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Oscillation of second order neutral dynamic equations with deviating arguments on time scales

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Abstract

In this paper, we consider the following second order neutral dynamic equations with deviating arguments on time scales:

$$\left(r(t)\left(z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta}+q(t)f\left(y\left(m(t)\right)\right)=0,$$

where $z(t)=y(t)+p(t)y(\tau(t))$, $m(t)\leq t$ or $m(t)\geq t$, and $\tau(t)\leq t$. Some new oscillatory criteria are obtained by means of the inequality technique and a Riccati transformation. Our results extend and improve many well-known results for oscillation of second order dynamic equations. Some examples are given to illustrate the main results.

MSC: 26E70; 34C10; 34K40

Keywords: Time scales; Oscillation; Neutral; Deviating arguments

1 Introduction

The study of dynamic equations on time scales which goes back to its founder Hilger [1] as an area of mathematics that has received a lot of attention. It has been created in order to unify the study of differential and difference equations. Many authors have contributed on various aspects of this theory, see the survey paper by Agarwal et al. [2] and the references cited therein.

The theory of time scales, which provides powerful new tools for exploring connections between the traditionally separated fields, has been developing rapidly and has received much attention. Dynamic equations cannot only unify the theories of differential equations and difference equations, but also extend these classical cases to cases "in between" and can be applied to other different types of time scales. The theory of dynamic equations on time scales is an adequate mathematical apparatus for the simulation of processes and phenomena observed in biotechnology, chemical technology, economy, neural networks, physics, social sciences etc.

With the rapid development of science, the contributions of mathematical researchers are more urgent than ever before. Thus, beyond the purely mathematical interest, all of the above make the theory of dynamic equation very attractive to researchers. As a result, in the last decades many of them have focused their interest on problems of this area. One of the most interesting problems is the study of the oscillation of solutions of dynamic equation with deviating arguments.

Dynamic equations with deviating argument are deemed to be adequate in modeling of the countless processes in all areas of science. As is well known, a distinguishing feature of delay dynamic equations under consideration is the dependence of the evolution rate of the processes described by such equations on the past history. This consequently results in predicting the future in a more reliable and efficient way, explaining at the same time many qualitative phenomena such as periodicity, oscillation or instability. The concept of the delay incorporation into systems plays an essential role in modeling to represent time taken to complete some hidden processes; see [3, 4]. Contrariwise, advanced dynamic equations can find use in many applied problems whose evolution rate depends not only on the present, but also on the future, it also play a vital role.

Saker [5] studied the oscillation of second order nonlinear neutral delay dynamic equation

$$\bigl(r(t)\bigl(\bigl[y(t)+p(t)y(t-\tau)\bigr]^{\Delta}\bigr)^{\alpha}\bigr)^{\Delta}+f\bigl(t,y(t-\delta)\bigr)=0,$$

under the condition $\int^{\infty} r^{-1/\alpha}(t)\Delta t=\infty$.

Baculíková [6] studied the oscillatory behavior of the second order advanced differential equations

$$y''(t)+p(t)y\bigl(\sigma(t)\bigr)=0,$$

where $\sigma(t)\geq t$, and provided some technique for studying oscillation of the second order advanced differential equation. The method is based on the monotonic properties of nonoscillatory solutions.

Motivate by the above articles, now, in this article, we consider the dynamic equation of the form

$$\bigl(r(t)\bigl(z^{\Delta}(t)\bigr)^{\alpha}\bigr)^{\Delta}+q(t)f\bigl(y\bigl(m(t)\bigr)\bigr)=0,\quad t\in I, \tag{1.1}$$

where $z(t)=y(t)+p(t)y(\tau(t))$, $I=[t_0,\infty)_{\mathbb{T}}$ and $\alpha\geq 1$ is the ratio of two odd positive integers. Assume that the following conditions are satisfied:

(H1) $r(t),m(t)\in C^1_{rd}(I,[0,\infty)_{\mathbb{T}})$, $p(t)\in C_{rd}(I,[0,\infty)_{\mathbb{T}})$, $q(t)\in C_{rd}(I,(0,\infty)_{\mathbb{T}})$, $\tau(t)\in C_{rd}(I,\mathbb{R})$;

(H2) $m(t)\leq\sigma(t)$, $\tau(t)\leq t$ and $m^{\Delta}(t)>0$ and $\lim_{t\to\infty}m(t)=\lim_{t\to\infty}\tau(t)=\infty$;

(H3) $f\in C(\mathbb{R},\mathbb{R})$ such that $xf(x)>0$ and $f(x)/x^{\alpha}\geq k>0$, for $x\neq 0$, k is a constant.

