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The existence and exponential stability of random impulsive fractional differential equations

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Abstract

In this paper, the existence and exponential stability of mild solutions of random impulsive fractional differential equations are studied. The results are obtained by using the Leray–Schauder alternative fixed point theorem.

Keywords: Random impulse; Fractional differential equations; Exponential stability; Leray–Schauder alternative fixed point

1 Introduction

For impulsive differential systems, most published papers deal with the problem with fixed time impulses [1–3]. However, actual jumps do not always happen at fixed points but usually at random points. The solutions of random impulsive differential equations is a stochastic process. It is different from deterministic impulsive differential equations and also it is different from stochastic differential equations. At present, the properties of solutions to some integer order differential equations with random impulses have been studied [4–8]. The existence, uniqueness, and stability of fractional differential equations without random impulses have been shown by many authors [9–18]. However, the properties of solutions to fractional delay differential equations with random impulses have not been studied. In [19], the authors study the existence, uniqueness, and stability through continuous dependence on initial conditions and Hyers–Ulam–Rassias stability for random impulsive fractional differential equations, but in that paper, the authors did not consider delay. In some applications, besides impulsive effects, delay effects cannot be ignored.

Motivated by the above consideration, we consider fractional differential equations with random impulses of the form

$$\begin{aligned} {}^c D_t^q x(t) &= Ax(t) + f(t, x_t), \quad t \neq \xi_k, t \geq t_0, \\ x(\xi_k) &= b_k(\tau_k)x(\xi_k^-), \quad k = 1, 2, \dots, \\ x_{t_0} &= \varphi. \end{aligned} \tag{1.1}$$

Let X be a real separable Hilbert space and Ω be a nonempty set. For the sake of simplicity, we denote $R_\tau = [\tau, +\infty)$, $R^+ = [0, +\infty)$. Assume that τ_k is a random variable defined from Ω to $D_k \stackrel{\text{def}}{=} (0, d_k)$ for $k = 1, 2, \dots$, where $0 < d_k < +\infty$. Furthermore, assume that τ_i

and τ_j are independent of each other as $i \neq j$ for $i, j = 1, 2, \dots$. A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $S(t)$ with domain $D(A) \subset X$; the functional $f : R^+ \times C \rightarrow X, C = C([-r, 0], X)$ is the set of piecewise continuous functions, $r > 0$; x_t is a function when t is fixed, $x_t(\theta) = x(t + \theta), \theta \in [-r, 0]$; $\xi_0 = t_0$ and $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1, 2, \dots$, here $t_0 \in R_\tau$ is an arbitrary given real number. The impulse moments ξ_k form a strictly increasing sequence, i.e., $t_0 = \xi_0 < \xi_1 < \xi_2 < \dots < \lim_{k \rightarrow \infty} \xi_k$; $b_k : D_k \rightarrow X$ for each $k = 1, 2, \dots$; $x(\xi_k^-) = \lim_{t \rightarrow \xi_k^-} x(t)$ according to their paths with the norm $\|x\|_t = \sup_{t-r \leq s \leq t} |x(s)|$ for each t satisfying $t \geq t_0$; $\|\cdot\|$ is any given norm in X ; φ is a function defined from $[-r, 0]$ to X ; ${}^c D^q$ is the Caputo fractional derivative of order $0 < q < 1$.

This paper studies the properties of solutions to system (1.1) by using the theory of functional differential equation, fractional differential equation, and stochastic analysis techniques. Firstly, we investigate the existence of a mild solution for (1.1) by using the Leray–Schauder alternative fixed point theorem. Secondly, we study the exponential stability in the quadratic mean of (1.1). Finally, an example is presented to illustrate our results.

2 Some preliminaries

In this section, we shall recall some basic definitions and lemma which will be used in this paper.

Denote by $\{B_t, t \geq 0\}$ the simple counting process generated by $\{\xi_n\}$, that is, $\{B_t \geq n\} = \{\xi_n \leq t\}$, and denote by F_t the σ -algebra generated by $\{B_t, t \geq 0\}$. Then $(\Omega, P, \{F_t\})$ is a probability space. Let $L_2 = L_2(\Omega, F_t, X)$ denote the Hilbert space of all F_t -measurable square integrable random variables with values in X .

Assume that $T > t_0$ is any fixed time to be determined later and β denotes the Banach space $\mathcal{B}([t_0 - r, T], L_2)$, the family of all F_t -measurable, C -valued random variables φ with the norm

$$\|\varphi\|_\beta^2 = \sup_{t_0 \leq t \leq T} E\|\varphi\|_t^2.$$

Let $L_2^0(\Omega, \beta)$ denote the family of all F_0 -measurable, β -valued random variables φ .

Definition 2.1 The fractional integral of order q with the lower limit 0 for a function f is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds, \quad t > 0, q > 0,$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the gamma function.

Definition 2.2 The Riemann–Liouville derivative of order q with the lower limit 0 for a function $f : [0, \infty) \rightarrow R$ can be written as

$${}^L D^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{q+1-n}} ds, \quad t > 0, n-1 < q < n.$$

Definition 2.3 The Caputo derivative of order q for a function $f : [0, \infty) \rightarrow R$ can be written as

$${}^c D^q f(t) = {}^L D^q \left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right], \quad t > 0, n - 1 < q < n.$$

Definition 2.4 A semigroup $\{S(t), t \geq t_0\}$ is said to be uniformly bounded if $\|S(t)\| \leq M$ for all $t \geq t_0$, where $M \geq 1$ is some constant. If $M = 1$, then the semigroup is said to be contraction semigroup.

Definition 2.5 A map $f : [\tau, T] \times C \rightarrow X$ is said to be L^2 -Carathéodory if

- (i) $t \rightarrow f(t, u)$ is measurable for each $u \in C$;
- (ii) $u \rightarrow f(t, u)$ is continuous for almost all $t \in [\tau, T]$;
- (iii) for each positive integer $m > 0$, there exists $\alpha_m \in L^1([\tau, T], R^+)$ such that

$$\sup_{\|u\|_{\mathbb{B}} \leq m} \|f(t, u)\|^2 \leq \alpha_m(t) \quad \text{for } t \in [\tau, T], \text{ a.e.}$$

Definition 2.6 For a given $T \in (t_0, +\infty)$, a stochastic process $\{x(t) \in \mathbb{B}, t_0 - r \leq t \leq T\}$ is called a mild solution to Eq. (1.1) in $(\Omega, P, \{F_t\})$ if

- (i) $x(t) \in \mathbb{B}$ is F_t -adapted for $t \geq t_0$;
- (ii) $x(t_0 + s) = \varphi(s) \in L^0_2(\Omega, \mathbb{B})$, when $s \in [-r, 0]$, and

$$\begin{aligned} x(t) = & \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) S(t - t_0) \varphi(0) \right. \\ & + \frac{1}{\Gamma(q)} \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t - s)^{q-1} S(t - s) f(s, x_s) ds \\ & \left. + \frac{1}{\Gamma(q)} \int_{\xi_k}^t (t - s)^{q-1} S(t - s) f(s, x_s) ds \right] I_{[\xi_k, \xi_{k+1})}(t), \quad t \in [t_0, T], \end{aligned}$$

where $\prod_{j=m}^n (\cdot) = 1$ as $m > n$, $\prod_{j=i}^k b_j(\tau_j) = b_k(\tau_k) b_{k-1}(\tau_{k-1}) \cdots b_i(\tau_i)$, and $I_A(\cdot)$ is the index function, i.e.,

$$I_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{if } t \notin A. \end{cases}$$

Our existence and exponential stability theorems are based on the following theorem, which is a version of the topological transversality theorem.

