}}{b_{21}\Delta_1 - b_{11}\Delta_2}, \tag{2.13}$$

from which we find that parameter l is dependent on μ, η, and p. In view of (2.11), we can see that μ is dependent on η and p.

Then we have derived the Riemann–theta function solution $f(n)$ of (2.2) and (2.3). Furthermore, the Riemann–theta function periodic solutions of (1.1) are obtained by using transformation (2.1).

2.2 Asymptotic behavior of the one-periodic wave solution

In what follows, we will prove that the soliton solution can be regarded as the limit of the following periodic solution. Therefore, we write $q = \exp \pi i\tau$ and take a limit $q \to 0$ (or $\operatorname{Im}\tau \to \infty$).

Theorem 1 *Under the condition $q \to 0$ (or $\operatorname{Im}\tau \to \infty$), the Riemann–theta function periodic solution* (2.7) *of* (2.2) *and* (2.3) *tends to the one-soliton solutions of* (1.1) *via* (2.1).

$$\begin{aligned}
u(n) &= \frac{(1 + e^{\widetilde{\zeta}+\widetilde{\eta}})(1 + e^{\widetilde{\zeta}-\widetilde{\eta}})}{(1 + e^{\widetilde{\zeta}})^2}, \\
v(n) &= -2\pi^2 p^2 e^{\frac{\widetilde{\zeta}}{2}} \operatorname{sech}\frac{\widetilde{\zeta}}{2}\frac{1 + e^{\widetilde{\eta}}}{1 + e^{\widetilde{\zeta}+\widetilde{\eta}}}, \\
w(n) &= 2\pi i p\frac{e^{\widetilde{\zeta}}(e^{\widetilde{\eta}} - 1)}{(1 + e^{\widetilde{\zeta}})(1 + e^{\widetilde{\zeta}+\widetilde{\eta}})},
\end{aligned} \tag{2.14}$$

where

$$\widetilde{\zeta} = 2\pi i(pt + ly + \mu z + \eta n) + \widetilde{\zeta}_0, \qquad \widetilde{\eta} = 2\pi i\eta, \qquad \widetilde{\zeta}_0 = \zeta_0 + \frac{1}{2}\tau,$$

$$\mu \to -2\pi p^2 \cot \pi\eta, \qquad l \to \mu + \frac{\cos^2 \pi\eta}{p\pi^2}, \qquad c_1 \to 0, \qquad c_2 \to 0.$$

Proof Utilizing $q = \exp \pi i\tau$, the quantities defined above are then expanded in powers of q

$$\Delta_1 = \sum_{k=-\infty}^{\infty} e^{2\pi i k^2\tau} = 1 + 2q^2 + o(q^2), \qquad \Delta_2 = \sum_{k=-\infty}^{\infty} e^{2\pi i[k^2+(k-1)^2]\tau} = 2q^2 + o(q^2),$$

$$a_{11} = \sum_{k=-\infty}^{\infty} 4\pi i k e^{2\pi i k\eta} e^{2\pi i k^2\tau} = -8\pi \sin(2\pi\eta)q^2 + o(q^2),$$

$$a_{11,\eta} = \sum_{k=-\infty}^{\infty} -8\pi^2 k^2 e^{2\pi i k\eta} e^{2\pi i k^2\tau} = -16\pi^2 \cos(2\pi\eta)q^2 + o(q^2),$$

$$a_{21} = \sum_{k=-\infty}^{\infty} 2\pi i(2k-1)e^{\pi i(2k-1)\eta} e^{\pi i[k^2+(k-1)^2]\tau} = -4\pi \sin(\pi\eta)q + o(q^5),$$

$$a_{21,\eta} = \sum_{k=-\infty}^{\infty} -2\pi^2(2k-1)^2 e^{\pi i(2k-1)\eta} e^{\pi i[k^2+(k-1)^2]\tau} = -4\pi^2 \cos(\pi\eta)q + o(q^5),$$

$$b_{11} = \sum_{k=-\infty}^{\infty} 16\pi^2 k^2 e^{2\pi i k^2 \tau} = 32\pi^2 q^2 + o(q^8),$$

$$b_{12} = \sum_{k=-\infty}^{\infty} \cosh(4\pi i k\eta) e^{2\pi i k^2 \tau} = 1 + 2\cos(4\pi\eta) q^2 + o(q^8),$$

$$b_{21} = \sum_{k=-\infty}^{\infty} 4\pi^2 (2k-1)^2 e^{\pi i [k^2 + (k-1)^2]\tau} = 8\pi^2 q + o(q^5),$$

$$b_{22} = \sum_{k=-\infty}^{\infty} \cosh 2\pi i(2k-1)\eta e^{\pi i [k^2 + (k-1)^2]\tau} = 2\cos(2\pi\eta) q + o(q^5).$$

Using (2.11) and (2.13), we have $\mu \to -2\pi p^2 \cot \pi\eta$, $l \to \mu + \frac{\cos 2\pi\eta}{2p\pi^2}$, $c_1 \to 0$, $c_2 \to 0$ for $q \to 0$.

