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On nonlocal Neumann boundary value problem for a second-order forward (α, β)-difference equation

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Abstract

In this paper, we present some properties of the forward (α, β)-difference operators, and the existence results of two nonlocal boundary value problems for second-order forward (α, β)-difference equations. The existence and uniqueness results are proved by using the Banach fixed point theorem, and the existence of at least one positive solution is established by using the Krasnoselskii' fixed point theorem.

MSC: 39A05; 39A12

Keywords: Forward (α, β)-difference equations; Neumann boundary value problem; Positive solution; Existence

1 Introduction

The difference calculus is known as the calculus without considering limits and deals with sets of non-differentiable functions. The difference calculus appears in many applications such as statistics, computing, electrical circuit analysis, dynamical systems, economics, biology, and other fields; see for example [1–7] and the references therein. There are two types of difference operators, one based on the forward (delta) difference operator $\Delta_h f(t) = \frac{f(t+1)-f(t)}{h}$, and one on the backward (nabla) difference operator $\nabla_h f(t) = \frac{f(t)-f(t-h)}{h}$; if $h = 1$, then Δ and ∇ are the standard difference operators. There is not much research involving the development of h-sums and h-difference operators (see [8–13]). Therefore, there is a gap in the literature as regards the details of this operation. In this paper, we aim to study the forward (α, β)-difference of f defined by

$$ {}_h\Delta_{(\alpha,\beta)} f(t) = \frac{\beta f(t+h) - \alpha f(t)}{h}, \quad \alpha, \beta, h > 0, $$

where the coefficient of $f(t)$ and $f(t+h)$ can be chosen [14].

In particular, since the boundary value problem for forward (α, β)-difference equations has not been studied, we attempt to fill this gap by considering the existence and uniqueness result of the nonlocal Neumann boundary value problem for a second-order forward

(α,β)-difference equation,

$$\begin{aligned}
&{}_h\Delta^2_{(\alpha,\beta)}u(t-h)+f\bigl(t,u(t),(\Psi u)(t)\bigr)=0, \quad t\in(h\mathbb{N})_{h,Th},\\
&{}_h\Delta_{(\alpha,\beta)}\left[\left(\frac{\beta}{\alpha}\right)^{\frac{t}{h}}u(t)\right]_{t=0}=\phi_1(u), \qquad {}_h\Delta_{(\alpha,\beta)}\left[\left(\frac{\beta}{\alpha}\right)^{\frac{t}{h}}u(t)\right]_{t=Th}=\phi_2(u),
\end{aligned} \tag{1.1}$$

where $(h\mathbb{N})_{h,Th}:=\{h,2h,\ldots,Th\}$, $\alpha,\beta,h>0$, $\alpha<\beta$, $f\in C((h\mathbb{N})_{0,(T+1)h}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $\phi_1,\phi_2: C((h\mathbb{N})_{0,(T+1)h},\mathbb{R})\to\mathbb{R}$, and for $\varphi\in((h\mathbb{N})_{0,(T+1)h}\times(h\mathbb{N})_{0,(T+1)h},[0,\infty))$, $(\Psi u)(t):=({}_h\Delta^{-1}_{(\alpha,\beta)}\varphi u)(t)=h\sum_{s=0}^{\frac{t}{h}-1}\varphi(t,s)u(hs)$.

Furthermore, we study the existence of at least one positive solution of boundary value problem for a second-order forward (α,β)-difference equation of the form

$$\begin{aligned}
&{}_h\Delta^2_{(\alpha,\beta)}u(t-h)+F\bigl(t,u(t)\bigr)=0, \quad t\in(h\mathbb{N})_{(1+\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h},\\
&{}_h\Delta_{(\alpha,\beta)}u\left(\left(\frac{\beta}{\beta-\alpha}\right)h\right)=\frac{\beta}{h}u\left(\left(\frac{\beta}{\beta-\alpha}+1\right)h\right),\\
&\left[{}_h\Delta_{(\alpha,\beta)}\left[\left(\frac{\beta}{\alpha}\right)^{t}u(t)\right]\right]_{t=(T+\frac{\beta}{\beta-\alpha})h}=\theta(u),
\end{aligned} \tag{1.2}$$

where $F\in C((h\mathbb{N})_{(1+\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}\times[0,\infty),[0,\infty))$ and $\theta: C((h\mathbb{N})_{(1+\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h},[0,\infty))\to[0,\infty)$.

In this paper, the plan is as follows. In Sect. 2 we recall some definitions and basic lemmas, and present some properties of the forward (α,β)-difference operators. In this section, we also derive a representation for the solution to (1.1) and (1.2) by converting the problem to equivalent summation equations. In Sect. 3, we show the existence and uniqueness result of problem (1.1). In Sects. 4 and 5, we show the properties of Green's function and the existence of at least one positive solution for problem (1.2) by using Krasnoselskii's fixed point theorem in a cone. Finally, some examples are provided to illustrate our results in the last section.

Theorem 1.1 ([15]) *Let E be a Banach space, and $K\subset E$ be a cone. Assume Ω_1, Ω_2 are open subsets of E with $0\in\Omega_1$, $\overline{\Omega}_1\subset\Omega_2$, and let*

$$A: K\cap(\overline{\Omega}_2\setminus\Omega_1)\longrightarrow K$$

be a completely continuous operator such that

(i) $\|Au\|\le\|u\|$, $u\in K\cap\partial\Omega_1$, *and* $\|Au\|\ge\|u\|$, $u\in K\cap\partial\Omega_2$, *or*

(ii) $\|Au\|\ge\|u\|$, $u\in K\cap\partial\Omega_1$, *and* $\|Au\|\le\|u\|$, $u\in K\cap\partial\Omega_2$.

Then A has a fixed point in $K\cap(\overline{\Omega}_2\setminus\Omega_1)$.

Lemma 1.1 ([16] (Arzelá–Ascoli theorem)) *A set of functions in $C[a,b]$ with the sup norm, is relatively compact if and only it is uniformly bounded and equicontinuous on $[a,b]$.*

Lemma 1.2 ([16]) *If a set is closed and relatively compact then it is compact.*

2 Preliminaries

In the following, there are notations, definitions, and lemmas which are used in the main results.

Let $\mathbb{N}_a := \{a, a+1, a+2, \ldots\}$, $\mathbb{N}_{a,b} := \{a, a+1, a+2, \ldots, b = a+k\}$, $(h\mathbb{N})_a := \{a, a+h, a+2h, \ldots\}$ and $(h\mathbb{N})_{a,b} := \{a, a+h, a+2h, \ldots, b = a+kh\}$ for some $k \in \mathbb{N}$, $a \in \mathbb{R}$.

Definition 2.1 For $\alpha, \beta, h > 0$ and f defined on $[0,\infty)$, the forward (α,β)-difference of f is defined by

$$_h\Delta_{(\alpha,\beta)} f(t) = \frac{\beta f(t+h) - \alpha f(t)}{h}.$$

Furthermore, we define the higher-order (α,β)-difference by

$$_h\Delta^n_{(\alpha,\beta)} f(t) := {_h\Delta_{(\alpha,\beta)}} {_h\Delta^{n-1}_{(\alpha,\beta)}} f(t) \quad \text{and} \quad {_h\Delta^0_{(\alpha,\beta)}} f(t) := f(t).$$

Example 2.1 $_h\Delta_{(\alpha,\beta)}(1) = \frac{\beta-\alpha}{h}$ and $_h\Delta_{(\alpha,\beta)}(t) = \frac{\beta(t+h)-\alpha t}{h} = (\frac{\beta-\alpha}{h})t + \beta$.

Remark 2.1

(i) If $\alpha = \beta = 1$, the difference operator $_h\Delta_{(1,1)}$ becomes the forward h-difference operator

$$\Delta_h f(t) = \frac{f(t+h) - f(t)}{h}.$$

(ii) Let $f(t) = \frac{1}{\alpha}(\frac{\alpha}{\beta})^{\frac{t}{h}} g(t)$ for $t \in (h\mathbb{N})_a$. Then

$$_h\Delta_{(\alpha,\beta)} f(t) = \frac{1}{h}\left[\frac{\beta}{\alpha}\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}+1} g(t+h) - \left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}} g(t)\right] = \left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}} \Delta_h g(t).$$

The right inverse of the operator $_h\Delta_{(\alpha,\beta)}$ or the (α,β)-sum operator is defined as follows.

Definition 2.2 Let $\alpha, \beta, h > 0$ and $(h\mathbb{N})_{a,b} \subset (h\mathbb{N})_c$. Assuming that $f, g : (h\mathbb{N})_c \to \mathbb{R}$ and $f(t) = \frac{1}{\alpha}(\frac{\alpha}{\beta})^{\frac{t}{h}} g(t)$, we define the (α,β)-sum of f by

$$_h\Delta^{-1}_{(\alpha,\beta)} f(t) := h \sum_{k=\frac{c}{h}}^{\frac{t}{h}-1} f(hk) = h \sum_{k=\frac{c}{h}}^{\frac{t}{h}-1} \frac{\alpha^{k-1}}{\beta^k} g(hk) \quad \text{for } t \in (h\mathbb{N})_{c+h},$$

and for $a, b \in (h\mathbb{N})_{c+h}$, $a < b$, the (α,β)-sum of f from a to b defined by

$$\left[{_h\Delta^{-1}_{(\alpha,\beta)}} f(t)\right]_a^b := h \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(hk) = h \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} \frac{\alpha^{k-1}}{\beta^k} g(hk).$$

Example 2.2

(i) If $f(t) = 1$ for $t \in (h\mathbb{N})_c$, then we have $g(t) = \alpha(\frac{\beta}{\alpha})^{\frac{t}{h}}$ and

$$_h\Delta^{-1}_{(\alpha,\beta)}(1) = h \sum_{k=\frac{c}{h}}^{\frac{t}{h}-1} \frac{\alpha^{k-1}}{\beta^k}\left(\frac{\beta^k}{\alpha^{k-1}}\right) = h \sum_{k=\frac{c}{h}}^{\frac{t}{h}-1} 1 = t - c.$$

(ii) If $f(t) = t$ for $t \in (h\mathbb{N})_c$, then we have $g(t) = \alpha(\frac{\beta}{\alpha})^{\frac{t}{h}} t$ and

$$ {}_h\Delta^{-1}_{(\alpha,\beta)}(t) = h \sum_{k=\frac{c}{h}}^{\frac{t}{h}-1} \frac{\alpha^{k-1}}{\beta^k} \left(\frac{\beta^k}{\alpha^{k-1}} hk \right) = h^2 \sum_{k=\frac{c}{h}}^{\frac{t}{h}-1} k = \frac{1}{2}(t-c)(t+c-h). $$

In the following lemma, we introduce the properties of forward (α,β)-operators.

