Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Solitary and periodic wave solutions of higher-dimensional conformable time-fractional differential equations using the $(\frac{G'}{G}, \frac{1}{G})$-expansion method

Altaf A. Al-Shawba[1], Farah A. Abdullah[1*], Khaled A. Gepreel[2,3] and Amirah Azmi[1]

*Correspondence:
farahaini@usm.my
[1]School of Mathematical Sciences, Universiti Sains Malaysia, USM Penang, Malaysia
Full list of author information is available at the end of the article

Abstract

In this paper, the two variables $(\frac{G'}{G}, \frac{1}{G})$-expansion method is applied to obtain new exact solutions with parameters of higher-dimensional nonlinear time-fractional differential equations (NTFDEs) in the sense of the conformable fractional derivative. To clarify the veracity of this method, it is implemented in nonlinear (2 + 1)-dimensional time-fractional biological population (BP) model and nonlinear (3 + 1)-dimensional KdV–Zakharov–Kuznetsov (KdV–ZK) equation with time-fractional derivative. When the parameters take some special values, the solitary and periodic solutions are obtained from the hyperbolic and trigonometric function solutions.

Keywords: $(\frac{G'}{G}, \frac{1}{G})$-expansion method; Conformable fractional derivative; Exact solutions; (2 + 1)-dimensional time-fractional biological population model; (3 + 1)-dimensional time-fractional KdV–Zakharov–Kuznetsov equation

1 Introduction

Fractional differential equations (FDEs) can be viewed as the generalized type of the ordinary differential equations (ODEs). The FDEs have attracted the researchers' attention over the past two decades because the effects in ODEs are neglected. Oldham and Spanier [1] are the first researchers who have taken the FDEs into consideration. The search for the exact solutions of FDEs plays an important role in understanding the qualitative and quantitative features of many physical phenomena, which are described by these equations. For instance, the nonlinear oscillation of an earthquake can be modeled by derivatives of fractional order. Actually, the physical phenomena may not depend only on the time moment but also on the former time history, which can be successfully modeled utilizing the theory of fractional integrals and derivatives [2–4]. Fractional evolution equations play a significant role in various fields like engineering, biology, physics, signal processing, rheology, fluid flow, finance, electrochemistry, and so on [5–9]. Several efficient methods have recently been developed to get analytical solutions for FDEs. For example, the generalized tanh-coth method [10], the auxiliary equation method [11], the $(\frac{G'}{G})$-expansion method [12–23], the improved F-expansion method [24], the exponential rational function method [25–27], the simplest equation method [28], the modified simple equation

method [29–32], the first integral method [33–37], the Kudryashov method [38–43], the modified extended tanh expansion method [44], the $(\frac{G'}{G}, \frac{1}{G})$-expansion method [45–51], etc. [52–56]. The basic idea of the $(\frac{G'}{G})$-expansion method is that the traveling wave solutions of nonlinear FDEs can be presented via a polynomial in one variable $(\frac{G'}{G})$, where G satisfies the equation $G'' + \lambda G' + \mu G = 0$. In this study, the $(\frac{G'}{G}, \frac{1}{G})$-expansion method is employed. It can be an extension of the $(\frac{G'}{G})$-expansion method. The main idea of the $(\frac{G'}{G}, \frac{1}{G})$-expansion method is that the traveling wave solutions for NTFDEs can be presented via a polynomial in the $(\frac{1}{G})$ and $(\frac{G'}{G})$, where G satisfies the equation $G'' + \lambda G = \mu$. Li et al. [45] are the first researchers, who proposed the $(\frac{G'}{G}, \frac{1}{G})$-expansion method to solve the Zakharov equations. Sar et al. [46], Guner et al. [47] and Topsakal et al. [48] have used this method to extract the exact solutions for some space-time NFDEs.

This research paper aims to implement the $(\frac{G'}{G}, \frac{1}{G})$-expansion method to obtain new exact solutions for some NTFDEs in biology and mathematical physics. The first considered model is a nonlinear $(2+1)$-dimensional BP model with time-fractional derivative [13, 41]:

$$D_t^\beta u = \left(u^2\right)_{xx} + \left(u^2\right)_{yy} + h\left(u^2 - r\right), \quad t > 0, 0 < \beta < 1, x, y \in R, \tag{1.1}$$

in which u denotes the density of population, $h(u^2 - r)$ shows the population supply because of deaths and births and h, r are constants. When $\beta \to 1$, the BP model assists us to understand the dynamical proceeding of population changes and provides valuable predictions. Recently, Zhang and Zhang [57], Lu [58], Bekir et al. [59], Bekir and Güner [13] and Manafian and Lakestani [10] have found the exact solutions of Eq. (1.1) using the fractional sub-equation method, the Bäcklund transformation of fractional Riccati equation, the exp-function method, the $(\frac{G'}{G})$-expansion method, and the generalized tanh-coth method, respectively. The comparison of the obtained results with the results obtained in [13, 57–59] will be discussed in the following sections of the paper.

The second studied model is nonlinear (3 + 1)-dimensional KdV–ZK equation with time-fractional derivative [60]:

$$D_t^\beta u + auu_x + u_{xxx} + c(u_{yyx} + u_{zzx}) = 0, \quad t > 0, a, c = \text{const.}, 0 < \beta < 1. \tag{1.2}$$

When $\beta \to 1$, the KdV–ZK equation is derived for plasma comprised of hot and cool electrons and fluid ions species. Recently, Sahoo et al. [60] have found the exact solutions of Eq. (1.2) utilizing the improved fractional sub-equation method, whereas Kaplan et al. [61] have obtained the exact solutions of Eq. (1.2) using the $\exp(-\phi(\xi))$ method. The study is organized as follows: In Sect. 2, the description of the conformable fractional derivative and its important properties are presented. In Sect. 3, the main ideas of the $(\frac{G'}{G}, \frac{1}{G})$-expansion method are discussed. In Sect. 4, the new exact solutions for the conformable BP model and KdV–ZK equation with time-fractional derivative by the $(\frac{G'}{G}, \frac{1}{G})$-expansion method are constructed. Finally, conclusions are presented in Sect. 5 of this paper.

2 Conformable fractional derivative and its important properties

The conformable fractional derivative of g of order β is defined as follows [62–64]:

$$\mathrm{T}_\beta(g)(t) = \lim_{x \to 0} \frac{g(t + xt^{1-\beta}) - g(t)}{x},$$

which $g : [0, \infty) \to R$, $t > 0$ and $\beta \in (0, 1)$. Some important properties of the above definition are given by

$$\mathrm{T}_\beta(ag + bf) = a\mathrm{T}_\beta(g) + b\mathrm{T}_\beta(f), \quad \forall a, b \in R.$$

$$\mathrm{T}_\beta\left(t^\mu\right) = \mu t^{\mu-\beta}, \quad \forall \mu \in R.$$

$$\mathrm{T}_\beta(g \circ f)(t) = t^{1-\beta} f'(t) g'\left(f(t)\right).$$

3 Key ideas of the $(\frac{G'}{G}, \frac{1}{G})$-expansion method to the NTFDEs

Li et al. [45] suggested the $(\frac{G'}{G}, \frac{1}{G})$-expansion method as follows:

For the auxiliary equation

$$G'' + \lambda G = \mu, \tag{3.1}$$

we set

$$\phi = \frac{G'(\xi)}{G(\xi)}, \qquad \psi = \frac{1}{G(\xi)}. \tag{3.2}$$

From Eqs. (3.1) and (3.2), we get

$$\phi' = -\phi^2 + \mu\psi - \lambda, \qquad \psi' = -\phi\psi. \tag{3.3}$$

The general solution of the ODE (3.1), in the following three distinct subcases:

Case 1 If $\lambda < 0$, the general solution of the ODE (3.1) is

$$G = A_1 \sinh\sqrt{-\lambda}\xi + A_2 \cosh\sqrt{-\lambda}\xi + \frac{\mu}{\lambda}, \tag{3.4}$$

and thus

$$\psi^2 = -\frac{\lambda}{\lambda^2\sigma + \mu^2}\left[\phi^2 - 2\mu\psi + \lambda\right], \tag{3.5}$$

where A_1, A_2 are two arbitrary constants and $\sigma = A_1^2 - A_2^2$.

Case 2 If $\lambda > 0$, the general solution of the ODE (3.1) is given by

$$G = A_1 \sin\sqrt{\lambda}\xi + A_2 \cos\sqrt{\lambda}\xi + \frac{\mu}{\lambda}, \tag{3.6}$$

therefore, we have

$$\psi^2 = \frac{\lambda}{\lambda^2\sigma - \mu^2}\left[\phi^2 - 2\mu\psi + \lambda\right], \tag{3.7}$$

where $\sigma = A_1^2 + A_2^2$.

Case 3 If $\lambda = 0$, the general solution of the ODE (3.1) is

$$G = \frac{1}{2}\mu\xi^2 + A_1\xi + A_2, \tag{3.8}$$

and hence

$$\psi^2 = \frac{1}{A_1^2 - 2\mu A_2}\left[\phi^2 - 2\mu\psi\right]. \tag{3.9}$$

The main steps of the two variables $(\frac{G'}{G}, \frac{1}{G})$-expansion method are described in the following steps.