Lemma 2.1 Let E be a convex subset of a Banach space X , and assume that $0 \in E$. Let $F : E \rightarrow E$ be a completely continuous operator, and let

$$U(F) = \{x \in E : x = \lambda Fx \text{ for some } 0 < \lambda < 1\},$$

then either $U(F)$ is unbounded or F has a fixed point.

3 Existence results

In this section, we prove the existence of a mild solution of system (1.1) by using the following hypotheses:

(H₁) There exist a continuous non-decreasing function $H : R^+ \rightarrow (0, \infty)$ and $\delta \in L^1([\tau, T], R^+)$ such that

$$E\|f(t, \psi_t)\|^2 \leq \delta(t)H(E\|\psi\|_t^2)$$

for every $t \in [\tau, T]$ and for any stochastic process $\psi \in C$.

(H₂) $\max_{i,k}\{\prod_{j=i}^k \|b_j(\tau_j)\|\}$ is uniformly bounded, that is, there is $B > 0$ such that

$$\max_{i,k} \left\{ \prod_{j=i}^k \|b_j(\tau_j)\| \right\} \leq B$$

for all $\tau_j \in D_j, j = 1, 2, \dots$

Theorem 3.1 Assume (H₁)–(H₂) hold, then system (1.1) has a mild solution $x(t)$, defined on $[t_0, T]$, provided that the following inequality is satisfied:

$$M_1 \int_{t_0}^T \delta(s) ds < \int_{c_1}^{\infty} \frac{ds}{H(s)}, \tag{3.1}$$

where $M_1 = 2M^2 \max\{1, B^2\} \frac{(T-t_0)^{2q-1}}{(2q-1)\Gamma(q)}$, $c_1 = 2M^2 B^2 E\|\varphi\|^2$, and $MC \geq \frac{1}{\sqrt{2}}$.

Proof Let T be an arbitrary positive number $t_0 < T < +\infty$. Define an operator $\Phi : \mathcal{B} \rightarrow \mathcal{B}$ as follows:

$$\Phi x(t) = \begin{cases} \varphi(t - t_0), & t \in [t_0 - r, t_0], \\ \sum_{k=0}^{+\infty} [\prod_{i=1}^k b_i(\tau_i) S(t - t_0) \varphi(0) \\ + \frac{1}{\Gamma(q)} \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t - s)^{q-1} S(t - s) f(s, x_s) ds \\ + \frac{1}{\Gamma(q)} \int_{\xi_k}^t (t - s)^{q-1} S(t - s) f(s, x_s) ds] I_{[\xi_k, \xi_{k+1})}(t), & t \in [t_0, T]. \end{cases}$$

First we establish the a priori estimates for the solutions of the integral equation and $\lambda \in (0, 1)$:

$$x(t) = \begin{cases} \lambda \varphi(t - t_0), & t \in [t_0 - r, t_0], \\ \lambda \sum_{k=0}^{+\infty} [\prod_{i=1}^k b_i(\tau_i) S(t - t_0) \varphi(0) \\ + \frac{1}{\Gamma(q)} \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t - s)^{q-1} S(t - s) f(s, x_s) ds \\ + \frac{1}{\Gamma(q)} \int_{\xi_k}^t (t - s)^{q-1} S(t - s) f(s, x_s) ds] I_{[\xi_k, \xi_{k+1})}(t), & t \in [t_0, T]. \end{cases}$$

Thus by (H₁)–(H₂), we have

$$\begin{aligned} & \|x(t)\|^2 \\ & \leq \lambda^2 \left[\sum_{k=0}^{+\infty} \left\| \prod_{i=1}^k b_i(\tau_i) \right\| \|S(t - t_0)\| \|\varphi(0)\| \right] \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{\Gamma(q)} \sum_{i=1}^k \left\| \prod_{j=i}^k b_j(\tau_j) \right\| \left\| \int_{\xi_{i-1}}^{\xi_i} (t-s)^{q-1} \|S(t-s)f(s, x_s)\| ds \right. \\
 & \left. + \frac{1}{\Gamma(q)} \int_{\xi_k}^t (t-s)^{q-1} \|S(t-s)f(s, x_s)\| ds \right] I_{[\xi_k, \xi_{k+1})}(t) \Big]^2 \\
 & \leq 2 \sum_{k=0}^{+\infty} \left[\left\| \prod_{i=1}^k b_i(\tau_i) \right\|^2 \|S(t-t_0)\|^2 \|\varphi(0)\|^2 I_{[\xi_k, \xi_{k+1})}(t) \right] \\
 & + 2 \left[\sum_{k=0}^{\infty} \left[\frac{1}{\Gamma(q)} \sum_{i=1}^k \left\| \prod_{j=i}^k b_j(\tau_j) \right\| \left\| \int_{\xi_{i-1}}^{\xi_i} (t-s)^{q-1} \|S(t-s)f(s, x_s)\| ds \right. \right. \right. \\
 & \left. \left. + \frac{1}{\Gamma(q)} \int_{\xi_k}^t (t-s)^{q-1} \|S(t-s)f(s, x_s)\| ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right]^2 \\
 & \leq 2M^2 B^2 \|\varphi(0)\|^2 + 2M^2 \max\{1, B^2\} \frac{(T-t_0)^{2q-1}}{\Gamma(q)(2q-1)} \int_{t_0}^t \|f(s, x_s)\|^2 ds, \\
 \|x\|_t^2 & \leq 2M^2 B^2 \|\varphi\|^2 + 2M^2 \max\{1, B^2\} \frac{(T-t_0)^{2q-1}}{\Gamma(q)(2q-1)} \int_{t_0}^t \|f(s, x_s)\|^2 ds, \\
 E\|x\|_t^2 & \\
 & \leq 2M^2 B^2 E[\|\varphi\|^2] + 2M^2 \max\{1, B^2\} \frac{(T-t_0)^{2q-1}}{\Gamma(q)(2q-1)} \int_{t_0}^t E[\|f(s, x_s)\|^2] ds \\
 & \leq 2M^2 B^2 E[\|\varphi\|^2] + 2M^2 \max\{1, B^2\} \frac{(T-t_0)^{2q-1}}{\Gamma(q)(2q-1)} \int_{t_0}^t \delta(s) H(E[\|x\|_s^2]) ds, \\
 \sup_{t_0 \leq v \leq t} E\|x\|_v^2 & \\
 & \leq 2M^2 B^2 E[\|\varphi\|^2] \\
 & + 2M^2 \max\{1, B^2\} \frac{(T-t_0)^{2q-1}}{\Gamma(q)(2q-1)} \int_{t_0}^t \delta(s) H\left(\sup_{t_0 \leq v \leq s} E[\|x\|_v^2]\right) ds.
 \end{aligned}$$

