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# Spin(7)-structure equation and the vector elliptic Liouville equation

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**Abstract**

The mapping between Belavin–Polyakov (BP) equation for the evolution of a unit tangent vector $T \in \mathbb{S}^2$ of a space curve in $\mathbb{R}^3$ and the elliptic Liouville equation has been shown by Balakrishnan (see Phys. Lett. A 204:243–246, 1995). In the present work, this result is effectively extended by mapping the BP equation for the unit tangent $T \in \mathbb{S}^6$ of a space curve in $\mathbb{R}^7$ to the vector elliptic Liouville equation. To show this correspondence, Spin(7)-frame field on the curve is used.

**MSC:** 53A04; 53C15; 35J15

**Keywords:** Spin(7)-structure equation; Octonions; Almost complex structure; The vector elliptic Liouville equation

## 1 Introduction

The two-dimensional elliptic Liouville equation [2]

$$\phi_{tt} + \phi_{ss} = -e^{2\phi}, \tag{1}$$

plays an important role in the modern field theory, namely in the strings theory, where the quantum Liouville field appears as a conformal anomaly (see Refs. [3, 4]). The Liouville equation arises in many contexts of complex analysis and differential geometry of Riemann surfaces, in particular in the prescribing curvature problem. The interplay between the geometric and analytic aspects makes the Liouville equation mathematically rich (see Ref. [5]).

By using differential geometry of surfaces and moving curves, the elliptic Liouville equation (1) has been shown [1, 6] to be equivalent to the Belavin–Polyakov equation [7]

$$T_t = T \times T_s, \quad T \in \mathbb{S}^2, \tag{2}$$

where $T = T(t,s)$ is the tangent vector of a space curve $\gamma(t,s)$ in $\mathbb{R}^3$ represented by a vector-valued function of time $t$ and arclength $s$, the subscript stands for the partial derivative with respect to the variable symbolized, and $\times$ denotes the cross product between vectors in $\mathbb{R}^3$. Using the moving space curve method, it has been shown [8, 9] that Eq. (2) can be mapped to the Lamb equation [10]

$$\sqrt{-1}\varphi_t + \varphi_s + 4\varphi \int_0^s |\varphi|^2 \, d\tilde{s} = 0 \tag{3}$$

for the complex Hasimoto function $\varphi = \frac{\kappa}{2} e^{\sqrt{-1}\int_0^s \tau \, d\tilde{s}}$. Here, $\kappa$ is the curvature and $\tau$ is the torsion of the curve. Furthermore, it has been shown [9] that the effective low-energy dynamics of the classical isotropic antiferromagnetic chain is described by the Belavin–Polyakov equation (2), which can be written in terms of $u$ as follows:

$$u_t = u \times u_s, \qquad u = (u_1, u_2, u_3) \in \mathbb{S}^2 \hookrightarrow \mathbb{R}^3 \tag{4}$$

for the "staggered" spin field $u(t, s)$.

On the other hand, the Euclidean 3-space $\mathbb{R}^3$ can be regarded as the imaginary part $\operatorname{Im}\mathbb{H}$ of the quaternions $\mathbb{H}$, and the cross product $\times$ in $\mathbb{R}^3 \cong \operatorname{Im}\mathbb{H}$ is created by the quaternion algebraic structure in $\operatorname{Im}\mathbb{H}$. It is also well known that the quaternion is contained in the octonions $\mathcal{O}$, and so the imaginary part $\operatorname{Im}\mathbb{H}$ in the imaginary part $\operatorname{Im}\mathcal{O}$. It is no surprise that there is a cross product between vectors in $\operatorname{Im}\mathcal{O} \cong \mathbb{R}^7$ induced by the octonional algebraic structure in $\operatorname{Im}\mathcal{O}$, which includes the cross product in $\operatorname{Im}\mathbb{H} \cong \mathbb{R}^3$ as a special case. Therefore, as a natural generalization of Eq. (2) in $\mathbb{R}^3$ in higher dimensions, we are interested in exploring the dynamical properties of the following evolving equation in $\operatorname{Im}\mathcal{O} \cong \mathbb{R}^7$:

$$T_t = T \times T_s, \qquad T(t, s) \in \mathbb{S}^6 \hookrightarrow \mathbb{R}^7, \tag{5}$$

where $T = T(t, s)$ is the tangent vector of a space curve $\gamma(t, s)$ in $\mathbb{R}^7$ represented by a vector-valued function of time $t$ and arclength $s$. It looks very interesting from purely mathematical point of view. Equation (5) may be regarded as the $G_2$-tangent motion in $\mathbb{S}^6$, as we shall see below. However, to our pity, we have not found physical applications of Eq. (5) in literature yet.

To our surprise, Eq. (2) and Eq. (5) can be equivalent to the maps $T$ from $\mathbb{R}$ to $\mathbb{S}^{2n}$ ($n = 1, 3$):