Lemma 2.1 *Letting $\alpha, \beta, h > 0$ and $f : (h\mathbb{N})_a \to \mathbb{R}$,*

$$ {}_h\Delta_{(\alpha,\beta)} {}_h\Delta^{-1}_{(\alpha,\beta)} f(t) = \beta f(t) + (\beta - \alpha) \sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} f(kh), $$

$$ {}_h\Delta^{-1}_{(\alpha,\beta)} {}_h\Delta_{(\alpha,\beta)} f(t) = \frac{\beta^2}{\alpha} \big[f(t) - f(a) \big] + \left(\frac{\beta^2 - \alpha^2}{\alpha h} \right) {}_h\Delta^{-1}_{(\alpha,\beta)} f(t). $$

Proof Letting $f(t) = \frac{1}{\alpha}(\frac{\alpha}{\beta})^{\frac{t}{h}} g(t)$ for $t \in (h\mathbb{N})_a$, we have

$$ \begin{aligned} {}_h\Delta_{(\alpha,\beta)} {}_h\Delta^{-1}_{(\alpha,\beta)} f(t) &= {}_h\Delta_{(\alpha,\beta)} \left[h \sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} \frac{\alpha^{k-1}}{\beta^k} g(hk) \right] \\ &= \beta \sum_{k=\frac{a}{h}}^{\frac{t}{h}} \frac{\alpha^{k-1}}{\beta^k} g(hk) - \alpha \sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} \frac{\alpha^{k-1}}{\beta^k} g(hk) \\ &= \beta \left[\frac{\alpha^{\frac{t}{h}-1}}{\beta^{\frac{t}{h}}} g(t) + \sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} \frac{\alpha^{k-1}}{\beta^k} g(hk) \right] - \alpha \sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} \frac{\alpha^{k-1}}{\beta^k} g(hk) \\ &= \left(\frac{\alpha}{\beta} \right)^{\frac{t}{h}-1} g(t) + (\beta - \alpha) \sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} \frac{\alpha^{k-1}}{\beta^k} g(hk) \\ &= \beta f(t) + (\beta - \alpha) \sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} f(kh). \end{aligned} $$

From Remark 2.1(ii), we obtain

$$ \begin{aligned} {}_h\Delta^{-1}_{(\alpha,\beta)} {}_h\Delta_{(\alpha,\beta)} f(t) &= {}_h\Delta^{-1}_{(\alpha,\beta)h} \left[\left(\frac{\alpha}{\beta} \right)^{\frac{t}{h}} \Delta_h g(t) \right] \\ &= {}_h\Delta^{-1}_{(\alpha,\beta)h} \left[\left(\frac{\alpha}{\beta} \right)^{\frac{t}{h}} \left(\frac{g(t+h) - g(t)}{h} \right) \right] \\ &= \frac{\beta}{h} {}_h\Delta^{-1}_{(\alpha,\beta)h} \left[\frac{1}{\alpha} \left(\frac{\alpha}{\beta} \right)^{\frac{t}{h}+1} g(t+h) \right] - \frac{\alpha}{h} {}_h\Delta^{-1}_{(\alpha,\beta)h} \left[\frac{1}{\alpha} \left(\frac{\alpha}{\beta} \right)^{\frac{t}{h}} g(t) \right] \\ &= \frac{\beta}{h} {}_h\Delta^{-1}_{(\alpha,\beta)h} f(t+h) - \frac{\alpha}{h} {}_h\Delta^{-1}_{(\alpha,\beta)h} f(t) \end{aligned} $$

$$
\begin{aligned}
&= \beta \sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} \frac{\alpha^{k-1}}{\beta^{k}} g\bigl((k+1)h\bigr) - \alpha \sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} \frac{\alpha^{k-1}}{\beta^{k}} g(kh) \\
&= \beta \sum_{k=\frac{a}{h}+1}^{\frac{t}{h}} \frac{\alpha^{k-2}}{\beta^{k-1}} g(kh) - \alpha \sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} \frac{\alpha^{k-1}}{\beta^{k}} g(kh) \\
&= \beta \left(\sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} \frac{\alpha^{k-2}}{\beta^{k-1}} g(kh) + \frac{\alpha^{\frac{t}{h}-2}}{\beta^{\frac{t}{h}-1}} g(t) - \frac{\alpha^{\frac{a}{h}-2}}{\beta^{\frac{a}{h}-1}} g(a) \right) - \alpha \sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} \frac{\alpha^{k-1}}{\beta^{k}} g(kh) \\
&= \frac{\beta^2}{\alpha} \sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} \frac{\alpha^{k-1}}{\beta^{k}} g(kh) + \frac{\alpha^{\frac{t}{h}-2}}{\beta^{\frac{t}{h}-2}} g(t) - \frac{\alpha^{\frac{a}{h}-2}}{\beta^{\frac{a}{h}-2}} g(a) - \alpha \sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} \frac{\alpha^{k-1}}{\beta^{k}} g(kh) \\
&= \frac{\beta^2}{\alpha} \bigl[f(t) - f(a)\bigr] + \left(\frac{\beta^2 - \alpha^2}{\alpha h} \right) {}_h\Delta^{-1}_{(\alpha,\beta)} f(t).
\end{aligned}
$$

This completes the proof. □

Example 2.3

(i) If $f(t) = 1$ for $t \in (h\mathbb{N})_a$, then we have

$$
\begin{aligned}
{}_h\Delta_{(\alpha,\beta)}\,{}_h\Delta^{-1}_{(\alpha,\beta)}(1) &= \beta + (\beta - \alpha) \sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} (1) = \beta + \left(\frac{\beta - \alpha}{h} \right)(t - a), \\
{}_h\Delta^{-1}_{(\alpha,\beta)}\,{}_h\Delta_{(\alpha,\beta)}(1) &= \beta(1 - 1) + \left(\frac{\beta - \alpha}{h} \right) {}_h\Delta^{-1}_{(\alpha,\beta)}(1) = \left(\frac{\beta - \alpha}{h} \right)(t - a).
\end{aligned}
$$

Note that if f is defined on $(h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h}$, we obtain

$$
\begin{aligned}
{}_h\Delta_{(\alpha,\beta)}\,{}_h\Delta^{-1}_{(\alpha,\beta)}(1) &= \left(\frac{\beta - \alpha}{h} \right) t, \\
{}_h\Delta^{-1}_{(\alpha,\beta)}\,{}_h\Delta_{(\alpha,\beta)}(1) &= \left(\frac{\beta - \alpha}{h} \right) t - \beta.
\end{aligned}
$$

(ii) If $f(t) = t$ for $t \in (h\mathbb{N})_a$, then we have

$$
\begin{aligned}
{}_h\Delta_{(\alpha,\beta)}\,{}_h\Delta^{-1}_{(\alpha,\beta)}(t) &= \beta t + (\beta - \alpha) \sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} (kh) = \beta t + \left(\frac{\beta - \alpha}{2h} \right)(t - a)(t + a - h), \\
{}_h\Delta^{-1}_{(\alpha,\beta)}\,{}_h\Delta_{(\alpha,\beta)}(t) &= \beta(t - a) + \left(\frac{\beta - \alpha}{h} \right) {}_h\Delta^{-1}_{(\alpha,\beta)}(t) \\
&= \beta(t - a) + \left(\frac{\beta - \alpha}{2h} \right)(t - a)(t + a - h).
\end{aligned}
$$

Note that if f is defined on $(h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h}$, we obtain

$$
\begin{aligned}
{}_h\Delta_{(\alpha,\beta)}\,{}_h\Delta^{-1}_{(\alpha,\beta)}(t) &= \frac{1}{2}\left(\frac{\beta-\alpha}{h}\right)t^2 + \left(\alpha+\frac{\beta}{2}\right)t - \frac{1}{2}\left(\frac{\alpha}{\beta-\alpha}\right)h, \\
{}_h\Delta^{-1}_{(\alpha,\beta)}\,{}_h\Delta_{(\alpha,\beta)}(t) &= \frac{1}{2}\left(\frac{\beta-\alpha}{h}\right)t^2 + \left(\alpha+\frac{\beta}{2}\right)t - \frac{1}{2}\left(\frac{2\beta^2+\alpha}{\beta-\alpha}\right)h.
\end{aligned}
$$

To study the solution of the boundary value problems (1.1), we need the following lemmas that deals with linear variant of the boundary value problems (1.1).

Lemma 2.2 *Let $\alpha,\beta,h>0$, $\alpha<\beta$, function $x\in C((h\mathbb{N})_{0,(T+1)h},[0,\infty))$ and functionals $\phi_1,\phi_2: C((h\mathbb{N})_{0,(T+1)h},[0,\infty))\to[0,\infty)$ be given. Then the problem*

$$
\begin{aligned}
&{}_h\Delta^2_{(\alpha,\beta)}u(t-h)+x(t)=0, \quad t\in(h\mathbb{N})_{h,Th}, \\
&{}_h\Delta_{(\alpha,\beta)}\left[\left(\frac{\beta}{\alpha}\right)^{\frac{t}{h}}u(t)\right]_{t=0}=\phi_1, \\
&{}_h\Delta_{(\alpha,\beta)}\left[\left(\frac{\beta}{\alpha}\right)^{\frac{t}{h}}u(t)\right]_{t=Th}=\phi_2,
\end{aligned}
\tag{2.1}
$$

has the unique solution

$$
\begin{aligned}
u(t) &= \frac{\beta h}{(\beta-\alpha)^2T}\Biggl\{\bigl[\bigl[(\beta-\alpha)T+\beta\bigr]\phi_1(u)-\beta\phi_2(u)\bigr]\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}} \\
&\quad -\frac{h}{\alpha\beta}\sum_{i=1}^{T}\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-i}\bigl[(T+1-i)\beta-(T-i)\alpha\bigr]x(ih)\Biggr\}-\frac{t}{(\beta-\alpha)T} \\
&\quad \times\Biggl\{\bigl[\phi_1(u)-\phi_2(u)\bigr]\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}}-\frac{h}{\alpha\beta}\sum_{i=1}^{T}\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-i}\bigl[(T+1-i)\beta-(T-i)\alpha\bigr]x(ih)\Biggr\} \\
&\quad -\frac{h^2}{\alpha\beta}\sum_{i=1}^{\frac{t}{h}-1}\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-i}\left[\frac{t}{h}-i\right]x(ih), \quad t\in(h\mathbb{N})_{0,(T+1)h}.
\end{aligned}
\tag{2.2}
$$

Proof From ${}_h\Delta^2_{(\alpha,\beta)}u(t-h)=\frac{1}{h}[\beta\,{}_h\Delta_{(\alpha,\beta)}u(t)-\alpha\,{}_h\Delta_{(\alpha,\beta)}u(t-h)]$ and the first equation of (2.1), we create the system of n equations

(A_1) $\quad \beta\,{}_h\Delta_{(\alpha,\beta)}u(t)-\alpha\,{}_h\Delta_{(\alpha,\beta)}u(t-h)=-hx(t),$

(A_2) $\quad \beta\,{}_h\Delta_{(\alpha,\beta)}u(t-h)-\alpha\,{}_h\Delta_{(\alpha,\beta)}u(t-2h)=-hx(t-h),$

(A_3) $\quad \beta\,{}_h\Delta_{(\alpha,\beta)}u(t-2h)-\alpha\,{}_h\Delta_{(\alpha,\beta)}u(t-3h)=-hx(t-2h),$

...