Let

$$\ell(t) = \sup_{t_0 \leq v \leq t} E[\|x\|_v^2], \quad t \in [t_0, T].$$

Then, for any $t \in [t_0, T]$, it follows that

$$\ell(t) \leq 2M^2 B^2 E[\|\varphi\|^2] + 2M^2 \max\{1, B^2\} \frac{(T-t_0)^{2q-1}}{\Gamma(q)(2q-1)} \int_{t_0}^t \delta(s) H(\ell(s)) ds.$$

Denoting by $u(t)$ the right-hand side of the above inequality, we obtain that

$$\ell(t) \leq u(t), \quad t \in [t_0, T],$$

$$u(t_0) = 2M^2 B^2 E[\|\varphi\|^2] = c_1$$

and

$$\begin{aligned}
 u'(t) &= 2M^2 \max\{1, B^2\} \frac{(T - t_0)^{2q-1}}{\Gamma(q)(2q - 1)} \delta(t)H(\ell(t)) \\
 &\leq 2M^2 \max\{1, B^2\} \frac{(T - t_0)^{2q-1}}{\Gamma(q)(2q - 1)} \delta(t)H(u(t)), \quad t \in [t_0, T].
 \end{aligned}$$

Then

$$\frac{u'(t)}{H(u(t))} \leq 2M^2 \max\{1, B^2\} \frac{(T - t_0)^{2q-1}}{\Gamma(q)(2q - 1)} \delta(t), \quad t \in [t_0, T].$$

Integrating the above inequality from t_0 to t and making use of the change of variable, we obtain

$$\begin{aligned}
 \int_{u(t_0)}^{u(t)} \frac{ds}{H(s)} &\leq 2M^2 \max\{1, B^2\} \frac{(T - t_0)^{2q-1}}{\Gamma(q)(2q - 1)} \int_{t_0}^t \delta(s) ds \\
 &\leq 2M^2 \max\{1, B^2\} \frac{(T - t_0)^{2q-1}}{\Gamma(q)(2q - 1)} \int_{t_0}^T \delta(s) ds \\
 &< \int_{c_1}^{\infty} \frac{ds}{H(s)} = \int_{u(t_0)}^{\infty} \frac{ds}{H(s)}.
 \end{aligned}$$

From the above inequality and by the mean value theorem, there is a constant K such that $u(t) \leq K$, and hence $\ell(t) \leq K$. Since $\sup_{t_0 \leq v \leq t} E[\|x\|_v^2] = \ell(t)$ holds for every $t \in [t_0, T]$, we have $\sup_{t_0 \leq v \leq T} E[\|x\|_v^2] \leq K$, where K only depends on T and the functions δ and H , consequently

$$E\|x\|_B^2 = \sup_{t_0 \leq v \leq T} E\|x\|_v^2 \leq K.$$

In the next steps, we will prove that Φ is continuous and completely continuous.

Step 1. We prove that Φ is continuous.

Let $\{x_n\}$ be a convergent sequence of elements of β . Then, for each $t \in [t_0, T]$, we have

$$\begin{aligned}
 \Phi x_n(t) &= \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) S(t - t_0) \varphi(0) \right. \\
 &\quad + \frac{1}{\Gamma(q)} \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t - s)^{q-1} S(t - s) f(s, x_{n_s}) ds \\
 &\quad \left. + \frac{1}{\Gamma(q)} \int_{\xi_k}^t (t - s)^{q-1} S(t - s) f(s, x_{n_s}) ds \right] I_{[\xi_k, \xi_{k+1})}(t).
 \end{aligned}$$

Thus

$$\Phi x_n(t) - \Phi x(t) = \sum_{k=0}^{+\infty} \left[\frac{1}{\Gamma(q)} \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t - s)^{q-1} S(t - s) \{f(s, x_{n_s}) - f(s, x_s)\} ds \right.$$

$$+ \frac{1}{\Gamma(q)} \int_{\xi_k}^t (t-s)^{q-1} S(t-s) \{f(s, x_{n_s}) - f(s, x_s)\} ds \Big] I_{[\xi_k, \xi_{k+1})}(t),$$

and

$$E \|\Phi x_n - \Phi x\|_t^2 \leq M^2 \max\{1, B^2\} \frac{(T-t_0)^{2q-1}}{\Gamma(q)(2q-1)} \int_{t_0}^t E \|f(s, x_{n_s}) - f(s, x_s)\|^2 ds.$$

Thus Φ is continuous.

Step 2. We prove that Φ is a completely continuous operator.

Denote

$$B_m = \{x \in \mathcal{B} \mid \|x\|_{\mathcal{B}}^2 \leq m\}$$

for some $m \geq 0$.

Step 2.1. We show that Φ maps B_m into an equicontinuous family.