In order to consider the convergence to the one-periodic wave solution (2.7) in the limit of $q \to 0$, under the transformation $\zeta_0 = \widetilde{\zeta}_0 - \frac{1}{2}\tau$, we can get the following convergent forms:

$$\begin{aligned}
&f(n) = 1 + \exp\widetilde{\zeta} + o(q^2),\\
&f(n-1) = 1 + \exp(\widetilde{\zeta} - \widetilde{\eta}) + o(q^2),\\
&f(n+1) = 1 + \exp(\widetilde{\zeta} + \widetilde{\eta}) + o(q^2),\\
&f_t(n) = 2\pi i p \exp\widetilde{\zeta} + o(q^2),\\
&f_{tt}(n) = -4\pi^2 p^2 \exp\widetilde{\zeta} + o(q^2),\\
&f_t(n+1) = 2\pi i p \exp(\widetilde{\zeta} + \widetilde{\eta}) + o(q^2),\\
&f_{tt}(n+1) = -4\pi^2 p^2 \exp(\widetilde{\zeta} + \widetilde{\eta}) + o(q^2).
\end{aligned} \tag{2.15}$$

After some tedious calculations, we derive (2.14). □

In what follows, Fig. 1, Fig. 2, and Fig. 3 describe the plots of $u(t,y,n)$, $v(t,y,n)$, and $w(t,y,n)$, respectively. From these figures, we find that the plots of $v(t,y,n)$ and $w(t,y,n)$ have similar forms.

3 Two-periodic wave solution and its asymptotic behavior

In what follows, similar to the one-periodic wave solution, we consider a two-periodic wave solution of the coupled two-dimensional lattice (1.1).

3.1 Construction of two-periodic wave solution

By letting $N = 2$ in (2.6), we have $f(n) = \sum_{k\in Z^2} e^{2\pi i\langle\zeta,k\rangle + \pi i\langle\tau k,k\rangle}$ and substitute it into (2.2). For convenience of calculations, we have introduced different forms of k and k'. Thus, we obtain

$$\begin{aligned}
F_1 f_n \cdot f_n &= \sum_{k,k'\in Z^2} F_1\bigl(D_z, D_t, e^{\frac{1}{2}D_n}\bigr) e^{2\pi i\langle\zeta,k\rangle + \pi i\langle\tau k,k\rangle} \cdot e^{2\pi i\langle\zeta,k'\rangle + \pi i\langle\tau k',k'\rangle}\\
&= \sum_{k,k'\in Z^2} F_1\bigl(2\pi i\langle k-k',\mu\rangle, 2\pi i\langle k-k',p\rangle, e^{2\pi i\langle k-k',\eta\rangle}\bigr)
\end{aligned}$$

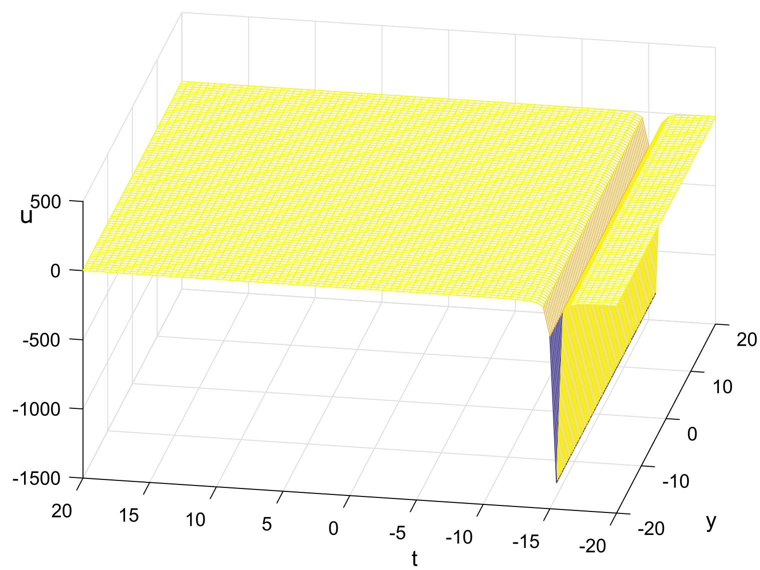

Figure 1 One-periodic solution plot of $u(t,y,n)$ for $z=0.6$, $n=2$, $\eta=0.1$, $\tilde{\zeta}_0=2$, $l=4$, $p=0.8$, $\tau=0.8i$, $t\in[-15,15]$, $y\in[-20,20]$

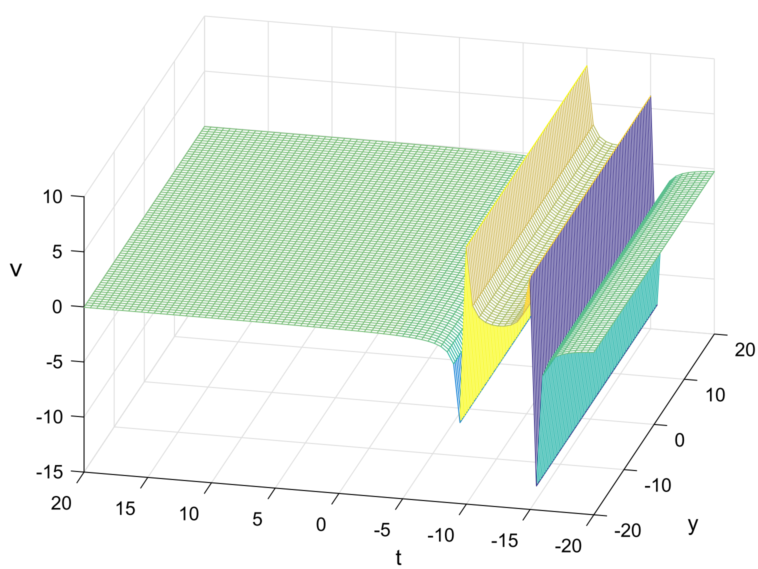

Figure 2 One-periodic solution plot of $v(t,y,n)$ for $z=0.6$, $n=2$, $\eta=0.1$, $\tilde{\zeta}_0=2$, $l=4$, $p=0.8$, $\tau=0.8i$, $t\in[-15,15]$, $y\in[-20,20]$

$$\begin{aligned}
&\times \exp\bigl(2\pi i\langle\zeta,k+k'\rangle\bigr)\exp\pi i\bigl(\langle\tau k',k'\rangle+\langle\tau k,k\rangle\bigr)\\
&=\sum_{s'\in Z^2}\sum_{k_1,k_2=-\infty}^{\infty}F_1\bigl(2\pi i\langle 2k-s',\mu\rangle,2\pi i\langle 2k-s',p\rangle,e^{2\pi i\langle 2k-s',\eta\rangle}\bigr)\\
&\quad\times\exp\pi i\bigl(\langle\tau(k-s'),k-s'\rangle+\langle\tau k,k\rangle\bigr)\exp\bigl(2\pi i\langle\zeta,s'\rangle\bigr)\\
&\equiv\sum_{s'\in Z^2}\tilde{F}_1(s_1',s_2')\exp\bigl(2\pi i\langle\zeta,s'\rangle\bigr)=0.
\end{aligned}\tag{3.1}$$