(A_{n-1}) $\quad \beta\,{}_h\Delta_{(\alpha,\beta)}u(2h)-\alpha\,{}_h\Delta_{(\alpha,\beta)}u(h)=-hx(2h),$

(A_n) $\quad \beta\,{}_h\Delta_{(\alpha,\beta)}u(h)-\alpha\,{}_h\Delta_{(\alpha,\beta)}u(0)=-hx(h).$

Considering $\beta \cdot (A_1) + \alpha \cdot (A_2)$, we obtain

$$(A_{n+1}) \quad \beta^2{}_h\Delta_{(\alpha,\beta)}u(t) - \alpha^2{}_h\Delta_{(\alpha,\beta)}u(t-2h) = -h\sum_{k=0}^{1}\alpha^k\beta^{1-k}x(t-kh).$$

For $\beta \cdot (A_{n+1}) + \alpha^2 \cdot (A_3)$, we have the next equation:

$$(A_{n+2}) \quad \beta^3{}_h\Delta_{(\alpha,\beta)}u(t) - \alpha^3{}_h\Delta_{(\alpha,\beta)}u(t-3h) = -h\sum_{k=0}^{2}\alpha^k\beta^{2-k}x(t-kh).$$

$\ldots$

Repeating the same process, we have the equation (A_{2n-2})

$$(A_{2n-2}) \quad \beta^{n-1}{}_h\Delta_{(\alpha,\beta)}u(t) - \alpha^{n-1}{}_h\Delta_{(\alpha,\beta)}u\bigl(t-(n-1)h\bigr) = -h\sum_{k=0}^{n-2}\alpha^k\beta^{n-2-k}x(t-kh).$$

Finally, for $\beta \cdot (A_{2n-2}) + \alpha^{n-1} \cdot (A_n)$, we get the equation (A_{2n-1})

$$(A_{2n-1}) \quad \beta^{n}{}_h\Delta_{(\alpha,\beta)}u(t) - \alpha^{n}{}_h\Delta_{(\alpha,\beta)}u(0) = -h\sum_{k=0}^{n-1}\alpha^k\beta^{n-1-k}x(t-kh).$$

So, we obtain

$$_h\Delta_{(\alpha,\beta)}u(t) = \left(\frac{\alpha}{\beta}\right)^n{}_h\Delta_{(\alpha,\beta)}u(0) - h\sum_{k=0}^{n-1}\frac{\alpha^k}{\beta^{k+1}}x(t-kh). \tag{2.3}$$

For $t = nh \in (h\mathbb{N})_{0,Th}$, $n \in \mathbb{N}$, we can write (2.3) in the form

$$_h\Delta_{(\alpha,\beta)}u(t) = \left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}}{}_h\Delta_{(\alpha,\beta)}u(0) - h\sum_{i=1}^{\frac{t}{h}}\frac{\alpha^{\frac{t}{h}-i}}{\beta^{\frac{t}{h}+1-i}}x(ih) \quad \text{and} \tag{2.4}$$

$$\beta u(t+h) - \alpha u(t) = h\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}}{}_h\Delta_{(\alpha,\beta)}u(0) - h^2\sum_{i=1}^{\frac{t}{h}}\frac{\alpha^{\frac{t}{h}-i}}{\beta^{\frac{t}{h}+1-i}}x(ih), \tag{2.5}$$

where $\sum_{s=p}^{q} y(s) = 0$; if $p < q$. By substituting $t = 0, h, \ldots, nh$, $n \in \mathbb{N}$ into (2.5), we have the system of n equations

$$(B_1) \quad \beta u(h) - \alpha u(0) = h{}_h\Delta_{(\alpha,\beta)}u(0),$$

$$(B_2) \quad \beta u(2h) - \alpha u(h) = h\left(\frac{\alpha}{\beta}\right){}_h\Delta_{(\alpha,\beta)}u(0) - \frac{h^2}{\beta}x(h),$$

$$(B_3) \quad \beta u(3h) - \alpha u(2h) = h\left(\frac{\alpha}{\beta}\right)^2{}_h\Delta_{(\alpha,\beta)}u(0) - h^2\sum_{i=1}^{2}\frac{\alpha^{2-i}}{\beta^{3-i}}x(ih),$$

$\ldots$

$$(B_n) \quad \beta u(nh) - \alpha u\bigl((n-1)h\bigr) = h\left(\frac{\alpha}{\beta}\right)^{n-1}{}_h\Delta_{(\alpha,\beta)}u(0) - h^2\sum_{i=1}^{n-1}\frac{\alpha^{n-1-i}}{\beta^{n-i}}x(ih).$$

Considering $\alpha \cdot (B_1) + \beta \cdot (B_2)$, we get

$$(B_{n+1}) \quad \beta^2 u(2h) - \alpha^2 u(0) = 2\alpha h_h\Delta_{(\alpha,\beta)} u(0) - h^2 x(h).$$

Next, for $\alpha \cdot (B_{n+1}) + \beta^2 \cdot (B_3)$, we get

$$(B_{n+2}) \quad \beta^3 u(3h) - \alpha^3 u(0) = 3\alpha^2 h_h\Delta_{(\alpha,\beta)} u(0) - h^2 \sum_{i=1}^{2} (3-i)\alpha^{2-i}\beta^{i-1} x(ih).$$

$$\ldots$$

Repeating this process, we have the equation (B_{2n})

$$(B_{2n}) \quad \beta^n u(nh) - \alpha^n u(0) = n\alpha^{n-1} h_h\Delta_{(\alpha,\beta)} u(0) - h^2 \sum_{i=1}^{n-1} (n-i)\alpha^{n-1-i}\beta^{i-1} x(ih).$$

For $t = nh$, $n \in \mathbb{N}$, we can write (B_{2n}) in the form

$$\begin{aligned} u(t) = {} & \left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}} u(0) + \frac{t}{\alpha}\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}} {}_h\Delta_{(\alpha,\beta)} u(0) \\ & - \frac{h^2}{\alpha\beta} \sum_{i=1}^{\frac{t}{h}-1} \left[\frac{t}{h} - i\right] \left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-i} x(ih), \quad t \in (h\mathbb{N})_{0,(T+1)h}. \end{aligned} \tag{2.6}$$

Hence

$$\begin{aligned} \left(\frac{\beta}{\alpha}\right)^{\frac{t}{h}} u(t) &= u(0) + \frac{t}{\alpha} {}_h\Delta_{(\alpha,\beta)} u(0) - \frac{h^2}{\alpha\beta} \sum_{i=1}^{\frac{t}{h}-1} \left[\frac{t}{h} - i\right] \left(\frac{\beta}{\alpha}\right)^{i} x(ih) \\ &=: A + \frac{t}{\alpha} B - \frac{h^2}{\alpha\beta} \sum_{i=1}^{\frac{t}{h}-1} \left[\frac{t}{h} - i\right] \left(\frac{\beta}{\alpha}\right)^{i} x(ih), \end{aligned} \tag{2.7}$$

for $t \in (h\mathbb{N})_{0,(T+1)h}$ and A, B are some constants.

Using the (α,β)-difference for (2.7), we have

$$\begin{aligned} {}_h\Delta_{(\alpha,\beta)} \left[\left(\frac{\beta}{\alpha}\right)^{\frac{t}{h}} u(t)\right] = {} & A\left(\frac{\beta-\alpha}{h}\right) + B\left[\left(\frac{\beta-\alpha}{h}\right)\frac{t}{\alpha} + \frac{\beta}{\alpha}\right] \\ & - \frac{h}{\alpha\beta} \sum_{i=1}^{\frac{t}{h}} \left[\left(\frac{t}{h} - i + 1\right)\beta - \left(\frac{t}{h} - i\right)\alpha\right] \left(\frac{\beta}{\alpha}\right)^{i} x(ih). \end{aligned} \tag{2.8}$$

By using the boundary conditions of (2.1), we obtain

$$\phi_1(u) = \left(\frac{\beta-\alpha}{h}\right) A + \left(\frac{\beta}{\alpha}\right) B, \tag{2.9}$$

$$\begin{aligned} \phi_2(u) = {} & \left(\frac{\beta-\alpha}{h}\right) A + \left(\frac{(\beta-\alpha)T + \beta}{\alpha}\right) B \\ & - \frac{h}{\alpha\beta} \sum_{i=1}^{T} \left[(T+1-i)\beta - (T-i)\alpha\right] \left(\frac{\beta}{\alpha}\right)^{i} x(ih). \end{aligned} \tag{2.10}$$

The constants A and B can be obtained by solving the system of equations (2.9) and (2.10),

$$\begin{aligned} A &= \frac{\beta h}{(\beta-\alpha)^2 T}\Bigg\{\big[(\beta-\alpha)T+\beta\big]\phi_1(u)-\beta\phi_2(u)-\frac{h}{\alpha\beta}\sum_{i=1}^{T}\left(\frac{\beta}{\alpha}\right)^i \\ &\quad\times\big[(T+1-i)\beta-(T-i)\alpha\big]x(ih)\Bigg\}, \\ B &= -\frac{\alpha}{(\beta-\alpha)T}\Bigg\{\phi_1(u)-\phi_2(u)-\frac{h}{\alpha\beta}\sum_{i=1}^{T}\left(\frac{\beta}{\alpha}\right)^i\big[(T+1-i)\beta-(T-i)\alpha\big]x(ih)\Bigg\}. \end{aligned}$$

Substituting all constants A, B into (2.7), we obtain (2.2). This completes the proof. □

Next, we present the following lemma that deals with linear variant of the boundary value problem (1.2).

Lemma 2.3 *Let $\alpha,\beta,h>0$, $\alpha<\beta$, function $y\in C((h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h},[0,\infty))$ and functional $\theta: C((h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h},[0,\infty))\to[0,\infty)$ be given. Then the problem*

$$\begin{aligned} &{}_h\Delta^2_{(\alpha,\beta)}u(t-h)+y(t)=0, \quad t\in(h\mathbb{N})_{(1+\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h} \\ &{}_h\Delta_{(\alpha,\beta)}u\left(\left(\frac{\beta}{\beta-\alpha}\right)h\right)=\frac{\beta}{h}u\left(\left(\frac{\beta}{\beta-\alpha}+1\right)h\right), \\ &\left[{}_h\Delta_{(\alpha,\beta)}\left[\left(\frac{\beta}{\alpha}\right)^t u(t)\right]\right]_{t=(T+\frac{\beta}{\beta-\alpha})h}=\theta(u), \end{aligned} \tag{2.11}$$

has the unique solution

$$\begin{aligned} u(t) &= \left[\frac{t}{(\beta-\alpha)T+2\beta}\right]\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}}\Bigg\{\theta(u)+\frac{h}{\alpha\beta}\sum_{i=\frac{\beta}{\beta-\alpha}+1}^{T+\frac{\beta}{\beta-\alpha}}\left(\frac{\beta}{\alpha}\right)^i \\ &\quad\times\big[(\beta-\alpha)(T-i)+2\beta\big]y(ih)\Bigg\}-\frac{h^2}{\alpha\beta}\sum_{i=\frac{\beta}{\beta-\alpha}+1}^{\frac{t}{h}-1}\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-i}\left[\frac{t}{h}-i\right]y(ih). \end{aligned} \tag{2.12}$$

Proof With the same argument of Lemma 2.2, we obtain

$$\begin{aligned} u(t) &= \left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}}u\left(\left(\frac{\beta}{\beta-\alpha}\right)h\right)+\frac{t}{\alpha}\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}}{}_h\Delta_{(\alpha,\beta)}u\left(\left(\frac{\beta}{\beta-\alpha}\right)h\right) \\ &\quad-\frac{h^2}{\alpha\beta}\sum_{i=\frac{\beta}{\beta-\alpha}+1}^{\frac{t}{h}-1}\left[\frac{t}{h}-i\right]\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-i}y(ih). \end{aligned} \tag{2.13}$$

Using the first boundary condition of (2.11), we obtain $u((\frac{\beta}{\beta-\alpha})h)=0$.

So, (2.13) can be written in the form

$$\left(\frac{\beta}{\alpha}\right)^{\frac{t}{h}} u(t) = \frac{t}{\alpha}\, {}_h\Delta_{(\alpha,\beta)} u\left(\left(\frac{\beta}{\beta-\alpha}\right)h\right) - \frac{h^2}{\alpha\beta}\sum_{i=\frac{\beta}{\beta-\alpha}+1}^{\frac{t}{h}-1}\left[\frac{t}{h}-i\right]\left(\frac{\beta}{\alpha}\right)^{i} y(ih)$$

$$=: \frac{t}{\alpha}C - \frac{h^2}{\alpha\beta}\sum_{i=\frac{\beta}{\beta-\alpha}+1}^{\frac{t}{h}-1}\left[\frac{t}{h}-i\right]\left(\frac{\beta}{\alpha}\right)^{i} y(ih), \tag{2.14}$$

for $t \in (h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}$; C is some constant.