Let $x \in B_m$ and $t_1, t_2 \in [t_0, T]$. If $t_0 < t_1 < t_2 < T$, then by using (H_1) – (H_2) and condition (3.1), we have

$$\begin{aligned} & \Phi x(t_2) - \Phi x(t_1) \\ &= \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) S(t_2 - t_0) \varphi(0) \right. \\ & \quad + \frac{1}{\Gamma(q)} \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t_2 - s)^{q-1} S(t_2 - s) f(s, x_s) ds \\ & \quad \left. + \frac{1}{\Gamma(q)} \int_{\xi_k}^{t_2} (t_2 - s)^{q-1} S(t_2 - s) f(s, x_s) ds \right] I_{[\xi_k, \xi_{k+1})}(t_2) \\ & \quad - \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) S(t_1 - t_0) \varphi(0) \right. \\ & \quad + \frac{1}{\Gamma(q)} \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t_1 - s)^{q-1} S(t_1 - s) f(s, x_s) ds \\ & \quad \left. + \frac{1}{\Gamma(q)} \int_{\xi_k}^{t_1} (t_1 - s)^{q-1} S(t_1 - s) f(s, x_s) ds \right] I_{[\xi_k, \xi_{k+1})}(t_1) \\ &= \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) S(t_2 - t_0) \varphi(0) \right. \\ & \quad + \frac{1}{\Gamma(q)} \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t_2 - s)^{q-1} S(t_2 - s) f(s, x_s) ds \\ & \quad \left. + \frac{1}{\Gamma(q)} \int_{\xi_k}^{t_2} (t_2 - s)^{q-1} S(t_2 - s) f(s, x_s) ds \right] (I_{[\xi_k, \xi_{k+1})}(t_2) - I_{[\xi_k, \xi_{k+1})}(t_1)) \\ & \quad + \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) [S(t_2 - t_0) - S(t_1 - t_0)] \varphi(0) \right. \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{\Gamma(q)} \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} [(t_2 - s)^{q-1} S(t_2 - s) - (t_1 - s)^{q-1} S(t_1 - s)] f(s, x_s) ds \\
 & + \frac{1}{\Gamma(q)} \int_{\xi_k}^{t_1} [(t_2 - s)^{q-1} S(t_2 - s) - (t_1 - s)^{q-1} S(t_1 - s)] f(s, x_s) ds \\
 & + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} S(t_2 - s) f(s, x_s) ds \Big] I_{[\xi_k, \xi_{k+1})}(t_1).
 \end{aligned}$$

Then

$$E \|\Phi x(t_2) - \Phi x(t_1)\|^2 \leq 2E \|I_1\|^2 + 2E \|I_2\|^2,$$

where

$$\begin{aligned}
 I_1 & = \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) S(t_2 - t_0) \varphi(0) \right. \\
 & + \frac{1}{\Gamma(q)} \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t_2 - s)^{q-1} S(t_2 - s) f(s, x_s) ds \\
 & \left. + \frac{1}{\Gamma(q)} \int_{\xi_k}^{t_2} (t_2 - s)^{q-1} S(t_2 - s) f(s, x_s) ds \right] (I_{[\xi_k, \xi_{k+1})}(t_2) - I_{[\xi_k, \xi_{k+1})}(t_1))
 \end{aligned}$$

and

$$\begin{aligned}
 I_2 & = \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) [S(t_2 - t_0) - S(t_1 - t_0)] \varphi(0) \right. \\
 & + \frac{1}{\Gamma(q)} \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} [(t_2 - s)^{q-1} S(t_2 - s) - (t_1 - s)^{q-1} S(t_1 - s)] f(s, x_s) ds \\
 & + \frac{1}{\Gamma(q)} \int_{\xi_k}^{t_1} [(t_2 - s)^{q-1} S(t_2 - s) - (t_1 - s)^{q-1} S(t_1 - s)] f(s, x_s) ds \\
 & \left. + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} S(t_2 - s) f(s, x_s) ds \right] I_{[\xi_k, \xi_{k+1})}(t_1).
 \end{aligned}$$

Furthermore,

$$\begin{aligned}
 E \|I_1\|^2 & \leq E \left(\sum_{k=0}^{+\infty} \left[\prod_{i=1}^k \|b_i(\tau_i)\| \|S(t_2 - t_0)\| \|\varphi(0)\| \right. \right. \\
 & + \frac{1}{\Gamma(q)} \sum_{i=1}^k \left\| \prod_{j=i}^k b_j(\tau_j) \right\| \left\| \int_{\xi_{i-1}}^{\xi_i} (t_2 - s)^{q-1} S(t_2 - s) \|f(s, x_s)\| ds \right. \\
 & \left. \left. + \frac{1}{\Gamma(q)} \int_{\xi_k}^{t_2} (t_2 - s)^{q-1} S(t_2 - s) \|f(s, x_s)\| ds \right] (I_{[\xi_k, \xi_{k+1})}(t_2) - I_{[\xi_k, \xi_{k+1})}(t_1)) \right)^2 \\
 & \leq 2M^2 B^2 E \|\varphi(0)\|^2 E (I_{[\xi_k, \xi_{k+1})}(t_2) - I_{[\xi_k, \xi_{k+1})}(t_1))
 \end{aligned}$$

$$\begin{aligned}
 &+ 2 \max\{1, B^2\} \frac{(T - t_0)^{2q-1}}{\Gamma(q)(2q - 1)} E \int_{t_0}^{t_2} \|S(t_2 - s)\| \|f(s, x_s)\| ds E(I_{[\xi_k, \xi_{k+1})}(t_2) \\
 &- I_{[\xi_k, \xi_{k+1})}(t_1)) \\
 &\leq 2M^2 B^2 E \|\varphi(0)\|^2 E(I_{[\xi_k, \xi_{k+1})}(t_2) - I_{[\xi_k, \xi_{k+1})}(t_1)) \\
 &+ 2M^2 \max\{1, B^2\} \frac{(T - t_0)^{2q-1}}{\Gamma(q)(2q - 1)} \int_{t_0}^{t_2} \delta(s) H(E\|x\|_s^2) ds E(I_{[\xi_k, \xi_{k+1})}(t_2) \\
 &- I_{[\xi_k, \xi_{k+1})}(t_1)) \\
 &\leq 2M^2 B^2 E \|\varphi(0)\|^2 E(I_{[\xi_k, \xi_{k+1})}(t_2) - I_{[\xi_k, \xi_{k+1})}(t_1)) \\
 &+ 2M^2 \max\{1, B^2\} \frac{(T - t_0)^{2q-1}}{\Gamma(q)(2q - 1)} \int_{t_0}^{t_2} M^* H(E(m)) ds E(I_{[\xi_k, \xi_{k+1})}(t_2) \\
 &- I_{[\xi_k, \xi_{k+1})}(t_1)) \\
 &\rightarrow 0, \quad \text{as } t_1 \rightarrow t_2,
 \end{aligned}$$

where $M^* = \sup\{\delta(t) : t \in [t_0, T]\}$.