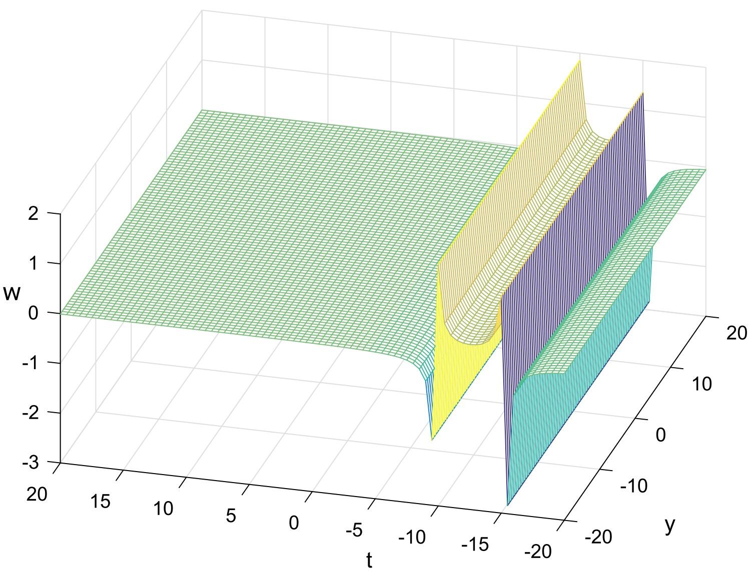

Figure 3 One-periodic solution plot of $w(t,y,n)$ for $z = 0.6$, $n = 2$, $\eta = 0.1$, $\tilde{\zeta}_0 = 2$, $l = 4$, $p = 0.8$, $\tau = 0.8i$, $t \in [-15,15]$, $y \in [-20,20]$

By introducing the new summation index $k + k' = s'$, $s' = (s'_1, s'_2)^T$, $k = (k_1,k_2)^T$, $\tilde{F}_1(s'_1,s'_2)$ is denoted by

$$\begin{aligned}
\tilde{F}_1(s'_1,s'_2) &= \sum_{k_1,k_2=-\infty}^{\infty} F_1\bigl[2\pi i\langle 2k-s',\mu\rangle, 2\pi i\langle 2k-s',p\rangle, e^{2\pi i\langle 2k-s',\eta\rangle}\bigr] \\
&\quad \times \exp\pi i\bigl(\bigl\langle\tau(k-s'),k-s'\bigr\rangle + \langle\tau k,k\rangle\bigr) \\
&= \sum_{k_j=-\infty}^{\infty} F_1\Biggl(2\pi i\sum_{j=1}^{2}\bigl(2k_j-(s'_j-2\delta_{jl})\bigr)\mu_j, 2\pi i\sum_{j=1}^{2}\bigl(2k_j-(s'_j-2\delta_{jl})\bigr)p_j, \\
&\quad 2\pi i\sum_{j=1}^{2}\bigl(2k_j-(s'_j-2\delta_{jl})\bigr)\eta_j\Biggr) \\
&\quad \times \exp\pi i\sum_{j,l=1}^{2}\bigl[(k_j+\delta_{jl})\tau_{jl}(k_j+\delta_{jl}) \\
&\quad + \bigl((s'_j-2\delta_{jl}-k_j)+\delta_{jl}\bigr)\tau_{jl}\bigl((s'_j-2\delta_{jl}-k_j)+\delta_{jl}\bigr)\bigr] \\
&= \begin{cases} \tilde{F}_1(s'_1-2,s'_2)e^{2\pi i(s'_1-1)\tau_{11}+2\pi i s'_2\tau_{12}}, & l \text{ is even}, \\ \tilde{F}_1(s'_1,s'_2-2)e^{2\pi i(s'_2-1)\tau_{22}+2\pi i s'_1\tau_{12}}, & l \text{ is odd}. \end{cases}
\end{aligned} \tag{3.2}$$

This relation implies that if $\tilde{F}_1(0,0) = \tilde{F}_1(0,1) = \tilde{F}_1(1,0) = \tilde{F}_1(1,1) = 0$, then $\tilde{F}_1(s'_1,s'_2) = 0$ for $s'_1, s'_2 \in Z$.