$$T_t = J_T T_s, \tag{6}$$

where $J_T$ stands for the standard (almost) complex structure of $\mathbb{S}^{2n}$ at $T$. It was proved by Borel and Serre in 1953 (see Ref. [11]) that $\mathbb{S}^{2n}$ admits an almost complex structure if and only if $n = 1$ or 3, see also the recent review paper [12]. In 1993, Calabi and Gluck [13] proved that the best almost-complex structure on $\mathbb{S}^6$ is the Kirchhoff's one from the octonions in the sense that it has the smallest volume in a class of sections of the bundle $O(8) = U(4)$ over $\mathbb{S}^6$. Ding et al. in [14] studied the almost complex structure on $\mathbb{S}^6$ and related Schrödinger flows. This indicates that Eq. (5) is not only a higher dimensional generalization of Eq. (2), but also it relates to almost complex structures on $\mathbb{S}^{2n}$ ($n = 1, 3$). This gives us further motivations on Eq. (5). The exploitation in this paper can also be regarded as an understanding of almost-complex structures on $\mathbb{S}^6$ and the octonional algebraic structure on $\operatorname{Im}\mathcal{O} = \mathbb{R}^7$ via Eq. (5) and Eq. (6). An approach similar to the one used in this paper could possibly give some insight in the 16-dimensional case, using the spinorial representation of Spin(9) in $\mathbb{R}^{16} = \mathcal{S}$ = sedenions in place of the spinorial representation of Spin(7) in $\mathbb{R}^8 = \mathcal{O}$ [15]. However, several pieces are missing: for instance, $G_2$ is related to Spin(7) by a cone construction ([16], Theorem B), but what is the analog of $G_2$ in the Spin(9) case is not clear [15].

The paper is organized as follows. Section 2 gives preliminaries about the octonions, Spin(7) and Spin(7)-structure equations. In Sect. 3 we construct complexified Spin(7)-frame field along curves and establish a related Frenet formula along curves. In Sect. 4, by using Spin(7)-frame field along curves, we give a proof of the correspondence between Eq. (5) and the vector elliptic Liouville equation.

## 2 Preliminaries

In this section, we will give some facts about the octonions, Spin(7) and Spin(7)-structure equations.

### 2.1 The octonions and Spin(7)

Let $\mathbb{H}$ be the skew field of all quaternions with canonical basis $\{1, i, j, k\}$, satisfying

$$i^2 = j^2 = k^2 = -1, \qquad ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j.$$

The octonions (or the Cayley numbers) $\mathcal{O}$ over $\mathbb{R}$ can be considered as a direct sum $\mathbb{H} \oplus \mathbb{H}$ with the following multiplication:

$$(a + b\epsilon)(c + d\epsilon) = ac - \bar{d}b + (da + b\bar{c})\epsilon,$$

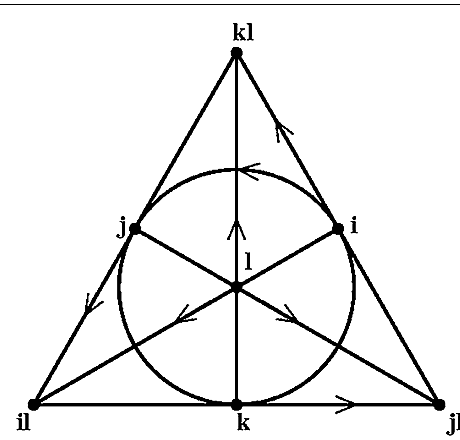
where $\epsilon = (0, 1) \in \mathbb{H} \oplus \mathbb{H}$, $a, b, c, d \in \mathbb{H}$ and the symbol $\bar{\cdot}$ denotes the conjugation of the quaternions. For any $x, y \in \mathcal{O}$, we have

$$\langle xy, xy\rangle = \langle x, y\rangle\langle y, y\rangle,$$

where $\langle\cdot\rangle$ is the canonical inner product of $\mathbb{R}^8 \cong \mathcal{O}$. This condition is called normed algebra condition (see Ref. [17]). The full multiplication table is summarized in Fig. 1 by means of the 7-point projective plane. Each point corresponds to an imaginary unit. Each line corresponds to a quaternionic triple, much like $\{i, j, k\}$ with the arrow giving the orientation. For example,

$$k\,l = kl, \qquad l\,(kl) = k, \qquad (kl)\,k = l,$$

**Figure 1** The octonionic multiplication table



and each of these products anticommutes, that is, reversing the order contributes a minus sign.

The octonions is a non-commutative, nonassociative alternative division algebra. The group of automorphisms of the octonions is the exceptional simple Lie group

$$G_2 = \{g \in \mathrm{SO}(8) \mid g(xy) = g(x)g(y), \forall x, y \in \mathcal{O}\}.$$

The cross product $x \times y$ and the scalar product $\langle x, y\rangle$ of $\mathcal{O}$ are defined respectively by

$$x \times y = (1/2)(\bar{y}x - \bar{x}y), \qquad \langle x, y\rangle = (1/2)(\bar{x}y + \bar{y}x),$$

where $\bar{x} = 2\langle x, 1\rangle - x$ is the conjugation of $x \in \mathcal{O}$. One may verify directly that $x \times y \in \operatorname{Im}\mathcal{O}$, $\forall x, y \in \operatorname{Im}\mathcal{O}$, where $\operatorname{Im}\mathcal{O} = \{x \in \mathcal{O} \mid \langle x, 1\rangle = 0\}$. This induces a cross product $\times$ among vectors in $\operatorname{Im}\mathcal{O} \cong \mathbb{R}^7$.

