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Symmetry analysis for three-dimensional dissipation Rossby waves

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Abstract

Rossby waves, belonging to the most important waves in the atmosphere and ocean, can affect the energy transfer of the atmosphere and ocean and have significant theoretical meaning and research value. In previous research performed with the theory and calculation method limit, the dissipation effect was commonly ignored. However, under the conditions of the weak linear approximation, the magnitude difference between nonlinear and dissipation is small, and the dissipation effect must be considered. In this paper, based on the classic Lie group approach, the (3 + 1)-dimensional quasi-geodetiophic vorticity equation with dissipation effect is solved. With the help of the solutions, we can better comprehend the influence of the dissipation effect on the propagation of Rossby waves.

Keywords: Rossby wave; Dissipation effect; Lie symmetry

1 Introduction

Rossby waves, which are caused by the rotation of the earth and the influence of the sphere effect, are long and large-scale permanent waves in the ocean and atmosphere, such as the huge red spots in Jupiter's atmosphere and eddy currents in the gulf of Mexico. Rossby waves determine the ocean's response to the climate and atmospheric change and have significant theoretical meaning and research value. However, in recent years, many researchers have focused on the traveling-wave solutions for handling nonlinear problems [1–3]. Few researchers have paid attention to the solution of the Rossby wave. Thus, with the development of theory, the study of Rossby waves is an important research direction [4–7].

Nonlinear partial differential equations [8–10] play an important role in the field of Rossby waves. Many models have been derived [11–14], and many methods have been used to solve the nonlinear partial differential equations, such as the algebro-geometric method [15, 16], Hirota method [17], Painlevé analysis method [18], Darboux transformations [19–21], Lie symmetry method [22–24] and so on [25–29]. Based on the classic Lie group method, the $(2+1)$-dimensional nonlinear inviscid barotropic nondivergent vorticity equation was studied by Huang and Lou [30], and the $(3+1)$-dimensional nonlinear Charney–Obukhov equation was studied by Kudryavtsev and Myagkov [31]. However, the dissipation effect was ignored in these studies of Rossby waves. Friction dissipation, one of the external forces in the atmosphere and ocean, plays an increasingly vital role in atmospheric circulation. Under the conditions of weak linear approximation, the magnitude difference between nonlinear and dissipation is very small, i.e., the dissipation effect should be considered in the research of Rossby waves.

In this paper, we consider the (3 + 1)-dimensional quasi-geodetiophic vorticity equation with dissipation effect

$$\frac{\partial}{\partial t}\Delta u + J(u,\Delta u) + \alpha\Delta u + \beta\frac{\partial}{\partial x}u = 0, \tag{1}$$

where the three-dimensional Laplacian can be expressed by $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, the dimensionless stream function can be described by u, the Jacobian operator can be introduced by $J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$, $\beta = \beta_0(L^2/U)$ and $\beta_0 = (\omega_0/R_0)\cos\phi_0$, in which ω_0 is the angular frequency of the Earth's rotation, L depicts the characteristic horizontal length, U expresses the velocity scales, ϕ_0 and R_0 are the latitude and the Earth's radius, respectively, and $\alpha\Delta u$ denotes the dissipation effect, in which α is the dissipation coefficient.

The structure of the full paper is as follows. We apply the classic Lie group method to acquire the solution of a (3 + 1)-dimensional dissipation Rossby wave in Sect. 2. In Sect. 3, we discuss the approximate analytical solution of a (2 + 1)-dimensional dissipation Rossby wave. Finally, the dissipation effect is researched, and some conclusions are reported in Sect. 4.

2 (3 + 1)-Dimensional dissipation Rossby wave

To discuss the dissipation effect of three-dimensional dissipation Rossby waves, we first study the solution of Eq. (1). In the following, we introduce the vector field

$$\begin{aligned} V &= \xi(x,y,z,t,u)\frac{\partial}{\partial x} + \eta(x,y,z,t,u)\frac{\partial}{\partial y} + \lambda(x,y,z,t,u)\frac{\partial}{\partial z} \\ &\quad + \tau(x,y,z,t,u)\frac{\partial}{\partial t} + \phi(x,y,z,t,u)\frac{\partial}{\partial u}. \end{aligned}$$