We only discuss these solutions of (1.1) which exist on some half-line $[t_0,\infty)_{\mathbb{T}}$ and satisfy

$$\sup\bigl\{\bigl|x(t)\bigr|:t_e\leq t<\infty\bigr\}>0$$

for any $t_e\geq t_0$. Such a solution of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called nonoscillatory. The equation itself is called oscillatory if all of its solutions are oscillatory.

In this paper, using a Riccati transformation and the inequality technique, we present some new sufficient conditions which ensure that any solution of (1.1) oscillates. We suppose that $m(t)\leq\sigma(t)$, when $m(t)\leq t$, Eq. (1.1) is a delay dynamic equation on time scales;

when $m(t) \geq t$, Eq. (1.1) is an advanced dynamic equation on time scales, such as the h-difference equation ($h > 0$), and it is also called a finite difference equation, etc. Our results extend, complement, or improve some of the existing results, such as [7–21]. Our results further develop the functional differential equations on time scales.

2 Preliminaries

In order to prove our main results, we establish some fundamental results in this section. Without loss of generality, we can only deal with the positive solutions of Eq. (1.1) since the proof of the other case is similar.

Lemma 2.1 *Let $\alpha \geq 1$ be a ratio of two odd numbers. Then*

$$A^{(1+\alpha)/\alpha} - \frac{B^{1/\alpha}}{\alpha}\bigl[(1+\alpha)A - B\bigr] \leq (A-B)^{(1+\alpha)/\alpha}, \quad AB \geq 0, \tag{2.1}$$

and

$$-Cv^{(1+\alpha)/\alpha} + Dv \leq \frac{\alpha^\alpha}{(\alpha+1)^{1+\alpha}}\frac{D^{1+\alpha}}{C^\alpha}, \quad C > 0. \tag{2.2}$$

The inequality (2.1) can be found in (3.1) in [19], Lemma 1 in [20], and Lemma 2.1 in [21]. The proof of (2.2) can be found in Zhang and Wang [22].

3 Oscillation results

In this section, we are ready to establish the main results of this paper.

Theorem 3.1 *Suppose that $\sigma(m(t)) = m(\sigma(t))$, and there exist two functions $\eta, \delta \in C^1_{rd}([t_0,\infty),(0,\infty))$ such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\eta(s)P\bigl(\sigma(s)\bigr) - \frac{r(m(s))(\eta^\Delta(s))_+^{\alpha+1}}{(\alpha+1)^{\alpha+1}(m^\Delta(s))^\alpha\eta^\alpha(s)}\right]\Delta s = \infty, \tag{3.1}$$

or

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\psi(s) - \frac{\delta^{\alpha+1}(s)r(s)((\varphi(s))_+)^{\alpha+1}}{\delta^\alpha(\sigma(s))(\alpha+1)^{\alpha+1}}\right]\Delta s = \infty, \tag{3.2}$$

where

$$P(t) = kq(t)\left(1 - p\bigl(m(t)\bigr)\frac{\pi(\tau(m(t)))}{\pi(m(t))}\right)^\alpha, \qquad \pi(t) = \int_t^\infty r^{-1/\alpha}(s)\Delta s,$$

$$p(t) < \frac{\pi(t)}{\pi(\tau(t))}, \qquad \psi(t) = \delta\bigl(\sigma(t)\bigr)\left[P(t) - \frac{\alpha}{r^{1/\alpha}(t)\pi^{\alpha+1}(\sigma(t))} + \frac{1}{r^{1/\alpha}(t)\pi^{\alpha+1}(t)}\right],$$

$$\varphi(t) = \frac{\delta^\Delta(t)}{\delta(t)} + \frac{\delta(\sigma(t))(\alpha+1)}{r^{1/\alpha}(t)\pi(t)\delta(t)}, \qquad \bigl(\varphi(t)\bigr)_+ = \max\bigl\{0, \varphi(t)\bigr\},$$

$$\bigl(\eta^\Delta(t)\bigr)_+ = \max\bigl\{0, \eta^\Delta(s)\bigr\}.$$

Then every solution $y(t)$ of (1.1) is oscillatory.