In this paper, we shall consider the Lie group Spin(7) which is defined by

$$\mathrm{Spin}(7) = \{g \in \mathrm{SO}(8) \mid g(xy) = g(x)\chi_g(y), \forall x, y \in \mathcal{O}\},$$

where $\chi_g(y) = g(g^{-1}(1)y)$. Note that $G_2$ is a Lie subgroup of Spin(7):

$$G_2 = \{g \in \mathrm{Spin}(7) \mid g(1) = 1\}.$$

The standard almost complex structure $J$ on $\mathbb{S}^6$ called the typical Kirchhoff's almost complex structure in the literature is explicitly given as follows: $\forall u \in \mathbb{S}^6$,

$$J_u : T_u\mathbb{S}^6 \to T_u\mathbb{S}^6, \qquad X \mapsto J_u(X) = u \times X.$$

### 2.2 Spin(7)-structure equations

In this section, we shall recall the structure equation of Spin(7) which was established by Bryant [18]. We fix a basis of the complexification of the octonions $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{O}$ over $\mathbb{C}$ given by

$$\begin{aligned}
N &= \frac{1}{\sqrt{2}}(1 - \sqrt{-1}l),\\
\bar{N} &= \frac{1}{\sqrt{2}}(1 + \sqrt{-1}l),\\
E_1 &= iN,\\
E_2 &= jN,\\
E_3 &= -kN,\\
\bar{E}_1 &= i\bar{N},\\
\bar{E}_2 &= j\bar{N},\\
\bar{E}_3 &= -k\bar{N}.
\end{aligned}$$

A frame $(n\, f\, n\, \bar{f})$ is said to be a Spin(7) admissible frame if there exists an element $g \in$ Spin(7) such that

$$(n \quad f \quad n \quad \bar{f}) = g(N \quad E \quad N \quad \bar{E}),$$

where $E = (E_1, E_2, E_3)$. Usually, $(n\, f\, n\, \bar{f})$ is called a complexified Spin(7)-frame.

**Theorem 1** (Bryant [18]) *For a complexified* Spin(7)*-frame* $(n\, f\, n\, \bar{f})$, *we have*

$$d\begin{pmatrix} n \\ f \\ n \\ \bar{f} \end{pmatrix} = \begin{pmatrix} \sqrt{-1}\lambda & \eta & 0 & \bar{\theta} \\ -\bar{\eta}^T & \kappa & -\bar{\theta}^T & [\theta] \\ 0 & \theta & -\sqrt{-1}\lambda & \bar{\eta} \\ -\theta^T & [\bar{\theta}] & -\eta^T & \bar{\kappa} \end{pmatrix}\begin{pmatrix} n \\ f \\ n \\ \bar{f} \end{pmatrix} := \Psi\begin{pmatrix} n \\ f \\ n \\ \bar{f} \end{pmatrix},$$

*where* $\Psi$ *is a* Spin(7)$(\subset M_{8\times 8}(\mathbb{C}))$*-valued* 1*-form*, $\lambda$ *is a real-valued* 1*-form*, $\theta = (\theta^1, \theta^2, \theta^3)$, $\eta = (\eta^1, \eta^2, \eta^3)$ *are* $M_{1\times 3}(\mathbb{C})$*-valued* 1*-forms* $\kappa$ *is a* $u(3)$*-valued* 1*-form*, *which satisfy* $\operatorname{tr}\kappa + \sqrt{-1}\lambda = 0$, *and*

$$[\theta] = \begin{pmatrix} 0 & -\theta^3 & \theta^2 \\ \theta^3 & 0 & -\theta^1 \\ -\theta^2 & \theta^1 & 0 \end{pmatrix}.$$

The above structure equations of the Spin(7)-frame will play a crucial role in the proof of Theorem 3, as we shall see below.

## 3 Spin(7)-frame field along curves

In this section, we describe the construction of $G_2$-frame field along curves in $\operatorname{Im}\mathcal{O} \cong \mathbb{R}^7$ and then take its complexification. One may refer to [19–21] for details. Based on the complexification, we present a Frenet formula of the complexified Spin(7)-frame field along a curve in $\mathcal{O} \cong \mathbb{R}^8$.

Let $\gamma(s)$ be a unit speed curve in $\operatorname{Im}\mathcal{O} \cong \mathbb{R}^7$. We set $k_1(s) = \|\gamma_{ss}(s)\|$ and assume that this function does not vanish everywhere. The $G_2$-frame field $(I_4\, I_1\, I_2\, I_3\, I_5\, I_6\, I_7)$ along curves is defined by

$$\begin{aligned}
I_4(s) &= \gamma_s(s), \\
I_1(s) &= \frac{1}{k_1} I_{4s}, \\
I_5(s) &= I_1 \times I_4, \\
I_2(s) &= \frac{1}{k_2}\bigl(I_{1s} - \langle I_{1s}, I_4\rangle I_4 - \langle I_{1s}, I_5\rangle I_5\bigr), \\
I_3(s) &= I_1 \times I_2, \\
I_6(s) &= I_2 \times I_4, \\
I_7(s) &= I_3 \times I_4,
\end{aligned}$$

where

$$k_2(s) = \sqrt{\|I_{1s}\| - \langle I_{1s}, I_4\rangle^2 - \langle I_{1s}, I_5\rangle^2} > 0$$

is assumed and $I_5 = I_1 \times I_4$ is usually regarded as the $G_2$-vector along the curve.