Step 1 Assume that we have the following general NFDE

$$F\left(u, \frac{\partial^\beta u}{\partial t^\beta}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial^{2\beta} u}{\partial t^{2\beta}}, \frac{\partial^2 u}{\partial x^2}, \ldots\right) = 0, \quad 0 < \beta < 1. \tag{3.10}$$

By introducing the transformation

$$u(x, y, z, t) = f(\xi), \quad \xi = ax + by + cz - l\frac{t^\beta}{\beta}, \tag{3.11}$$

where l, a, b and c are nonzero arbitrary constants. Equation (3.10) can be reduced into ODE in the following form:

$$P\left(f, f', f'', f''', \ldots\right) = 0. \tag{3.12}$$

Step 2 Assume that the solution of ODE (3.12) can be expressed by a polynomial in ϕ, ψ as follows:

$$f(\xi) = \sum_{i=0}^{m} \alpha_i \phi^i + \sum_{i=1}^{m} \beta_i \phi^{i-1}\psi, \tag{3.13}$$

where α_i $(i = 0, \ldots, m)$ and β_i $(i = 1, \ldots, m)$ are constants, and m in (3.13) can be determined by utilizing the homogeneous balance between the nonlinear terms and the highest-order derivative in (3.12).

Step 3 Substituting (3.13) into Eq. (3.12) utilizing (3.3) and (3.5), we obtain a polynomial in ψ and ϕ, where the degree of ψ is not bigger than one. Setting the coefficients of ϕ^i $(i = 0, 1, \ldots)$ and ψ^j $(j = 0, 1)$ to be zero, yields a set of algebraic equations, which can be solved with the help of Mathematica or Maple software package to obtain the values of α_i, β_i, l, a, b and c in which $(\lambda < 0)$.

Step 4 Substituting (3.13) into (3.12) with (3.3) and (3.7) or ((3.3) and (3.9)), we get the exact solutions of Eq. (3.12).

4 Applications of $(\frac{G'}{G}, \frac{1}{G})$-expansion method to NTFDEs in mathematical physics

In this part, new exact solutions of the (2 + 1)-dimensional BP model and (3 + 1)-dimensional KdV–ZK equation with time-fractional derivative are extricated by utilizing the $(\frac{G'}{G}, \frac{1}{G})$-expansion method.

4.1 (2 + 1)-dimensional BP model with time-fractional derivative

Using the fractional traveling wave variable,

$$u(x, y, t) = f(\xi), \quad \xi = x + y - l\frac{t^\beta}{\beta}, \tag{4.1}$$

the nonlinear BP model can be reduced into ODE:

$$lf' + 4ff'' + 4\left(f'\right)^2 + hf^2 - hr = 0. \tag{4.2}$$

To get the exact solution, we utilize the transformation

$$f = V^{-1}, \tag{4.3}$$

in Eq. (4.2) to find a new equation,

$$-lV^2V' - 4VV'' + 12\left(V'\right)^2 + hV^2 - hrV^4 = 0. \tag{4.4}$$

By utilizing the homogeneous balance principle, we get $m = 1$. Therefore, Eq. (1.1) has the formal solution

$$V(\xi) = \alpha_0 + \alpha_1\phi + \beta_1\psi, \tag{4.5}$$

where α_0, α_1 and β_1 are constants.

Case 1 When $\lambda < 0$ (hyperbolic function solutions)

By substituting (4.5) into (4.4) and utilizing (3.3) and (3.5), we obtain a polynomial in ψ and ϕ. Equating the coefficients of the equation to zero, we get a system of algebraic equations for α_0, α_1, β_1 and l as follows:

$$\phi^4 : -\frac{\beta_1^4h\lambda^2r}{(\lambda^2\sigma+\mu^2)^2} - \frac{4\beta_1^2\lambda}{\lambda^2\sigma+\mu^2} - \frac{3\alpha_1\beta_1^2l\lambda}{\lambda^2\sigma+\mu^2} + \frac{6\alpha_1^2\beta_1^2h\lambda r}{\lambda^2\sigma+\mu^2} - \alpha_1^4hr + \alpha_1^3l + 4\alpha_1^2 = 0,$$

$$\phi^3\psi : \frac{4\alpha_1\beta_1^3h\lambda r}{\lambda^2\sigma+\mu^2} - \frac{\beta_1^3l\lambda}{\lambda^2\sigma+\mu^2} + 8\alpha_1\beta_1 + 3\alpha_1^2\beta_1l - 4\alpha_1^3\beta_1hr = 0,$$

$$\phi^3 : -\frac{2\beta_1^3l\lambda^2\mu}{(\lambda^2\sigma+\mu^2)^2} + \frac{8\alpha_1\beta_1^3h\lambda^2r\mu}{(\lambda^2\sigma+\mu^2)^2} + \frac{12\alpha_0\alpha_1\beta_1^2h\lambda r}{\lambda^2\sigma+\mu^2} + \frac{8\alpha_1\beta_1\mu\lambda}{\lambda^2\sigma+\mu^2}$$
$$+ \frac{2\alpha_1^2\beta_1l\mu\lambda}{\lambda^2\sigma+\mu^2} - \frac{2\alpha_0\beta_1^2l\lambda}{\lambda^2\sigma+\mu^2} + 2\alpha_0\alpha_1^2l - 4\alpha_0\alpha_1^3hr - 8\alpha_0\alpha_1 = 0,$$

$$\phi^2\psi : \frac{4\beta_1^4h\lambda^2r\mu}{(\lambda^2\sigma+\mu^2)^2} + \frac{4\alpha_0\beta_1^3h\lambda r}{\lambda^2\sigma+\mu^2} - \frac{12\alpha_1^2\beta_1^2h\lambda r\mu}{\lambda^2\sigma+\mu^2} + \frac{4\beta_1^2\lambda\mu}{\lambda^2\sigma+\mu^2}$$
$$+ \frac{7\alpha_1\beta_1^2l\lambda\mu}{\lambda^2\sigma+\mu^2} - \alpha_1^3l\mu - 8\alpha_0\beta_1 - 12\alpha_1^2\mu - 12\alpha_0\alpha_1^2\beta_1hr + 4\alpha_0\alpha_1\beta_1l = 0,$$

$$\phi^2 : \frac{4\beta_1^4h\lambda^3r\mu^2}{(\lambda^2\sigma+\mu^2)^3} + \frac{2\alpha_1\beta_1^2l\lambda^2\mu^2}{(\lambda^2\sigma+\mu^2)^2} + \frac{8\alpha_0\beta_1^3h\lambda^2r\mu}{(\lambda^2\sigma+\mu^2)^2} - \frac{8\beta_1^2\lambda^2\mu^2}{(\lambda^2\sigma+\mu^2)^2} - \frac{2\beta_1^4h\lambda^3r}{(\lambda^2\sigma+\mu^2)^2}$$
$$+ \frac{2\alpha_0\alpha_1\beta_1l\mu\lambda}{\lambda^2\sigma+\mu^2} + \frac{6\alpha_1^2\beta_1^2h\lambda^2r}{\lambda^2\sigma+\mu^2} - \frac{\beta_1^2h\lambda}{\lambda^2\sigma+\mu^2} - \frac{4\alpha_0\beta_1\mu\lambda}{\lambda^2\sigma+\mu^2} + \frac{6\alpha_0^2\beta_1^2h\lambda r}{\lambda^2\sigma+\mu^2} - \frac{12\alpha_1^2\mu^2\lambda}{\lambda^2\sigma+\mu^2}$$
$$- \frac{4\alpha_1\beta_1^2l\lambda^2}{\lambda^2\sigma+\mu^2} - 6\alpha_0^2\alpha_1^2hr + \alpha_1^3l\lambda + \alpha_1^2h + \alpha_0^2\alpha_1l + 16\alpha_1^2\lambda = 0,$$

$$\phi\psi : \frac{4\beta_1^3l\lambda^2\mu^2}{(\lambda^2\sigma+\mu^2)^2} - \frac{16\alpha_1\beta_1^3h\lambda^2r\mu^2}{(\lambda^2\sigma+\mu^2)^2} - \frac{4\alpha_1^2\beta_1l\mu^2\lambda}{\lambda^2\sigma+\mu^2} + \frac{4\alpha_1\beta_1^3h\lambda^2r}{\lambda^2\sigma+\mu^2} - \frac{16\alpha_1\beta_1\mu^2\lambda}{\lambda^2\sigma+\mu^2}$$
$$+ \frac{4\alpha_0\beta_1^2l\lambda\mu}{\lambda^2\sigma+\mu^2} - \frac{24\alpha_0\alpha_1\beta_1^2hr\mu\lambda}{\lambda^2\sigma+\mu^2} - \frac{\beta_1^3l\lambda^2}{\lambda^2\sigma+\mu^2} + 2\alpha_1\beta_1h - 12\alpha_0^2\alpha_1\beta_1hr$$