Using the (α,β)-difference for (2.14), we have

$$ {}_h\Delta_{(\alpha,\beta)}\left[\left(\frac{\beta}{\alpha}\right)^{\frac{t}{h}} u(t)\right] = C\left[\left(\frac{\beta-\alpha}{h}\right)\frac{t}{\alpha} + \frac{\beta}{\alpha}\right] - \frac{h}{\alpha\beta}\sum_{i=\frac{\beta}{\beta-\alpha}+1}^{\frac{t}{h}}\left[\left(\frac{t}{h}-i+1\right)\beta - \left(\frac{t}{h}-i\right)\alpha\right]\left(\frac{\beta}{\alpha}\right)^{i} y(ih). \tag{2.15}$$

By using the second boundary conditions of (2.11), we obtain

$$\theta(u) = \frac{C}{\alpha}\left[(\beta-\alpha)T + 2\beta\right] - \frac{h}{\alpha\beta}\sum_{i=\frac{\beta}{\beta-\alpha}+1}^{T+\frac{\beta}{\beta-\alpha}}\left[(T+2-i)\beta - (T-i)\alpha\right]\left(\frac{\beta}{\alpha}\right)^{i} y(ih). \tag{2.16}$$

Solving the above equation, we have

$$C = \frac{\alpha}{(\beta-\alpha)T+2\beta}\left\{\theta(u) + \frac{h}{\alpha\beta}\sum_{i=\frac{\beta}{\beta-\alpha}+1}^{T+\frac{\beta}{\beta-\alpha}}\left(\frac{\beta}{\alpha}\right)^{i}\left[(\beta-\alpha)(T-i)+2\beta\right]y(ih)\right\}.$$

Substituting constants C into (2.14), we obtain (2.12). This completes the proof. □

Lemma 2.4 *Problem* (2.11) *has the unique solution*

$$u(t) = \mathcal{C}(t)\theta(u) + \sum_{s=1}^{T} G\left(\frac{t}{h}, s+\frac{\beta}{\beta-\alpha}\right) y\left(\left(s+\frac{\beta}{\beta-\alpha}\right)h\right), \tag{2.17}$$

where

$$\mathcal{C}(t) = \left[\frac{t}{(\beta-\alpha)T+2\beta}\right]\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}}, \tag{2.18}$$

and

$$G\left(\frac{t}{h}, s\right) = \frac{h^2}{\alpha\beta}\begin{cases} g_1(\frac{t}{h}, s), & s \in \mathbb{N}_{\frac{\beta}{\beta-\alpha}+1, \frac{t}{h}-1}, \\ g_2(\frac{t}{h}, s), & s \in \mathbb{N}_{\frac{t}{h}, T+\frac{\beta}{\beta-\alpha}}, \end{cases} \tag{2.19}$$

with

$$g_1\left(\frac{t}{h},s\right) = \left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-s}\left\{\left(\frac{\frac{t}{h}}{(\beta-\alpha)T+2\beta}\right)\left[(\beta-\alpha)(T-s)+2\beta\right]-\left[\frac{t}{h}-s\right]\right\},$$

$$g_2\left(\frac{t}{h},s\right) = \left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-s}\left(\frac{\frac{t}{h}}{(\beta-\alpha)T+2\beta}\right)\left[(\beta-\alpha)(T-s)+2\beta\right].$$

Proof The unique solution of problem (2.1) can be written as

$$\begin{aligned}
u(t) &= \left[\frac{t}{(\beta-\alpha)T+2\beta}\right]\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}}\theta(u) + \left[\frac{t}{(\beta-\alpha)T+2\beta}\right]\frac{h}{\alpha\beta} \\
&\quad\times\left\{\sum_{s=\frac{\beta}{\beta-\alpha}+1}^{\frac{t}{h}-1}\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-s}\left[(\beta-\alpha)(T-s)+2\beta\right]y(sh) + \sum_{s=\frac{t}{h}}^{T+\frac{\beta}{\beta-\alpha}}\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-s}\right. \\
&\quad\left.\times\left[(\beta-\alpha)(T-s)+2\beta\right]y(sh)\right\} - \frac{h^2}{\alpha\beta}\sum_{s=\frac{\beta}{\beta-\alpha}+1}^{\frac{t}{h}-1}\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-s}\left[\frac{t}{h}-s\right]x(sh) \\
&= \mathcal{C}(t)\theta(u) + \frac{h^2}{\alpha\beta}\sum_{s=\frac{\beta}{\beta-\alpha}+1}^{\frac{t}{h}-1}\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-s}\left\{\left(\frac{\frac{t}{h}}{(\beta-\alpha)T+2\beta}\right)\right. \\
&\quad\left.\times\left[(\beta-\alpha)(T-s)+2\beta\right] - \sum_{s=\frac{\beta}{\beta-\alpha}+1}^{\frac{t}{h}-1}\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-s}\left[\frac{t}{h}-s\right]\right\}y(ih) \\
&\quad+\frac{h^2}{\alpha\beta}\sum_{s=\frac{t}{h}}^{T+\frac{\beta}{\beta-\alpha}}\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-s}\left(\frac{\frac{t}{h}}{(\beta-\alpha)T+2\beta}\right)\left[(\beta-\alpha)(T-s)+2\beta\right]y(sh) \\
&= \mathcal{C}(t)\theta(u) + \sum_{s=\frac{\beta}{\beta-\alpha}+1}^{T+\frac{\beta}{\beta-\alpha}} G\left(\frac{t}{h},s\right)y(sh) \\
&= \mathcal{C}(t)\theta(u) + \sum_{s=1}^{T} G\left(\frac{t}{h},s+\frac{\beta}{\beta-\alpha}\right)y\left(\left(s+\frac{\beta}{\beta-\alpha}\right)h\right).
\end{aligned}$$

This completes the proof. □

3 Existence and uniqueness of a solution for problem (1.1)

In this section, we present the existence and uniqueness result for problem (1.1). Let $E = C((h\mathbb{N})_{0,(T+1)h},\mathbb{R})$ be a Banach space of all function u with the norm defined by

$$\|u\| = \max_{t\in(h\mathbb{N})_{0,(T+1)h}}\left\{\left|u(t)\right|\right\},$$

and also define an operator $\mathcal{F}: E \to E$

$$(\mathcal{F}u)(t) = \frac{\beta h}{(\beta-\alpha)^2 T}\Biggl\{\bigl[\bigl[(\beta-\alpha)T+\beta\bigr]\phi_1(u)-\beta\phi_2(u)\bigr]\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}} - \frac{h}{\alpha\beta}\sum_{i=1}^{T}\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-i}\bigl[(T+1-i)\beta-(T-i)\alpha\bigr]f\bigl(ih,u(ih),(\Psi u)(ih)\bigr)\Biggr\} - \frac{t}{(\beta-\alpha)T}\Biggl\{\bigl[\phi_1(u)-\phi_2(u)\bigr]\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}} - \frac{h}{\alpha\beta}\sum_{i=1}^{T}\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-i}\bigl[(T+1-i)\beta-(T-i)\alpha\bigr]f\bigl(ih,u(ih),(\Psi u)(ih)\bigr)\Biggr\} - \frac{h^2}{\alpha\beta}\sum_{i=1}^{\frac{t}{h}-1}\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-i}\left[\frac{t}{h}-i\right]f\bigl(ih,u(ih),(\Psi u)(ih)\bigr). \tag{3.1}$$

Obviously, problem (1.1) has solutions if and only if the operator $\mathcal{F}$ has fixed points.

Theorem 3.1 *Assume that function $f \in C((h\mathbb{N})_{0,(T+1)h} \times [0,\infty),[0,\infty))$, functionals $\phi_1,\phi_2 : C((h\mathbb{N})_{0,(T+1)h},[0,\infty)) \to [0,\infty)$, and for $\varphi \in ((h\mathbb{N})_{0,(T+1)h} \times (h\mathbb{N})_{0,(T+1)h},[0,\infty))$, with $\varphi_0 = \max\{\varphi(t,s) : (t,s) \in (h\mathbb{N})_{0,(T+1)h} \times (h\mathbb{N})_{0,(T+1)h}\}$. In addition, suppose that:*

(H_1) *There exist constants $\lambda_1,\lambda_2 > 0$, such that*

$$\bigl|f(t,u,\Psi u)-f(t,v,\Psi v)\bigr| \le \lambda_1\|u-v\| + \lambda_2\|\Psi u-\Psi v\|,$$

for each $t \in (h\mathbb{N})_{0,(T+1)h}$ and $u,v \in E$.

(H_2) *There exist constants $\ell_1,\ell_2 > 0$, such that*

$$\bigl|\phi_1(u)-\phi_1(v)\bigr| \le \ell_1\|u-v\|,$$
$$\bigl|\phi_2(u)-\phi_2(v)\bigr| \le \ell_2\|u-v\|,$$

for each $u,v \in E$.

(H_3) $\Theta := \ell\Omega + (\lambda_1 + \lambda_2\frac{h\varphi_0(T+1)}{\alpha})\Phi < 1$, *where*

$$\ell := \max\{\ell_1,\ell_2\}, \tag{3.2}$$

$$\Omega := \frac{h}{(\beta-\alpha)}\left\{1+\beta+\frac{2\beta^2}{(\beta-\alpha)T}\right\}, \tag{3.3}$$

$$\Phi := \frac{h^2 T}{2\alpha\beta}\left(\frac{\beta}{\alpha}\right)^T\left\{\left[T+\frac{\beta}{\beta-\alpha}\right]\left[1+\frac{\beta}{\beta-\alpha}\right]+T+1\right\}. \tag{3.4}$$

Then problem (1.1) has a unique solution in $(h\mathbb{N})_{0,(T+1)h}$.

Proof Denote

$$\mathcal{H}|u-v|(t) := \bigl|f\bigl(t,u(t),(\Psi u)(t)\bigr)-f\bigl(t,v(t),(\Psi v)(t)\bigr)\bigr|.$$

For each $t \in (h\mathbb{N})_{0,(T+1)h}$ and $u, v \in \mathcal{C}$, we obtain

$$\big|(\Psi u)(t) - (\Psi v)(t)\big| \le h\varphi_0 \sum_{s=0}^{\frac{t}{h}-1} \big|u(sh) - v(sh)\big| = \frac{h\varphi_0(T+1)}{\alpha}\|u - v\|$$