Proof If $y(t)$ is an eventually positive solution of (1.1), and there exists a $t_1 > t_0$ such that $y(t) > 0$, $y(\tau(t)) > 0$, $y(m(t)) > 0$, $t \geq t_1$. By the definition of $z(t)$ and (H1) we have $z(t) \geq y(t) > 0$, and from (1.1) we know that

$$\bigl(r(t)\bigl(z^{\Delta}(t)\bigr)^{\alpha}\bigr)^{\Delta} = -q(t)f\bigl(y\bigl(m(t)\bigr)\bigr) \leq -kq(t)y^{\alpha}\bigl(m(t)\bigr) < 0, \quad t \in I. \tag{3.3}$$

Thus $r(t)(z^{\Delta}(t))^{\alpha}$ is decreasing for all $t \geq t_1$, then, for any $s \geq t \geq t_1$, we have

$$r(t)\bigl(z^{\Delta}(t)\bigr)^{\alpha} \geq r(s)\bigl(z^{\Delta}(s)\bigr)^{\alpha}, \tag{3.4}$$

thus

$$z^{\Delta}(s) \leq \left(\frac{r(t)}{r(s)}\right)^{1/\alpha} z^{\Delta}(t). \tag{3.5}$$

Integrating (3.5) from t to v, then

$$z(v) - z(t) \leq r^{1/\alpha}(t)z^{\Delta}(t)\int_t^v r^{-1/\alpha}(s)\Delta s. \tag{3.6}$$

Letting $v \to \infty$, we have

$$z(t) \geq -r^{1/\alpha}(t)z^{\Delta}(t)\pi(t). \tag{3.7}$$

Then

$$\left(\frac{z(t)}{\pi(t)}\right)^{\Delta} = \frac{z^{\Delta}(t)\pi(t) - z(t)\pi^{\Delta}(t)}{\pi(\sigma(t))\pi(t)} = \frac{z^{\Delta}(t)\pi(t) + z(t)r^{-1/\alpha}(t)}{\pi(\sigma(t))\pi(t)} \geq 0, \tag{3.8}$$

thus $\frac{z(t)}{\pi(t)} \geq \frac{z(\tau(t))}{\pi(\tau(t))}$, i.e.,

$$\frac{\pi(\tau(t))}{\pi(t)}z(t) \geq z\bigl(\tau(t)\bigr). \tag{3.9}$$

Moreover, by the definition of $z(t)$ and (3.9) we have $z(t) \geq y(t)$, then

$$\begin{aligned} y(t) &= z(t) - p(t)y\bigl(\tau(t)\bigr) \geq z(t) - p(t)z\bigl(\tau(t)\bigr) \geq z(t) - z(t)p(t)\frac{\pi(\tau(t))}{\pi(t)} \\ &= z(t)\left(1 - p(t)\frac{\pi(\tau(t))}{\pi(t)}\right). \end{aligned} \tag{3.10}$$

From (3.3) and (3.10), we get

$$\begin{aligned} \bigl(r(t)\bigl(z^{\Delta}(t)\bigr)^{\alpha}\bigr)^{\Delta} &\leq -kq(t)y^{\alpha}\bigl(m(t)\bigr) \\ &\leq -kq(t)z^{\alpha}\bigl(m(t)\bigr)\left(1 - p\bigl(m(t)\bigr)\frac{\pi(\tau(m(t)))}{\pi(m(t))}\right)^{\alpha}, \end{aligned} \tag{3.11}$$

i.e.,

$$\bigl(r(t)\bigl(z^{\Delta}(t)\bigr)^{\alpha}\bigr)^{\Delta} + kq(t)z^{\alpha}\bigl(m(t)\bigr)\left(1 - p\bigl(m(t)\bigr)\frac{\pi(\tau(m(t)))}{\pi(m(t))}\right)^{\alpha} \leq 0. \tag{3.12}$$

Let $P(t) = kq(t)(1 - p(m(t))\frac{\pi(\tau(m(t)))}{\pi(m(t))})^{\alpha}$. Then

$$\bigl(r(t)\bigl(z^{\Delta}(t)\bigr)^{\alpha}\bigr)^{\Delta} + P(t)z^{\alpha}\bigl(m(t)\bigr) \le 0. \tag{3.13}$$

We know that $r(t)(z^{\Delta}(t))^{\alpha}$ is decreasing for all $t \ge t_1$, which implies that $r(t)(z^{\Delta}(t))^{\alpha}$ does not change sign eventually, so as $z^{\Delta}(t)$, then there exists a $t_2 \ge t_1$ such that either $z^{\Delta}(t) < 0$ or $z^{\Delta}(t) > 0$ for any $t \ge t_2$. We shall show that in each case we are led to a contradiction.