$$
\begin{aligned}
&+ 2\alpha_1^2\beta_1 l\lambda - 2\alpha_0\alpha_1^2 l\mu + \alpha_0^2\beta_1 l + 12\alpha_1\beta_1\lambda + 12\alpha_0\alpha_1\mu = 0,\\
\phi:\ &\frac{8\alpha_1\beta_1^3 h\lambda^3 r\mu}{(\lambda^2\sigma+\mu^2)^2} - \frac{2\beta_1^3 l\lambda^3\mu}{(\lambda^2\sigma+\mu^2)^2} + \frac{12\alpha_0\alpha_1\beta_1^2 h\lambda^2 r}{\lambda^2\sigma+\mu^2} + \frac{2\alpha_1^2\beta_1 l\mu\lambda^2}{\lambda^2\sigma+\mu^2} - \frac{2\alpha_0\beta_1^2 l\lambda^2}{\lambda^2\sigma+\mu^2}\\
&+ \frac{8\alpha_1\beta_1\mu\lambda^2}{\lambda^2\sigma+\mu^2} - 4\alpha_1\alpha_0^3 hr - 8\alpha_0\alpha_1\lambda + 2\alpha_0\alpha_1^2 l\lambda + 2\alpha_0\alpha_1 h = 0,\\
\psi:\ &-\frac{8\beta_1^4 h\lambda^3 r\mu^3}{(\lambda^2\sigma+\mu^2)^3} - \frac{4\alpha_1\beta_1^2 l\lambda^2\mu^3}{(\lambda^2\sigma+\mu^2)^2} - \frac{16\alpha_0\beta_1^3 h\lambda^2 r\mu^2}{(\lambda^2\sigma+\mu^2)^2} + \frac{16\beta_1^2\lambda^2\mu^3}{(\lambda^2\sigma+\mu^2)^2} + \frac{4\beta_1^4 h\lambda^3 r\mu}{(\lambda^2\sigma+\mu^2)^2}\\
&+ \frac{3\alpha_1\beta_1^2 l\lambda^2\mu}{\lambda^2\sigma+\mu^2} - \frac{4\alpha_0\alpha_1\beta_1 l\mu^2\lambda}{\lambda^2\sigma+\mu^2} + \frac{8\alpha_0\beta_1\mu^2\lambda}{\lambda^2\sigma+\mu^2} + \frac{4\alpha_0\beta_1^3 h\lambda^2 r}{\lambda^2\sigma+\mu^2} + \frac{24\alpha_1^2\mu^3\lambda}{\lambda^2\sigma+\mu^2}\\
&- \frac{12\alpha_0^2\beta_1^2 h\lambda r\mu}{\lambda^2\sigma+\mu^2} - \frac{12\beta_1^2\lambda^2\mu}{\lambda^2\sigma+\mu^2} + \frac{2\beta_1^2 h\lambda\mu}{\lambda^2\sigma+\mu^2} + 2\alpha_0\beta_1 h - 4\alpha_0\beta_1\lambda\\
&- 24\alpha_1^2\lambda\mu - \alpha_0^2\alpha_1 l\mu - 4\alpha_0^3\beta_1 hr + 2\alpha_0\alpha_1\beta_1 l\lambda = 0,\\
\phi^0:\ &\frac{4\beta_1^4 h\lambda^4 r\mu^2}{(\lambda^2\sigma+\mu^2)^3} + \frac{8\alpha_0\beta_1^3 h\lambda^3 r\mu}{(\lambda^2\sigma+\mu^2)^2} - \frac{\beta_1^4 h\lambda^4 r}{(\lambda^2\sigma+\mu^2)^2} - \frac{8\beta_1^2\lambda^3\mu^2}{(\lambda^2\sigma+\mu^2)^2} + \frac{2\alpha_1\beta_1^2 l\lambda^3\mu^2}{(\lambda^2\sigma+\mu^2)^2}\\
&+ \frac{2\alpha_0\alpha_1\beta_1 l\mu\lambda^2}{\lambda^2\sigma+\mu^2} + \frac{4\beta_1^2\lambda^3}{\lambda^2\sigma+\mu^2} - \frac{\alpha_1\beta_1^2 l\lambda^3}{\lambda^2\sigma+\mu^2} - \frac{4\alpha_0\beta_1\mu\lambda^2}{\lambda^2\sigma+\mu^2} + \frac{6\alpha_0^2\beta_1^2 h\lambda^2 r}{\lambda^2\sigma+\mu^2}\\
&- \frac{\beta_1^2 h\lambda^2}{\lambda^2\sigma+\mu^2} - \frac{12\alpha_1^2\lambda^2\mu^2}{\lambda^2\sigma+\mu^2} + 12\alpha_1^2\lambda^2 + \alpha_0^2 h + \alpha_0^2\alpha_1 l\lambda - \alpha_0^4 hr = 0.
\end{aligned}
$$

By solving the algebraic equations mentioned above utilizing the Maple software package, the following results are obtained.

Result 1

$$
\begin{aligned}
&\alpha_0 = \mp\frac{1}{2\sqrt{r}}, \qquad \alpha_1 = \mp\frac{2\sqrt{2}}{\sqrt{-hr}}, \qquad \beta_1 = 0,\\
&l = \pm\frac{6}{\sqrt{2}}\sqrt{-h}\sqrt{r}, \qquad h = 32\lambda, \qquad \mu = 0.
\end{aligned}
\tag{4.6}
$$

By substituting (4.6) into (4.5) with (4.3) and (4.1), utilizing (3.2) and (3.4), we get the exact solutions of Eq. (1.1) as follows:

$$
u_1(x,y,t) = \mp 2\sqrt{r}\left(1 + \frac{A_1\cosh(\sqrt{-\lambda}\xi) + A_2\sinh(\sqrt{-\lambda}\xi)}{A_1\sinh(\sqrt{-\lambda}\xi) + A_2\cosh(\sqrt{-\lambda}\xi)}\right)^{-1}. \tag{4.7}
$$

In particular, if we put $A_1 = 0$ and $A_2 > 0$ in Eq. (4.7), we get the solitary solution

$$
u_{1_1}(x,y,t) = \mp\frac{2\sqrt{r}}{1+\tanh(\sqrt{-\lambda}\xi)}, \tag{4.8}
$$

while if we set $A_2 = 0$ and $A_1 > 0$, then we get the solitary solution

$$
u_{1_2}(x,y,t) = \mp\frac{2\sqrt{r}}{1+\coth(\sqrt{-\lambda}\xi)}, \tag{4.9}
$$

where $\xi = x + y \pm 24\sqrt{-\lambda}r\frac{t^\beta}{\beta}$.

Result 2

$$\alpha_0 = \mp\frac{1}{2\sqrt{r}}, \qquad \alpha_1 = \mp\frac{\sqrt{2}}{\sqrt{-hr}}, \qquad \beta_1 = \mp\sqrt{\frac{h^2\sigma + 64\mu^2}{4h^2r}},$$
$$l = \pm\frac{6}{\sqrt{2}}\sqrt{-h}\sqrt{r}, \qquad h = 8\lambda. \tag{4.10}$$

Based on Result 2, an exact solution of Eq. (1.1) is obtained. We have

$$u_2(x,y,t) = \left(\mp\frac{1}{2\sqrt{r}} \mp \frac{\frac{1}{2\sqrt{r}}(A_1\cosh(\sqrt{-\lambda}\xi) + A_2\sinh(\sqrt{-\lambda}\xi)) + \sqrt{\frac{\lambda^2\sigma+\mu^2}{4r\lambda^2}}}{A_1\sinh(\sqrt{-\lambda}\xi) + A_2\cosh(\sqrt{-\lambda}\xi) + \frac{\mu}{\lambda}}\right)^{-1}. \tag{4.11}$$

In particular, if we put $\mu = 0, A_1 = 0$ and $A_2 > 0$ in Eq. (4.11), we get the solitary solution

$$u_{2_1}(x,y,t) = \left(\mp\frac{1}{2\sqrt{r}} \mp \frac{1}{2\sqrt{r}}\tanh(\sqrt{-\lambda}\xi) \mp \frac{1}{2\sqrt{-r}}\operatorname{sech}(\sqrt{-\lambda}\xi)\right)^{-1}, \tag{4.12}$$

but if we set $\mu = 0, A_2 = 0$ and $A_1 > 0$, then we get the solitary solution

$$u_{2_2}(x,y,t) = \mp\frac{2\sqrt{r}}{1 + \coth(\sqrt{-\lambda}\xi) + \operatorname{csch}(\sqrt{-\lambda}\xi)}, \tag{4.13}$$

where $\xi = x + y \pm 12\sqrt{-\lambda r}\frac{t^\beta}{\beta}$.