$$\begin{aligned}
 E\|I_2\|^2 &\leq E \left(\sum_{k=0}^{+\infty} \left[\prod_{i=1}^k \|b_i(\tau_i)\| \|S(t_2 - t_0) - S(t_1 - t_0)\| \|\varphi(0)\| \right. \right. \\
 &+ \frac{1}{\Gamma(q)} \sum_{i=1}^k \prod_{j=i}^k \|b_j(\tau_j)\| \int_{\xi_{i-1}}^{\xi_i} \|(t_2 - s)^{q-1} S(t_2 - s) \\
 &- (t_1 - s)^{q-1} S(t_1 - s)\| \|f(s, x_s)\| ds \\
 &+ \frac{1}{\Gamma(q)} \int_{\xi_k}^{t_1} \|(t_2 - s)^{q-1} S(t_2 - s) - (t_1 - s)^{q-1} S(t_1 - s)\| \|f(s, x_s)\| ds \\
 &\left. \left. + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} \|S(t_2 - s)\| \|f(s, x_s)\| ds \right] I_{[\xi_k, \xi_{k+1})}(t_1) \right)^2 \\
 &\leq 3B^2 \|S(t_2 - t_0) - S(t_1 - t_0)\|^2 E \|\varphi(0)\|^2 \\
 &+ 3 \max\{1, B^2\} (t_1 - t_0) \frac{1}{\Gamma(q)} E \int_{t_0}^{t_1} \|(t_2 - s)^{q-1} S(t_2 - s) \\
 &- (t_1 - s)^{q-1} S(t_1 - s)\|^2 \|f(s, x_s)\|^2 ds \\
 &+ 3M^2 \frac{(T - t_0)^{2q-1}}{\Gamma(q)(2q - 1)} E \int_{t_1}^{t_2} \|f(s, x_s)\|^2 ds \\
 &\leq 3B^2 \|S(t_2 - t_0) - S(t_1 - t_0)\|^2 E \|\varphi(0)\|^2 \\
 &+ 3 \max\{1, B^2\} (t_1 - t_0) \frac{1}{\Gamma(q)} \int_{t_0}^{t_1} \|(t_2 - s)^{q-1} S(t_2 - s) \\
 &- (t_1 - s)^{q-1} S(t_1 - s)\|^2 M^* H(m) ds \\
 &+ 3M^2 \frac{(T - t_0)^{2q-1}}{\Gamma(q)(2q - 1)} \int_{t_1}^{t_2} M^* H(m) ds \\
 &\rightarrow 0, \quad \text{as } t_1 \rightarrow t_2.
 \end{aligned}$$

Thus, Φ maps B_m into an equicontinuous family of functions.

Step 2.2. We show that ΦB_m is uniformly bounded.

From (3.1), $\|x\|_B^2 \leq m$, by (H_1) – (H_2) , we obtain

$$\begin{aligned} \|\Phi x(t)\|^2 &\leq \left(\sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) \|S(t-t_0)\| \|\varphi(0)\| \right. \right. \\ &\quad \left. \left. + \frac{1}{\Gamma(q)} \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{q-1} \|S(t-s)\| \|f(s, x_s)\| ds \right. \right. \\ &\quad \left. \left. + \frac{1}{\Gamma(q)} \int_{\xi_k}^t (t-s)^{q-1} \|S(t-s)\| \|f(s, x_s)\| ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right)^2 \\ &\leq 2M^2 B^2 \|\varphi(0)\|^2 + 2M^2 \max\{1, B^2\} \frac{(T-t_0)^{2q-1}}{\Gamma(q)(2q-1)} \int_{t_0}^t \|f(s, x_s)\|^2 ds. \end{aligned}$$

Thus

$$\begin{aligned} E\|\Phi x(t)\|^2 &\leq 2M^2 B^2 E\|\varphi(0)\|^2 + 2M^2 \max\{1, B^2\} \frac{(T-t_0)^{2q-1}}{\Gamma(q)(2q-1)} \int_{t_0}^t E\|f(s, x_s)\|^2 ds \\ &\leq 2M^2 B^2 E\|\varphi(0)\|^2 + 2M^2 \max\{1, B^2\} \frac{(T-t_0)^{2q}}{\Gamma(q)(2q-1)} \|\alpha_m\|_{L^1}. \end{aligned}$$

This yields that the set $\{(\Phi x)(t), \|x\|_B^2 \leq m\}$ is uniformly bounded, so $\{\Phi B_m\}$ is uniformly bounded, by the Arzela–Ascoli theorem, Φ maps B_m into a precompact set in X .

Step 2.3. We show that ΦB_m is compact.

Let $t_0 < t \leq T$ be fixed, and let ϵ be a real number satisfying $\epsilon \in (0, t - t_0)$, for $x \in B_m$, we define

$$\begin{aligned} (\Phi_\epsilon x)(t) &= \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) S(t-t_0) \varphi(0) \right. \\ &\quad \left. + \frac{1}{\Gamma(q)} \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{q-1} S(t-s) f(s, x_s) ds \right. \\ &\quad \left. + \frac{1}{\Gamma(q)} \int_{\xi_k}^{t-\epsilon} (t-s)^{q-1} S(t-s) f(s, x_s) ds \right] I_{[\xi_k, \xi_{k+1})}(t), \quad t \in (t_0, t-\epsilon). \end{aligned}$$

Since $S(t)$ is a compact operator, the set

$$H_\epsilon(t) = \{(\Phi_\epsilon x)(t) : x \in B_m\}$$

is precompact in X for every $\epsilon \in (0, t - t_0)$. Moreover, for every $x \in B_m$, we have

$$\begin{aligned} (\Phi x)(t) - (\Phi_\epsilon x)(t) &= \sum_{k=0}^{+\infty} \left[\frac{1}{\Gamma(q)} \int_{\xi_k}^t (t-s)^{q-1} S(t-s) f(s, x_s) ds \right] I_{[\xi_k, \xi_{k+1})}(t) \\ &\quad - \sum_{k=0}^{+\infty} \left[\frac{1}{\Gamma(q)} \int_{\xi_k}^{t-\epsilon} (t-s)^{q-1} S(t-s) f(s, x_s) ds \right] I_{[\xi_k, \xi_{k+1})}(t). \end{aligned}$$

By using (H_1) – (H_2) , condition (3.1), and $\|x\|_{\mathbb{B}}^2 \leq m$, we have

$$E\|(\Phi x)(t) - (\Phi_{\epsilon} x)(t)\|_t^2 \leq M^2 \frac{(T - t_0)^{2q-1}}{\Gamma(q)(2q-1)} \int_{t-\epsilon}^t M^* H(m) ds.$$

Therefore, there are precompact sets arbitrarily close to the set $\{(\Phi x)(t) : x \in B_m\}$. Hence the set $\{(\Phi x)(t) : x \in B_m\}$ is precompact in X . Therefore, Φ is a completely continuous operator.