Case (1): Assume first that $z^{\Delta}(t) > 0$ for all $t \ge t_2$. Define the following Riccati transformation:

$$w(t) = \eta(t)\frac{r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\alpha}}{z^{\alpha}(m(t))}. \tag{3.14}$$

Then $w(t) > 0$, and

$$\begin{aligned} w^{\Delta}(t) &= \left[\eta(t)\frac{r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\alpha}}{z^{\alpha}(m(t))}\right]^{\Delta} \\ &= \eta^{\Delta}(t)\frac{r(\sigma(\sigma(t)))(z^{\Delta}(\sigma(\sigma(t))))^{\alpha}}{z^{\alpha}(m(\sigma(t)))} + \eta(t)\left[\frac{r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\alpha}}{z^{\alpha}(m(t))}\right]^{\Delta} \\ &= \frac{\eta^{\Delta}(t)}{\eta(\sigma(t))}w\bigl(\sigma(t)\bigr) + \eta(t)\frac{[r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\alpha}]^{\Delta}}{z^{\alpha}(m(\sigma(t)))} \\ &\quad - \eta(t)\frac{r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\alpha}(z^{\alpha}(m(t)))^{\Delta}}{z^{\alpha}(m(t))z^{\alpha}(m(\sigma(t)))}. \end{aligned} \tag{3.15}$$

By the corollary of the Keller chain rule [23] and the condition $\sigma(m(t)) = m(\sigma(t))$ ([24] Lemma 2.2), for $\alpha \ge 1$, we have

$$\begin{aligned} \bigl(\bigl(z\bigl(m(t)\bigr)\bigr)^{\alpha}\bigr)^{\Delta} &= \alpha\int_0^1 \bigl[hz\bigl(m\bigl(\sigma(t)\bigr)\bigr) + (1-h)z\bigl(m(t)\bigr)\bigr]^{\alpha-1}\bigl(z\bigl(m(t)\bigr)\bigr)^{\Delta}\,dh \\ &\ge \alpha\int_0^1 \bigl[hz\bigl(m(t)\bigr) + (1-h)z\bigl(m(t)\bigr)\bigr]^{\alpha-1}\bigl(z\bigl(m(t)\bigr)\bigr)^{\Delta}\,dh \\ &= \alpha z^{\alpha-1}\bigl(m(t)\bigr)z^{\Delta}\bigl(m(t)\bigr)m^{\Delta}(t). \end{aligned} \tag{3.16}$$

Thus we get

$$\begin{aligned} w^{\Delta}(t) &\le \frac{\eta^{\Delta}(t)}{\eta(\sigma(t))}w\bigl(\sigma(t)\bigr) + \eta(t)\frac{-P(\sigma(t))z^{\alpha}(m(\sigma(t)))}{z^{\alpha}(m(\sigma(t)))} \\ &\quad - \eta(t)\frac{r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\alpha}\alpha z^{\Delta}(m(t))m^{\Delta}(t)}{z(m(t))z^{\alpha}(m(\sigma(t)))} \\ &\le \frac{\eta^{\Delta}(t)}{\eta(\sigma(t))}w\bigl(\sigma(t)\bigr) - \eta(t)P\bigl(\sigma(t)\bigr) - \frac{\alpha\eta(t)m^{\Delta}(t)}{r^{1/\alpha}(m(t))\eta^{\frac{\alpha+1}{\alpha}}(\sigma(t))}w^{\frac{\alpha+1}{\alpha}}\bigl(\sigma(t)\bigr) \\ &\le \bigl(\eta^{\Delta}(t)\bigr)_{+}\frac{w(\sigma(t))}{\eta(\sigma(t))} - \eta(t)P\bigl(\sigma(t)\bigr) - \frac{\alpha\eta(t)m^{\Delta}(t)}{r^{1/\alpha}(m(t))}\left(\frac{w(\sigma(t))}{\eta(\sigma(t))}\right)^{\frac{\alpha+1}{\alpha}}. \end{aligned} \tag{3.17}$$

Set

$$F(v) = \bigl(\eta^{\Delta}(t)\bigr)_{+}v - \frac{\alpha\eta(t)m^{\Delta}(t)}{r^{1/\alpha}(m(t))}v^{\frac{\alpha+1}{\alpha}}. \tag{3.18}$$