Denoting

$$\delta_j(n) = e^{\pi i\langle\tau(k-m^{(j)}),k-m^{(j)}\rangle+\pi i\langle\tau k,k\rangle},$$

$$m^{(1)} = (0,0)^T, \qquad m^{(2)} = (1,0)^T, \qquad m^{(3)} = (0,1)^T, \qquad m^{(4)} = (1,1)^T,$$

we have

$$\begin{aligned}
\tilde{F}_1(0,0) &= \sum_{k_1,k_2=-\infty}^{\infty} \bigl[2\pi i\langle 2k-(0,0)^T,\mu\rangle e^{\pi i\langle 2k-(0,0)^T,\eta\rangle} \\
&\quad + 4\pi^2\langle 2k-(0,0)^T,p\rangle^2 e^{\pi i\langle 2k-(0,0)^T,\eta\rangle} + c_1\bigr] e^{2\pi i\langle \tau k,k\rangle}, \\
\tilde{F}_1(0,1) &= \sum_{k_1,k_2=-\infty}^{\infty} \bigl[2\pi i\langle 2k-(0,1)^T,\mu\rangle e^{\pi i\langle 2k-(0,1)^T,\eta\rangle} \\
&\quad + 4\pi^2\langle 2k-(0,1)^T,p\rangle^2 e^{\pi i\langle 2k-(0,1)^T,\eta\rangle} + c_1\bigr] e^{\pi i\langle \tau(k-(0,1)^T),k-(0,1)^T\rangle+\pi i\langle \tau k,k\rangle}, \\
\tilde{F}_1(1,0) &= \sum_{k_1,k_2=-\infty}^{\infty} \bigl[2\pi i\langle 2k-(1,0)^T,\mu\rangle e^{\pi i\langle 2k-(1,0)^T,\eta\rangle} \\
&\quad + 4\pi^2\langle 2k-(1,0)^T,p\rangle^2 e^{\pi i\langle 2k-(1,0)^T,\eta\rangle} + c_1\bigr] e^{\pi i\langle \tau(k-(1,0)^T),k-(1,0)^T\rangle+\pi i\langle \tau k,k\rangle}, \\
\tilde{F}_1(1,1) &= \sum_{k_1,k_2=-\infty}^{\infty} \bigl[2\pi i\langle 2k-(1,1)^T,\mu\rangle e^{\pi i\langle 2k-(1,1)^T,\eta\rangle} \\
&\quad + 4\pi^2\langle 2k-(1,1)^T,p\rangle^2 e^{\pi i\langle 2k-(1,1)^T,\eta\rangle} + c_1\bigr] e^{\pi i\langle \tau(k-(1,1)^T),k-(1,1)^T\rangle+\pi i\langle \tau k,k\rangle}.
\end{aligned} \tag{3.3}$$

Denote

$$\begin{aligned}
a_{j1} &= \sum_{k_1,k_2=-\infty}^{\infty} 2\pi i\bigl(2k_1 - m_1^{(j)}\bigr) e^{\pi i\langle k-m^{(j)},\eta\rangle}\delta_1, \\
a_{j2} &= \sum_{k_1,k_2=-\infty}^{\infty} 2\pi i\bigl(2k_2 - m_2^{(j)}\bigr) e^{\pi i\langle k-m^{(j)},\eta\rangle}\delta_2, \\
a_{j3} &= \sum_{k_1,k_2=-\infty}^{\infty} 4\pi^2\bigl(2k_1 - m_1^{(j)}\bigr)^2 e^{\pi i\langle k-m^{(j)},\eta\rangle}\delta_3, \\
a_{j4} &= \sum_{k_1,k_2=-\infty}^{\infty} 4\pi^2\bigl(2k_2 - m_2^{(j)}\bigr)^2 e^{\pi i\langle k-m^{(j)},\eta\rangle}\delta_4, \\
b_j &= -c_1\sum_{k_1,k_2=-\infty}^{\infty} \delta_j - 8\pi^2\bigl(2k_1 - m_1^{(j)}\bigr)\bigl(2k_2 - m_2^{(j)}\bigr)p_1p_2 e^{\pi i\langle k-m_1^{(j)},\eta\rangle}\delta_j,
\end{aligned}$$

then (3.3) can be written as

$$A\begin{pmatrix}\mu_1\\ \mu_2\\ p_1^2\\ p_2^2\end{pmatrix} = \vec{b},$$

from which we have $\mu_1 = \frac{\triangle_1}{\triangle}$, $\mu_2 = \frac{\triangle_2}{\triangle}$, $p_1^2 = \frac{\triangle_3}{\triangle}$, $p_2^2 = \frac{\triangle_4}{\triangle}$, where $\triangle = \det A$ and $\triangle_1$, $\triangle_2$, $\triangle_3$, $\triangle_4$ are given $\triangle$ by replacing 1st, 2nd, 3rd, 4th columns with $\vec{b}$, respectively.

Similarly, by letting $N = 2$ in (2.6), then we have $f(n) = \sum_{k\in Z^2} e^{2\pi i\langle \zeta,k\rangle+\pi i\langle \tau k,k\rangle}$ and substitute it into (2.3). For convenience of calculations, we have introduced k and k' of different form. We have derived

$$\begin{aligned}
F_2(f_n\cdot f_n) &= \sum_{k,k'\in Z^2} F_2(D_z,D_t,D_y,\cosh D_n)e^{2\pi i\langle\zeta,k\rangle+\pi i\langle\tau k,k\rangle}\cdot e^{2\pi i\langle\zeta,k'\rangle+\pi i\langle\tau k',k'\rangle}\\
&= \sum_{k,k'\in Z^2} F_2\bigl(2\pi i\langle k-k',\mu\rangle, 2\pi i\langle k-k',p\rangle, 2\pi i\langle k-k',l\rangle, \cosh 2\pi i\langle k-k',\eta\rangle\bigr)\\
&\quad\times \exp\bigl(2\pi i\langle\zeta,k+k'\rangle\bigr)\exp(\pi i\bigl(\langle\tau k',k'\rangle+\langle\tau k,k\rangle\bigr)\\
&= \sum_{s'\in Z^2}\sum_{k_1,k_2=-\infty}^{\infty} F_2\bigl(2\pi i\langle 2k-s',\mu\rangle, 2\pi i\langle 2k-s',p\rangle, 2\pi i\langle 2k-s',l\rangle,\\
&\quad\cosh 2\pi i\langle 2k-s',\eta\rangle\bigr)\\
&\quad\times \exp\pi i\bigl(\langle\eta(k-s'),k-s'\rangle+\langle\tau k,k\rangle\bigr)\exp\bigl(2\pi i\langle\zeta,s'\rangle\bigr)\\
&\equiv \sum_{s'\in Z^2}\tilde{F}_2\bigl(s_1',s_2'\bigr)\exp\bigl(2\pi i\langle\zeta,s'\rangle\bigr)=0.
\end{aligned}\tag{3.4}$$