The multiplication table of $(I_4\ I_1\ I_2\ I_3\ I_5\ I_6\ I_7)$ coincides with that of $(l\ i\ j\ k\ il\ jl\ kl)$, in other words, there exists a $G_2$-valued function $g$ such that

$$(I_4 \quad I_1 \quad I_2 \quad I_3 \quad I_5 \quad I_6 \quad I_7) = \bigl(g(l) \quad g(i) \quad g(j) \quad g(k) \quad g(il) \quad g(jl) \quad g(kl)\bigr).$$

If $k_2(s) = 0$, we take $I_2(s) \in (\mathrm{span}_{\mathbb{R}}\{I_4, I_1, I_5\})^{\perp}$ with $|I_2(s)| = 1$, $I_3(s) = I_1 \times I_2$, $I_6(s) = I_2 \times I_4$, $I_7(s) = I_3 \times I_4$, then $(I_4\ I_1\ I_2\ I_3\ I_5\ I_6\ I_7)$ also consists of a $G_2$-frame field along the curve in $\mathrm{Im}\,\mathcal{O} \cong \mathbb{R}^7$, in which $\{I_4,\ I_1,\ I_5\}$ consists of an autonomy system, i.e., it satisfies the formula

$$\begin{pmatrix} I_4 \\ I_1 \\ I_5 \end{pmatrix}_s = \begin{pmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & \rho_1 \\ 0 & -\rho_1 & 0 \end{pmatrix} \begin{pmatrix} I_4 \\ I_1 \\ I_5 \end{pmatrix},$$

where $\rho_1 = \langle I_{1s}, I_5\rangle$.

**Proposition 1** ([20, 21]) *Let $\gamma : I = (0,1) \to \mathrm{Im}\,\mathcal{O}$ be a curve with $k_1 > 0$. The associated $G_2$-frame field $(I_4\ I_1\ I_2\ I_3\ I_5\ I_6\ I_7)$ satisfies the following differential equation*:

$$\begin{pmatrix} I_4 \\ I_1 \\ I_2 \\ I_3 \\ I_5 \\ I_6 \\ I_7 \end{pmatrix}_s = \left(\begin{array}{c|ccc|ccc} 0 & k_1 & 0 & 0 & 0 & 0 & 0 \\ \hline -k_1 & 0 & k_2 & 0 & \rho_1 & 0 & 0 \\ 0 & -k_2 & 0 & \alpha & 0 & \rho_2 & \beta_1 \\ 0 & 0 & -\alpha & 0 & 0 & \beta_2 & \rho_3 \\ \hline 0 & -\rho_1 & 0 & 0 & 0 & k_2 & 0 \\ 0 & 0 & -\rho_2 & -\beta_2 & -k_2 & 0 & \alpha \\ 0 & 0 & -\beta_1 & -\rho_3 & 0 & -\alpha & 0 \end{array}\right) \begin{pmatrix} I_4 \\ I_1 \\ I_2 \\ I_3 \\ I_5 \\ I_6 \\ I_7 \end{pmatrix} \tag{7}$$

*with $\rho_1 = \langle I_{1s}, I_5\rangle$, $\rho_2 = \langle I_{2s}, I_6\rangle$, $\rho_3 = \langle I_{3s}, I_7\rangle$, $\alpha = \langle I_{2s}, I_3\rangle$, $\beta_1 = \langle I_{2s}, I_7\rangle$, $\beta_2 = \langle I_{3s}, I_6\rangle$. These functions satisfy the following*:

$$\rho_1 + \rho_2 + \rho_3 = 0, \tag{8}$$

$$\beta_1 - \beta_2 + k_1 = 0. \tag{9}$$

*Remark* 1 One notes from (7) that Eq. (5) can be rewritten as $I_{4t} = -k_1 I_5$, so Eq. (5) may be regarded as the $G_2$-tangent motion in $\mathbb{S}^6$.

The six functions $(k_1,\ k_2,\ \rho_1,\ \rho_3,\ \alpha,\ \beta_1)$ are the complete $G_2$-invariants of the curve $\gamma(s)$. Now, we give the complexification of the $G_2$-frame field $(I_4\ I_1\ I_2\ I_3\ I_5\ I_6\ I_7)$ along $\gamma(s)$

according to Bryant in [18] as follows. Let

$$\begin{aligned} n &= \frac{1}{\sqrt{2}}(1-\sqrt{-1}I_4), & \bar{n} &= \frac{1}{\sqrt{2}}(1+\sqrt{-1}I_4), \\ e_1 &= r(I_1-\sqrt{-1}I_5), & \bar{e}_1 &= \bar{r}(I_1+\sqrt{-1}I_5), \\ e_2 &= q(I_2-\sqrt{-1}I_6), & \bar{e}_2 &= \bar{q}(I_2+\sqrt{-1}I_6), \\ e_3 &= -p(I_3-\sqrt{-1}I_7), & \bar{e}_3 &= -\bar{p}(I_3+\sqrt{-1}I_7), \end{aligned} \tag{10}$$

where

$$\begin{aligned} r &= \frac{1}{\sqrt{2}}\exp\biggl(-\sqrt{-1}\int_0^s \rho_1\,d\tilde{s}\biggr), \\ q &= \frac{1}{\sqrt{2}}\exp\biggl(-\sqrt{-1}\int_0^s \rho_2\,d\tilde{s}\biggr), \\ p &= \sqrt{2}\bar{q}\bar{r}, \end{aligned} \tag{11}$$

and $\bar{\cdot}$ denotes the complex conjugation of elements in $\mathbb{C}\otimes_{\mathbb{R}}\mathcal{O}$, namely

$$\overline{x_1+\sqrt{-1}x_2} = x_1-\sqrt{-1}x_2, \quad \forall x_1, x_2 \in \mathcal{O}.$$

One notes that the complex conjugation $\bar{\cdot}$ here is different from the conjugation over $\mathcal{O}$. The complex-conjugate of $\mathbb{C}\otimes_{\mathbb{R}}\mathcal{O}$ satisfies

$$\overline{xy} = \bar{x}\quad\bar{y}, \qquad \forall x, y \in \mathbb{C}\otimes_{\mathbb{R}}\mathcal{O}.$$

In the sequel, we use the symbol $\bar{\cdot}$ to denote the complex conjugation of $\mathbb{C}\otimes_{\mathbb{R}}\mathcal{O}$, unless otherwise specified.