Moreover, the set $U(\Phi) = \{x \in \mathbb{B} : x = \lambda \Phi x \text{ for some } 0 < \lambda < 1\}$ is bounded. Consequently, by Lemma 2.1, the operator Φ has a fixed point in \mathbb{B} . Therefore, system (1.1) has a mild solution. □

4 Exponential stability in the quadratic mean

In this section, we will study the exponential stability of the second moment of a mild solution of system (1.1). For an F_t -adapted process, $\Phi(t) : [-r, \infty) \rightarrow R$ is almost surely continuous in t . For the purposes of stability, we may assume that $f(t, 0) \equiv 0$ for any $t \geq t_0$ so that system (1.1) admits a trivial solution. Moreover, $\Phi(t) = \varphi(t - t_0)$ for $t \in [t_0 - r, t_0]$ and $E\|\Phi\|_t^2 \rightarrow 0$ as $t \rightarrow \infty$.

Definition 4.1 System (1.1) is said to be exponentially stable in the quadratic mean if there exist positive constants $C_1 > 0$ and $\lambda > 0$ such that

$$E\|x(t)\| \leq C_1 E\|\varphi\|^2 e^{-\lambda(t-t_0)}, \quad t \geq t_0.$$

Now we introduce the following hypotheses used in our discussion:

- (H_3) $\mu H(\psi) \leq H(\mu\psi)$ for all $\psi \in R^+$, where $\mu > 1$.
- (H_4) $\|S(t)\| \leq Me^{-r(t-t_0)}$, $t \geq t_0$, where $M \geq 1, r > 0$.

Theorem 4.1 Assume that the hypotheses of Theorem 3.1 and (H_3) – (H_4) hold. If the following inequality is satisfied, then system (1.1) is exponentially stable in the quadratic mean:

$$M_2 \int_{t_0}^T \delta(s) ds < \int_{c_2}^{\infty} \frac{ds}{H(s)}, \tag{4.1}$$

where $M_2 = 2M^2 \max\{1, B^2\} \frac{(T-t_0)^{2q-1}}{\Gamma(q)(2q-1)}$, $c_2 = 2M^2 B^2 E[\|\varphi\|^2]$, and $MB \geq \frac{1}{\sqrt{2}}$.

Proof Let Φ be the map defined in Theorem 3.1. Using hypotheses (H_1) – (H_4) , we have

$$\begin{aligned} & \|x(t)\|^2 \\ & \leq \lambda^2 \left[\sum_{k=0}^{+\infty} \left[\left\| \prod_{i=1}^k b_i(\tau_i) \right\| \|S(t-t_0)\| \|\varphi(0)\| \right. \right. \\ & \quad + \frac{1}{\Gamma(q)} \sum_{i=1}^k \left\| \prod_{j=i}^k b_j(\tau_j) \right\| \int_{\xi_{i-1}}^{\xi_i} (t-s)^{q-1} \|S(t-s)f(s, x_s)\| ds \\ & \quad \left. \left. + \frac{1}{\Gamma(q)} \int_{\xi_k}^t (t-s)^{q-1} \|S(t-s)f(s, x_s)\| ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right]^2 \end{aligned}$$

$$\begin{aligned}
 &\leq 2 \sum_{k=0}^{+\infty} \left[\left\| \prod_{i=1}^k b_i(\tau_i) \right\|^2 M^2 e^{-2r(t-t_0)} \|\varphi(0)\|^2 I_{[\xi_k, \xi_{k+1})}(t) \right] \\
 &\quad + 2 \left[\sum_{k=0}^{\infty} \left[\frac{1}{\Gamma(q)} \sum_{i=1}^k \left\| \prod_{j=i}^k b_j(\tau_j) \right\| \int_{\xi_{i-1}}^{\xi_i} (t-s)^{q-1} M e^{-r(t-s)} \|f(s, x_s)\| ds \right. \right. \\
 &\quad \left. \left. + \frac{1}{\Gamma(q)} \int_{\xi_k}^t (t-s)^{q-1} M e^{-r(t-s)} \|f(s, x_s)\| ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right]^2 \\
 &\leq 2 \left[\max_k \left\{ \prod_{j=i}^k \|b_j(\tau_j)\|^2 \right\} \right] M^2 e^{-2r(t-t_0)} \|\varphi(0)\|^2 \\
 &\quad + 2 \left[\max_{i,k} \left\{ 1, \prod_{j=i}^k \|b_j(\tau_j)\| \right\} \right]^2 \frac{1}{\Gamma(q)} \sum_{k=0}^{+\infty} \int_{t_0}^t (t-s)^{q-1} M e^{-r(t-s)} \|f(s, x_s)\| ds I_{[\xi_k, \xi_{k+1})}(t)^2 \\
 &\leq 2M^2 B^2 e^{-2r(t-t_0)} \|\varphi(0)\|^2 + 2M^2 \max\{1, B^2\} \frac{(T-t_0)^{2q-1}}{\Gamma(q)(2q-1)} \int_{t_0}^t e^{-2r(t-s)} \|f(s, x_s)\|^2 ds, \\
 \|\mathbf{x}\|_t^2 &\leq 2M^2 B^2 e^{-2r(t-t_0)} \|\varphi\|^2 + 2M^2 \max\{1, B^2\} \frac{(T-t_0)^{2q-1}}{\Gamma(q)(2q-1)} \int_{t_0}^t e^{-2r(t-s)} \|f(s, x_s)\|^2 ds,
 \end{aligned}$$

$$\begin{aligned}
 E\|\mathbf{x}\|_t^2 &\leq 2M^2 B^2 e^{-2r(t-t_0)} E[\|\varphi\|^2] \\
 &\quad + 2M^2 \max\{1, B^2\} \frac{(T-t_0)^{2q-1}}{\Gamma(q)(2q-1)} \int_{t_0}^t e^{-2r(t-s)} \delta(s) H(E[\|\mathbf{x}\|_s^2]) ds \\
 &= 2M^2 B^2 e^{-2r(t-t_0)} E[\|\varphi\|^2] \\
 &\quad + 2M^2 \max\{1, B^2\} \frac{(T-t_0)^{2q-1}}{\Gamma(q)(2q-1)} e^{-2r(t-t_0)} \int_{t_0}^t e^{2r(s-t_0)} \delta(s) H(E[\|\mathbf{x}\|_s^2]) ds, \\
 e^{2r(t-t_0)} E\|\mathbf{x}\|_t^2 &\leq 2M^2 B^2 E[\|\varphi\|^2] \\
 &\quad + 2M^2 \max\{1, B^2\} \frac{(T-t_0)^{2q-1}}{\Gamma(q)(2q-1)} \int_{t_0}^t \delta(s) H(e^{2r(s-t_0)} E[\|\mathbf{x}\|_s^2]) ds, \\
 \sup_{t_0 \leq v \leq t} e^{2r(v-t_0)} E\|\mathbf{x}\|_v^2 &\leq 2M^2 B^2 E[\|\varphi\|^2] \\
 &\quad + 2M^2 \max\{1, B^2\} \frac{(T-t_0)^{2q-1}}{\Gamma(q)(2q-1)} \int_{t_0}^t \delta(s) H\left(\sup_{t_0 \leq v \leq s} e^{2r(v-t_0)} E[\|\mathbf{x}\|_v^2]\right) ds.
 \end{aligned}$$