By calculating, we have

$$\nu_0 = \frac{1}{(\alpha+1)^\alpha}\frac{r(m(t))}{(m^\Delta(t))^\alpha}\left(\frac{(\eta^\Delta(t))_+}{\eta(t)}\right)^\alpha, \tag{3.19}$$

and when $\nu < \nu_0$, we have $F^\Delta(\nu) > 0$, when $\nu > \nu_0$, $F^\Delta(\nu) < 0$, then $F(\nu)$ obtains its maximum. So,

$$\begin{aligned} F(\nu) \le F(\nu_0) &= \frac{r(m(t))(\eta^\Delta(t))_+^{\alpha+1}}{(\alpha+1)^\alpha(m^\Delta(t))^\alpha\eta^\alpha(t)} - \frac{\alpha r(m(t))(\eta^\Delta(t))_+^{\alpha+1}}{(\alpha+1)^{\alpha+1}(m^\Delta(t))^\alpha\eta^\alpha(t)} \\ &= \frac{(\alpha+1)r(m(t))(\eta^\Delta(t))_+^{\alpha+1} - \alpha r(m(t))(\eta^\Delta(t))_+^{\alpha+1}}{(\alpha+1)^{\alpha+1}(m^\Delta(t))^\alpha\eta^\alpha(t)} \\ &= \frac{r(m(t))(\eta^\Delta(t))_+^{\alpha+1}}{(\alpha+1)^{\alpha+1}(m^\Delta(t))^\alpha\eta^\alpha(t)}. \end{aligned} \tag{3.20}$$

Therefore,

$$w^\Delta(t) \le -\eta(t)P(\sigma(t)) + \frac{r(m(t))(\eta^\Delta(t))_+^{\alpha+1}}{(\alpha+1)^{\alpha+1}(m^\Delta(t))^\alpha\eta^\alpha(t)}. \tag{3.21}$$

Integrating the above inequality from T_0 to t, we have

$$0 < w(t) \le w(T_0) - \int_{T_0}^t \left[\eta(s)P(\sigma(s)) - \frac{r(m(s))(\eta^\Delta(s))_+^{\alpha+1}}{(\alpha+1)^{\alpha+1}(m^\Delta(s))^\alpha\eta^\alpha(s)}\right]\Delta s. \tag{3.22}$$

Let $t \to \infty$ in the above inequality, which contradicts (3.1).

Case (2): Assume now that $z^\Delta(t) < 0$ for all $t \ge t_2$. Define the following Riccati transformation:

$$w(t) = \delta(t)\left[\frac{r(t)(z^\Delta(t))^\alpha}{z^\alpha(t)} + \frac{1}{\pi^\alpha(t)}\right]. \tag{3.23}$$

Since (3.7), we have $w(t) > 0$, then

$$\begin{aligned} w^\Delta(t) &= \left\{\delta(t)\left[\frac{r(t)(z^\Delta(t))^\alpha}{z^\alpha(t)} + \frac{1}{\pi^\alpha(t)}\right]\right\}^\Delta \\ &= \frac{\delta^\Delta(t)}{\delta(t)}w(t) + \delta(\sigma(t))\left[\frac{[r(t)(z^\Delta(t))^\alpha]^\Delta}{z^\alpha(\sigma(t))} - \frac{r(t)(z^\Delta(t))^\alpha[z^\alpha(t)]^\Delta}{z^\alpha(t)z^\alpha(\sigma(t))} \right. \\ &\quad \left. + \frac{-[\pi^\alpha(t)]^\Delta}{\pi^\alpha(t)\pi^\alpha(\sigma(t))}\right]. \end{aligned} \tag{3.24}$$

By the corollary of the Keller chain rule [23], for $\alpha \ge 1$, we have

$$\begin{aligned} \left(z^\alpha(t)\right)^\Delta &= \alpha\int_0^1 \left[hz(\sigma(t)) + (1-h)z(t)\right]^{\alpha-1}z^\Delta(t)\,dh \\ &\ge \alpha\int_0^1 \left[hz(\sigma(t)) + (1-h)z(\sigma(t))\right]^{\alpha-1}z^\Delta(t)\,dh \\ &= \alpha z^{\alpha-1}(\sigma(t))z^\Delta(t) \end{aligned} \tag{3.25}$$

and

$$
\begin{aligned}
\left(\pi^{\alpha}(t)\right)^{\Delta} &= \alpha \int_0^1 \left[h\pi\left(\sigma(t)\right) + (1-h)\pi(t)\right]^{\alpha-1} \pi^{\Delta}(t)\,dh \\
&\geq \alpha \int_0^1 \left[h\pi\left(\sigma(t)\right) + (1-h)\pi\left(\sigma(t)\right)\right]^{\alpha-1} \pi^{\Delta}(t)\,dh \\
&= \alpha\pi^{\alpha-1}\left(\sigma(t)\right)\pi^{\Delta}(t) = -\alpha\pi^{\alpha-1}\left(\sigma(t)\right)r^{-1/\alpha}(t).
\end{aligned}
\tag{3.26}
$$