Note that Eq. (10) can be rewritten as

$$\begin{pmatrix} n \\ f \\ \bar{n} \\ \bar{f} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{\sqrt{-1}}{\sqrt{2}} & 0 \\ 0 & A & 0 & -\sqrt{-1}A \\ \frac{1}{\sqrt{2}} & 0 & \frac{\sqrt{-1}}{\sqrt{2}} & 0 \\ 0 & \bar{A} & 0 & \sqrt{-1}\quad\bar{A} \end{pmatrix} \begin{pmatrix} 1 \\ J_1 \\ I_4 \\ J_2 \end{pmatrix} := N_1 \begin{pmatrix} 1 \\ J_1 \\ I_4 \\ J_2 \end{pmatrix}, \tag{12}$$

where

$$f = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \qquad J_1 = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} I_5 \\ I_6 \\ I_7 \end{pmatrix}, \qquad A = \begin{pmatrix} r & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & -p \end{pmatrix}. \tag{13}$$

**Theorem 2** *For the complexified* Spin(7)*-frame field* $(n\, e_1\, e_2\, e_3\, \bar{n}\, \bar{e}_1\, \bar{e}_2\, \bar{e}_3)$ *along the curve* $\gamma$, *we have the following Frenet formula*:

$$\begin{pmatrix} n \\ e_1 \\ e_2 \\ e_3 \\ \bar{n} \\ \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{pmatrix}_s = \left(\begin{array}{c|ccc|c|ccc} 0 & \varphi_1 & 0 & 0 & 0 & -\bar{\varphi}_1 & 0 & 0 \\ \hline -\bar{\varphi}_1 & 0 & \varphi_2 & 0 & \bar{\varphi}_1 & 0 & 0 & 0 \\ 0 & -\bar{\varphi}_2 & 0 & \varphi_3 & 0 & 0 & 0 & \varphi_1 \\ 0 & 0 & -\bar{\varphi}_3 & 0 & 0 & 0 & -\varphi_1 & 0 \\ \hline 0 & -\varphi_1 & 0 & 0 & 0 & \bar{\varphi}_1 & 0 & 0 \\ \hline \varphi_1 & 0 & 0 & 0 & -\varphi_1 & 0 & \bar{\varphi}_2 & 0 \\ 0 & 0 & 0 & \bar{\varphi}_1 & 0 & -\varphi_2 & 0 & \bar{\varphi}_3 \\ 0 & 0 & -\bar{\varphi}_1 & 0 & 0 & 0 & -\varphi_3 & 0 \end{array}\right) \begin{pmatrix} n \\ e_1 \\ e_2 \\ e_3 \\ \bar{n} \\ \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{pmatrix}, \tag{14}$$

*where*

$$\varphi_1 = -\frac{\sqrt{-1}}{\sqrt{2}} k_1 \bar{r}, \qquad \varphi_2 = 2k_2 r\bar{q}, \qquad \varphi_3 = -\sqrt{2} q^2 r\left[2\alpha + \sqrt{-1}(\beta_1 + \beta_2)\right]. \tag{15}$$

*Proof* First of all, we can rewrite (7) as

$$\begin{pmatrix} 1 \\ J_1 \\ I_4 \\ J_2 \end{pmatrix}_s = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & B & -u^T & C \\ 0 & u & 0 & 0 \\ 0 & -C^T & 0 & B \end{pmatrix} \begin{pmatrix} 1 \\ J_1 \\ I_4 \\ J_2 \end{pmatrix} := N_2 \begin{pmatrix} 1 \\ J_1 \\ I_4 \\ J_2 \end{pmatrix}, \tag{16}$$

where

$$u = \begin{pmatrix} k_1 \\ 0 \\ 0 \end{pmatrix}^T, \qquad B = \begin{pmatrix} 0 & k_2 & 0 \\ -k_2 & 0 & \alpha \\ 0 & -\alpha & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} \rho_1 & 0 & 0 \\ 0 & \rho_2 & \beta_1 \\ 0 & \beta_2 & \rho_3 \end{pmatrix}. \tag{17}$$

From (12) and (16), we obtain

$$(n \quad f \quad \bar{n} \quad \bar{f})_s^T = (N_{1s} + N_1 N_2) N_1^{-1} (n \quad f \quad \bar{n} \quad \bar{f})^T, \tag{18}$$

where

$$(N_{1s} + N_1 N_2) N_1^{-1} = \begin{pmatrix} 0 & -\frac{\sqrt{-1}}{\sqrt{2}} u\bar{A} & 0 & -\frac{\sqrt{-1}}{\sqrt{2}} uA \\ -\frac{\sqrt{-1}}{\sqrt{2}} Au^T & \eta_1 & \frac{\sqrt{-1}}{\sqrt{2}} Au^T & \eta_2 \\ 0 & \frac{\sqrt{-1}}{\sqrt{2}} u\bar{A} & 0 & \frac{\sqrt{-1}}{\sqrt{2}} uA \\ -\frac{\sqrt{-1}}{\sqrt{2}} \bar{A}u^T & \bar{\eta}_2 & \frac{\sqrt{-1}}{\sqrt{2}} \bar{A}u^T & \bar{\eta}_1 \end{pmatrix}, \tag{19}$$

$$\eta_1 = \left[2A_s + 2AB + \sqrt{-1}A\left(C^T + C\right)\right]\bar{A},$$

$$\eta_2 = -\sqrt{-1}A\left(C - C^T\right)A.$$

It follows from (8), (9), (13), and (17) that

$$-\frac{\sqrt{-1}}{\sqrt{2}} u\bar{A} = \left(-\frac{\sqrt{-1}}{\sqrt{2}} k_1 \bar{r} \qquad 0 \qquad 0\right),$$