The first-order propagator is defined as

$$Pr^{(1)}V = V + \phi^x\frac{\partial}{\partial u_x} + \phi^y\frac{\partial}{\partial u_y} + \phi^z\frac{\partial}{\partial u_z} + \phi^t\frac{\partial}{\partial u_t},$$

and the second-order propagator is defined as

$$\begin{aligned} Pr^{(2)}V &= Pr^{(1)}V + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xy}\frac{\partial}{\partial u_{xy}} + \phi^{yy}\frac{\partial}{\partial u_{xx}} + \phi^{xz}\frac{\partial}{\partial u_{pz}} + \phi^{zz}\frac{\partial}{\partial u_{zz}} + \phi^{yz}\frac{\partial}{\partial u_{yz}} \\ &\quad + \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{yt}\frac{\partial}{\partial u_{\theta t}} + \phi^{zt}\frac{\partial}{\partial u_{zt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}}. \end{aligned}$$

Similarly, the third-order propagator has the form

$$\begin{aligned} Pr^{(3)}V &= Pr^{(2)}V + \phi^{xxx}\frac{\partial}{\partial u_{xxx}} + \phi^{xxy}\frac{\partial}{\partial u_{xxy}} + \phi^{xyy}\frac{\partial}{\partial u_{xyy}} + \phi^{yyy}\frac{\partial}{\partial u_{yyy}} + \phi^{\rho\rho z}\frac{\partial}{\partial u_{xxz}} \\ &\quad + \phi^{xzz}\frac{\partial}{\partial u_{xzz}} + +\phi^{zzz}\frac{\partial}{\partial u_{zzz}} + \cdots + \phi^{yyz}\frac{\partial}{\partial u_{yyz}} + \cdots. \end{aligned} \tag{2}$$

According to the Lie group method, by substituting (2) into (1), we obtain

$$\phi^{xxt} + \phi^{yyt} + \phi^{xxy} + \phi^{yyy}u_x + \phi^{yzz}u_x + \phi^x(u_{xxy} + u_{yyy} + u_{yzz}) - \left(\phi^{xxx} + \phi^{xyy} + \phi^{xzz}\right)u_y + \alpha\left(\phi^{xx} + \phi^{yy} + \phi^{zz}\right) + \beta\left(\phi^x + \phi^y + \phi^z\right) = 0, \tag{3}$$

where

$$\phi^x = D_x(\phi - \xi u_x - \eta u_y - \lambda u_z - \tau u_t) + \xi u_{xx} + \eta u x_{xy} + \lambda u_{xz} + \tau u_{xt},$$
$$\phi^{xx} = D_x^2(\phi - \xi u_x - \eta u_y - \lambda u_z - \tau u_t) + \xi u_{xxx} + \eta u x_{xxy} + \lambda u_{xxz} + \tau u_{xxt},$$
$$\phi^{xxx} = D_x^3(\phi - \xi u_x - \eta u_y - \lambda u_z - \tau u_t) + \xi u_{xxxx} + \eta u x_{xxxy} + \lambda u_{xxxz} + \tau u_{xxxt}.$$

It is important to emphasize that

$$\frac{\partial^3 u}{\partial x^2\partial t} + \frac{\partial^3 u}{\partial y^2\partial t} + \frac{\partial^3 u}{\partial z^2\partial t} = -\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^3\partial y} - \frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial y^3} - \frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial y\partial z^2} + \frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial x^3} + \frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial x\partial z^2} - \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) - \beta\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

and

$$\frac{\partial^3 u}{\partial x^2\partial t} = -\frac{\partial^3 u}{\partial y^2\partial t} - \frac{\partial^3 u}{\partial z^2\partial t} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^3\partial y} - \frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial y^3} - \frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial y\partial z^2} + \frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial x^3} + \frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial x\partial z^2} - \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) - \beta\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right),$$

and that $u_z, u_y, u_x, u_t, u_{xt}, u_{xz}, u_{xy}, u_{xyz}, \ldots$ are not independent.

According to a complicated calculation and the above transformation, we obtain

$$\begin{cases}
\frac{\partial\xi}{\partial u} = 0, & \frac{\partial\eta}{\partial u} = 0, \quad \frac{\partial\lambda}{\partial u} = 0, \quad \frac{\partial\tau}{\partial u} = 0,\\
\frac{\partial\xi}{\partial y} = 0, & \frac{\partial\xi}{\partial z} = 0, \quad \frac{\partial\eta}{\partial x} = 0, \quad \frac{\partial\eta}{\partial z} = 0,\\
\frac{\partial\lambda}{\partial x} = 0, & \frac{\partial\lambda}{\partial y} = 0, \quad \frac{\partial\lambda}{\partial t} = 0,\\
\frac{\partial\tau}{\partial x} = 0, & \frac{\partial\tau}{\partial y} = 0, \quad \frac{\partial\tau}{\partial z} = 0,\\
\frac{\partial^2\phi}{\partial u^2} = 0, & \frac{\partial^2\phi}{\partial u\partial x} = 0, \quad \frac{\partial^2\phi}{\partial uy} = 0, \quad \frac{\partial^2\phi}{\partial u\partial z} = 0, \quad \frac{\partial^2\phi}{\partial u\partial t} = 0,\\
\frac{\partial^2\xi}{\partial x^2} = 0, & \frac{\partial^2\eta}{\partial y^2} = 0, \quad \frac{\partial^2\lambda}{\partial z^2} = 0.
\end{cases}$$