By introducing the new summation index $k+k'=s'$, $k=(k_1,k_2)^T$, $\tilde{F}_2(s_1',s_2')$ is denoted by

$$\begin{aligned}
\tilde{F}_2\bigl(s_1',s_2'\bigr) &= \sum_{k_1,k_2=-\infty}^{\infty} F_2\bigl[2\pi i\langle 2k-s',\mu\rangle, 2\pi i\langle 2k-s',p\rangle, 2\pi i\langle 2k-s',l\rangle, \cosh 2\pi i\langle 2k-s',\eta\rangle\bigr]\\
&\quad\times\exp\pi i\bigl(\bigl\langle\tau\bigl(k-s'\bigr),\bigl(k-s'\bigr)\bigr\rangle+\langle\tau k,k\rangle\bigr)\\
&= \begin{cases}\tilde{F}_2(s_1'-2,s_2')e^{2\pi i(s_1'-1)\tau_{11}+2\pi i s_2'\tau_{12}}, & l \text{ is even},\\ \tilde{F}_2(s_1',s_2'-2)e^{2\pi i(s_2'-1)\tau_{22}+2\pi i s_1'\tau_{12}}, & l \text{ is odd},\end{cases}
\end{aligned}\tag{3.5}$$

which means that if $\tilde{F}_2(m^{(j)})=0$, thus all $\tilde{F}_2(s_1',s_2')=0$. Through direct calculations, we have derived

$$\begin{aligned}
\tilde{F}_2\bigl(m^{(j)}\bigr) &= \sum_{k_1,k_2=-\infty}^{\infty}\bigl[-4\pi^2\bigl\langle 2k-m^{(j)},p\bigr\rangle\bigl\langle 2k-m^{(j)},\mu\bigr\rangle+4\pi^2\bigl\langle 2k-m^{(j)},p\bigr\rangle\bigl\langle 2k-m^{(j)},l\bigr\rangle\\
&\quad-2\cosh 2\pi i\bigl\langle 2k-m^{(j)},\eta\bigr\rangle+2+c_2\bigr]\\
&\quad\times e^{\pi i\langle\tau k,k\rangle+\pi i\langle\tau(k-m^{(j)}),k-m^{(j)}\rangle}.
\end{aligned}\tag{3.6}$$

Let

$$\begin{aligned}
c_{j1} &= \sum_{k_1,k_2=-\infty}^{\infty} 4\pi^2\bigl(2k_1-m_1^{(j)}\bigr)^2\delta_j(n),\\
c_{j2} &= \sum_{k_1,k_2=-\infty}^{\infty} 4\pi^2\bigl(2k_2-m_2^{(j)}\bigr)^2\delta_j(n),\\
c_{j3} &= \sum_{k_1,k_2=-\infty}^{\infty} 4\pi^2\bigl(2k_1-m_1^{(j)}\bigr)\bigl(2k_2-m_2^{(j)}\bigr)\delta_j(n),
\end{aligned}\tag{3.7}$$

$$c_{j4} = \sum_{k_1,k_2=-\infty}^{\infty} \delta_j(n),$$

$$d_j = \sum_{k_1,k_2=-\infty}^{\infty} 4\cosh 2\pi i\langle 2k - m^{(j)}, \eta\rangle \delta_j(n),$$

then $\tilde{F}_2(m^{(j)}) = 0$ can be rewritten as

$$C\begin{pmatrix} p_1(l_1-\mu_1) \\ p_2(l_2-\mu_2) \\ p_1(l_2-\mu_2)+p_2(l_1-\mu_1) \\ 2+c_2 \end{pmatrix} = \vec{d},$$

from which we have $p_1(l_1-\mu_1) = \frac{\triangle_1}{\triangle}$, $p_2(l_2-\mu_2) = \frac{\triangle_2}{\triangle}$, $p_1(l_2-\mu_2)+p_2(l_1-\mu_1) = \frac{\triangle_3}{\triangle}$, $2+c_2 = \frac{\triangle_4}{\triangle}$, where $\triangle = \det C$ and $\triangle_1$, $\triangle_2$, $\triangle_3$, $\triangle_4$ are given $\triangle$ by replacing 1st, 2nd, 3rd, 4th columns with $\vec{d}$.

3.2 Asymptotic behavior of the two-periodic wave solution

In what follows, we can verify the asymptotic behavior of the two-periodic wave solution to be the well-known two-soliton solution given by the Hirota method.

Theorem 2 *Let* $\lambda_1 = \exp\tau_{11} \to 0$, $\lambda_2 = \exp\tau_{22} \to 0$, *the periodic solution* (2.1) *of* (1.1) *tends to the two-soliton solution*

$$\begin{aligned}
u(n) &= \frac{(1+e^{\tilde{\zeta}_1+\tilde{\eta}_1}+e^{\tilde{\zeta}_2+\tilde{\eta}_2}+e^{\tilde{\zeta}_1+\tilde{\eta}_1+\tilde{\zeta}_2+\tilde{\eta}_2+2\pi i\tau_{12}})(1+e^{\tilde{\zeta}_1-\tilde{\eta}_1}+e^{\tilde{\zeta}_2-\tilde{\eta}_2}+e^{\tilde{\zeta}_1-\tilde{\eta}_1+\tilde{\zeta}_2-\tilde{\eta}_2+2\pi i\tau_{12}})}{(1+\exp\tilde{\zeta}_1+\exp\tilde{\zeta}_2+\exp(\tilde{\zeta}_1+\tilde{\zeta}_2+2\pi i\tau_{12}))^2}, \\
v(n) &= \frac{f_{tt}(n)f(n+1)-2f_t(n)f_t(n+1)+f(n)f_{tt}(n+1)}{f(n)f(n+1)}, \\
w(n) &= \frac{f_t(n+1)f(n)-f_t(n)f(n+1)}{f(n)f(n+1)},
\end{aligned} \tag{3.8}$$