**Table 1** The multiplication table *AB*

| $A\backslash B$ | n | $e_1$ | $e_2$ | $e_3$ | $\bar{n}$ | $\bar{e}_1$ | $\bar{e}_2$ | $\bar{e}_3$ |
|---|---|---|---|---|---|---|---|---|
| $n$ | $\sqrt{2}n$ | 0 | 0 | 0 | 0 | $\sqrt{2}\bar{e}_1$ | $\sqrt{2}\bar{e}_2$ | $\sqrt{2}\bar{e}_3$ |
| $e_1$ | $\sqrt{2}e_1$ | 0 | $-\sqrt{2}\bar{e}_3$ | $\sqrt{2}\bar{e}_2$ | 0 | $-\sqrt{2}\bar{n}$ | 0 | 0 |
| $e_2$ | $\sqrt{2}e_2$ | $\sqrt{2}\bar{e}_3$ | 0 | $-\sqrt{2}\bar{e}_1$ | 0 | 0 | $-\sqrt{2}\bar{n}$ | 0 |
| $e_3$ | $\sqrt{2}e_3$ | $-\sqrt{2}\bar{e}_2$ | $\sqrt{2}\bar{e}_1$ | 0 | 0 | 0 | 0 | $-\sqrt{2}\bar{n}$ |
| $\bar{n}$ | 0 | $\sqrt{2}e_1$ | $\sqrt{2}e_2$ | $\sqrt{2}e_3$ | $\sqrt{2}\bar{n}$ | 0 | 0 | 0 |
| $\bar{e}_1$ | 0 | $-\sqrt{2}n$ | 0 | 0 | $\sqrt{2}\bar{e}_1$ | 0 | $-\sqrt{2}e_3$ | $\sqrt{2}e_2$ |
| $\bar{e}_2$ | 0 | 0 | $-\sqrt{2}n$ | 0 | $\sqrt{2}\bar{e}_2$ | $\sqrt{2}e_3$ | 0 | $-\sqrt{2}e_1$ |
| $\bar{e}_3$ | 0 | 0 | 0 | $-\sqrt{2}n$ | $\sqrt{2}\bar{e}_3$ | $-\sqrt{2}e_2$ | $\sqrt{2}e_1$ | 0 |

$$\eta_1 = \begin{pmatrix} 0 & 2k_2 r\bar{q} & 0 \\ -2k_2 q\bar{r} & 0 & -\sqrt{2}q^2 r[2\alpha + \sqrt{-1}(\beta_1 + \beta_2)] \\ 0 & \sqrt{2}\bar{q}^2\bar{r}[2\alpha - \sqrt{-1}(\beta_1 + \beta_2)] & 0 \end{pmatrix},$$

$$\eta_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{-1}}{\sqrt{2}}k_1\bar{r} \\ 0 & \frac{\sqrt{-1}}{\sqrt{2}}k_1\bar{r} & 0 \end{pmatrix}.$$

By setting

$$\varphi_1 = -\frac{\sqrt{-1}}{\sqrt{2}}k_1\bar{r}, \qquad \varphi_2 = 2k_2 r\bar{q}, \qquad \varphi_3 = -\sqrt{2}q^2 r\big[2\alpha + \sqrt{-1}(\beta_1 + \beta_2)\big],$$

we arrive at Eq. (14). □

Based on the complexified Spin(7)-frame field $(n\, f\, \bar{n}\, \bar{f})^T$ along the curve $\gamma(s)$, we shall establish the multiplication table with respect to $(n\, e_1\, e_2\, e_3\, \bar{n}\, \bar{e}_1\, \bar{e}_2\, \bar{e}_3)$, which is well-suited for the development in the next section.

Furthermore, the complexified Spin(7)-frame field $(n\, f\, \bar{n}\, \bar{f})^T$ along the curves satisfies

$$\langle n, n\rangle = 0, \qquad \langle n, \bar{n}\rangle = 1, \qquad \langle n, e_i\rangle = 0, \qquad \langle n, \bar{e}_i\rangle = 0, \qquad \langle e_i, e_j\rangle = 0, \tag{20}$$

$$\langle e_i, \bar{e}_j\rangle = \delta_{ij}, \qquad e_i n = \sqrt{2}e_i, \qquad \langle e_1 e_2, e_3\rangle = -\sqrt{2}, \tag{21}$$

for any $i \in \{1, 2, 3\}$. By Table 1, (20) and (21), Theorem 1 may be directly deduced.

## 4 The vector elliptic Liouville equation

In this section, we shall transform the $G_2$-tangent motion (5) to the vector elliptic Liouville equation.

At any given instant of time t, the unit vector $T(t,s) = I_4(t,s)$ may be associated with the unit tangent of a space curve $\gamma$ in $\mathbb{R}^7$.