In addition, we can obtain the following coefficients:

$$\begin{cases}
1: & \frac{\partial^3\phi}{\partial x^2\partial t} + \frac{\partial^3\phi}{\partial y^2\partial t} + \frac{\partial^3\phi}{\partial z^2\partial t} + \alpha(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}) + \beta\frac{\partial\phi}{\partial x} = 0,\\
u_x: & \beta(\frac{\partial\xi}{\partial x} + \frac{\partial\tau}{\partial t}) = 0,\\
u_{yyt}: & \frac{\partial\xi}{\partial x} - \frac{\partial\eta}{\partial y} = 0,\\
u_{zzt}: & \frac{\partial\xi}{\partial x} - \frac{\partial\lambda}{\partial z} = 0,\\
u_x u_{yyy}: & \frac{\partial\phi}{\partial u} + \frac{\partial\xi}{\partial x} - 3\frac{\partial\eta}{\partial y} + \frac{\partial\tau}{\partial t} = 0,\\
u_x u_{xxy}: & \frac{\partial\phi}{\partial u} - \frac{\partial\xi}{\partial x} - \frac{\partial\eta}{\partial y} + \frac{\partial\tau}{\partial t} = 0,\\
u_y u_{xxx}: & -\frac{\partial\phi}{\partial u} + \frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} - \frac{\partial\tau}{\partial t} = 0,\\
u_y u_{xyy}: & -\frac{\partial\phi}{\partial u} - \frac{\partial\xi}{\partial x} + 3\frac{\partial\eta}{\partial y} - \frac{\partial\tau}{\partial t} = 0.
\end{cases}$$

By comparing the coefficients of $u_z, u_y, u_x, u_t, u_{xt}, u_{xz}, u_{xy}, u_{xyz}, \ldots$, the general solutions can be written as

$$\begin{cases} \xi = C_1 + C_5 x + C_7 \int h_2(t)\,dt, \\ \eta = C_2 + C_5 y + C_{10} \int h_5(t)\,dt, \\ \lambda = C_3 + C_5 z, \\ \tau = C_4 - C_5 t, \\ \phi = 3C_5 u + C_6 h_1(t) - C_7 h_2(t) y + C_8 h_3(z) + C_9 h_4(t) z \\ \qquad + C_{10}[h_5(t)x - \frac{\beta}{2} h_5(t) z^2 + \frac{\alpha}{2} e^{-t} z^2], \end{cases} \tag{4}$$

where $C_1, C_2, \ldots, C_{10}$ are arbitrary constants, and $h_1(t), h_2(t), \ldots, h_5(t)$ are arbitrary functions of t.

Thus, we obtain the Lie algebra basis of the classic symmetry group for Eq. (1):

$$\begin{cases} V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial y}, \qquad V_3 = \frac{\partial}{\partial z}, \qquad V_4 = \frac{\partial}{\partial t}, \\ V_5 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - t\frac{\partial}{\partial t} + 3u\frac{\partial}{\partial u}, \qquad V_6 = h_1(t)\frac{\partial}{\partial u}, \\ V_7 = (\int h_2(t)\,dt)\frac{\partial}{\partial x} - h_2(t) y \frac{\partial}{\partial u}, \\ V_8 = h_3(z)\frac{\partial}{\partial u}, \qquad V_9 = h_4(t) z \frac{\partial}{\partial u}, \\ V_{10} = (\int h_5(t)\,dt)\frac{\partial}{\partial y} + [h_5(t)x - \frac{\beta}{2} h_5(t) z^2 + \frac{\alpha}{2} e^{-t} z^2]\frac{\partial}{\partial u}, \end{cases} \tag{5}$$

where h_1, h_2, h_3, h_4, h_5 are continuous functions.

If z is eliminated in V_5, operators V_1, V_2, V_3, V_4, V_5, V_6, V_7 are the Lie algebra basis of the classic symmetry group for the (2 + 1)-dimensional case. Specifically, V_8, V_9, V_{10} can be extended to the classic symmetry group for the (3 + 1)-dimensional case.