and

$$\begin{aligned}
&\big|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\big| \\
&\quad\le \frac{\beta h}{(\beta-\alpha)^2 T}\Bigg\{\Big[\big[(\beta-\alpha)T+\beta\big]\big|\phi_1(u)-\phi_1(v)\big| + \beta\big|\phi_2(u)-\phi_2(v)\big|\Big]\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}} \\
&\qquad + \frac{h}{\alpha\beta}\sum_{i=1}^{T}\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-i}\big[(T+1-i)\beta - (T-i)\alpha\big]\mathcal{H}|u-v|(ih)\Bigg\} \\
&\qquad + \frac{t}{(\beta-\alpha)T}\Bigg\{\Big[\big|\phi_1(u)-\phi_1(v)\big| + \big|\phi_2(u)-\phi_2(v)\big|\Big]\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}} \\
&\qquad + \frac{h}{\alpha\beta}\sum_{i=1}^{T}\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-i}\big[(T+1-i)\beta - (T-i)\alpha\big]\mathcal{H}|u-v|(ih)\Bigg\} \\
&\qquad + \frac{h^2}{\alpha\beta}\sum_{i=1}^{\frac{t}{h}-1}\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-i}\left[\frac{t}{h}-i\right]\mathcal{H}|u-v|(ih) \\
&\quad\le \frac{\beta h}{(\beta-\alpha)^2 T}\Bigg\{\Big[\big[(\beta-\alpha)T+\beta\big]\ell_1\|u-v\| + \beta\ell_2\|u-v\|\Big] \\
&\qquad + \frac{h}{\alpha\beta}\big(\lambda_1\|u-v\| + \lambda_2\|\Psi u - \Psi v\|\big)\sum_{i=1}^{T}\left(\frac{\beta}{\alpha}\right)^{i}\big[(T+1-i)\beta - (T-i)\alpha\big]\Bigg\} \\
&\qquad + \frac{h}{(\beta-\alpha)T}\Bigg\{\big[\ell_1\|u-v\| + \ell_2\|u-v\|\big] \\
&\qquad + \frac{h}{\alpha\beta}\big(\lambda_1\|u-v\| + \lambda_2\|\Psi u - \Psi v\|\big)\sum_{i=1}^{T}\left(\frac{\beta}{\alpha}\right)^{i}\big[(T+1-i)\beta - (T-i)\alpha\big]\Bigg\} \\
&\qquad + \frac{h^2}{\alpha\beta}\big(\lambda_1\|u-v\| + \lambda_2\|\Psi u - \Psi v\|\big)\sum_{i=1}^{T}\left(\frac{\beta}{\alpha}\right)^{i}[T+1-i] \\
&\quad\le \frac{\beta h}{(\beta-\alpha)^2 T}\Bigg\{\ell\big[(\beta-\alpha)T+2\beta\big]\|u-v\| + \frac{h}{\alpha\beta}\left(\frac{\beta}{\alpha}\right)^{T} \\
&\qquad \times\left(\lambda_1 + \lambda_2\frac{h\varphi_0(T+1)}{\alpha}\right)\frac{T}{2}\big[(\beta-\alpha)T+\beta\big]\|u-v\|\Bigg\} \\
&\qquad + \frac{h}{(\beta-\alpha)T}\Bigg\{\ell\|u-v\| + \frac{h}{\alpha\beta}\left(\frac{\beta}{\alpha}\right)^{T}\left(\lambda_1 + \lambda_2\frac{h\varphi_0(T+1)}{\alpha}\right) \\
&\qquad \times\frac{T}{2}\big[(\beta-\alpha)T+\beta\big]\|u-v\|\Bigg\} \\
&\qquad + \frac{h^2}{\alpha\beta}\left(\frac{\beta}{\alpha}\right)^{T}\left(\lambda_1 + \lambda_2\frac{h\varphi_0(T+1)}{\alpha}\right)\frac{1}{2}T(T+1)\|u-v\|
\end{aligned}$$

$$
\begin{aligned}
&= \left[\ell\Omega + \left(\lambda_1 + \lambda_2 \frac{h\varphi_0(T+1)}{\alpha}\right)\Phi\right]\|u-v\| \\
&= \Theta\|u-v\|.
\end{aligned}
$$

This implies that $\mathcal{F}$ is a contraction. Therefore, by using the Banach fixed point theorem, $\mathcal{F}$ has a fixed point which is a unique solution of problem (1.1) on $t \in (h\mathbb{N})_{0,(T+1)h}$. □

4 Properties of Green's function for problem (1.2)

The necessary for considering the existence of a positive solution to problem (1.2) is to prove that Green's function $G(t,s)$ in (2.19) satisfies a variety of properties. Firstly, we prove some necessary preliminary lemmas as follows.

Lemma 4.1 *The coefficient function $\mathcal{C}(t)$ given in* (2.18) *is positive and strictly increasing in t, for $t \in (h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}$. In addition,*

$$
\min_{t\in(h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}} \mathcal{C}(t) \geq \left[\frac{\beta h}{(\beta-\alpha)[(\beta-\alpha)T+2\beta]}\right]\left(\frac{\alpha}{\beta}\right)^{T+1+\frac{\beta}{\beta-\alpha}}, \quad \textit{and}
$$

$$
\max_{t\in(h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}} \mathcal{C}(t) \leq \left[\frac{[(\beta-\alpha)(T+1)+\beta]h}{(\beta-\alpha)[(\beta-\alpha)T+2\beta]}\right]\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\beta-\alpha}}.
$$

Proof It is clear that $\mathcal{C}(t) \geq 0$ for $t \in (h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}$. Next, we prove that $\mathcal{C}(t)$ is strictly increasing in $t \in (h\mathbb{N})_{0,(T+1)h}$. Note that the forward (α,β)-difference with respect to t for $\mathcal{C}(t)$ is

$$
\begin{aligned}
\Delta_{(\alpha,\beta)}\mathcal{C}(t) &= \left[\frac{1}{(\beta-\alpha)T+2\beta}\right]\Delta_{(\alpha,\beta)}\left\{t\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}}\right\} \\
&= \left[\frac{1}{(\beta-\alpha)T+2\beta}\right]\alpha\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}} = \left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}}\Delta_h\mathcal{C}(t).
\end{aligned}
$$

From $\alpha < \beta$, it implies that

$$
\Delta_h\mathcal{C}(t) = \frac{\alpha}{(\beta-\alpha)T+2\beta} > 0.
$$

Hence $\mathcal{C}(t)$ is strictly increasing in $t \in (h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}$.

Finally, observe that

$$
\begin{aligned}
\min_{t\in(h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}} \mathcal{C}(t) &= \min_{t\in(h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}} \left[\frac{t}{(\beta-\alpha)T+2\beta}\right]\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}} \\
&\geq \left[\frac{\beta h}{(\beta-\alpha)[(\beta-\alpha)T+2\beta]}\right]\left(\frac{\alpha}{\beta}\right)^{T+1+\frac{\beta}{\beta-\alpha}}, \quad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
\max_{t\in(h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}} \mathcal{C}(t) &= \max_{t\in(h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}} \left[\frac{t}{(\beta-\alpha)T+2\beta}\right]\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}} \\
&\le \left[\frac{[(\beta-\alpha)(T+1)+\beta]h}{(\beta-\alpha)[(\beta-\alpha)T+2\beta]}\right]\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\beta-\alpha}}.
\end{aligned}
$$

The proof is complete. □

Corollary 4.1 *Let* $I := [\frac{h}{4}(T+1+\frac{\beta}{\beta-\alpha}), \frac{3h}{4}(T+1+\frac{\beta}{\beta-\alpha})]$. *There is a constant* $M_{\mathcal{C}} \in (0,1)$ *such that* $\min_{t\in I}\mathcal{C}(t) = M_{\mathcal{C}}\|\mathcal{C}\|$ *where* $\|\cdot\|$ *is the usual maximum norm.*

Proof Since $\mathcal{C}(t)$ is strictly increasing in $t \in (h\mathbb{N})_{0,(T+1)h}$, it follows that there exists a positive constant $M_{\mathcal{C}}$, such that

$$
\min_{t\in I}\mathcal{C}(t) = M_{\mathcal{C}}\left[\frac{[(\beta-\alpha)(T+1)+\beta]h}{(\beta-\alpha)[(\beta-\alpha)T+2\beta]}\right]\left(\frac{\alpha}{\beta}\right)^{T+1+\frac{\beta}{\beta-\alpha}} = M_{\mathcal{C}}\|\mathcal{C}\|.
$$

It is clear that $M_{\mathcal{C}} \in (0,1)$. The proof is complete. □

Lemma 4.2 *Let* $\alpha < \beta$ *and* $G(\frac{t}{h},s)$ *be Green's function given in* (2.19). *Then* $G(\frac{t}{h},s) \ge 0$ *for each* $(t,s) \in (h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h} \times \mathbb{N}_{\frac{\beta}{\beta-\alpha}+1,T+\frac{\beta}{\beta-\alpha}}$.

Proof We aim to show that $g_i(\frac{t}{h},s) > 0$, $i = 1,2$, for each admissible pair (t,s).

Firstly, we consider the function

$$
g_2\left(\frac{t}{h},s\right) = \left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}}\left(\frac{\frac{t}{h}}{(\beta-\alpha)T+2\beta}\right)\big[(\beta-\alpha)(T-s)+2\beta\big], \quad s \in \mathbb{N}_{\frac{t}{h},T+\frac{\beta}{\beta-\alpha}}.
$$

To guarantee that $g_2(\frac{t}{h},s) > 0$, it suffices to show that

$$
(\beta-\alpha)(T-s)+2\beta \ge (\beta-\alpha)\left[T-\left(T+\frac{\beta}{\beta-\alpha}\right)\right]+2\beta = \beta > 0. \tag{4.1}
$$

Thus, we conclude that $g_2(\frac{t}{h},s) > 0$ for their respective domains.

Next, we consider the function $g_1(\frac{t}{h},s) > 0$ for $s \in \mathbb{N}_{\frac{\beta}{\beta-\alpha}+1,\frac{t}{h}-1}$ where

$$
g_1\left(\frac{t}{h},s\right) = \left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}}\left\{\left(\frac{\frac{t}{h}}{(\beta-\alpha)T+2\beta}\right)\big[(\beta-\alpha)(T-s)+2\beta\big]-\left(\frac{t}{h}-s\right)\right\}.
$$

To guarantee that $g_1(\frac{t}{h},s) > 0$, it suffices to show that

$$
\begin{aligned}
&\frac{\frac{t}{h}[(\beta-\alpha)(T-s)+2\beta]}{(\frac{t}{h}-s)[(\beta-\alpha)T+2\beta]} \\
&\quad = \frac{\frac{t}{h}[(\beta-\alpha)(T-s)+2\beta]}{\frac{t}{h}([(\beta-\alpha)T+2\beta]-s[\frac{(\beta-\alpha)T+2\beta}{\frac{t}{h}}])}
\end{aligned}
$$

$$= \frac{[(\beta-\alpha)T+2\beta]-s(\beta-\alpha)}{[(\beta-\alpha)T+2\beta]-s(\beta-\alpha)[\frac{T+\frac{2\beta}{\beta-\alpha}}{\frac{t}{h}}]}$$

$$\geq \frac{(T-s)(\beta-\alpha)+2\beta}{[(\beta-\alpha)T+2\beta]-s(\beta-\alpha)[\frac{T+\frac{2\beta}{\beta-\alpha}}{T+\frac{2\beta-\alpha}{\beta-\alpha}}]} > 1.$$

So, we conclude that $g_1(\frac{t}{h},s) > 0$ for their respective domains.

Consequently, it follows that $g_i(\frac{t}{h},s) > 0$ for each $i = 1,2$. Therefore, $G(\frac{t}{h},s) > 0$. □

Lemma 4.3 *Let $\alpha < \beta$ and $G(\frac{t}{h},s)$ be Green's function given in* (2.19). *Then it follows that*

$$\max_{t,\in(h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}} G\left(\frac{t}{h},s\right) = G(s,s) \quad \textit{for each } s \in \mathbb{N}_{\frac{\beta}{\beta-\alpha}+1,T+\frac{\beta}{\beta-\alpha}}. \tag{4.2}$$

Proof Our aim is to show that the forward (α,β)-difference with respect to t satisfies with

$${}_{t,h}\Delta_{(\alpha,\beta)}g_1\left(\frac{t}{h},s\right) = \left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}} {}_t\Delta_h g_1\left(\frac{t}{h},s\right) < 0$$

and

$${}_{t,h}\Delta_{(\alpha,\beta)}g_2\left(\frac{t}{h},s\right) = \left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}} {}_t\Delta_h g_2\left(\frac{t}{h},s\right) > 0.$$

This implies that g_1 is decreasing and g_2 is increasing in t. So, $G(\frac{t}{h},s) \leq G(s,s)$ for all $(t,s) \in \mathbb{N}_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h} \times \mathbb{N}_{\frac{\beta}{\beta-\alpha}+1,T+\frac{\beta}{\beta-\alpha}}$.