with

$$p_1 = \frac{1-\cosh\tilde{\eta}_1}{2\pi^2(l_1-\mu_1)}, \qquad p_2 = \frac{1-\cosh\tilde{\eta}_2}{2\pi^2(l_2-\mu_2)},$$

$$\tilde{\zeta}_i = 2\pi i(p_i t + l_i y + \mu_i z + \eta_i n) + \tilde{\zeta}_{0i}, \qquad \tilde{\zeta}_{0i} = \zeta_{0i} + \frac{1}{2}\tau_{ii}, \qquad \tilde{\eta}_i = 2\pi i\eta_i, \quad i = 1,2,$$

$$e^{2\pi i\tau_{12}} = -\frac{2\pi^2(p_1-p_2)(l_1-\mu_1+\mu_2-l_2)-\cosh(\tilde{\eta}_1-\tilde{\eta}_2)+1}{2\pi^2(p_1+p_2)(l_1-\mu_1+l_2-\mu_2)-\cosh(\tilde{\eta}_1+\tilde{\eta}_2)+1},$$

$$2\pi p_2^2\cos\pi\eta_2 - \mu_2\sin\pi\eta_2 = 0, \qquad 2\pi p_1^2\cos\pi\eta_1 - \mu_1\sin\pi\eta_1 = 0,$$

$$\begin{aligned}
&\frac{2\pi(p_1-p_2)^2\cos\pi(\eta_1-\eta_2)-(\mu_1-\mu_2)\sin\pi(\eta_1-\eta_2)}{2\pi(p_1+p_2)^2\cos\pi(\eta_1+\eta_2)-(\mu_1+\mu_2)\sin\pi(\eta_1+\eta_2)} \\
&\quad = \frac{2\pi^2(p_1-p_2)(l_1-l_2-\mu_1+\mu_2)-\cos 2\pi(\eta_1-\eta_2)+1}{2\pi^2(p_1+p_2)(l_1+l_2-\mu_1-\mu_2)-\cos 2\pi(\eta_1+\eta_2)+1},
\end{aligned}$$

$$f(n) \to 1+\exp\tilde{\zeta}_1+\exp\tilde{\zeta}_2+\exp(\tilde{\zeta}_1+\tilde{\zeta}_2+2\pi i\tau_{12}),$$

$$f_t(n) \to 2\pi i p_1 e^{\tilde{\zeta}_1} + 2\pi i p_2 e^{\tilde{\zeta}_2} + 2\pi i(p_1+p_2)e^{\tilde{\zeta}_1+\tilde{\zeta}_2+2\pi i\tau_{12}},$$

$$f_{tt}(n) \to -4\pi^2 p_1^2 e^{\tilde{\zeta_1}} - 4\pi^2 p_2^2 e^{\tilde{\zeta_2}} - 4\pi^2 (p_1+p_2)^2 e^{\tilde{\zeta_1}+\tilde{\zeta_2}+2\pi i\tau_{12}},$$
$$f_t(n+1) \to 2\pi i p_1 e^{\tilde{\zeta_1}+\tilde{\eta_1}} + 2\pi i p_2 e^{\tilde{\zeta_2}+\tilde{\eta_1}} + 2\pi i(p_1+p_2) e^{\tilde{\zeta_1}+\tilde{\zeta_2}+\tilde{\eta_1}+\tilde{\eta_2}+2\pi i\tau_{12}},$$
$$f_{tt}(n+1) \to -4\pi^2 p_1^2 e^{\tilde{\zeta_1}+\tilde{\eta_1}} - 4\pi^2 p_2^2 e^{\tilde{\zeta_2}+\tilde{\eta_2}} - 4\pi^2 (p_1+p_2)^2 e^{\tilde{\zeta_1}+\tilde{\zeta_2}+\tilde{\eta_1}+\tilde{\eta_2}+2\pi i\tau_{12}}.$$

Proof Let $\tilde{\zeta_i} = 2\pi i\zeta_i + \pi i\frac{\tau_{ii}}{2}$, $\tilde{\eta_i} = 2\pi i\eta_i$ for $i = 1, 2$. We expand the two-periodic wave solution (2.6) ($N = 2$) of (2.2) and (2.3):

$$\begin{aligned} f(n) &= 1 + \exp(2\pi i\zeta_1 + \pi i\tau_{11}) + \exp(-2\pi i\zeta_1 + \pi i\tau_{11}) + \exp(2\pi i\zeta_2 + \pi i\tau_{22}) \\ &\quad + \exp(-2\pi i\zeta_2 + \pi i\tau_{22}) + \exp\bigl(2\pi i(\zeta_1+\zeta_2) + \pi i(\tau_{11} + 2\tau_{12} + \tau_{22})\bigr) \\ &\quad + \exp\bigl(-2\pi i(\zeta_1+\zeta_2) + \pi i(\tau_{11} + 2\tau_{12} + \tau_{22})\bigr) + \cdots \\ &\to 1 + \exp\tilde{\zeta_1} + \exp\tilde{\zeta_2} + \exp(\tilde{\zeta_1} + \tilde{\zeta_2} + 2\pi i\tau_{12}), \end{aligned} \tag{3.9}$$