Case 2 When $\lambda > 0$ (trigonometric function solutions)

By substituting (4.5) into (4.4) and utilizing (3.3) and (3.7), we obtain a polynomial in ψ and ϕ. Equating the coefficients of the equation to zero to obtain a set of algebraic equations for α_0, α_1, β_1 and l as follows:

$$\phi^4 : -\frac{\beta_1^4 h\lambda^2 r}{(\lambda^2\sigma-\mu^2)^2} + \frac{4\beta_1^2\lambda}{\lambda^2\sigma-\mu^2} + \frac{3\alpha_1\beta_1^2 l\lambda}{\lambda^2\sigma-\mu^2} - \frac{6\alpha_1^2\beta_1^2 h\lambda r}{\lambda^2\sigma-\mu^2} - \alpha_1^4 hr + \alpha_1^3 l + 4\alpha_1^2 = 0,$$

$$\phi^3\psi : -\frac{4\alpha_1\beta_1^3 h\lambda r}{\lambda^2\sigma-\mu^2} + \frac{\beta_1^3 l\lambda}{\lambda^2\sigma-\mu^2} + 8\alpha_1\beta_1 + 3\alpha_1^2\beta_1 l - 4\alpha_1^3\beta_1 hr = 0,$$

$$\phi^3 : -\frac{2\beta_1^3 l\lambda^2\mu}{(\lambda^2\sigma-\mu^2)^2} + \frac{8\alpha_1\beta_1^3 h\lambda^2 r\mu}{(\lambda^2\sigma-\mu^2)^2} - \frac{12\alpha_0\alpha_1\beta_1^2 h\lambda r}{\lambda^2\sigma-\mu^2} - \frac{8\alpha_1\beta_1\mu\lambda}{\lambda^2\sigma-\mu^2}$$
$$- \frac{2\alpha_1^2\beta_1 l\mu\lambda}{\lambda^2\sigma-\mu^2} + \frac{2\alpha_0\beta_1^2 l\lambda}{\lambda^2\sigma-\mu^2} + 2\alpha_0\alpha_1^2 l - 4\alpha_0\alpha_1^3 hr - 8\alpha_0\alpha_1 = 0,$$

$$\phi^2\psi : \frac{4\beta_1^4 h\lambda^2 r\mu}{(\lambda^2\sigma-\mu^2)^2} - \frac{4\alpha_0\beta_1^3 h\lambda r}{\lambda^2\sigma-\mu^2} + \frac{12\alpha_1^2\beta_1^2 h\lambda r\mu}{\lambda^2\sigma-\mu^2} - \frac{4\beta_1^2\lambda\mu}{\lambda^2\sigma-\mu^2}$$
$$- \frac{7\alpha_1\beta_1^2 l\lambda\mu}{\lambda^2\sigma-\mu^2} - \alpha_1^3 l\mu - 8\alpha_0\beta_1 - 12\alpha_1^2\mu - 12\alpha_0\alpha_1^2\beta_1 hr + 4\alpha_0\alpha_1\beta_1 l = 0,$$

$$\phi^2 : -\frac{4\beta_1^4 h\lambda^3 r\mu^2}{(\lambda^2\sigma-\mu^2)^3} + \frac{2\alpha_1\beta_1^2 l\lambda^2\mu^2}{(\lambda^2\sigma-\mu^2)^2} + \frac{8\alpha_0\beta_1^3 h\lambda^2 r\mu}{(\lambda^2\sigma-\mu^2)^2} - \frac{8\beta_1^2\lambda^2\mu^2}{(\lambda^2\sigma-\mu^2)^2} - \frac{2\beta_1^4 h\lambda^3 r}{(\lambda^2\sigma-\mu^2)^2}$$
$$- \frac{2\alpha_0\alpha_1\beta_1 l\mu\lambda}{\lambda^2\sigma-\mu^2} - \frac{6\alpha_1^2\beta_1^2 h\lambda^2 r}{\lambda^2\sigma-\mu^2} + \frac{\beta_1^2 h\lambda}{\lambda^2\sigma-\mu^2} + \frac{4\alpha_0\beta_1\mu\lambda}{\lambda^2\sigma-\mu^2} - \frac{6\alpha_0^2\beta_1^2 h\lambda r}{\lambda^2\sigma 0-\mu^2} + \frac{12\alpha_1^2\mu^2\lambda}{\lambda^2\sigma-\mu^2}$$
$$+ \frac{4\alpha_1\beta_1^2 l\lambda^2}{\lambda^2\sigma-\mu^2} - 6\alpha_0^2\alpha_1^2 hr + \alpha_1^3 l\lambda + \alpha_1^2 h + \alpha_0^2\alpha_1 l + 16\alpha_1^2\lambda = 0,$$

$$
\begin{aligned}
\phi\psi :{}& \frac{4\beta_1^3 l\lambda^2\mu^2}{(\lambda^2\sigma-\mu^2)^2}-\frac{16\alpha_1\beta_1^3 h\lambda^2 r\mu^2}{(\lambda^2\sigma-\mu^2)^2}+\frac{4\alpha_1^2\beta_1 l\mu^2\lambda}{\lambda^2\sigma-\mu^2}-\frac{4\alpha_1\beta_1^3 h\lambda^2 r}{\lambda^2\sigma-\mu^2}+\frac{16\alpha_1\beta_1\mu^2\lambda}{\lambda^2\sigma-\mu^2}\\
&-\frac{4\alpha_0\beta_1^2 l\lambda\mu}{\lambda^2\sigma-\mu^2}+\frac{24\alpha_0\alpha_1\beta_1^2 hr\mu\lambda}{\lambda^2\sigma-\mu^2}+\frac{\beta_1^3 l\lambda^2}{\lambda^2\sigma-\mu^2}+2\alpha_1\beta_1 h-12\alpha_0^2\alpha_1\beta_1 hr\\
&+2\alpha_1^2\beta_1 l\lambda-2\alpha_0\alpha_1^2 l\mu+\alpha_0^2\beta_1 l+12\alpha_1\beta_1\lambda+12\alpha_0\alpha_1\mu=0,\\
\phi :{}& \frac{8\alpha_1\beta_1^3 h\lambda^3 r\mu}{(\lambda^2\sigma-\mu^2)^2}-\frac{2\beta_1^3 l\lambda^3\mu}{(\lambda^2\sigma-\mu^2)^2}-\frac{12\alpha_0\alpha_1\beta_1^2 h\lambda^2 r}{\lambda^2\sigma-\mu^2}-\frac{2\alpha_1^2\beta_1 l\mu\lambda^2}{\lambda^2\sigma-\mu^2}+\frac{2\alpha_0\beta_1^2 l\lambda^2}{\lambda^2\sigma-\mu^2}\\
&-\frac{8\alpha_1\beta_1\mu\lambda^2}{\lambda^2\sigma-\mu^2}-4\alpha_1\alpha_0^3 hr-8\alpha_0\alpha_1\lambda+2\alpha_0\alpha_1^2 l\lambda+2\alpha_0\alpha_1 h=0,\\
\psi :{}& \frac{8\beta_1^4 h\lambda^3 r\mu^3}{(\lambda^2\sigma-\mu^2)^3}-\frac{4\alpha_1\beta_1^2 l\lambda^2\mu^3}{(\lambda^2\sigma-\mu^2)^2}-\frac{16\alpha_0\beta_1^3 h\lambda^2 r\mu^2}{(\lambda^2\sigma-\mu^2)^2}+\frac{16\beta_1^2\lambda^2\mu^3}{(\lambda^2\sigma-\mu^2)^2}+\frac{4\beta_1^4 h\lambda^3 r\mu}{(\lambda^2\sigma-\mu^2)^2}\\
&-\frac{3\alpha_1\beta_1^2 l\lambda^2\mu}{\lambda^2\sigma-\mu^2}+\frac{4\alpha_0\alpha_1\beta_1 l\mu^2\lambda}{\lambda^2\sigma-\mu^2}-\frac{8\alpha_0\beta_1\mu^2\lambda}{\lambda^2\sigma-\mu^2}-\frac{4\alpha_0\beta_1^3 h\lambda^2 r}{\lambda^2\sigma+\mu^2}-\frac{24\alpha_1^2\mu^3\lambda}{\lambda^2\sigma-\mu^2}\\
&+\frac{12\alpha_0^2\beta_1^2 h\lambda r\mu}{\lambda^2\sigma-\mu^2}+\frac{12\beta_1^2\lambda^2\mu}{\lambda^2\sigma-\mu^2}-\frac{2\beta_1^2 h\lambda\mu}{\lambda^2\sigma-\mu^2}+2\alpha_0\beta_1 h-4\alpha_0\beta_1\lambda\\
&-24\alpha_1^2\lambda\mu-\alpha_0^2\alpha_1 l\mu-4\alpha_0^3\beta_1 hr+2\alpha_0\alpha_1\beta_1 l\lambda=0,\\
\phi^0 :{}& -\frac{4\beta_1^4 h\lambda^4 r\mu^2}{(\lambda^2\sigma-\mu^2)^3}+\frac{8\alpha_0\beta_1^3 h\lambda^3 r\mu}{(\lambda^2\sigma-\mu^2)^2}-\frac{\beta_1^4 h\lambda^4 r}{(\lambda^2\sigma-\mu^2)^2}-\frac{8\beta_1^2\lambda^3\mu^2}{(\lambda^2\sigma-\mu^2)^2}+\frac{2\alpha_1\beta_1^2 l\lambda^3\mu^2}{(\lambda^2\sigma-\mu^2)^2}\\
&-\frac{2\alpha_0\alpha_1\beta_1 l\mu\lambda^2}{\lambda^2\sigma-\mu^2}-\frac{4\beta_1^2\lambda^3}{\lambda^2\sigma-\mu^2}+\frac{\alpha_1\beta_1^2 l\lambda^3}{\lambda^2\sigma-\mu^2}+\frac{4\alpha_0\beta_1\mu\lambda^2}{\lambda^2\sigma-\mu^2}-\frac{6\alpha_0^2\beta_1^2 h\lambda^2 r}{\lambda^2\sigma-\mu^2}+\frac{\beta_1^2 h\lambda^2}{\lambda^2\sigma-\mu^2}\\
&+\frac{12\alpha_1^2\lambda^2\mu^2}{\lambda^2\sigma-\mu^2}+12\alpha_1^2\lambda^2+\alpha_0^2 h+\alpha_0^2\alpha_1 l\lambda-\alpha_0^4 hr=0.
\end{aligned}
$$