When we know a particular solution, a new solution of the differential equation can be acquired by the classic Lie symmetry group method [32–34]. Suppose that a solution of Eq. (1) is expressed by $u_s(t,x,y,z)$. It is easy to infer that operators V_6, V_7, V_8, V_9, V_{10} have the following formulas for the new solution $u_{\text{new}}(t,x,y,z)$:

$$\begin{cases} u_{\text{new}}(t,x,y,z) = h_1(t) + u_s(t,x,y,z), & \text{(a)} \\ u_{\text{new}}(t,x,y,z) = -h_2(t)y + u_s(t, x + \int h_2(t)\,dt, y, z), & \text{(b)} \\ u_{\text{new}}(t,x,y,z) = h_3(z) + u_s(t,x,y,z), & \text{(c)} \\ u_{\text{new}}(t,x,y,z) = h_4(t)z + u_s(t,x,y,z), & \text{(d)} \\ u_{\text{new}}(t,x,y,z) = h_5(t)x - \frac{\beta}{2} h_5(t) z^2 + \frac{\alpha}{2} e^{-t} z^2 + u_s(t, x, y + \int h_5(t)\,dt, z). & \text{(e)} \end{cases} \tag{6}$$

According to the nontrivial transformations (6b), (6d), and (6e), we obtain

$$u_{\text{new}}(t,\rho,\theta,z) = h_5(t)x - h_2(t)y + h_4(t)z - \frac{\beta}{2} h_5(t) z^2 + \frac{\alpha}{2} e^{-t} z^2 + u_s\left(t, x + \int h_2(t)\,dt, y + \int h_5(t)\,dt, z\right). \tag{7}$$

Clearly, Eq. (7) has the following transformations for the component velocities:

$$\begin{cases} l_{\text{new}}(t,x,y,z) = -h_2(t) + l_s(t, x + \int h_2(t)\,dt, y + \int h_5(t)\,dt, z), & \text{(a)} \\ m_{\text{new}}(t,x,y,z) = -h_5(t) + m_s(t, x + \int h_2(t)\,dt, y + \int h_5(t)\,dt, z), & \text{(b)} \\ n_{\text{new}}(t,x,y,z) = -[h_4(t) - \frac{\beta}{2}h_5(t)z + \frac{\alpha}{2}e^{-t}z] & \\ \qquad\qquad + n_s(t, x + \int h_2(t)\,dt, y + \int h_5(t)\,dt, z), & \text{(c)} \end{cases} \tag{8}$$

where

$$\begin{cases} l_s(t,x,y,z) = -\frac{\partial u_s(t,x,y,z)}{\partial y}, \\ m_s(t,x,y,z) = \frac{\partial u_s(t,x,y,z)}{\partial x}, \\ n_s(t,x,y,z) = -(\frac{\partial}{\partial t} + l_s(x,y,z)\frac{\partial}{\partial x} + m_s(x,y,z)\frac{\partial}{\partial y})\frac{\partial u_s(t,x,y,z)}{\partial z}. \end{cases}$$

Based on Eqs. (7)–(8), we can note that in order to get the new form of a (3 + 1)-dimensional dissipation Rossby wave, we need to find the solution of a (2 + 1)-dimensional dissipation Rossby wave.

3 (2 + 1)-Dimensional dissipation Rossby wave

In this section, we consider the (2 + 1)-dimensional approximate analytical solution of Eq. (1)

$$\frac{\partial}{\partial t}\nabla^2 u + J(u, \nabla^2 u) + \alpha\nabla^2 u + \beta\frac{\partial}{\partial x}u = 0, \tag{9}$$

where the Jacobian operator can be introduced by $J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$, and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ depicts the two-dimensional Laplacian.

First, when dissipation does not exist, we introduce the following transform:

$$\xi = x - y - ct, \tag{10}$$

where c expresses the phase speed of the wave. Substituting (10) into (9), we obtain

$$-c\frac{\partial}{\partial \xi}\left[2\frac{\partial^2 u}{\partial \xi^2} - \frac{\beta}{c}u\right] = 0. \tag{11}$$

It is easy to write the general solution of (11)

$$u = C_1 e^{\frac{\sqrt{2\beta}\xi}{\sqrt{c}}} + C_2 e^{-\frac{\sqrt{2\beta}\xi}{\sqrt{c}}}. \tag{12}$$

Equation (12) can be rewritten as

$$\begin{aligned} u = {}& C_1\left(\cosh\left(\frac{\sqrt{2\beta}(x-y-ct)}{\sqrt{c}}\right) + \operatorname{sech}\left(\frac{\sqrt{2\beta}(x-y-ct)}{\sqrt{c}}\right)\right) \\ & + C_2\left(\cosh\left(\frac{\sqrt{2\beta}(x-y-ct)}{\sqrt{c}}\right) - \operatorname{sech}\left(\frac{\sqrt{2\beta}(x-y-ct)}{\sqrt{c}}\right)\right), \end{aligned} \tag{13}$$

where C_1, C_2 are constants.