then we have

$$\begin{aligned} &f_t(n) \to 2\pi i p_1 e^{\tilde{\zeta_1}} + 2\pi i p_2 e^{\tilde{\zeta_2}} + 2\pi i(p_1+p_2) e^{\tilde{\zeta_1}+\tilde{\zeta_2}+2\pi i\tau_{12}}, \\ &f_{tt}(n) \to -4\pi^2 p_1^2 e^{\tilde{\zeta_1}} - 4\pi^2 p_2^2 e^{\tilde{\zeta_2}} - 4\pi^2 (p_1+p_2)^2 e^{\tilde{\zeta_1}+\tilde{\zeta_2}+2\pi i\tau_{12}}, \\ &f_t(n+1) \to 2\pi i p_1 e^{\tilde{\zeta_1}+\tilde{\eta_1}} + 2\pi i p_2 e^{\tilde{\zeta_2}+\tilde{\eta_1}} + 2\pi i(p_1+p_2) e^{\tilde{\zeta_1}+\tilde{\zeta_2}+\tilde{\eta_1}+\tilde{\eta_2}+2\pi i\tau_{12}}, \\ &f_{tt}(n+1) \to -4\pi^2 p_1^2 e^{\tilde{\zeta_1}+\tilde{\eta_1}} - 4\pi^2 p_2^2 e^{\tilde{\zeta_2}+\tilde{\eta_2}} - 4\pi^2 (p_1+p_2)^2 e^{\tilde{\zeta_1}+\tilde{\zeta_2}+\tilde{\eta_1}+\tilde{\eta_2}+2\pi i\tau_{12}}. \end{aligned} \tag{3.10}$$

For convenience, we denote $\lambda_1 = e^{\pi i\tau_{11}}$, $\lambda_2 = e^{\pi i\tau_{22}}$. In what follows, we expand each function in $\tilde{F}_1(0,0) = \tilde{F}_1(0,1) = \tilde{F}_1(1,0) = \tilde{F}_1(1,1) = 0$, $\tilde{F}_2(0,0) = \tilde{F}_2(0,1) = \tilde{F}_2(1,0) = \tilde{F}_2(1,1) = 0$ into series of λ_1, λ_2,

$$\begin{aligned} \tilde{F}_1(0,0) &= c_1 + \bigl[4\pi i\mu_1\bigl(e^{2\pi i\eta_1} - e^{-2\pi i\eta_1}\bigr) + 16\pi^2 p_1^2\bigl(e^{2\pi i\eta_1} + e^{-2\pi i\eta_1}\bigr) + c_1\bigr]e^{2\pi i\tau_{11}} \\ &\quad + \bigl[4\pi i\mu_2\bigl(e^{2\pi i\eta_2} - e^{-2\pi i\eta_2}\bigr) + 16\pi^2 p_2^2\bigl(e^{2\pi i\eta_2} + e^{-2\pi i\eta_2}\bigr) + c_1\bigr]e^{2\pi i\tau_{22}} \\ &\quad + o\bigl(\lambda_1^{r_1}\lambda_2^{r_2}\bigr), \end{aligned} \tag{3.11}$$

when $r_1 + r_2 \geq 4$, it is easy to see that $c_1 \to 0$.

$$\begin{aligned} \tilde{F}_1(0,1) &= \bigl[\bigl(2\pi i\mu_2\bigl(e^{\pi i\eta_2} - e^{-\pi i\eta_2}\bigr) + 4\pi^2 p_2^2\bigl(e^{\pi i\eta_2} + e^{-\pi i\eta_2}\bigr) + c_1\bigr) \\ &\quad + \bigl(2\pi i(2\mu_1 - \mu_2) + 4\pi^2(2p_1 - p_2)^2 + c_1\bigr)e^{2\pi i(\tau_{11}-\tau_{12})} \\ &\quad + \bigl(2\pi i(2\mu_1 + \mu_2) + 4\pi^2(2p_1 + p_2)^2 + c_1\bigr)e^{2\pi i(\tau_{11}+\tau_{12})}\bigr]e^{\pi i\tau_{22}} \\ &\quad + o\bigl(\lambda_1^{r_1}\lambda_2^{r_2}\bigr) \end{aligned} \tag{3.12}$$

in view of $c_1 \to 0$, from which we have $2\pi p_2^2 \cos\pi\eta_2 - \mu_2 \sin\pi\eta_2 = 0$.

$$\begin{aligned} \tilde{F}_1(1,0) &= \bigl[\bigl(2\pi i\mu_1\bigl(e^{\pi i\eta_1} - e^{-\pi i\eta_1}\bigr) + 4\pi^2 p_1^2\bigl(e^{\pi i\eta_1} + e^{-\pi i\eta_1}\bigr) + c_1\bigr) \\ &\quad + \bigl(2\pi i(2\mu_2 - \mu_1) + 4\pi^2(2p_2 - p_1)^2 + c_1\bigr)e^{2\pi i(\tau_{22}-\tau_{12})} \\ &\quad + \bigl(2\pi i(2\mu_2 + \mu_1) + 4\pi^2(2p_2 + p_1)^2 + c_1\bigr)e^{2\pi i(\tau_{22}+\tau_{12})}\bigr]e^{\pi i\tau_{11}} \\ &\quad + o\bigl(\lambda_1^{r_1}\lambda_2^{r_2}\bigr). \end{aligned} \tag{3.13}$$

In view of $c_1 \to 0$, from (3.13), we have $2\pi p_1^2 \cos\pi\eta_1 - \mu_1 \sin\pi\eta_1 = 0$.