Let

$$\ell_1(t) = \sup_{t_0 \leq v \leq t} e^{2r(v-t_0)} E\|\mathbf{x}\|_v^2, \quad t \in [t_0, T].$$

Then, for any $t \in [t_0, T]$, it follows that

$$\ell_1(t) \leq 2M^2 B^2 E[\|\varphi\|^2] + 2M^2 \max\{1, B^2\} \frac{(T-t_0)^{2q-1}}{\Gamma(q)(2q-1)} \int_{t_0}^t \delta(s) H(\ell_1(s)) ds.$$

Denoting by $u_1(t)$ the right-hand side of the above inequality, we obtain that

$$\begin{aligned} \ell_1(t) &\leq u_1(t), \quad t \in [t_0, T], \\ u_1(t_0) &= 2M^2 B^2 E \|\varphi\|^2 = c_2 \end{aligned}$$

and

$$\begin{aligned} u_1'(t) &= 2M^2 \max\{1, B^2\} \frac{(T - t_0)^{2q-1}}{\Gamma(q)(2q - 1)} \delta(t) H(\ell_1(t)) \\ &\leq 2M^2 \max\{1, B^2\} \frac{(T - t_0)^{2q-1}}{\Gamma(q)(2q - 1)} \delta(t) H(u_1(t)), \quad t \in [t_0, T]. \end{aligned}$$

Then

$$\frac{u_1'(t)}{H(u_1(t))} \leq 2M^2 \max\{1, B^2\} \frac{(T - t_0)^{2q-1}}{\Gamma(q)(2q - 1)} \delta(t), \quad t \in [t_0, T].$$

Integrating the above inequality from t_0 to t and making use of the change of variable, we obtain

$$\begin{aligned} \int_{u_1(t_0)}^{u_1(t)} \frac{ds}{H(s)} &\leq 2M^2 \max\{1, B^2\} \frac{(T - t_0)^{2q-1}}{\Gamma(q)(2q - 1)} \int_{t_0}^t \delta(s) ds \\ &\leq 2M^2 \max\{1, B^2\} \frac{(T - t_0)^{2q-1}}{\Gamma(q)(2q - 1)} \int_{t_0}^T \delta(s) ds \\ &< \int_{c_2}^\infty \frac{ds}{H(s)} = \int_{u_1(t_0)}^\infty \frac{ds}{H(s)}, \quad t \in [t_0, T]. \end{aligned}$$

From the above inequality and by the mean value theorem, there is a constant K_1 such that $u_1(t) \leq K_1$, and hence $\ell_1(t) \leq K_1$. Since $\sup_{t_0 \leq v \leq t} e^{2r(v-t_0)} E \|x\|_v^2 = \ell_1(t)$ holds for every $t \in [t_0, T]$, we have $\sup_{t_0 \leq v \leq t} e^{2r(v-t_0)} E \|x\|_v^2 \leq K_1$, where K_1 depends only on δ and H . Consequently,

$$e^{2r(t-t_0)} E \|x\|_t^2 = \sup_{t_0 \leq v \leq T} e^{2r(v-t_0)} E \|x\|_v^2 \leq K_1.$$

In the following, we proceed as in the previous theorem. Now, we will show that Φ is a completely continuous operator using a two-step proof.

Step 1. We prove that Φ is continuous.

Let $\{x_n\}$ be a convergent sequence of elements of x in β . Then, for each $t \in [t_0, T]$, we have

$$\begin{aligned} E \|\Phi x_n - \Phi x\|_t^2 &\leq M^2 \max\{1, B^2\} \frac{(T - t_0)^{2q-1}}{\Gamma(q)(2q - 1)} e^{-2r(t-t_0)} \int_{t_0}^t e^{2r(s-t_0)} E \|f(s, x_{n_s}) - f(s, x_s)\|^2 ds \\ &\rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus Φ is continuous.

Step 2. We prove that Φ is a completely continuous operator.

Denote

$$B_{m_1} = \{x \in B \mid \|x\|_B^2 \leq m_1\}$$

for some $m_1 \geq 0$.

Step 2.1. We show that Φ maps B_{m_1} into an equicontinuous family.

Let $x \in B_m$ and let $t_1, t_2 \in [t_0, T]$. If $t_0 < t_1 < t_2 < T$, then by using (H_1) – (H_4) and condition (4.1) and then by following a process similar to Step 2.1 of Theorem 3.1, we get

$$E\|\Phi x(t_2) - \Phi x(t_1)\|^2 \rightarrow 0 \quad \text{as } t_2 \rightarrow t_1.$$

Thus, Φ maps B_{m_1} into an equicontinuous family of functions.

Step 2.2. We show that ΦB_{m_1} is uniformly bounded.

From condition (4.1) and (H_1) – (H_4) , we obtain

$$\begin{aligned} \|\Phi x(t)\|^2 &\leq 2 \left[\max_k \left\{ \prod_{j=i}^k \|b_j(\tau_j)\|^2 \right\} \right] M^2 e^{-2r(t-t_0)} \|\varphi(0)\|^2 \\ &\quad + 2 \left[\max_{i,k} \left\{ 1, \prod_{j=i}^k \|b_j(\tau_j)\| \right\} \right]^2 \\ &\quad \times \left(\frac{1}{\Gamma(q)} \sum_{k=0}^{+\infty} \int_{t_0}^t (t-s)^{q-1} M e^{-r(t-s)} \|f(s, x_s)\| ds \right) I_{[\xi_k, \xi_{k+1})}(t)^2. \end{aligned}$$

Thus

$$\begin{aligned} E\|\Phi x(t)\|^2 &\leq 2M^2 B^2 e^{-2r(t-t_0)} E\|\varphi(0)\|^2 \\ &\quad + 2M^2 \max\{1, B^2\} \frac{(T-t_0)^{2q-1}}{\Gamma(q)(2q-1)} e^{-2r(t-t_0)} \\ &\quad \times \int_{t_0}^t e^{2r(s-t_0)} M^* H(m) ds, \end{aligned}$$

where $M^* = \sup\{\delta(t) : t \in [t_0, T]\}$. Since $e^{-2r(t-t_0)} \rightarrow 0$, the right-hand side of the above inequality tends to 0 as $t \rightarrow \infty$. That is,

$$\|(\Phi x)\|_B^2 \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

This yields that the set $\{(\Phi x)(t), \|x\|_B^2 \leq m_1\}$ is uniformly bounded, so $\{\Phi B_{m_1}\}$ is uniformly bounded.