By solving the algebraic equations mentioned above utilizing the Maple software package, the following results are obtained.

Result 1

$$
\begin{aligned}
&\alpha_0=\mp\frac{1}{2\sqrt{r}},\qquad \alpha_1=\mp\frac{2\sqrt{2}}{\sqrt{-hr}},\qquad \beta_1=0,\\
&l=\pm\frac{6}{\sqrt{2}}\sqrt{h}\sqrt{-r},\qquad h=32\lambda,\qquad \mu=0.
\end{aligned}
\tag{4.14}
$$

By substituting (4.14) into (4.5) with (4.3) and (4.1), utilizing (3.2) and (3.6), we get the exact solutions of Eq. (1.1) as follows:

$$
u_1(x,y,t)=\left(\mp\frac{1}{2\sqrt{r}}\mp\frac{1}{2\sqrt{-r}}\left(\frac{A_1\cos(\sqrt{\lambda}\xi)-A_2\sin(\sqrt{\lambda}\xi)}{A_1\sin(\sqrt{\lambda}\xi)+A_2\cos(\sqrt{\lambda}\xi)}\right)\right)^{-1}.
\tag{4.15}
$$

In particular, if we put $A_1=0$ and $A_2>0$ in Eq. (4.15), we get the periodic solution

$$
u_{1_1}(x,y,t)=\left(\mp\frac{1}{2\sqrt{r}}\pm\frac{1}{2\sqrt{-r}}\tan(\sqrt{\lambda}\xi)\right)^{-1},
\tag{4.16}
$$

whereas if we set $A_2 = 0$ and $A_1 > 0$, then we get the periodic solution

$$u_{1_2}(x,y,t) = \left(\mp\frac{1}{2\sqrt{r}} \mp \frac{1}{2\sqrt{-r}}\cot(\sqrt{\lambda}\xi)\right)^{-1}, \tag{4.17}$$

where $\xi = x + y \pm 24\sqrt{-\lambda r}\frac{t^\beta}{\beta}$.

Result 2

$$\begin{aligned} &\alpha_0 = \mp\frac{1}{2\sqrt{r}}, \qquad \alpha_1 = \mp\frac{\sqrt{2}}{\sqrt{-hr}}, \qquad \beta_1 = \mp\sqrt{\frac{64\mu^2 - h^2\sigma}{4h^2 r}}, \\ &l = \pm\frac{6}{\sqrt{2}}\sqrt{h}\sqrt{-r}, \qquad h = 8\lambda. \end{aligned} \tag{4.18}$$

Based on Result 2, exact solutions of Eq. (1.1) are obtained. We have

$$u_2(x,y,t) = \left(\mp\frac{1}{2\sqrt{r}} \mp \frac{\frac{1}{2\sqrt{-r}}(A_1\cos(\sqrt{\lambda}\xi) - A_2\sin(\sqrt{\lambda}\xi)) + \sqrt{\frac{\mu^2-\lambda^2\sigma}{4\lambda^2 r}}}{A_1\sin(\sqrt{\lambda}\xi) + A_2\cos(\sqrt{\lambda}\xi) + \frac{\mu}{\lambda}}\right)^{-1}. \tag{4.19}$$

In particular, if we put $\mu = 0, A_1 = 0$ and $A_2 > 0$ in Eq. (4.19), we get the periodic solution

$$u_{2_1}(x,y,t) = \left(\mp\frac{1}{2\sqrt{r}} \pm \frac{1}{2\sqrt{-r}}\tan(\sqrt{\lambda}\xi) \mp \frac{1}{2\sqrt{-r}}\sec(\sqrt{\lambda}\xi)\right)^{-1}, \tag{4.20}$$

but if we set $\mu = 0, A_2 = 0$ and $A_1 > 0$, then we get the periodic solution

$$u_{2_2}(x,y,t) = \left(\mp\frac{1}{2\sqrt{r}} \mp \frac{1}{2\sqrt{-r}}\cot(\sqrt{\lambda}\xi) \mp \frac{1}{2\sqrt{-r}}\csc(\sqrt{\lambda}\xi)\right)^{-1}, \tag{4.21}$$

where $\xi = x + y \pm 12\sqrt{-\lambda r}\frac{t^\beta}{\beta}$.

Remark 1 By comparing our results with the results obtained by Lu [58], Zhang and Zhang [57], Bekir et al. [59], Bekir and Güner[13] and Manafian and Lakestani [10], we conclude that all our solutions of Eq. (1.1) are new and satisfy the equation.

4.2 (3 + 1)-dimensional KdV–ZK equation with time-fractional derivative

The fractional traveling wave variable

$$u(x,y,z,t) = f(\xi), \quad \xi = x + y + z - l\frac{t^\beta}{\beta}, \tag{4.22}$$

reduces Eq. (1.2) to the following ODE:

$$-lf + \frac{a}{2}f^2 + (2c+1)f'' + \xi_0 = 0. \tag{4.23}$$

By balancing f'' with f^2 in Eq. (4.23), we get $m = 2$. Thus, we get

$$f(\xi) = \alpha_0 + \alpha_1\phi + \alpha_2\phi^2 + \beta_1\psi + \beta_2\phi\psi, \tag{4.24}$$

in which α_0, α_1, α_2, β_1 and β_2 are constants.

Case 1 Hyperbolic function solution when $\lambda < 0$.

If $\lambda < 0$ substituting (4.24) into (4.23) and using (3.3) and (3.5), we obtain a polynomial in ψ and ϕ. Equating the coefficients of this equation to zero yields a set of algebraic equations in α_0, α_1, α_2, β_1, β_2, σ, λ and μ which can be solved by applying Maple to get the following values:

$$\begin{aligned} &\alpha_0 = -\frac{5\lambda(2c+1)-l}{a}, \qquad \alpha_1 = 0, \qquad \alpha_2 = -\frac{6(2c+1)}{a}, \\ &\beta_1 = \frac{6\mu(2c+1)}{a}, \qquad \beta_2 = \mp\frac{6(2c+1)}{a}\sqrt{-\frac{\lambda^2\sigma+\mu^2}{\lambda}}, \\ &\xi_0 = -\frac{\lambda^2(2c+1)^2-l^2}{2a}. \end{aligned} \tag{4.25}$$

From (3.4), (4.24) and (4.25), we get the exact solution of Eq. (1.2) as follows:

$$\begin{aligned} u_1(x,y,z,t) = {} & \frac{l-10c\lambda-5\lambda}{a} + \frac{6\mu(2c+1)}{a(A_1\sinh\sqrt{-\lambda}\xi + A_2\cosh\sqrt{-\lambda}\xi + \frac{\mu}{\lambda})} \\ & + \frac{(12c+6)(A_1\cosh\sqrt{-\lambda}\xi + A_2\sinh\sqrt{-\lambda}\xi)}{a(A_1\sinh\sqrt{-\lambda}\xi + A_2\cosh\sqrt{-\lambda}\xi + \frac{\mu}{\lambda})^2} \\ & \times \left(A_1\lambda\cosh\sqrt{-\lambda}\xi + A_2\lambda\sinh\sqrt{-\lambda}\xi \mp \sqrt{\lambda^2\sigma+\mu^2}\right). \end{aligned} \tag{4.26}$$

In particular, if we put $\mu = 0, A_1 = 0$ and $A_2 > 0$ in Eq. (4.26), we get the solitary solution

$$\begin{aligned} &u_{1_1}(x,y,z,t) \\ &\quad = \frac{l-10c\lambda-5\lambda}{a} + \frac{6\lambda(2c+1)}{a}(\tanh\sqrt{-\lambda}\xi)(\tanh\sqrt{-\lambda}\xi \mp i\,\mathrm{sech}\sqrt{-\lambda}\xi), \end{aligned} \tag{4.27}$$

while if we set $\mu = 0$, $A_2 = 0$ and $A_1 > 0$, then we get the solitary solution

$$\begin{aligned} &u_{1_2}(x,y,z,t) \\ &\quad = \frac{l-10c\lambda-5\lambda}{a} + \frac{6\lambda(2c+1)}{a}(\coth\sqrt{-\lambda}\xi)(\coth\sqrt{-\lambda}\xi \mp \mathrm{csch}\sqrt{-\lambda}\xi), \end{aligned} \tag{4.28}$$

in which $\xi = x + y + z - l\frac{t^\beta}{\beta}$.