**Proposition 2** *Let $T = T(t,s)$ be a solution of Eq. (5). If $T(0,s)$ parametrized by arclength, then so is $T = T(t,s)$ for all $t$.*

*Proof* It suffices to prove that $\frac{d}{dt}|T|^2 = 0$ for the solutions of Eq. (5). In fact,

$$\frac{d}{dt}|T|^2 = \langle T, T\rangle_t = 2\langle T_t, T\rangle = 2\langle T \times T_s, T\rangle = 0.$$ □

**Theorem 3** *The $G_2$-tangent motion $I_{4t} = I_4 \times I_{4s}$ in $\mathbb{S}^6$ is equivalent to the coupled Lamb equations*:

$$\begin{aligned} \sqrt{-1}\varphi_{1t} &= \varphi_{1s} + 2\varphi_1 \int_0^s \left(2|\varphi_1|^2 - |\varphi_2|^2\right) d\tilde{s}, \\ \sqrt{-1}\varphi_{2t} &= \varphi_{2s} + 2\varphi_2 \int_0^s \left(2|\varphi_2|^2 - 3|\varphi_1|^2 - |\varphi_3|^2\right) d\tilde{s}, \\ \sqrt{-1}\varphi_{3t} &= \varphi_{3s} + 2\varphi_3 \int_0^s \left(2|\varphi_3|^2 - |\varphi_2|^2\right) d\tilde{s}. \end{aligned} \tag{22}$$

*Proof* From $\gamma_s = I_4$, $I_{4s} = k_1 I_1$, $I_4 \times I_1 = -I_5$, $I_5 = \sqrt{-1}(\bar{r}e_1 - r\bar{e}_1)$, $\varphi_1 = -\frac{\sqrt{-1}}{\sqrt{2}}k_1\bar{r}$, and Table 1, we have

$$n_t = \frac{1}{\sqrt{2}}(1 - \sqrt{-1}I_4)_t = -\sqrt{-1}\varphi_1 e_1 - \sqrt{-1}\bar{\varphi}_1\bar{e}_1.$$

Hence, the complexified Spin(7)-frame field by $(n\, f\ \bar{n}\, \bar{f})^T$ admits

$$\begin{pmatrix} n \\ f \\ \bar{n} \\ \bar{f} \end{pmatrix}_t = \begin{pmatrix} 0 & \omega & 0 & -\bar{\omega} \\ -\bar{\omega}^T & \kappa & \bar{\omega}^T & [\Omega] \\ 0 & -\omega & 0 & \bar{\omega} \\ \omega^T & [\bar{\Omega}] & -\omega^T & \bar{\kappa} \end{pmatrix} \begin{pmatrix} n \\ f \\ \bar{n} \\ \bar{f} \end{pmatrix},$$

where

$$\kappa = \begin{pmatrix} \sqrt{-1}R_1 & a_1 & a_2 \\ -\bar{a}_1 & \sqrt{-1}R_2 & a_3 \\ -\bar{a}_2 & -\bar{a}_3 & \sqrt{-1}R_3 \end{pmatrix}, \qquad [\Omega] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sqrt{-1}\varphi_1 \\ 0 & \sqrt{-1}\varphi_1 & 0 \end{pmatrix},$$

$\omega = (\omega^1\ \omega^2\ \omega^3) = (-\sqrt{-1}\varphi_1\ 0\ 0)$, $R_1 + R_2 + R_3 = 0$, and $R_i \in \mathbb{R}$, $a_i \in \mathbb{C}$, $i \in \{1, 2, 3\}$ are functions of $t$ and $s$, which will be determined later.

On the other hand, Eq. (14) can be rewritten as

$$\begin{pmatrix} n \\ f \\ \bar{n} \\ \bar{f} \end{pmatrix}_s = \begin{pmatrix} 0 & g & 0 & -\bar{g} \\ -\bar{g}^T & M & \bar{g}^T & [G] \\ 0 & -g & 0 & \bar{g} \\ g^T & [\bar{G}] & -g^T & \bar{M} \end{pmatrix} \begin{pmatrix} n \\ f \\ \bar{n} \\ \bar{f} \end{pmatrix},$$

where

$$g = \begin{pmatrix} \varphi_1 \\ 0 \\ 0 \end{pmatrix}^T, \qquad [G] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \varphi_1 \\ 0 & -\varphi_1 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & \varphi_2 & 0 \\ -\bar{\varphi}_2 & 0 & \varphi_3 \\ 0 & -\bar{\varphi}_3 & 0 \end{pmatrix}.$$

From the integrability condition

$$\begin{pmatrix} n \\ f \\ \bar{n} \\ \bar{f} \end{pmatrix}_{ts} = \begin{pmatrix} n \\ f \\ \bar{n} \\ \bar{f} \end{pmatrix}_{st},$$

we have

$$g_t = \omega_s + \omega M + \bar{g}[\bar{\Omega}] - \bar{\omega}[\bar{G}] - g\kappa, \tag{23a}$$

$$M_t = \kappa_s - 2\bar{\omega}^T g + \kappa M + [\Omega][\bar{G}] + 2\bar{g}^T\omega - M\kappa - [G][\bar{\Omega}]. \tag{23b}$$

From (23a) and (23b), we have

$$\begin{cases} \varphi_{1t} = -\sqrt{-1}\varphi_{1s} - \sqrt{-1}R_1\varphi_1, \\ a_1 = -\sqrt{-1}\varphi_2, \\ a_2 = 0, \end{cases} \tag{24}$$

and

$$\begin{cases} \varphi_{2t} = a_{1s} + \sqrt{-1}\varphi_2(R_1 - R_2) - a_2\bar{\varphi}_3, \\ \varphi_{3t} = a_{3s} + \sqrt{-1}\varphi_3(R_2 - R_3) + \bar{\varphi}_2 a_2, \\ \sqrt{-1}R_{1s} = 4\sqrt{-1}|\varphi_1|^2 - \varphi_2\bar{a}_1 + \bar{\varphi}_2 a_1, \\ \sqrt{-1}R_{2s} = -2\sqrt{-1}|\varphi_1|^2 - \bar{\varphi}_2 a_1 + \varphi_2\bar{a}_1 + \bar{\varphi}_3 a_3 - \varphi_3\bar{a}_3, \\ \sqrt{-1}R_{3s} = -2\sqrt{-1}|\varphi_1|^2 - \bar{\varphi}_3 a_3 + \varphi_3\bar{a}_3, \\ \varphi_2 a_3 = a_{2s} + a_1\varphi_3. \end{cases} \tag{25}$$