Then, we consider the impact of dissipation on Eq. (9). Assume $\alpha \ll 1$ and $\alpha \ll \beta$, and take a new space coordinate:

$$\rho = x + y - \int_0^t \frac{au_0}{2}\,dt. \tag{14}$$

Suppose that $u_0 = u_0(\alpha t)$ varies slowly with time; we then obtain

$$2\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial \rho^2}\right) - au_0\frac{\partial}{\partial \rho}\left(\frac{\partial^2 u}{\partial \rho^2}\right) + 2\alpha\frac{\partial^2 u}{\partial \rho^2} + \beta\frac{\partial u}{\partial \rho} = 0, \tag{15}$$

by substituting (14) into (9). Let

$$\tau = t, \qquad \eta = \alpha t, \tag{16}$$

and the solution has the following form:

$$u(\rho, t) = u_1(\rho, \tau, \eta) + \alpha u_2(\rho, \tau) + \cdots. \tag{17}$$

Substituting (17) into (15), we have

$$\alpha^0: \quad 2\frac{\partial}{\partial \tau}\left(\frac{\partial^2 u_1}{\partial \rho^2}\right) - au_0\frac{\partial}{\partial \rho}\left(\frac{\partial^2 u_1}{\partial \rho^2}\right) + \beta\frac{\partial u_1}{\partial \rho} = 0, \tag{18}$$

$$\alpha^1: \quad 2\frac{\partial}{\partial \tau}\left(\frac{\partial^2 u_2}{\partial \rho^2}\right) - au_0\frac{\partial}{\partial \rho}\left(\frac{\partial^2 u_2}{\partial \rho^2}\right) + 2\frac{\partial^2 u_1}{\partial \rho^2} + \beta\frac{\partial u_2}{\partial \rho} = -2\frac{\partial}{\partial \eta}\left(\frac{\partial^2 u_1}{\partial \rho^2}\right). \tag{19}$$

Then, let

$$\zeta = \rho + \frac{au_0}{2}\tau. \tag{20}$$

We have

$$\alpha^0: \quad 2\frac{\partial}{\partial \tau}\left(\frac{\partial^2 u_1}{\partial \rho^2}\right) + \beta\frac{\partial u_1}{\partial \rho} = 0, \tag{21}$$

$$\alpha^1: \quad 2\frac{\partial}{\partial \tau}\left(\frac{\partial^2 u_2}{\partial \rho^2}\right) + \beta\frac{\partial u_2}{\partial \rho} = -2\frac{\partial}{\partial \eta}\left(\frac{\partial^2 u_1}{\partial \rho^2}\right) - 2\frac{\partial^2 u_1}{\partial \rho^2}. \tag{22}$$

The solution of (21) is

$$\begin{aligned} u = {}& \frac{u_0}{C_2}\left(\cosh\left(\frac{\sqrt{2\beta}(\zeta - \frac{au_0}{2}\tau)}{\sqrt{c}}\right) + \operatorname{sech}\left(\frac{\sqrt{2\beta}(\zeta - \frac{au_0}{2}\tau)}{\sqrt{c}}\right)\right) \\ & + \frac{u_0}{C_1}\left(\cosh\left(\frac{\sqrt{2\beta}(\zeta - \frac{au_0}{2}\tau)}{\sqrt{c}}\right) - \operatorname{sech}\left(\frac{\sqrt{2\beta}(\zeta - \frac{au_0}{2}\tau)}{\sqrt{c}}\right)\right), \end{aligned} \tag{23}$$

where $u_0 = C_1C_2$. By using (14) and (20), we obtain

$$\begin{aligned} u = {}& \frac{u_0}{C_2}\left(\cosh\left(\frac{\sqrt{2\beta}(x + y - \int_0^t \frac{au_0}{2}\,dt)}{\sqrt{c}}\right) + \operatorname{sech}\left(\frac{\sqrt{2\beta}(x + y - \int_0^t \frac{au_0}{2}\,dt)}{\sqrt{c}}\right)\right) \\ & + \frac{u_0}{C_1}\left(\cosh\left(\frac{\sqrt{2\beta}(x + y - \int_0^t \frac{au_0}{2}\,dt)}{\sqrt{c}}\right) - \operatorname{sech}\left(\frac{\sqrt{2\beta}(x + y - \int_0^t \frac{au_0}{2}\,dt)}{\sqrt{c}}\right)\right). \end{aligned} \tag{24}$$