Firstly, for $g_2(\frac{t}{h},s)$, we have

$$\begin{aligned}
{}_{t,h}\Delta_{(\alpha,\beta)}g_2\left(\frac{t}{h},s\right) &= \left[\frac{(\beta-\alpha)(T-s)+2\beta}{(\beta-\alpha)T+2\beta}\right] {}_{t,h}\Delta_{(\alpha,\beta)}\left\{\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}}\frac{t}{h}\right\} \\
&= \left[\frac{(\beta-\alpha)(T-s)+2\beta}{(\beta-\alpha)T+2\beta}\right]\left\{\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}}\frac{1}{h}\right\} \\
&= \left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}} {}_t\Delta_h g_2\left(\frac{t}{h},s\right) > 0,
\end{aligned}$$

for all $(t,s) \in \mathbb{N}_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h} \times \mathbb{N}_{\frac{\beta}{\beta-\alpha}+1,T+\frac{\beta}{\beta-\alpha}}$.

Next, for $g_1(\frac{t}{h},s)$, we note that

$$g_1\left(\frac{t}{h},s\right) = \left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}}\left\{\left(\frac{\frac{t}{h}}{(\beta-\alpha)T+2\beta}\right)\left[(\beta-\alpha)(T-s)+2\beta\right]-\left(\frac{t}{h}-s\right)\right\}.$$

Here, we obtain

$$\begin{aligned}
{}_{t,h}\Delta_{(\alpha,\beta)}g_1\left(\frac{t}{h},s\right) &= \left[\frac{(\beta-\alpha)(T-s)+2\beta}{(\beta-\alpha)T+2\beta}\right]\left\{\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}}\frac{1}{h}\right\} - \left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}}\frac{1}{h} \\
&= \left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}}\frac{1}{h}\left[\frac{(\beta-\alpha)(T-s)+2\beta}{(\beta-\alpha)T+2\beta}-1\right]
\end{aligned}$$

$$
\begin{aligned}
&= -\left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}} \frac{1}{h}\left[\frac{(\beta-\alpha)s}{(\beta-\alpha)T+2\beta}\right] \\
&= \left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}} {}_{t}\Delta_h g_1\left(\frac{t}{h}, s\right) < 0,
\end{aligned}
$$

for all $(t,s) \in \mathbb{N}_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h} \times \mathbb{N}_{\frac{\beta}{\beta-\alpha}+1,T+\frac{\beta}{\beta-\alpha}}$.

Now, note that

$$
G\left(\frac{\beta}{\beta-\alpha}+1, s\right) \leq G(s,s) \quad \text{for all } s \in \mathbb{N}_{\frac{\beta}{\beta-\alpha}+1,T+\frac{\beta}{\beta-\alpha}}.
$$

Consequently, this implies that

$$
\max_{t,\in(h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}} G\left(\frac{t}{h}, s\right) = G(s,s) \quad \text{for each } s \in \mathbb{N}_{\frac{\beta}{\beta-\alpha}+1,T+\frac{\beta}{\beta-\alpha}}.
$$

Observe that $G(s,s) = \frac{h^2}{\alpha} g_2(s,s) = \frac{h^2}{\alpha}(\frac{\alpha}{\beta})^s(\frac{s}{(\beta-\alpha)T+2\beta})[(\beta-\alpha)(T-s)+2\beta]$.

Thus, by the discussion in the first paragraph of this proof, we deduce that (4.2) holds. The proof is complete. □

Lemma 4.4 *Let $\alpha < \beta$ and $G(\frac{t}{h}, s)$ be Green's function given in (2.19). There exists a number $0 < \sigma < 1$ such that, for $s \in \mathbb{N}_{\frac{\beta}{\beta-\alpha}+1,T+\frac{\beta}{\beta-\alpha}}$,*

$$
\begin{aligned}
\min_{\frac{h}{4}(T+1+\frac{\beta}{\beta-\alpha})\leq t\leq \frac{3h}{4}(T+1+\frac{\beta}{\beta-\alpha})} G\left(\frac{t}{h}, s\right) &\geq \sigma \max_{t\in(h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}} G\left(\frac{t}{h}, s\right) \\
&= \sigma G(s,s).
\end{aligned} \tag{4.3}
$$

Proof We define the following notation:

$$
\tilde{g}_i\left(\frac{t}{h}, s\right) := \frac{g_i(\frac{t}{h}, s)}{g_2(s,s)}, \quad i = 1, 2.
$$

For $t < sh$ and $\frac{h}{4}(T+1+\frac{\beta}{\beta-\alpha}) \leq t \leq \frac{3h}{4}(T+1+\frac{\beta}{\beta-\alpha})$, we find that

$$
\tilde{g}_2\left(\frac{t}{h}, s\right) = \left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-s} \frac{t}{sh} \geq \frac{1}{4}\left(\frac{\beta}{\alpha}\right)^{\frac{1}{4}[3(T+\frac{\beta}{\beta-\alpha})-1]} \left[\frac{T+1+\frac{\beta}{\beta-\alpha}}{T+\frac{\beta}{\beta-\alpha}}\right] := \sigma_1. \tag{4.4}
$$

For $t > sh$, since $g_1(\frac{t}{h}, s)$ is decreasing with respect to t, we have

$$
\begin{aligned}
\tilde{g}_1\left(\frac{t}{h}, s\right) &= \left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-s} \left\{\frac{\frac{t}{h}[(\beta-\alpha)(T-s)+2\beta] - (\frac{t}{h}-s)[(\beta-\alpha)T+2\beta]}{s\bigl[(\beta-\alpha)(T-s)+2\beta\bigr]}\right\} \\
&= \left(\frac{\alpha}{\beta}\right)^{\frac{t}{h}-s} \left\{\frac{(\beta-\alpha)(T-\frac{t}{h})+2\beta}{(\beta-\alpha)(T-s)+2\beta}\right\} \\
&\geq \frac{1}{4}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{4}(T+1+\frac{\beta}{\beta-\alpha})} \left[\frac{\alpha}{(\beta-\alpha)(T-1)+\beta}\right] := \sigma_2.
\end{aligned} \tag{4.5}
$$

Finally, since $\sigma_1 > \frac{1}{4}$ and $\sigma_2 < 1$, it follows that

$$\sigma = \min\{\sigma_1, \sigma_2\} < 1. \tag{4.6}$$

We can conclude that (4.3) holds. □

Lemma 4.5 *Let φ be a nonnegative functional. Then there is $\sigma^* \in (0,1)$ such that*

$$\begin{aligned}&\min_{\frac{h}{4}(T+1+\frac{\beta}{\beta-\alpha})\leq t\leq \frac{3h}{4}(T+1+\frac{\beta}{\beta-\alpha})}\left\{\mathcal{C}(t)\theta(u) + \sum_{s=\frac{\beta}{\beta-\alpha}+1}^{T+\frac{\beta}{\beta-\alpha}} G\left(\frac{t}{h}, s\right)F\bigl(sh, u(sh)\bigr)\right\} \\ &\quad \geq \sigma^* \max_{t\in(h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}}\left\{\mathcal{C}(t)\theta(u) + \sum_{s=\frac{\beta}{\beta-\alpha}+1}^{T+\frac{\beta}{\beta-\alpha}} G\left(\frac{t}{h}, s\right)F\bigl(sh, u(sh)\bigr)\right\}.\end{aligned} \tag{4.7}$$

Proof By Lemma 4.4 and Corollary 4.1, we find that there exist constants $\sigma, M_{\mathcal{C}} \in (0,1)$ such that

$$\min_{\frac{h}{4}(T+1+\frac{\beta}{\beta-\alpha})\leq t\leq \frac{3h}{4}(T+1+\frac{\beta}{\beta-\alpha})} G\left(\frac{t}{h}, s\right) \geq \sigma \max_{t\in(h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}} G\left(\frac{t}{h}, s\right), \tag{4.8}$$

$$\min_{\frac{h}{4}(T+1+\frac{\beta}{\beta-\alpha})\leq t\leq \frac{3h}{4}(T+1+\frac{\beta}{\beta-\alpha})} \mathcal{C}(t) \geq M_{\mathcal{C}} \max_{t\in(h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}} \mathcal{C}(t). \tag{4.9}$$

We define

$$\sigma^* := \min\{\sigma, M_{\mathcal{C}}\} \in (0,1). \tag{4.10}$$

Hence, we obtain (4.7). This completes the proof. □

5 Existence of positive solution to problem (1.2)

In this section, we consider the existence of at least one positive solution for problem (1.2) by using the Krasnoselskii fixed point theorem in a cone as mentioned in Sect. 1. Let $\mathcal{E} = C((h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}, \mathbb{R})$ be a Banach space of all function u with the norm defined by $\|u\| = \max_{t\in(h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}}\{|u(t)|\}$. Define the cone $\mathcal{P} \subseteq \mathcal{E}$ by

$$\mathcal{P} := \left\{u \in \mathcal{E} : u(t) \geq 0, \min_{\frac{h}{4}(T+1+\frac{\beta}{\beta-\alpha})\leq t\leq \frac{3h}{4}(T+1+\frac{\beta}{\beta-\alpha})} u(t) \geq \sigma^*\|u\| \text{ and } \theta(u) \geq 0\right\},$$

where σ^* is the number defined by (4.10).

From the nonlinear equation (1.2), we note that there exists a solution u of (1.2) if and only if u is a fixed point of the operator $\mathcal{A} : \mathcal{P} \to \mathcal{P}$ which is defined by

$$(\mathcal{A}u)(t) := \mathcal{C}(t)\theta(u) + \sum_{s=\frac{\beta}{\beta-\alpha}+1}^{T+\frac{\beta}{\beta-\alpha}} G\left(\frac{t}{h}, s\right)F\bigl(sh, u(sh)\bigr), \tag{5.1}$$

where G is Green's function for problem (1.2).

Lemma 5.1 *Suppose that* $F \in C((h\mathbb{N})_{(1+\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h} \times [0,\infty),[0,\infty))$ *and* $\theta : C((h\mathbb{N})_{(1+\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h},[0,\infty)) \to [0,\infty)$. *Then the operator* $\mathcal{A}:\mathcal{P}\to\mathcal{P}$ *is completely continuous.*

Proof Since $G(\frac{t}{h},s) \geq 0$ for all (t,s), we have $\mathcal{A}\geq 0$ for all $u\in\mathcal{P}$. For a constant $R>0$, we define

$$B_R = \{u \in \mathcal{P} : \|u\| < R\},$$

and let $M = \max_{(t,u)\in(h\mathbb{N})_{(1+\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}\times B_R} |F(t,u(t))|$, $N = \sup_{u\in B_R}|\theta(u)|$. Then, for $u \in B_R$, we obtain

$$\begin{aligned}
\big|(\mathcal{A}u)(t)\big| &= \left|\mathcal{C}(t)\theta(u) + \sum_{s=\frac{\beta}{\beta-\alpha}+1}^{T+\frac{\beta}{\beta-\alpha}} G\left(\frac{t}{h},s\right)F\big(sh,u(sh)\big)\right| \\
&\leq N\left[\frac{[(\beta-\alpha)(T+1)+\beta]h}{(\beta-\alpha)[(\beta-\alpha)T+2\beta]}\right]\left(\frac{\alpha}{\beta}\right)^{T+1+\frac{\beta}{\beta-\alpha}} + \frac{Mh^2}{\alpha\beta}\sum_{s=\frac{\beta}{\beta-\alpha}+1}^{T+\frac{\beta}{\beta-\alpha}} g_2(s,s) \\
&=: \mathcal{K}.
\end{aligned}$$

Therefore, $\|\mathcal{A}u\| = \mathcal{K}$, and hence $\mathcal{A}(B_R)$ is uniformly bounded.