By (3.25), (3.26), Eq. (3.24) can be written as

$$
\begin{aligned}
&w^{\Delta}(t) \\
&\quad= \frac{\delta^{\Delta}(t)}{\delta(t)}w(t) + \delta\left(\sigma(t)\right)\frac{[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta}}{z^{\alpha}(\sigma(t))} - \delta\left(\sigma(t)\right)\frac{r(t)(z^{\Delta}(t))^{\alpha}[z^{\alpha}(t)]^{\Delta}}{z^{\alpha}(t)z^{\alpha}(\sigma(t))} \\
&\qquad - \delta\left(\sigma(t)\right)\frac{[\pi^{\alpha}(t)]^{\Delta}}{\pi^{\alpha}(t)\pi^{\alpha}(\sigma(t))} \\
&\quad\leq \frac{\delta^{\Delta}(t)}{\delta(t)}w(t) + \delta\left(\sigma(t)\right)\frac{[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta}}{z^{\alpha}(\sigma(t))} - \alpha\delta\left(\sigma(t)\right)r(t)\frac{(z^{\Delta}(t))^{\alpha+1}}{z^{\alpha+1}(t)} + \frac{\alpha\delta(\sigma(t))}{r^{1/\alpha}(t)\pi^{\alpha+1}(\sigma(t))} \\
&\quad= \frac{\delta^{\Delta}(t)}{\delta(t)}w(t) + \delta\left(\sigma(t)\right)\frac{[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta}}{z^{\alpha}(\sigma(t))} - \alpha\delta\left(\sigma(t)\right)r(t)\left[\frac{w(t)}{\delta(t)r(t)} - \frac{1}{r(t)\pi^{\alpha}(t)}\right]^{\frac{\alpha+1}{\alpha}} \\
&\qquad + \frac{\alpha\delta(\sigma(t))}{r^{1/\alpha}(t)\pi^{\alpha+1}(\sigma(t))}.
\end{aligned}
\tag{3.27}
$$

Let $A := w(t)/(\delta(t)r(t))$ and $B := 1/(r(t)\pi^{\alpha}(t))$. Using inequality (2.1), we conclude that

$$
\begin{aligned}
&\left(\frac{w(t)}{\delta(t)r(t)} - \frac{1}{r(t)\pi^{\alpha}(t)}\right)^{(\alpha+1)/\alpha} \\
&\quad\geq \left(\frac{w(t)}{\delta(t)r(t)}\right)^{(\alpha+1)/\alpha} - \frac{1}{\alpha r^{1/\alpha}(t)\pi(t)}\left[(\alpha+1)\frac{w(t)}{\delta(t)r(t)} - \frac{1}{r(t)\pi^{\alpha}(t)}\right].
\end{aligned}
\tag{3.28}
$$

On the other hand, we get by (3.11), $[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta} < 0$, $z^{\Delta} < 0$, and $m(t) \leq \sigma(t)$ that

$$
\begin{aligned}
\frac{[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta}}{z^{\alpha}(\sigma(t))} &\leq \frac{[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta}}{z^{\alpha}(m(t))} \leq -kq(t)\left(1 - p\left(m(t)\right)\frac{\pi(\tau(m(t)))}{\pi(m(t))}\right)^{\alpha} \\
&= -P(t).
\end{aligned}
\tag{3.29}
$$

Thus, (3.27) yields

$$
\begin{aligned}
w^{\Delta}(t) &\leq \frac{\delta^{\Delta}(t)}{\delta(t)}w(t) - \delta\left(\sigma(t)\right)P(t) + \frac{\alpha\delta(\sigma(t))}{r^{1/\alpha}(t)\pi^{\alpha+1}(\sigma(t))} \\
&\quad - \alpha\delta\left(\sigma(t)\right)r(t)\left\{\left(\frac{w(t)}{\delta(t)r(t)}\right)^{(\alpha+1)/\alpha}\right. \\
&\quad \left. - \frac{1}{\alpha r^{1/\alpha}(t)\pi(t)}\left[(\alpha+1)\frac{w(t)}{\delta(t)r(t)} - \frac{1}{r(t)\pi^{\alpha}(t)}\right]\right\} \\
&= \frac{\delta^{\Delta}(t)}{\delta(t)}w(t) - \delta\left(\sigma(t)\right)P(t) + \frac{\alpha\delta(\sigma(t))}{r^{1/\alpha}(t)\pi^{\alpha+1}(\sigma(t))}
\end{aligned}
$$