Next, setting

$$u_2 = M(N), \qquad N = \zeta - \nu\tau. \tag{25}$$

By substituting (25) into (22), we obtain

$$-2\nu\frac{\partial^3 M}{\partial N^3} + \beta\frac{\partial M}{\partial N} = W(u_1), \tag{26}$$

where

$$W(u_1) = -2\frac{\partial}{\partial\eta}\left(\frac{\partial^2 u_1}{\partial N^2}\right) - 2\frac{\partial^2 u_1}{\partial N^2}.$$

The solvability condition of Eq. (26) is

$$\int_{-\infty}^{+\infty} F(N)W(u_1)\,dN = 0, \tag{27}$$

where

$$-2\nu\frac{\partial^3 F}{\partial N^3} + \beta\frac{\partial F}{\partial N} = 0. \tag{28}$$

The solution of Eq. (28) is easy to obtain:

$$\begin{aligned} F = {} & \frac{u_0}{C_2}\left(\cosh\left(\frac{\sqrt{2\beta}N}{\sqrt{c}}\right) + \operatorname{sech}\left(\frac{\sqrt{2\beta}N}{\sqrt{c}}\right)\right) \\ & + \frac{u_0}{C_1}\left(\cosh\left(\frac{\sqrt{2\beta}N}{\sqrt{c}}\right) - \operatorname{sech}\left(\frac{\sqrt{2\beta}N}{\sqrt{c}}\right)\right), \end{aligned} \tag{29}$$

in the case of $G(\pm\infty) = 0$. Equation (29) can be rewritten as

$$\begin{aligned} F = {} & \frac{u_0}{C_2}\left(\cosh\left(\frac{\sqrt{2\beta}(\zeta - \nu\tau)}{\sqrt{c}}\right) + \operatorname{sech}\left(\frac{\sqrt{2\beta}(\zeta - \nu\tau)}{\sqrt{c}}\right)\right) \\ & + \frac{u_0}{C_1}\left(\cosh\left(\frac{\sqrt{2\beta}(\zeta - \nu\tau)}{\sqrt{c}}\right) - \operatorname{sech}\left(\frac{\sqrt{2\beta}(\zeta - \nu\tau)}{\sqrt{c}}\right)\right). \end{aligned} \tag{30}$$

Substituting (30) into (27), we obtain

$$u_0 = \bar{u}_0 e^{-\alpha t}, \tag{31}$$

where $\bar{u}_0 = u_0$. Therefore, we get

$$\begin{aligned} u = {} & \frac{\bar{u}_0 e^{-\alpha t}}{C_2}\left(\cosh\left(\frac{\sqrt{2\beta}(x + y - \int_0^t \frac{a\bar{u}_0 e^{-\alpha t}}{2}\,dt)}{\sqrt{c}}\right) + \operatorname{sech}\left(\frac{\sqrt{2\beta}(x + y - \int_0^t \frac{a\bar{u}_0 e^{-\alpha t}}{2}\,dt)}{\sqrt{c}}\right)\right) \\ & + \frac{\bar{u}_0 e^{-\alpha t}}{C_1}\left(\cosh\left(\frac{\sqrt{2\beta}(x + y - \int_0^t \frac{a\bar{u}_0 e^{-\alpha t}}{2}\,dt)}{\sqrt{c}}\right)\right. \\ & \left. - \operatorname{sech}\left(\frac{\sqrt{2\beta}(x + y - \int_0^t \frac{a\bar{u}_0 e^{-\alpha t}}{2}\,dt)}{\sqrt{c}}\right)\right). \end{aligned} \tag{32}$$

It is easy to cheek that (32) is the approximate analytical solution of Eq. (9). According to (7) and (32), we obtain the new solution of the (3 + 1)-dimensional quasi-geodetiophic vorticity equation with dissipation

$$
\begin{aligned}
u_{\text{new}}(t,x,y,z) = {} & h_5(t)x - h_2(t)y + h_4(t)z - \frac{\beta}{2}h_5(t)z^2 + \frac{\alpha}{2}e^{-t}z^2 \\
& + \frac{\bar{u}_0 e^{-\alpha t}}{C_2}\left(\cosh\left(\frac{\sqrt{2\beta}(x + y - \int_0^t \frac{a\bar{u}_0 e^{-\alpha t}}{2}\,dt)}{\sqrt{c}}\right)\right. \\
& \left. + \operatorname{sech}\left(\frac{\sqrt{2\beta}(x + y - \int_0^t \frac{a\bar{u}_0 e^{-\alpha t}}{2}\,dt)}{\sqrt{c}}\right)\right) \\
& + \frac{\bar{u}_0 e^{-\alpha t}}{C_1}\left(\cosh\left(\frac{\sqrt{2\beta}(x + y - \int_0^t \frac{a\bar{u}_0 e^{-\alpha t}}{2}\,dt)}{\sqrt{c}}\right)\right. \\
& \left. - \operatorname{sech}\left(\frac{\sqrt{2\beta}(x + y - \int_0^t \frac{a\bar{u}_0 e^{-\alpha t}}{2}\,dt)}{\sqrt{c}}\right)\right).
\end{aligned}
\tag{33}
$$