One notes that $R_1 + R_2 + R_3 = 0$ in (25), which is just compatible with the requirement. Furthermore, from (24) and (25), we obtain

$$\begin{cases} \varphi_{1t} = -\sqrt{-1}\varphi_{1s} - \sqrt{-1}\varphi_1 \int_0^s (4|\varphi_1|^2 - 2|\varphi_2|^2)\,d\tilde{s} - \sqrt{-1}\varphi_1 R_{10}(t), \\ \varphi_{2t} = -\sqrt{-1}\varphi_{2s} - \sqrt{-1}\varphi_2 \int_0^s (4|\varphi_2|^2 - 6|\varphi_1|^2 - 2|\varphi_3|^2)\,d\tilde{s} + \sqrt{-1}\varphi_2(R_{10}(t) - R_{20}(t)), \\ \varphi_{3t} = -\sqrt{-1}\varphi_{3s} - \sqrt{-1}\varphi_3 \int_0^s (4|\varphi_3|^2 - 2|\varphi_2|^2)\,d\tilde{s} + \sqrt{-1}\varphi_3(2R_{20}(t) + R_{10}(t)), \end{cases}$$

where $R_{10}$, $R_{20}$ are the functions depending only on $t$. Now, under the transformations

$$\varphi_1 \mapsto \varphi_1 \exp\left(-\sqrt{-1}\int_0^t R_{10}\,d\tilde{t}\right),$$

$$\varphi_2 \mapsto \varphi_2 \exp\left(\sqrt{-1}\int_0^t (R_{10} - R_{20})\,d\tilde{t}\right),$$

$$\varphi_3 \mapsto \varphi_3 \exp\left(\sqrt{-1}\int_0^t (R_{10} + 2R_{20})\,d\tilde{t}\right),$$

we arrive at Eqs. (22). □

*Remark* 2 If $\varphi_2 = \varphi_3 = 0$, Eqs. (22) return to the Lamb equation

$$\sqrt{-1}\varphi_{1t} = \varphi_{1s} + 4\varphi_1 \int_0^s |\varphi_1|^2 \, d\tilde{s}. \tag{26}$$

A short calculation verifies that equating the imaginary parts of Eq. (26) yields $\rho_1 = \frac{k_{1t}}{k_1}$ and equating the real parts returns to the two-dimensional elliptic Liouville equation (1), where $\phi = \ln k_1$.

Let $2\alpha + \sqrt{-1}(\beta_1 + \beta_2) := \sqrt{4\alpha^2 + (\beta_1 + \beta_2)^2} \exp(-\sqrt{-1} \int_0^s \widetilde{\theta} \, d\tilde{s})$. From Eq. (11) and Eq. (15), we have

$$\begin{cases} \varphi_1 = -\frac{\sqrt{-1}}{2} k_1 \exp(\sqrt{-1} \int_0^s \rho_1 \, d\tilde{s}), \\ \varphi_2 = k_2 \exp(\sqrt{-1} \int_0^s (\rho_2 - \rho_1) \, d\tilde{s}), \\ \varphi_3 = k_3 \exp(-\sqrt{-1} \int_0^s \theta_1 \, d\tilde{s}), \end{cases}$$

where $k_3 = -\sqrt{\alpha^2 + (\frac{\beta_1+\beta_2}{2})^2}$, $\theta_1 = \widetilde{\theta} + \rho_1 + 2\rho_2$. If $k_2, k_3 \neq 0$, a short calculation verifies that equating the imaginary parts of Eqs. (22) yields $\rho_1 = \frac{k_{1t}}{k_1}$, $\rho_2 - \rho_1 = \frac{k_{2t}}{k_2}$, $\theta_1 = -\frac{k_{3t}}{k_3}$. A straightforward computation shows that the real parts of Eqs. (22) give the following equations:

$$\begin{cases} (\frac{k_{1t}}{k_1})_t + (\frac{k_{1t}}{k_1})_s = -k_1^2 + 2k_2^2, \\ (\frac{k_{2t}}{k_2})_t + (\frac{k_{2t}}{k_2})_s = -4k_2^2 + \frac{3}{2}k_1^2 + 2k_3^2, \\ (\frac{k_{3t}}{k_3})_t + (\frac{k_{3t}}{k_3})_s = -4k_3^2 + 2k_2^2. \end{cases}$$

Here set $\phi_1 = \ln k_1$, $\phi_2 = \ln k_2$, $\phi_3 = \ln(-k_3)$, we have showed the following.

**Theorem 4** *The $G_2$-tangent motion $I_{4t} = I_4 \times I_{4s}$ in $\mathbb{S}^6$ is also equivalent to the vector elliptic Liouville equation*:

$$U_{tt} + U_{ss} = Ve^{2U},$$

*where*

$$U = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \qquad V = \begin{pmatrix} -1 & 2 & 0 \\ \frac{3}{2} & -4 & 2 \\ 0 & 2 & -4 \end{pmatrix}, \quad e^U := \begin{pmatrix} e^{\phi_1} \\ e^{\phi_2} \\ e^{\phi_3} \end{pmatrix}.$$

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## Publisher's Note



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