We next prove that $\mathcal{A}(B_R)$ is equicontinuous. For any $\epsilon > 0$, there exists a positive constant $\delta^* = \max\{\delta_1,\delta_2,\delta_3\}$ such that, for $u\in B_R$ and $t_1,t_2 \in (h\mathbb{N})_{(1+\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}$ with $t_1 < t_2$,

$$\begin{aligned}
&\left|\left(\frac{\alpha}{\beta}\right)^{\frac{t_2}{h}}t_2 - \left(\frac{\alpha}{\beta}\right)^{\frac{t_1}{h}}t_1\right| < \frac{\epsilon(\beta-\alpha)T+2\beta}{3N} \quad \text{whenever } |t_2-t_1| < \delta_1, \\
&\left|t_2^2 - t_1^2\right| < \frac{\epsilon\beta^2}{3M} \quad \text{whenever } |t_2-t_1| < \delta_2, \\
&|t_2 - t_1| < \frac{\epsilon\alpha\beta}{3hM(T+2+\frac{\beta}{\beta-\alpha})} = \delta_3.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\big|(\mathcal{A}u)(t_2) - (\mathcal{A}u)(t_1)\big| \\
&\quad\leq \big|\mathcal{C}(t_2)-\mathcal{C}(t_1)\big|N + M\left|\sum_{s=\frac{\beta}{\beta-\alpha}+1}^{T+\frac{\beta}{\beta-\alpha}}\left[G\left(\frac{t_2}{h},s\right) - G\left(\frac{t_1}{h},s\right)\right]\right| \\
&\quad\leq \frac{N}{(\beta-\alpha)T+2\beta}\left|\left(\frac{\alpha}{\beta}\right)^{\frac{t_2}{h}}t_2 - \left(\frac{\alpha}{\beta}\right)^{\frac{t_1}{h}}t_1\right|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{h^2M}{\alpha\beta}\left|\sum_{s=\frac{\beta}{\beta-\alpha}+1}^{\frac{t_2}{h}-1} g_1\left(\frac{t_2}{h},s\right)-\sum_{s=\frac{\beta}{\beta-\alpha}+1}^{\frac{t_1}{h}-1} g_1\left(\frac{t_1}{h},s\right)\right| \\
&+\frac{h^2M}{\alpha\beta}\left|\sum_{s=\frac{t_2}{h}}^{T+1+\frac{\beta}{\beta-\alpha}} g_2\left(\frac{t_2}{h},s\right)-\sum_{s=\frac{t_1}{h}}^{T+1+\frac{\beta}{\beta-\alpha}} g_2\left(\frac{t_1}{h},s\right)\right| \\
&<\frac{N}{(\beta-\alpha)T+2\beta}\left|\left(\frac{\alpha}{\beta}\right)^{\frac{t_2}{h}}t_2-\left(\frac{\alpha}{\beta}\right)^{\frac{t_1}{h}}t_1\right| \\
&+\frac{h^2M}{\alpha\beta}\left|\left(\frac{\alpha}{\beta}\right)^{\frac{t_2}{h}}\sum_{s=\frac{\beta}{\beta-\alpha}+1}^{\frac{t_2}{h}-1}\left(\frac{\alpha}{\beta}\right)^{-s}\left[\frac{t_2}{h}\cdot\frac{(\beta-\alpha)(T-s)+2\beta}{(\beta-\alpha)T+2\beta}-\left(\frac{t_2}{h}-s\right)\right]\right. \\
&\left.-\left(\frac{\alpha}{\beta}\right)^{\frac{t_1}{h}}\sum_{s=\frac{\beta}{\beta-\alpha}+1}^{\frac{t_1}{h}-1}\left(\frac{\alpha}{\beta}\right)^{-s}\left[\frac{t_1}{h}\cdot\frac{(\beta-\alpha)(T-s)+2\beta}{(\beta-\alpha)T+2\beta}-\left(\frac{t_1}{h}-s\right)\right]\right| \\
&+\frac{h^2M}{\alpha\beta}\left|\left(\frac{\alpha}{\beta}\right)^{\frac{t_2}{h}}\frac{t_2}{h}\sum_{s=\frac{t_2}{h}}^{T+1+\frac{\beta}{\beta-\alpha}}\left(\frac{\alpha}{\beta}\right)^{-s}\left[\frac{(\beta-\alpha)(T-s)+2\beta}{(\beta-\alpha)T+2\beta}\right]\right. \\
&\left.-\left(\frac{\alpha}{\beta}\right)^{\frac{t_1}{h}}\frac{t_1}{h}\sum_{s=\frac{t_1}{h}}^{T+1+\frac{\beta}{\beta-\alpha}}\left(\frac{\alpha}{\beta}\right)^{-s}\left[\frac{(\beta-\alpha)(T-s)+2\beta}{(\beta-\alpha)T+2\beta}\right]\right| \\
&<\frac{N}{(\beta-\alpha)T+2\beta}\left|\left(\frac{\alpha}{\beta}\right)^{\frac{t_2}{h}}t_2-\left(\frac{\alpha}{\beta}\right)^{\frac{t_1}{h}}t_1\right| \\
&+\frac{h^2M}{\alpha\beta}\left|\frac{\alpha}{\beta}\left(\frac{t_2}{h}-1\right)\left(\frac{t_2}{h}-1-\frac{\beta}{\beta-\alpha}\right)-\frac{\alpha}{\beta}\left(\frac{t_1}{h}-1\right)\left(\frac{t_1}{h}-1-\frac{\beta}{\beta-\alpha}\right)\right| \\
&+\frac{h^2M}{\alpha\beta}\left|\frac{t_2}{h}\left(T+2+\frac{\beta}{\beta-\alpha}\right)-\frac{t_1}{h}\left(T+2+\frac{\beta}{\beta-\alpha}\right)\right| \\
&<\frac{N}{(\beta-\alpha)T+2\beta}\left|\left(\frac{\alpha}{\beta}\right)^{\frac{t_2}{h}}t_2-\left(\frac{\alpha}{\beta}\right)^{\frac{t_1}{h}}t_1\right|+\frac{M}{\beta}\left|t_2^2-t_1^2\right| \\
&+\frac{hM}{\alpha\beta}\left(T+2+\frac{\beta}{\beta-\alpha}\right)|t_2-t_1| \\
&<\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.
\end{aligned}
$$

This implies that the set $\mathcal{A}(B_R)$ is an equicontinuous set.

Finally, we apply Lemmas 4.2–4.4 to obtain

$$
(\mathcal{A}x)(t)\geq 0 \quad \text{for all } t\in(h\mathbb{N})_{(1+\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}, \tag{5.2}
$$

and for $F\in\mathcal{P}$

$$
\min_{t\in I}(\mathcal{A}u)(t)\geq\min_{t\in I}\left(\sum_{s=\frac{\beta}{\beta-\alpha}+1}^{T+\frac{\beta}{\beta-\alpha}} G\left(\frac{t}{h},s\right)F\big(sh,u(sh)\big)\right)+\min_{t\in I}\mathcal{C}(t)\theta(u)
$$

$$\begin{aligned}
&\geq \sigma^* \max_{t\in(h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}} \left(\sum_{s=\frac{\beta}{\beta-\alpha}+1}^{T+\frac{\beta}{\beta-\alpha}} G\left(\frac{t}{h},s\right) F\bigl(sh,u(sh)\bigr) \right) + \sigma^* \|\mathcal{C}\| \theta(u) \\
&\geq \sigma^* \|\mathcal{A}u\|. \qquad (5.3)
\end{aligned}$$

So, $\mathcal{A}\mathcal{P} \subset \mathcal{P}$.

Consequently, from the Arzelá–Ascoli theorem, it follows that $\mathcal{A} : \mathcal{P} \to \mathcal{P}$ is the completely continuous operator. □

We next define

$$\mathcal{M} := \left[\sum_{s=\frac{\beta}{\beta-\alpha}+1}^{T+\frac{\beta}{\beta-\alpha}} G(s,s) \right]^{-1},$$

$$\mathcal{N} := \left[\sigma^* \sum_{s=\frac{\beta}{\beta-\alpha}+1}^{T+\frac{\beta}{\beta-\alpha}} G(s,s) \right]^{-1},$$

$$\Upsilon_1 := \left[\frac{[(\beta-\alpha)(T+1)+\beta]h}{(\beta-\alpha)[(\beta-\alpha)T+2\beta]} \right] \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\beta-\alpha}},$$

$$\Upsilon_2 := \frac{1}{\sigma^*} \left[\frac{[(\beta-\alpha)(T+1)+\beta]h}{(\beta-\alpha)[(\beta-\alpha)T+2\beta]} \right] \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\beta-\alpha}},$$

and introduce some assumptions that will be used in the sequel.

(X_1) There exists a constant $r_1 > 0$ such that

$$F\bigl(t,u(t)\bigr) \leq \frac{1}{2}\mathcal{M}r_1 \quad \text{whenever } 0 \leq u \leq r_1.$$

(X_2) There exists a constant $r_2 > 0$ with $r_2 < r_1$ such that

$$F\bigl(t,u(t)\bigr) \geq \frac{1}{2}\mathcal{N}r_2 \quad \text{whenever } \sigma^* r_2 \leq u \leq r_2.$$

(X_3) There exists a constant $r_1 > 0$ such that for each $u \in \mathcal{P}$ and $0 \leq \|u\| \leq r_1$,

$$\theta(u) \leq \frac{1}{2}\Upsilon_1 r_1.$$

(X_4) There exists a constant $r_2 > 0$ such that for each $u \in \mathcal{P}$ and $\sigma^* r_2 \leq \|u\| \leq r_2$,

$$\theta(u) \geq \frac{1}{2}\Upsilon_2 r_2.$$

The following theorem presents the proof of the existence result of at least one positive solution.

Theorem 5.1 *Suppose that the conditions* (X_1)–(X_4) *hold. Then problem* (1.2) *has at least one positive solution,* u^* *where* $r_2 \leq \|u^*\| \leq r_1$.