$$\begin{aligned}
&-\frac{\alpha\delta(\sigma(t))}{\delta^{(\alpha+1)/\alpha}(t)r^{1/\alpha}(t)}w^{(\alpha+1)/\alpha}(t)+\frac{\delta(\sigma(t))(\alpha+1)}{r^{1/\alpha}(t)\pi(t)\delta(t)}w(t)-\frac{\delta(\sigma(t))}{r^{1/\alpha}(t)\pi^{\alpha+1}(t)}\\
&=-\delta\bigl(\sigma(t)\bigr)\left[P(t)-\frac{\alpha}{r^{1/\alpha}(t)\pi^{\alpha+1}(\sigma(t))}+\frac{1}{r^{1/\alpha}(t)\pi^{\alpha+1}(t)}\right]\\
&\quad+\left[\frac{\delta^{\Delta}(t)}{\delta(t)}+\frac{\delta(\sigma(t))(\alpha+1)}{r^{1/\alpha}(t)\pi(t)\delta(t)}\right]w(t)-\frac{\alpha\delta(\sigma(t))}{\delta^{(\alpha+1)/\alpha}(t)r^{1/\alpha}(t)}w^{(\alpha+1)/\alpha}(t)\\
&\le-\psi(t)+\bigl(\varphi(t)\bigr)_{+}w(t)-\frac{\alpha\delta(\sigma(t))}{\delta^{(\alpha+1)/\alpha}(t)r^{1/\alpha}(t)}w^{(\alpha+1)/\alpha}(t),
\end{aligned}\tag{3.30}$$

where

$$\begin{aligned}
&\psi(t)=\delta\bigl(\sigma(t)\bigr)\left[P(t)-\frac{\alpha}{r^{1/\alpha}(t)\pi^{\alpha+1}(\sigma(t))}+\frac{1}{r^{1/\alpha}(t)\pi^{\alpha+1}(t)}\right],\\
&\varphi(t)=\frac{\delta^{\Delta}(t)}{\delta(t)}+\frac{\delta(\sigma(t))(\alpha+1)}{r^{1/\alpha}(t)\pi(t)\delta(t)},\qquad \bigl(\varphi(t)\bigr)_{+}=\max\bigl\{0,\varphi(t)\bigr\}.
\end{aligned}$$

Define now $C:=\frac{\alpha\delta(\sigma(t))}{\delta^{(\alpha+1)/\alpha}(t)r^{1/\alpha}(t)}$, $D:=(\varphi(t))_{+}$, and $v:=w(t)$. Applying inequality (2.2), we obtain

$$-\frac{\alpha\delta(\sigma(t))}{\delta^{(\alpha+1)/\alpha}(t)r^{1/\alpha}(t)}w^{(\alpha+1)/\alpha}(t)+\bigl(\varphi(t)\bigr)_{+}w(t)\le\frac{\delta^{\alpha+1}(t)r(t)((\varphi(t))_{+})^{\alpha+1}}{\delta^{\alpha}(\sigma(t))(\alpha+1)^{\alpha+1}}.\tag{3.31}$$

By (3.30) and (3.31),

$$w^{\Delta}(t)\le-\psi(t)+\frac{\delta^{\alpha+1}(t)r(t)((\varphi(t))_{+})^{\alpha+1}}{\delta^{\alpha}(\sigma(t))(\alpha+1)^{\alpha+1}}.\tag{3.32}$$

Integrating the latter inequality from t_2 to t, we have

$$\int_{t_2}^{t}\left[\psi(s)-\frac{\delta^{\alpha+1}(s)r(s)((\varphi(s))_{+})^{\alpha+1}}{\delta^{\alpha}(\sigma(s))(\alpha+1)^{\alpha+1}}\right]\Delta s\le-w(t)+w(t_2)\le w(t_2),\tag{3.33}$$

which contradicts (3.2). Therefore, Eq. (1.1) is oscillatory. □

4 Examples

Example 4.1 As an illustrative example, we consider the following equation:

$$\left(t^2\left(y(t)+\frac{1}{3}y\left(\frac{t}{2}\right)\right)'\right)'+y\left(\frac{t}{3}\right)=0.\tag{4.1}$$