Case 2 Trigonometric function solution when $\lambda > 0$.

If $\lambda > 0$ substituting (4.24) into (4.23) and using (3.3) and (3.7), we obtain a polynomial in ψ and ϕ. Equating the coefficients of this equation to zero yields a set of algebraic equations, which can be solved by applying Maple to get the following values:

$$\begin{aligned} &\alpha_0 = -\frac{5\lambda(2c+1)-l}{a}, \qquad \alpha_1 = 0, \qquad \alpha_2 = -\frac{6(2c+1)}{a}, \\ &\beta_1 = \frac{6\mu(2c+1)}{a}, \qquad \beta_2 = \mp\frac{6(2c+1)}{a}\sqrt{\frac{\lambda^2\sigma-\mu^2}{\lambda}}, \\ &\xi_0 = -\frac{\lambda^2(2c+1)^2-l^2}{2a}. \end{aligned} \tag{4.29}$$

From (3.6), (4.24) and (4.29), we get the exact solution of Eq. (1.2) as follows:

$$\begin{aligned} u_1(x,y,z,t) = {} & \frac{l-10c\lambda-5\lambda}{a} + \frac{6\mu(2c+1)}{a(A_1\sin\sqrt{\lambda}\xi + A_2\cos\sqrt{\lambda}\xi + \frac{\mu}{\lambda})} \\ & - \frac{6(2c+1)(A_1\cos\sqrt{\lambda}\xi - A_2\sin\sqrt{\lambda}\xi)}{a(A_1\sin\sqrt{\lambda}\xi + A_2\cos\sqrt{\lambda}\xi + \frac{\mu}{\lambda})^2} \\ & \times \left(A_1\lambda\cos\sqrt{\lambda}\xi - A_2\lambda\sin\sqrt{\lambda}\xi \pm \sqrt{\lambda^2\sigma-\mu^2}\right). \end{aligned} \tag{4.30}$$

Particularly, if we put $\mu = 0, A_1 = 0$ and $A_2 > 0$ in Eq. (4.30), we get the periodic solution

$$u_{1_1}(x,y,z,t) = \frac{l-10c\lambda-5\lambda}{a} - \frac{6\lambda(2c+1)}{a}(\tan\sqrt{\lambda}\xi)(\tan\sqrt{\lambda}\xi \mp \sec\sqrt{\lambda}\xi), \tag{4.31}$$

but if we set $\mu = 0, A_2 = 0$ and $A_1 > 0$, then we get the periodic solution

$$u_{1_2}(x,y,z,t) = \frac{l-10c\lambda-5\lambda}{a} - \frac{6\lambda(2c+1)}{a}(\cot\sqrt{\lambda}\xi)(\cot\sqrt{\lambda}\xi \pm \csc\sqrt{\lambda}\xi), \tag{4.32}$$

where $\xi = x + y + z - l\frac{t^\beta}{\beta}$.

Case 3 Rational function solution when $\lambda = 0$.

If $\lambda = 0$ substituting (4.24) into (4.23) and using (3.3) and (3.9), we obtain a polynomial in ψ and ϕ. Equating the coefficients of this equation to zero yields a set of algebraic equations, which can be solved by applying Maple to get the following values:

$$\begin{aligned} &\alpha_0 = \alpha_0, \qquad \alpha_1 = 0, \qquad \alpha_2 = -\frac{6(2c+1)}{a}, \qquad \beta_1 = \frac{6\mu(2c+1)}{a}, \\ &l = a\alpha_0, \qquad \beta_2 = \mp\frac{6(2c+1)}{a}\sqrt{A_1^2 - 2A_2\mu}, \qquad \xi_0 = \frac{1}{2}a\alpha_0^2. \end{aligned} \tag{4.33}$$

From (3.8), (4.24) and (4.33), we get the exact solution of Eq. (1.2) as follows:

$$\begin{aligned} u_1(x,y,z,t) = {} & \alpha_0 + \frac{6\mu(2c+1)}{a(\mu/2\xi^2 + A_1\xi + A_2)} - \frac{6(2c+1)(\mu\xi + A_1)}{a(\mu/2\xi^2 + A_1\xi + A_2)^2} \\ & \times \left(\mu\xi + A_1 \pm \sqrt{A_1^2 - 2A_2\mu}\right), \end{aligned} \tag{4.34}$$

where $\xi = x + y + z - a\alpha_0\frac{t^\beta}{\beta}$.

Remark 2 If we compare our results with the results obtained by [60, 61], we can see that our solutions of Eq. (1.2) are new and satisfy the equation.

4.2.1 Graphical presentation of some exact solutions

We presented some graphs to illustrate the behavior of exact solutions of Eqs. (1.1) and (1.2). Figures 1–5 show the solitary and periodic wave forms.

5 Conclusion

The $(\frac{G'}{G}, \frac{1}{G})$-expansion method is used to discuss the exact solutions to NFDEs. The $(\frac{G'}{G}, \frac{1}{G})$ is successfully implemented to solve two NTFDEs. As applications, new exact solutions

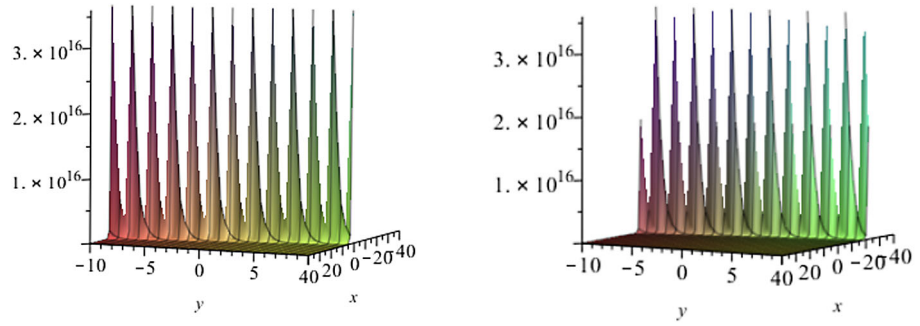

Figure 1 Solitary wave solution of Eq. (4.8) with $\beta = 0.5$ and $\beta = 0.9$, respectively, when $\lambda = -1$

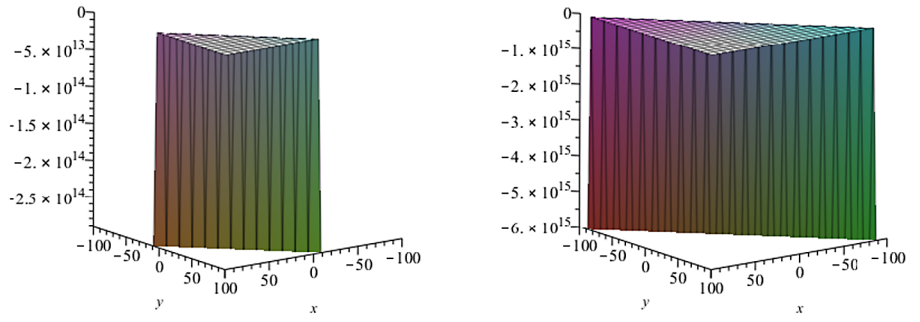

Figure 2 Solitary wave solution of Eq. (4.9) with $\beta = 0.25$ and $\beta = 0.5$, respectively, when $\lambda = -1$

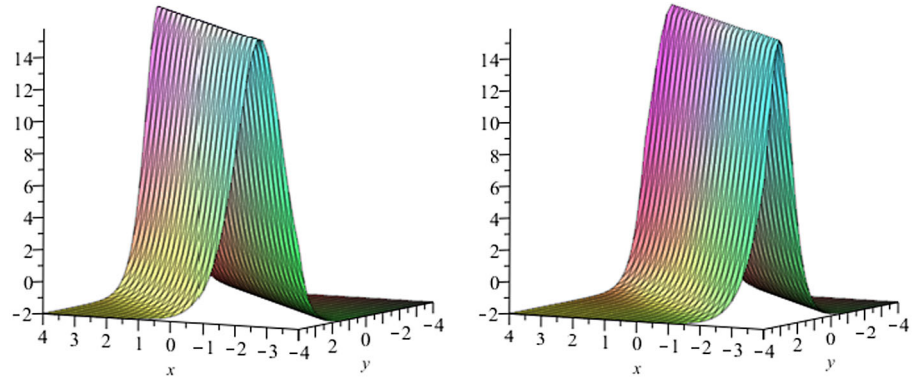

Figure 3 Solitary wave solution of real part for Eq. (4.27) with $\beta = 0.25$ and $\beta = 0.9$, respectively, when $\lambda = -1$

for (2 + 1)-dimensional BP model and (3 + 1)-dimensional KdV–ZK equation with time-fractional derivative are obtained. When the parameters μ, A_1, and A_2 are given special values, the solitary wave solutions (4.8), (4.9), (4.12), (4.13), (4.27) and (4.28) and the periodic solutions (4.16), (4.17), (4.20), (4.21), (4.31) and (4.32) are obtained. When $\mu = 0$ and $\beta_i = 0$ in Eq. (3.1) and Eq. (3.13), respectively, the $(\frac{G'}{G}, \frac{1}{G})$-expansion method is reduced to the $(\frac{G'}{G})$-expansion method. Therefore, it can be concluded that the $(\frac{G'}{G}, \frac{1}{G})$-expansion method is more general and efficient than the $(\frac{G'}{G})$-expansion method. In comparison with other methods, the key feature of this method is that it possesses all three types of solu-