Proof Set $\Omega_1 = \{u \in \mathcal{E} : \|u\| < r_1\}$. Then, for $u \in \mathcal{P} \cap \partial\Omega_1$, we have

$$\begin{aligned}
\|\mathcal{A}u\| &\leq \max_{t\in(h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}} \left\{\mathcal{C}(t)\theta(u) + \sum_{s=\frac{\beta}{\beta-\alpha}+1}^{T+\frac{\beta}{\beta-\alpha}} G\left(\frac{t}{h},s\right)F\bigl(sh,u(sh)\bigr)\right\} \\
&\leq \left[\frac{[(\beta-\alpha)(T+1)+\beta]h}{(\beta-\alpha)[(\beta-\alpha)T+2\beta]}\right]\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\beta-\alpha}} \cdot \frac{1}{2}\Upsilon_1 r_1 + \frac{1}{2}\mathcal{M}r_1 \sum_{s=\frac{\beta}{\beta-\alpha}+1}^{T+\frac{\beta}{\beta-\alpha}} G(s,s) \\
&= \frac{r_1}{2} + \frac{r_1}{2} = r_1.
\end{aligned}$$

Further, we let $\Omega_2 = \{u \in \mathcal{E} : \|u\| < r_2\}$. Then, for $u \in \mathcal{P} \cap \partial\Omega_2$ and by using Lemma 4.4, we find that

$$\begin{aligned}
(\mathcal{A}u)(t) &\geq \min_{t\in\frac{h}{4}(T+1+\frac{\beta}{\beta-\alpha})\leq t\leq\frac{3h}{4}(T+1+\frac{\beta}{\beta-\alpha})} \bigl\{\mathcal{C}(t)\theta(u)\bigr\} \\
&\quad + \sum_{s=\frac{\beta}{\beta-\alpha}+1}^{T+\frac{\beta}{\beta-\alpha}} \left[\min_{\frac{h}{4}(T+1+\frac{\beta}{\beta-\alpha})\leq t\leq\frac{3h}{4}(T+1+\frac{\beta}{\beta-\alpha})} G\left(\frac{t}{h},s\right)F\bigl(sh,u(sh)\bigr)\right] \\
&\geq \frac{1}{2}\Upsilon_2 r_2\sigma^* \max_{t\in(h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}} \bigl\{\mathcal{C}(t)\theta(u)\bigr\} \\
&\quad + \frac{1}{2}\mathcal{N}r_2\sigma^* \max_{t\in(h\mathbb{N})_{(\frac{\beta}{\beta-\alpha})h,(T+1+\frac{\beta}{\beta-\alpha})h}} \sum_{s=\frac{\beta}{\beta-\alpha}+1}^{T+\frac{\beta}{\beta-\alpha}} G\left(\frac{t}{h},s\right) \\
&= \frac{1}{2}\Upsilon_2 r_2\sigma^*\left[\frac{[(\beta-\alpha)(T+1)+\beta]h}{(\beta-\alpha)[(\beta-\alpha)T+2\beta]}\right]\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\beta-\alpha}} + \frac{1}{2}\mathcal{N}r_2\sigma^* \sum_{s=\frac{\beta}{\beta-\alpha}+1}^{T+\frac{\beta}{\beta-\alpha}} G(s,s) \\
&= \frac{r_2}{2} + \frac{r_2}{2} = r_2.
\end{aligned}$$

We can conclude by Theorem 1.1 that the operator $\mathcal{A}$ has a fixed point. This implies that problem (1.2) has a positive solution, u^*, where $r_2 \leq \|u^*\| \leq r_1$. □

6 Some examples

In this section, we present some examples to illustrate our results.

Example 6.1 Consider the following boundary value problem for the second-order (α,β)-difference equation:

$$\begin{aligned}
&{}_{\frac{1}{2}}\Delta^2_{(\frac{1}{3},\frac{3}{2})}u\left(t-\frac{1}{2}\right) + \frac{1}{(t+100)^{10}}\left[\frac{e^{-t/2}(|u|+1)}{1+\cos^2 u} + (\Psi u)(t)\right] = 0, \quad t \in \left(\frac{1}{2}\mathbb{N}\right)_{\frac{1}{2},10}, \\
&{}_{\frac{1}{2}}\Delta_{(\frac{1}{3},\frac{3}{2})}u(0) = \frac{|u|}{(10e)^2}\sin^2(\pi u), \\
&{}_{\frac{1}{2}}\Delta_{(\frac{1}{3},\frac{3}{2})}\left(\frac{9}{2}\right)^{20} u(10) = \frac{|u|}{(10\pi)^2}\cos^2(\pi u),
\end{aligned} \tag{6.1}$$

where $(\Psi u)(t) = \frac{1}{2}\sum_{s=0}^{2t-1} \frac{e^{-4|s-t|}}{100\sqrt{\pi}}u(sh)$.

Set $h = \frac{1}{2}$, $\alpha = \frac{1}{3}$, $\beta = \frac{3}{2}$, $T = 20$, $\phi_1(u) = \frac{|u|}{(10e)^2}\sin^2(\pi u)$, $\phi_2(u) = \frac{|u|}{(10\pi)^2}\cos^2(\pi u)$, $\varphi(t,s) = \frac{e^{-4|s-t|}}{100\sqrt{\pi}}$ and

$$f\bigl(t,u(t),(\Psi u)(t)\bigr) = \frac{1}{(t+100)^{10}}\left[\frac{e^{-t/2}(|u|+1)}{1+\cos^2 u} + \frac{1}{2}\sum_{s=0}^{2t-1}\frac{e^{-4|s-t|}}{100\sqrt{\pi}}u(sh)\right].$$

We find that $\varphi_0 = \frac{1}{100\sqrt{\pi}} = 0.00564$, and

$$\bigl|f\bigl(t,u(t),(\Psi u)(t)\bigr) - f\bigl(t,v(t),(\Psi v)(t)\bigr)\bigr| \leq \frac{e^{-10}}{120^{10}}|u-v| + \frac{1}{120^{10}}|\Psi u - \Psi v|.$$

Thus, $(H1)$ holds with $\lambda_1 = 7.332 \times 10^{-26}$ and $\lambda_2 = 1.615 \times 10^{-21}$. Since

$$\bigl|\phi_1(u) - \phi_1(v)\bigr| \leq \frac{1}{100e^2}\|u-v\| \quad \text{and} \quad \bigl|\phi_2(u) - \phi_2(v)\bigr| \leq \frac{1}{100\pi^2}\|u-v\|,$$

$(H2)$ holds with $\ell_1 = 0.00135$ and $\ell_2 = 0.00101$.

Clearly,

$$\ell = \max\{\ell_1,\ell_2\} = 0.00135, \qquad \Omega = 1.1541 \quad \text{and} \quad \Phi = 2.171 \times 10^{15}.$$

Then we find that

$$\Theta = \ell\Omega + \left(\lambda_1 + \lambda_2\frac{h\varphi_0(T+1)}{\alpha}\right)\Phi = 0.00156 < 1.$$

Hence, by Theorem 3.1 problem (6.1) has a unique solution.

Example 6.2 Consider the following boundary value problem for the second-order (α,β)-difference equation:

$$\begin{aligned}
&{}_{\frac{3}{2}}\Delta^2_{(\frac{1}{2},\frac{4}{3})}u\left(t-\frac{3}{2}\right) + \frac{e^{-3t}|u|}{(1+|u|)(t+100)^5} = 0, \quad t \in \left(\frac{3}{2}\mathbb{N}\right)_{\frac{39}{10},\frac{99}{10}},\\
&{}_{\frac{3}{2}}\Delta_{(\frac{1}{2},\frac{4}{3})}u\left(\frac{12}{5}\right) = \frac{8}{9}u\left(\frac{39}{10}\right),\\
&{}_{\frac{3}{2}}\Delta_{(\frac{1}{2},\frac{4}{3})}\left(\left(\frac{8}{3}\right)^{\frac{99}{10}}u\left(\frac{99}{10}\right)\right) = \sum_{i=0}^{6}\frac{C_i|u(t_i)|}{1+|u(t_i)|}, \quad t_i = \left(\frac{8}{5}+i\right)\frac{3}{2},
\end{aligned} \tag{6.2}$$

where C_i are given positive constants with $\frac{1}{\pi^2} \leq \sum_{i=0}^{6} C_i \leq \frac{3}{\pi^2}$.

Set $h = \frac{3}{2}$, $\alpha = \frac{1}{2}$, $\beta = \frac{4}{3}$, $T = 5$, $\theta(u) = \sum_{i=0}^{6}\frac{C_i|u(t_i)|}{1+|u(t_i)|}$ and $F(t,u(t)) = \frac{e^{-3t}|u|}{(1+|u|)(t+100)^5}$.

We find

$$\begin{aligned}
&\sigma_1 = 0.00471, \qquad \sigma_2 = 0.1067, \qquad M_{\mathcal{C}} = 0.000274,\\
&\sigma = \min\{\sigma_1,\sigma_2\} = 0.00471, \qquad \sigma^* = \min\{\sigma, M_{\mathcal{C}}\} = 0.000274,\\
&\mathcal{M} = 3.972, \qquad \mathcal{N} = 14495.004, \qquad \Upsilon_1 = 0.2476, \qquad \Upsilon_2 = 903.832.
\end{aligned}$$

Clearly,

$$\left|F\big(t,u(t)\big)\right| \leq 9.3761 \times 10^{-6} = 4.721 \times 10^{-6}\frac{\mathcal{M}}{2},$$

for $0 \leq u \leq r_1 \leq 4.721 \times 10^{-6}$,

$$\left|F\big(t,u(t)\big)\right| \geq 7.2879 \times 10^{-21} = 1.006 \times 10^{-24}\frac{\mathcal{N}}{2},$$

for $2.755 \times 10^{-28} = \sigma^* r_2 \leq u \leq r_2 \leq 1.006 \times 10^{-24}$,

$$\theta(u) \leq 0.304 = 2.465\frac{\Upsilon_1}{2} \quad \text{for } 0 \leq u \leq r_1 \leq 2.465,$$

and

$$\theta(u) \geq 0.101 = 0.000223\frac{\Upsilon_2}{2} \quad \text{for } 6.124 \times 10^{-8} = \sigma^* r_2 \leq u \leq r_2 \leq 0.000223.$$

Therefore, conditions (X_1)–(X_4) are satisfied. Consequently, by Theorem 5.1 problem (6.2) has at least one positive solution u^* such that

$$r_2 = 6.124 \times 10^{-8} \leq \left\|u^*\right\| \leq 4.721 \times 10^{-6} = r_1.$$

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References

1. Goodrich, C.S., Peterson, A.C.: Discrete Fractional Calculus. Springer, New York (2015)
2. Elaydi, S.N.: An Introduction to Difference Equations. Undergraduate Texts in Mathematics. Springer, New York (1996)
3. Kelley, W.G., Peterson, A.C.: Difference Equations. An Introduction with Applications, 2nd edn. Academic Press, Tokyo (1991)
4. Lakshmikantham, V., Trigiante, D.: Theory of Difference Equations Numerical Methods and Applications. Mathematics in Science and Engineering, vol. 181. Academic Press, Boston (1988)
5. Agarwal, R.P.: Difference Equations and Inequalities: Theory, Methods, and Applications. CRC Press, New York (2000)

6. Agarwal, R.P.: Focal Boundary Value Problems for Differential and Difference Equations. Kluwer Academic, Dordrecht (1998)
7. Agarwal, R.P., O'Regan, D., Wong, P.J.Y.: Positive Solutions of Differential, Difference and Integral Equations. Kluwer Academic, Dordrecht (1999)
8. Bastos, N.R.O., Ferreira, R.A.C., Torres, D.F.M.: Necessary optimality conditions for fractional difference problems of the calculus of variations. Discrete Contin. Dyn. Syst. **29**(2), 417–437 (2011)
9. Ferreira, R.A.C., Torres, D.F.M.: Fractional *h*-difference equations arising from the calculus of variations. Appl. Anal. Discrete Math. **5**(1), 110–121 (2011)
10. Mozyrska, D., Girejko, E., Wyrwas, M.: Comparison of *h*-difference fractional operators. In: Advances in the Theory and Applications of Non-Integer Order Systems. Lecture Notes in Electrical Engineering, vol. 257, pp. 191–197. Springer, New York (2013)
11. Mozyrska, D., Girejko, E.: Overview of the fractional *h*-difference operators. In: Advances in Harmonic Analysis and Operator Theory. Operator Theory: Advances and Applications, vol. 229, pp. 253–268. Springer, New York (2013)
12. Wyrwas, M., Mozyrska, D., Girejko, E.: On solutions to fractional discrete systems with sequential *h*-differences. Abstr. Appl. Anal. **2013**, Article ID 475350 (2013)
13. Mozyrska, D., Wyrwas, M.: Explicit criteria for stability of fractional *h*-difference two-dimensional systems. Int. J. Dyn. Control **5**, 4–9 (2017)
14. Xavier, G.B.A., Rajiniganth, P., Manuel, M.M.S., Chandrasekar, V.: Forward (α,β)-differerence operator and its some applications in number theory. Int. J. Appl. Math. **25**(1), 109–124 (2012)
15. Guo, D., Lakshmikantham, V.: Nonlinear Problems in Abstract Cone. Academic Press, Orlando (1988)
16. Griffel, D.H.: Applied Functional Analysis. Ellis Horwood, Chichester (1981)