Here $\mathbb{T}=\mathbb{R}^{+}$, and $\alpha=1$, $t_0=2$, $r(t)=t^2$, $p(t)=\frac{1}{3}$, $\tau(t)=\frac{t}{2}$, $q(t)=1$, $f(t)=t$, $m(t)=\frac{t}{3}$. By taking $\eta(t)=k=1$, then $\pi(t)=\frac{1}{t}$, $\frac{\pi(t)}{\pi(\frac{t}{2})}=\frac{1}{2}>\frac{1}{3}$, and $P(t)=\frac{1}{3}$.

$$\limsup_{t\to\infty}\int_{2}^{t}\left[\eta(s)P(s)-\frac{r(m(s))(\eta'(s))_{+}^{\alpha+1}}{(\alpha+1)^{\alpha+1}(m'(s))^{\alpha}\eta^{\alpha}(s)}\right]ds=\limsup_{t\to\infty}\int_{2}^{t}\frac{1}{3}\,ds=\infty.\tag{4.2}$$

It is easy to check that all hypotheses of Theorem 3.1 are satisfied, so we see that Eq. (4.1) is oscillatory.

Example 4.2 Consider the equation

$$\Delta^2\left(y(t)+\frac{1}{4}y(t-2)\right)+y(t+1)=0. \tag{4.3}$$

Here $\mathbb{T}=\mathbb{N}^+$, and $\alpha=1$, $t_0=2$, $r(t)=1$, $p(t)=\frac{1}{4}$, $\tau(t)=t-2$, $q(t)=1$, $f(t)=t$, $m(t)=t+1$. By taking $\eta(t)=k=1$, then $P(t)=\frac{3}{4}$.

$$\limsup_{t\to\infty}\sum_{2}^{t}\left[\eta(s)P(s+1)-\frac{r(m(s))(\Delta\eta(s))_+^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\Delta m(s))^{\alpha}\eta^{\alpha}(s)}\right]=\limsup_{t\to\infty}\sum_{2}^{t}\frac{3}{4}=\infty. \tag{4.4}$$

It is easy to check that all hypotheses of Theorem 3.1 are satisfied, so we see that Eq. (4.3) is oscillatory.

Example 4.3 Consider the following equation:

$$D^q\left(t\left(D^q\left(y(t)+\frac{1}{2}y(t-1)\right)\right)\right)+\frac{1}{3}y(t-2)=0. \tag{4.5}$$

Here $0<q<1$, $\alpha=1$, $t_0=2$, $r(t)=t$, $p(t)=\frac{1}{2}$, $\tau(t)=t-1$, $q(t)=\frac{1}{3}$, $f(t)=t$, $m(t)=t-2$. By taking $\eta(t)=k=1$, then $\pi(t)=\int_t^\infty r^{-1/\alpha}(s)\,d_qs=\int_t^\infty s^{-1}\,d_qs=\sum_{k=1}^\infty tq^{-k}(1-q)(tq^{-k})^{-1}=\sum_{k=1}^\infty(1-q)$, $\frac{\pi(t)}{\pi(\frac{t}{2})}=1>\frac{1}{2}$, and $P(t)=\frac{1}{6}$.

$$\limsup_{t\to\infty}\int_2^t\left[\eta(s)P(s)-\frac{r(m(s))(D^q\eta(s))_+^{\alpha+1}}{(\alpha+1)^{\alpha+1}(m'(s))^{\alpha}\eta^{\alpha}(s)}\right]d_qs=\limsup_{t\to\infty}\int_2^t\frac{1}{6}\,d_qs=\infty. \tag{4.6}$$

It is easy to check that all hypotheses of Theorem 3.1 are satisfied, so we see that Eq. (4.5) is oscillatory.

5 Conclusion and future direction

The results of this article are presented in a form which is essentially new and of high degree of generality. In this article, using a Riccati transformation and the inequality technique, we offer some new sufficient conditions which ensure that any solution of Eq. (1.1) oscillates. We suppose that $m(t)\le\sigma(t)$, when $m(t)\le t$, Eq. (1.1) is a delay dynamic equation on time scales; when $m(t)\ge t$, Eq. (1.1) is an advanced dynamic equation on time scales. Further, we can consider the case of $m(t)\ge\sigma(t)$, and we can try to get some oscillation criteria of Eq. (1.1) if $p(t)<0$ and $q(t)<0$ in the future work.

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Competing interests
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Authors' contributions
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