Step 2.3. We show that ΦB_{m_1} is compact.

Let $t_0 < t \leq T$ be fixed and let ϵ be a real number satisfying $\epsilon \in (0, t - t_0)$, for $x \in B_{m_1}$, we define

$$(\Phi_\epsilon x)(t) = \sum_{k=0}^{+\infty} \prod_{i=1}^k b_i(\tau_i) S(t - t_0) \varphi(0)$$

$$\begin{aligned}
 & + \frac{1}{\Gamma(q)} \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{q-1} S(t-s) f(s, x_s) ds \\
 & + \frac{1}{\Gamma(q)} \int_{\xi_k}^{t-\epsilon} (t-s)^{q-1} S(t-s) f(s, x_s) ds \Big] I_{[\xi_k, \xi_{k+1})}(t), \quad t \in (t_0, t-\epsilon).
 \end{aligned}$$

Since $S(t)$ is a compact operator, the set

$$H_\epsilon(t) = \{(\Phi_\epsilon x)(t) : x \in B_{m_1}\}$$

is precompact in X for every $\epsilon \in (0, t - t_0)$.

Making use of hypotheses (H_1) – (H_4) , condition (4.1), and the fact that $\|x\|_\beta^2 \leq m_1$, we have

$$E\|(\Phi x)(t) - (\Phi_\epsilon x)(t)\|_t^2 \leq M^2 \frac{(T - t_0)^{2q-1}}{\Gamma(q)(2q - 1)} e^{-2r(t-t_0)} \int_{t-\epsilon}^t e^{2r(s-t_0)} M^* H(E\|x\|_s^2) ds.$$

Therefore, there are precompact sets arbitrarily close to the set $\{(\Phi x)(t) : x \in B_{m_1}\}$. Hence the set $\{(\Phi x)(t) : x \in B_{m_1}\}$ is precompact in X . Therefore, Φ is a completely continuous operator.

Moreover, the set $U(\Phi) = \{x \in \beta : x = \lambda \Phi x \text{ for some } 0 < \lambda < 1\}$ is bounded. Consequently, by Lemma 2.1, the operator Φ has a fixed point in β . Therefore, system (1.1) has a mild solution with $\Phi(t) = \varphi(t - t_0)$ when $t \in [t_0 - r, t_0]$ and $E\|\Phi\|_t^2 \rightarrow 0$ as $t \rightarrow \infty$. This completes the proof. □

5 Example

In this section, we provide an example to illustrate our main results. Consider the following random impulsive fractional differential system:

$$\begin{cases}
 {}^c D_t^q u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x) + au(x, t - r)u(x, t), & t \neq \xi_k, \\
 u(x, \xi_k) = q(k)\tau_k u(x, \xi_k^-), & t = \xi_k, \\
 u(t, 0) = u(t, \pi) = 0, \\
 u(t, x) = \phi(t, x), & -r < t \leq 0, 0 \leq x \leq \pi.
 \end{cases} \tag{5.1}$$

Let $X = L^2([0, \pi])$ and τ_k be a random variable defined on $D_k \equiv (0, d_k)$ for $k = 1, 2, \dots$, where $0 < d_k < +\infty$. Furthermore, assume that τ_k follows the Erlang distribution, where $k = 1, 2, \dots$ and τ_i and τ_j are independent of each other as $i \neq j$ for $i, j = 1, 2, \dots$; q is a function of k ; $\xi_0 = t_0$ and $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1, 2, \dots$, here $t_0 \in R_r$ is an arbitrary given real number.

Define A an operator on X by $Au = \frac{\partial^2 u}{\partial x^2}$ with the domain $D(A) = \{u \in X \mid u \text{ and } \frac{\partial u}{\partial x} \text{ are absolutely continuous, } \frac{\partial^2 u}{\partial x^2} \in X, u(0) = 0 = u(\pi)\}$.

It is well known that A generates a strongly continuous semigroup $S(t)$ which is compact, analytic, and self-adjoint. Moreover, the operator A can be expressed as

$$Au = \sum_{n=1}^\infty n^2(u, u_n)u_n, \quad u \in D(A),$$

where $u_n(\zeta) = \sqrt{\frac{2}{\pi}} \sin n\zeta$, $n \in N$ is the orthogonal set of eigenvectors of A , and for every $u \in X, S(t)u = \sum_{n=1}^{\infty} \exp(-n^2 t)(u, u_n)u_n$, which satisfies

$$\|S(t)\| \leq \exp(-\pi^2(t - t_0)), \quad t \geq t_0.$$

Hence $S(t)$ is a contraction semigroup.

Equation (5.1) can be reformulated as the following abstract equation in $X = L^2([0, \pi])$:

$${}^c D_t^q z(t) = Az(t) + f(t, z_t), \quad t \neq \xi_k, t \geq \tau,$$

$$z(\xi_k) = b_k(\tau_k)z(\xi_k^-), \quad k = 1, 2, \dots,$$

$$z_{t_0} = \varphi,$$

where $z(t) = u(\cdot, t)$, i.e., $z(t)(x) = u(x, t), z_t(\theta)(x) = u(x, t + \theta), t \in [t_0, T], \theta \in [-r, 0], z(t, x) = \varphi(t, x), t \in [-r, 0], x \in [0, \pi]$. The operator A is defined as above. The functions $f : [\tau, T] \times C \rightarrow X$ and $b_k : D_k \rightarrow X$ are given by

$$f(t, z_t) = au(x, t - r)u(x, t), \quad b_k(\tau_k) = q(k)\tau_k,$$

where $a > 0$.

It is easy to check that system (5.1) has a mild solution if hypotheses $(H_1), (H_2)$ and the inequality

$$2 \max\{1, B^2\} \frac{(T - t_0)^{2q-1}}{(2q - 1)\Gamma(q)} \int_{t_0}^T \delta(s) ds < \int_{c_1}^{\infty} \frac{ds}{H(s)}$$

hold, where $c_1 = 2B^2E\|\varphi\|^2$.

Furthermore, by Theorem 4.1, if the following conditions hold

$$2 \max\{1, B^2\} \frac{(T - t_0)^{2q-1}}{\Gamma(q)(2q - 1)} \int_{t_0}^T \delta(s) ds < \int_{c_2}^{\infty} \frac{ds}{H(s)},$$

where $c_2 = 2B^2E[\|\varphi\|^2]$, then system (5.1) is exponentially stable in the quadratic mean.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

SPZ carried out the main part of this article. Both authors read and approved the final manuscript.

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