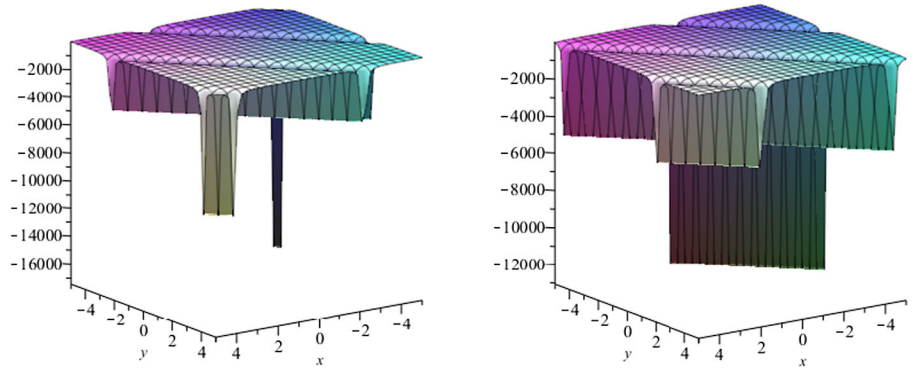

Figure 4 Periodic wave solution of Eq. (4.31) with $\beta = 0.25$ and $\beta = 0.9$, respectively, when $\lambda = 1$

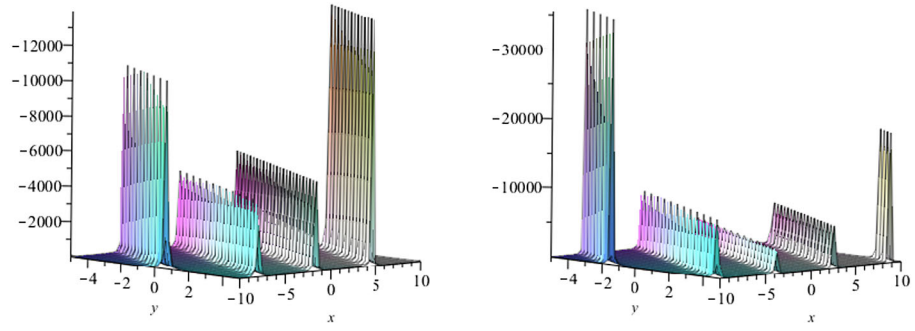

Figure 5 Periodic wave solution of Eq. (4.32) with $\beta = 0.25$ and $\beta = 0.9$, respectively, when $\lambda = 1$

tions. Some diagrams have been given in three dimensions for fractional order to illustrate the behavior of the solutions when the parameters take some special values.

Acknowledgements
F.A. Abdullah would like to acknowledge the financial support of the Fundamental Research Grant Scheme (FRGS 203/PMATHS/6711570) by the Ministry of Higher Education, Malaysia, and Research Creativity and Management Office (RCMO), Universiti Sains Malaysia. The first, second, and fourth authors would like to thank the School of Mathematical Sciences, Universiti Sains Malaysia, Penang Malaysia, for providing computing facilities.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare to have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Author details
[1]School of Mathematical Sciences, Universiti Sains Malaysia, USM Penang, Malaysia. [2]Department Faculty of Science, Taif University, Al Hawiyah, Saudi Arabia. [3]Mathematics Department, Faculty of Science, Zagazig University, Zagazig, Egypt.

Publisher's Note

Received: 23 July 2018 Accepted: 24 September 2018 Published online: 10 October 2018

References

1. Oldham, K.B., Spanier, J.: The Fractional Calculus. Academic Press, New York (1974)
2. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
3. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives Theory and Applications. Gordonand Breach, New York (1993)
4. Sahadevan, R., Bakkyaraj, T.: Invariant analysis of time fractional generalized Burgers and Korteweg–de Vries equations. J. Math. Anal. Appl. **393**(2), 341–347 (2012)
5. Tarasov, V.E.: Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media. Higher Education Press, Beijing (2011)
6. Mainardi, F.: Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models. Imperial College Press, London (2010)
7. Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000). https://doi.org/10.1142/3779
8. Podlubny, I.: Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. Academic Press, San Diego (1998)
9. West, B., Bologna, M., Grigolini, P.: Physics of Fractal Operators. Springer, New York (2012)
10. Manafian, J., Lakestani, M.: A new analytical approach to solve some of the fractional-order partial differential equations. Indian J. Phys. **91**(3), 243–258 (2017)
11. Akbulut, A., Kaplan, M.: Auxiliary equation method for time-fractional differential equations with conformable derivative. Comput. Math. Appl. **75**(1), 876–882 (2018). https://doi.org/10.1016/j.camwa.2017.10.016
12. Bin, Z.: (G'/G)-expansion method for solving fractional partial differential equations in the theory of mathematical physics. Commun. Theor. Phys. **58**(5), Article ID 623 (2012). https://doi.org/10.1088/0253-6102/58/5/02
13. Bekir, A., Güner, Ö.: Exact solutions of nonlinear fractional differential equations by (G'/G)-expansion method. Chin. Phys. B **22**(11), 110202 (2013)
14. Shang, N., Zheng, B.: Exact solutions for three fractional partial differential equations by the (G'/G) method. Int. J. Appl. Math. **43**(3), 114–119 (2013)
15. Akram, G., Batool, F.: A class of traveling wave solutions for space-time fractional biological population model in mathematical physics. Indian J. Phys. **91**(10), 1145–1148 (2017)
16. Zayed, E., Al-Nowehy, A.-G.: Exact solutions for nonlinear foam drainage equation. Indian J. Phys. **91**(2), 209–218 (2017)
17. Al-Shawba, A., Gepreel, K., Abdullah, F., Azmia, A.: Abundant closed form solutions of conformable time fractional Sawada–Kotera–Ito equation using (G'/G)-expansion method. Results Phys. **9**, 337–343 (2018). https://doi.org/10.1016/j.rinp.2018.02.012
18. Das, A., Ghosh, N., Ansari, K.: Bifurcation and exact traveling wave solutions for dual power Zakharov–Kuznetsov–Burgers equation with fractional temporal evolution. Comput. Math. Appl. **75**(1), 59–69 (2018). https://doi.org/10.1016/j.camwa.2017.08.043
19. Al-Shawba, A.A., Abdullah, F.A., Azmi, A.: Travelling wave solutions for fractional Boussinesq equation using modified (G'/G) expansion method. AIP Conf. Proc. **1974**, Article ID 020036 (2018). https://doi.org/10.1063/1.5041567
20. Islam, T., Akbar, M.A., Azad, A.K.: Traveling wave solutions to some nonlinear fractional partial differential equations through the rational (G'/G)-expansion method. J. Ocean Eng. Sci. **3**(1), 76–81 (2018)
21. Roshid, H., Rahman, N., Akbar, M.: Traveling waves solutions of nonlinear Klein Gordon equation by extended (G'/G)-expasion method. Ann. Math. Pures Appl. **3**, 10–16 (2013)
22. Hafez, M., Alam, M.N., Akbar, M.A.: Exact traveling wave solutions to the Klein–Gordon equation using the novel (G'/G)-expansion method. Results Phys. **4**, 177–184 (2014)
23. Akbar, M.A., Ali, N.H.M., Roy, R.: Closed form solutions of two time fractional nonlinear wave equations. Results Phys. **9**, 1031–1039 (2018)
24. Ali Akbar, M., Ali, N.H.M.: The improved F-expansion method with Riccati equation and its applications in mathematical physics. Cogent Math. **4**(1), Article ID 1282577 (2017)
25. Mohyud-Din, S.T. Bibi, S.: Exact solutions for nonlinear fractional differential equations using exponential rational function method. Opt. Quantum Electron. **49**(2), Article ID 64 (2017). https://doi.org/10.1007/s11082-017-0895-9
26. Aksoy, E., Kaplan, M., Bekir, A.: Exponential rational function method for space-time fractional differential equations. Waves Random Complex Media **26**(2), 142–151 (2016)
27. Guner, O., Bekir, A.: Exact solutions to the time-fractional differential equations via local fractional derivatives. Waves Random Complex Media **28**(1), 139–149 (2018). https://doi.org/10.1080/17455030.2017.1332442
28. Chen, C., Jiang, Y.-L.: Simplest equation method for some time-fractional partial differential equations with conformable derivative. Comput. Math. Appl. **75**(8), 2978–2988 (2018). https://doi.org/10.1016/j.camwa.2018.01.025
29. Kaplan, M., Bekir, A., Akbulut, A., et al.: The modified simple equation method for nonlinear fractional differential equations. Rom. J. Phys. **60**(9–10), 1374–1383 (2015)
30. Khan, K., Akbar, M.A.: Traveling wave solutions of the (2 + 1)-dimensional Zoomeron equation and the Burgers equations via the MSE method and the exp-function method. Ain Shams Eng. J. **5**(1), 247–256 (2014)
31. Khan, K., Akbar, M.A.: Exact and solitary wave solutions for the Tzitzeica–Dodd–Bullough and the modified KdV–Zakharov–Kuznetsov equations using the modified simple equation method. Ain Shams Eng. J. **4**(4), 903–909 (2013)
32. Irshad, A., Mohyud-Din, S.T., Ahmed, N., Khan, U.: A new modification in simple equation method and its applications on nonlinear equations of physical nature. Results Phys. **7**, 4232–4240 (2017). https://doi.org/10.1016/j.rinp.2017.10.048
33. Aminikhah, H., Sheikhani, A.R., Rezazadeh, H.: Exact solutions for the fractional differential equations by using the first integral method. Nonlinear Eng. **4**(1), 15–22 (2015)
34. Mirzazadeh, M.: Analytical study of solitons to nonlinear time fractional parabolic equations. Nonlinear Dyn. **85**(4), 2569–2576 (2016)
35. Mirzazadeh, M., Eslami, M., Biswas, A.: Solitons and periodic solutions to a couple of fractional nonlinear evolution equations. Pramāna **82**(3), 465–476 (2014)