$$\begin{aligned}\tilde{F}_1(1,1) = {} & \big[\big[2\pi i(\mu_1+\mu_2)\big(e^{\pi i(\eta_1+\eta_2)} - e^{-\pi i(\eta_1+\eta_2)}\big) \\ & + 4\pi^2(p_1+p_2)^2\big(e^{\pi i(\eta_1+\eta_2)} + e^{-\pi i(\eta_1+\eta_2)}\big)\big]e^{2\pi i\tau_{12}} \\ & + 2\pi i(\mu_1-\mu_2)\big(e^{\pi i(\eta_1-\eta_2)} - e^{\pi i(\eta_1-\eta_2)}\big) \\ & + 4\pi^2(p_1-p_2)^2\big(e^{\pi i(\eta_1-\eta_2)} + e^{-\pi i(\eta_1+\eta_2)}\big)\big]e^{\pi i(\tau_{11}+\tau_{22})} + o\big(\lambda_1^{r_1}\lambda_2^{r_2}\big). \end{aligned} \tag{3.14}$$

From (3.14), we have

$$e^{2\pi i\tau_{12}} = -\frac{2\pi i(\mu_1-\mu_2)(e^{\pi i(\eta_1-\eta_2)} - e^{-\pi i(\eta_1-\eta_2)}) + 4\pi^2(p_1-p_2)^2(e^{\pi i(\eta_1-\eta_2)} + e^{-\pi i(\eta_1-\eta_2)})}{2\pi i(\mu_1+\mu_2)(e^{\pi i(\eta_1+\eta_2)} - e^{-\pi i(\eta_1+\eta_2)}) + 4\pi^2(p_1+p_2)^2(e^{\pi i(\eta_1+\eta_2)} + e^{-\pi i(\eta_1+\eta_2)})}. \tag{3.15}$$

In the following, we will consider

$$\begin{aligned}\tilde{F}_2(0,0) = {} & c_2 + \big[16\pi^2 p_2(l_2-\mu_2) - 2\cosh 4\pi i\eta_2 + 2 + c_2\big]e^{2\pi i\tau_{22}} \\ & + \big[16\pi^2 p_1(l_1-\mu_1) - 2\cosh 4\pi i\eta_1 + 2 + c_2\big]e^{2\pi i\tau_{11}} + o\big(\lambda_1^{r_1}\lambda_2^{r_2}\big). \end{aligned} \tag{3.16}$$

From $\tilde{F}_2(0,0) \to 0$, we have $c_2 \to 0$.

$$\begin{aligned}\tilde{F}_2(0,1) = {} & \big(8\pi^2 p_2(l_2-\mu_2) - 4\cosh 2\pi i\eta_2 + 4 + 2c_2\big)e^{\pi i\tau_{22}} \\ & + \big[4\pi^2(2p_1-p_2)(2l_1-2\mu_1+l_2-\mu_2) - 2\cosh 2\pi i(2\eta_1-\eta_2) + 2 + c_2\big] \\ & \times e^{\pi i(2\tau_{11}-2\tau_{12}+\tau_{22})} \\ & + \big[4\pi^2(2p_1+p_2)(2l_1+2\mu_1-l_2-\mu_2) - 2\cosh 2\pi i(2\eta_1+\eta_2) + 2 + c_2\big] \\ & \times e^{\pi i(2\tau_{11}+2\tau_{12}+\tau_{22})} + o\big(\lambda_1^{r_1}\lambda_2^{r_2}\big). \end{aligned} \tag{3.17}$$

From $\tilde{F}_2(0,1) \to 0$, in view of $c_2 \to 0$, we have $2\pi^2 p_2(\mu_2 - l_2) + \cosh 2\pi i\eta_2 - 1 = 0$.

$$\begin{aligned}\tilde{F}_2(1,0) = {} & \big(8\pi^2 p_1(l_1-\mu_1) - 4\cosh 2\pi i\eta_1 + 4 + 2c_2\big)e^{\pi i\tau_{11}} \\ & + \big[4\pi^2(p_2-p_1)(2l_2-2\mu_2-l_1+\mu_1) - 2\cosh 2\pi i(2\eta_2-\eta_1) + 2 + c_2\big] \\ & \times e^{\pi i(2\tau_{22}-2\tau_{12}+\tau_{11})} \\ & + \big[4\pi^2(p_1+2p_2)(2l_2-2\mu_2+l_1-\mu_1) - 2\cosh 2\pi i(2\eta_2+\eta_1) + 2 + c_2\big] \\ & \times e^{\pi i(\tau_{11}+2\tau_{12}+2\tau_{22})} + o\big(\lambda_1^{r_1}\lambda_2^{r_2}\big). \end{aligned} \tag{3.18}$$

For $\tilde{F}_2(1,0) \to 0$, we have $2\pi^2 p_1(\mu_1 - l_1) + \cosh 2\pi i\eta_1 - 1 = 0$.

$$\begin{aligned}\tilde{F}_2(1,1) = {} & 2\big[4\pi^2(p_1+p_2)(l_1+l_2-\mu_1-\mu_2) - 2\cosh 2\pi i(\eta_1+\eta_2) + 2 + c_2\big]e^{2\pi i\tau_{12}} \\ & + 2\big[4\pi^2(p_1-p_2)(l_1-l_2-\mu_1+\mu_2) - 2\cosh 2\pi i(\eta_1-\eta_2) + 2 + c_2\big]e^{\pi i(\tau_{11}+\tau_{22})} \\ & + o\big(\lambda_1^{r_1}\lambda_2^{r_2}\big). \end{aligned} \tag{3.19}$$

In view of $\tilde{F}_2(1,1) \to 0$, we have

$$e^{2\pi i\tau_{12}} = -\frac{2\pi^2(p_1-p_2)(l_1-\mu_1+\mu_2-l_2)-2\cosh 2\pi i(\eta_1-\eta_2)+1}{2\pi^2(p_1+p_2)(l_1-\mu_1-\mu_2+l_2)-2\cosh 2\pi i(\eta_1+\eta_2)+1}. \tag{3.20}$$

This completes the proof of Theorem 2. □

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