4 Results and discussion

Based on the classic Lie group method, the solution of the (3 + 1)-dimensional quasi-geodetiophic vorticity equation with dissipation is derived. On the one hand, the small dissipation effect can result in a decrease in amplitude $e^{-\alpha t}$, where α is the dissipation coefficient from (33). On the other hand, the small dissipation effect can result in a decrease in velocity of a Rossby wave in the process of propagation. Hence, with the help of the new solution (33) and Fig. 1, we can better comprehend the influence of the dissipation effect on the propagation of Rossby waves.

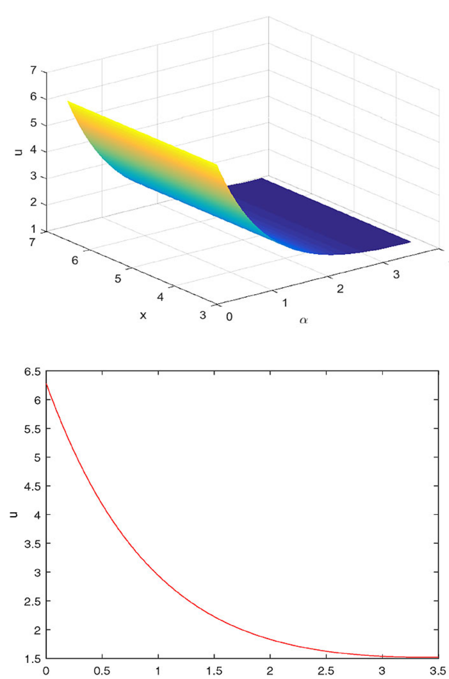

Figure 1 The solution of (3 + 1)-dimensional quasi-geodetiophic vorticity equation with dissipation represented by (33), where $h_5(t) = \cos(t)$, $h_2(t) = \sin(t)$, $h_4(t) = \cos(t)$

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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References