36. Eslami, M., Khodadad, F.S., Nazari, F., Rezazadeh, H.: The first integral method applied to the Bogoyavlenskii equations by means of conformable fractional derivative. Opt. Quantum Electron. **49**(12), Article ID 391 (2017). https://doi.org/10.1007/s11082-017-1224-z
37. Rezazadeh, H., Manafian, J., Khodadad, F.S., et al.: Traveling wave solutions for density-dependent conformable fractional diffusion–reaction equation by the first integral method and the improved $\tan(\frac{1}{2}\varphi(\xi))$-expansion method. Opt. Quantum Electron. **50**(3), Article ID 121 (2018). https://doi.org/10.1007/s11082-018-1388-1
38. Ali, K.K., Nuruddeen, R., Hadhoud, A.R.: New exact solitary wave solutions for the extended (3 + 1)-dimensional Jimbo–Miwa equations. Results Phys. **9**, 12–16 (2018)
39. Nuruddeen, R., Nass, A.M.: Exact solitary wave solution for the fractional and classical GEW-Burgers equations: an application of Kudryashov method. J. Taibah Univ. Sci. **12**(3), 309–314 (2018)
40. Hosseini, K., Ansari, R.: New exact solutions of nonlinear conformable time-fractional Boussinesq equations using the modified Kudryashov method. Waves Random Complex Media **27**(4), 628–636 (2017). https://doi.org/10.1080/17455030.2017.1296983
41. Ege, S.M., Misirli, E.: The modified Kudryashov method for solving some fractional-order nonlinear equations. Adv. Differ. Equ. **2014**(1), Article ID 135 (2014). https://doi.org/10.1186/1687-1847-2014-135
42. Korkmaz, A., Hosseini, K.: Exact solutions of a nonlinear conformable time-fractional parabolic equation with exponential nonlinearity using reliable methods. Opt. Quantum Electron. **49**(8), Article ID 278 (2017). https://doi.org/10.1007/s11082-017-1116-2
43. Kumar, D., Seadawy, A.R., Joardar, A.K.: Modified Kudryashov method via new exact solutions for some conformable fractional differential equations arising in mathematical biology. Chin. J. Phys. **56**(1), 75–85 (2018)
44. Ali, K.K., Nuruddeen, R., Raslan, K.: New hyperbolic structures for the conformable time-fractional variant Bussinesq equations. Opt. Quantum Electron. **50**(2), Article ID 61 (2018). https://doi.org/10.1007/s11082-018-1330-6
45. Li, L., Li, E., Wang, M.: The $(G'/G, 1/G)$-expansion method and its application to travelling wave solutions of the Zakharov equations. Appl. Math. J. Chin. Univ. **25**(4), 454–462 (2010). https://doi.org/10.1007/s11766-010-2128-x
46. Yąsar, E.Y., Giresunlu, I.B.: The $(G'/G, 1/G)$-expansion method for solving nonlinear space–time fractional differential equations. Pramana J. Phys. **87**(2), Article ID 17 (2016)
47. Guner, O., Bekir, A., Ünsal, Ö.: Two reliable methods for solving the time fractional Clannish Random Walker's Parabolic equation. Optik, Int. J. Light Electron Opt. **127**(20), 9571–9577 (2016)
48. Topsakal, M., Guner, O., Bekir, A., Unsal, O.: Exact solutions of some fractional differential equations by various expansion methods. J. Phys. Conf. Ser. **766**(1), Article ID 012035 (2016). https://doi.org/10.1088/1742-6596/766/1/012035
49. Miah, M.M., Ali, H.S., Ali Akbar, M., et al.: Some applications of the $(G'/G, 1/G)$-expansion method to find new exact solutions of NLEEs. Eur. Phys. J. Plus **132**(6), Article ID 252 (2017). https://doi.org/10.1140/epjp/i2017-11571-0
50. Miah, M.M., Shahadat Ali, H.M., Ali Akbar, M.: An investigation of abundant traveling wave solutions of complex nonlinear evolution equations: the perturbed nonlinear Schrodinger equation and the cubic-quintic Ginzburg–Landau equation. Cogent Math. **3**(1), Article ID 1277506 (2016)
51. Huda, M.A., Akbar, M.A., Shanta, S.S.: The new types of wave solutions of the Burger's equation and the Benjamin–Bona–Mahony equation. J. Ocean Eng. Sci. **3**(1), 1–10 (2018)
52. Babolian, E., Vahidi, A., Shoja, A.: An efficient method for nonlinear fractional differential equations: combination of the Adomian decomposition method and spectral method. Indian J. Pure Appl. Math. **45**(6), 1017–1028 (2014)
53. Aksoy, E., Çevikel, A.C., Bekir, A.: Soliton solutions of (2 + 1)-dimensional time-fractional Zoomeron equation. Optik, Int. J. Light Electron Opt. **127**(17), 6933–6942 (2016)
54. Ekici, M.: Soliton and other solutions of nonlinear time fractional parabolic equations using extended G'/G-expansion method. Optik, Int. J. Light Electron Opt. **130**, 1312–1319 (2017)
55. Gepreel, K.A.: Explicit Jacobi elliptic exact solutions for nonlinear partial fractional differential equations. Adv. Differ. Equ. **2014**(1), Article ID 286 (2014). https://doi.org/10.1186/1687-1847-2014-286
56. Zheng, B.: A new fractional Jacobi elliptic equation method for solving fractional partial differential equations. Adv. Differ. Equ. **2014**(1), Article ID 228 (2014)
57. Zhang, S., Zhang, H.-Q.: Fractional sub-equation method and its applications to nonlinear fractional PDEs. Phys. Lett. A **375**(7), 1069–1073 (2011)
58. Lu, B.: Bäcklund transformation of fractional Riccati equation and its applications to nonlinear fractional partial differential equations. Phys. Lett. A **376**(28), 2045–2048 (2012)
59. Bekir, A., Güner, Ö., Cevikel, A.C.: Fractional complex transform and exp-function methods for fractional differential equations. Abstr. Appl. Anal. **2013**, Article ID 426462 (2013). https://doi.org/10.1155/2013/426462
60. Sahoo, S., Ray, S.S.: Improved fractional sub-equation method for (3 + 1)-dimensional generalized fractional KdV–Zakharov–Kuznetsov equations. Comput. Math. Appl. **70**(2), 158–166 (2015)
61. Kaplan, M., Bekir, A.: A novel analytical method for time-fractional differential equations. Optik, Int. J. Light Electron Opt. **127**(20), 8209–8214 (2016)
62. Khalil, R., Al Horani, M., Yousef, A., et al.: A new definition of fractional derivative. J. Comput. Appl. Math. **264**, 65–70 (2014)
63. Jarad, F., Uğurlu, E., Abdeljawad, T., Baleanu, D.: On a new class of fractional operators. Adv. Differ. Equ. **2017**, Article ID 247 (2017). https://doi.org/10.1186/s13662-017-1306-z
64. Abdeljawad, T.: On conformable fractional calculus. J. Comput. Appl. Math. **279**, 57–66 (2015)