1. Yang, X.J., Gao, F., Srivastava, H.M.: Exact travelling wave solutions for the local fractional two-dimensional Burgers-type equations. Comput. Math. Appl. **73**, 203–210 (2017)
2. Baleanu, D., Inc, M., Yusuf, A., Aliyu, A.I.: Travelling wave solutions and conservation laws for nonlinear evolution equation. J. Math. Phys. **59**, 023506 (2018)
3. Yang, X.J., Machado, J.A.T., Baleanu, D.: Exact traveling-wave solution for local fractional Boussinesq equation in fractal domain. Fractals **25**, 1740006 (2017)
4. Lu, C.N., Fu, C., Yang, H.W.: Time-fractional generalized Boussinesq equation for Rossby solitary waves with dissipation effect in stratified fluid and conservation laws as well as exact solutions. Appl. Math. Comput. **327**, 104–116 (2018)
5. Zhang, R.G., Yang, L.G., Song, J., Yang, H.L.: $(2 + 1)$-Dimensional Rossby waves with complete Coriolis force and its solution by homotopy perturbation method. Comput. Math. Appl. **73**, 1996–2003 (2017)
6. Zhao, B.J., Wang, R.Y., Sun, W.J., Yang, H.W.: Combined ZK–mZK equation for Rossby solitary waves with complete Coriolis force and its conservation laws as well as exact solutions. Adv. Differ. Equ. **2018**, 42 (2018)
7. Yang, H.W., Xu, Z.H., Yang, D.Z., Feng, X.R., Yin, B.S., Dong, H.H.: ZK–Burgers equation for three-dimensional Rossby solitary waves and its solutions as well as chirp effect. Adv. Differ. Equ. **2016**, 167 (2016)
8. Yong, X.L., Ma, W.X., Huang, Y.H., Liu, Y.: Lump solutions to the Kadomtsev–Petviashvili I equation with a self-consistent source. Comput. Math. Appl. **75**, 3414–3419 (2018)
9. Ma, W.X., Yong, X.L., Zhang, H.Q.: Diversity of interaction solutions to the $(2 + 1)$-dimensional Ito equation. Comput. Math. Appl. **75**, 289–295 (2018)
10. Zhang, J.B., Ma, W.X.: Mixed lump-kink solutions to the BKP equation. Comput. Math. Appl. **74**, 591–596 (2017)
11. Guo, M., Zhang, Y., Wang, M., Chen, Y.D., Yang, H.W.: A new ZK-ILW equation for algebraic gravity solitary waves in finite depth stratified atmosphere and the research of squall lines formation mechanism. Comput. Math. Appl. **75**, 5468–5478 (2018)
12. Ma, W.X., Zhou, Y.: Lump solutions to nonlinear partial differential equations via Hirota bilinear forms. J. Differ. Equ. **264**, 2633–2659 (2018)
13. Fu, C., Lu, C.N., Yang, H.W.: Time–space fractional $(2 + 1)$-dimensional nonlinear Schrödinger equation for envelope gravity waves in baroclinic atmosphere and conservation laws as well as exact solutions. Adv. Differ. Equ. **2018**, 56 (2018)
14. McAnally, M., Ma, W.X.: An integrable generalization of the D-Kaup–Newell soliton hierarchy and its bi-Hamiltonian reduced hierarchy. Appl. Math. Comput. **323**, 220–227 (2018)
15. Liu, Y., Dong, H.H., Zhang, Y.: Solutions of a discrete integrable hierarchy by straightening out of its continuous and discrete constrained flows. Anal. Math. Phys. (2018). https://doi.org/10.1007/s13324-018-0209-9
16. Tao, M.S., Dong, H.H.: Algebro-geometric solutions for a discrete integrable equation. Discrete Dyn. Nat. Soc. **2017**, 5258375 (2017)
17. Zhou, Y., Ma, W.X.: Complexiton solutions to soliton equations by the Hirota method. J. Math. Phys. **58**, 101511 (2017)
18. Gordoa, P.R., Pickering, A., Zhu, Z.N.: On matrix Painlevé hierarchies. J. Differ. Equ. **261**, 1128–1175 (2016)
19. Xu, X.X.: A deformed reduced semi-discrete Kaup–Newell equation, the related integrable family and Darboux transformation. Appl. Math. Comput. **251**, 275–283 (2015)
20. Zhang, H.Q., Wang, Y., Ma, W.X.: Binary Darboux transformation for the coupled Sasa–Satsuma equations. Chaos **27**, 073102 (2017)
21. Zhao, Q.L., Li, X.Y.: Two integrable lattice hierarchies and their respective Darboux transformations. Appl. Math. Comput. **219**, 5693–5705 (2013)
22. Baleanu, D., Inc, M., Yusuf, A., Aliyu, A.I.: Space–time fractional rosenou-haynam equation: Lie symmetry analysis, explicit solutions and conservation laws. Adv. Differ. Equ. **2018**, 46 (2018)
23. Inc, M., Yusuf, A., Aliyu, A.I., Baleanu, D.: Lie symmetry analysis, explicit solutions and conservation laws for the space–time fractional nonlinear evolution equations. Phys. A **496**, 371–383 (2018)
24. Baleanu, D., Inc, M., Yusuf, A., Aliyu, A.I.: Time fractional third-order evolution equation: symmetry analysis, explicit solutions, and conservation laws. J. Comput. Nonlinear Dyn. **13**, 021011 (2018)
25. Chen, C.S., Song, H.X., Yang, H.W.: Liouville type theorems for stable solutions of p-Laplace equation in $\mathbb{R}^N$. Nonlinear Anal. **160**, 44–52 (2017)

26. Khalique, C.M., Magalakwe, G.: Combined sinh–cosh-Gordon equation: symmetry reductions, exact solutions and conservation laws. Quaest. Math. **37**, 199 (2014)
27. Ma, W.X.: Conservation laws by symmetries and adjoint symmetries. Discrete Contin. Dyn. Syst., Ser. S **11**, 707–721 (2018)
28. Xu, X.X., Sun, Y.P.: Two symmetry constraints for a generalized Dirac integrable hierarchy. J. Math. Anal. Appl. **458**, 1073–1090 (2018)
29. Gu, X., Ma, W.X., Zhang, W.Y.: Two integrable Hamiltonian hierarchies in sl$(2,\mathbb{R})$ and so$(3,\mathbb{R})$ with three potentials. Appl. Math. Comput. **14**, 053512 (2017)
30. Huang, F., Lou, S.Y.: Analytical investigation of Rossby waves in atmospheric dynamics. Phys. Lett. A **320**, 428–437 (2004)
31. Kudryavtsev, A.G., Myagkov, N.N.: Symmetry group application for the $(3+1)$-dimensional Rossby waves. Phys. Lett. A **375**, 586–588 (2011)
32. Ovsiannikov, L.V.: Group Analysis of Differential Equations. Academic Press, New York (1978)
33. Olver, P.: Applications of Lie Groups to Differential Equations. Springer, Berlin (1986)
34. Peslosky, J.: Geophysical Fluid Dynamics. Springer, New York (1987)