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# A Crank–Nicolson linear difference scheme for a BBM equation with a time fractional nonlocal viscous term

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## Abstract

In this paper, we construct a Crank–Nicolson linear finite difference scheme for a Benjamin–Bona–Mahony equation with a time fractional nonlocal viscous term. The stability and convergence of the proposed numerical scheme are rigorously derived. Theoretical analysis shows that the numerical scheme is convergent in the order of  $O(\tau^{\frac{3}{2}} + h^2)$ , where  $\tau$  and  $h$  are the time and space step sizes. Two numerical experiments are presented to verify that the theoretical analysis is accurate and to demonstrate that the numerical scheme is effective.

**MSC:** 65N15; 65N30

**Keywords:** BBM equation; Time fractional nonlocal viscous term; Finite difference scheme; Stability; Convergence

## 1 Introduction

In many nonlinear partial differential equations, the most common method to stimulate the nonlinear dispersion wave is by using the Korteweg–de Vries (KdV) equation [1]

$$u_t + u_x + u_{xxx} + uu_x = 0, \quad (1)$$

because it obeys conservation laws and has been widely applied in solid, liquid, and other subject fields [2]. As an improvement of the KdV equation, generalized regularized long-wave (GRLW) equation [3, 4], Rosenau–Burgers (R–B) equation [5, 6], and Benjamin–Bona–Mahony (BBM) equation [7] can better simulate shallow water waves. Among them, the BBM equation, which describes the unidirectional propagation of surface water in a nonlinear dispersive medium with small amplitude and long wave, is the most typical model [8–13].

Generally, it is difficult to obtain the analytic solution of the aforementioned water wave equation. Therefore, the investigation of using numerical methods to simulate the solutions of the classical water wave equation and the regularized water wave equation has been the subject of many studies in the recent literature (see, for example, [14–20] and the references therein). More recently, Zhang and Xu [10] constructed two linearized schemes for the time fractional water wave equation in a BBM form. Li [21] discussed the linearized difference scheme for a BBM equation with a space fractional nonlocal viscous term.

When simulating the nonlinear differential equation, it is effective to construct the linear difference scheme, which reduces the complexity in computing and the difficulties in the analysis of convergence. Zhang and Xu [10] used the linearized difference method to solve the nonlinear time fractional water wave equation in a BBM form. However, the different method that is constructed above is a three-layer linear scheme, which does not minimize the amount of calculation. Inspired by the work [21], we find that the Crank–Nicolson method can be combined with the technique of linearization for a nonlinear equation to establish a two-level linear difference scheme, which can further reduce the computational complexity while maintaining the accuracy. Consequently, in this paper, we consider a Crank–Nicolson linear difference scheme for the initial and boundary value problems of the following BBM equation with a time fractional nonlocal viscous term [12]:

$$u_t + u_x - \beta u_{xxt} + \epsilon \frac{1}{2} {}^C D_t^{\frac{1}{2}} u + \gamma uu_x - \alpha u_{xx} = 0, \quad x \in (a, b), t \in (0, T], \tag{2}$$

and an initial condition

$$u(x, 0) = u_0(x), \quad x \in (a, b), \tag{3}$$

and the boundary conditions

$$u(a, t) = u(b, t) = 0, \quad t \in (0, T], \tag{4}$$

where  $\beta, \epsilon, \gamma$ , and  $\alpha$  are positive parameters,  ${}^C D_t^{\frac{1}{2}}$  is  $\frac{1}{2}$ -order Caputo’s fractional derivative operator. Ordinarily, the  $\alpha$ -order Caputo time fractional derivative  ${}^C D_t^\alpha u(x, t)$  is defined as follows [22]:

$${}^C D_t^\alpha u(x, t) := \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} \frac{\partial^m u(x,s)}{\partial s^m} ds, & m-1 < \alpha < m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \alpha = m \in N, \end{cases} \tag{5}$$

where  $\Gamma(\cdot)$  is a gamma function.

In this paper, we use the  $L1$  formula to approximate  $\frac{1}{2}$ -order Caputo’s fractional derivative; we use the central difference to approximate both the first and second space derivative; we use the linear difference to approximate the nonlinear term; and we use the Crank–Nicolson difference formula to approximate the first-order time derivatives. Then, we construct a Crank–Nicolson- $L1$  (C-N- $L1$ ) linear difference scheme for problems (2)–(4). The stability and convergence of the C-N- $L1$  numerical scheme are rigorously derived. Numerical experiments show that the theoretical analysis is accurate and that the numerical scheme is effective. The contribution of this paper is to propose, analyze, and verify the combination scheme of a Crank–Nicolson scheme and linearization of nonlinear term.

The outline of this paper is as follows. Section 2 is devoted to the design of the C-N- $L1$  linear difference scheme for the time fractional nonlinear BBM equation. Sections 3 and 4 prove that the scheme is stable and convergent, respectively. Two numerical examples are presented to verify the theoretical result and the effectiveness of the numerical scheme and to demonstrate the impact of various terms by changing the parameters of the corresponding terms in Sect. 5. Finally, the conclusions of this paper are provided.

## 2 The derivation of the C-N-L1 linear difference scheme

In this section, we describe the derivation of the C-N-L1 linear difference scheme for the time fractional BBM equation (2).

We consider the difference scheme for equation (2) in a finite domain  $(x, t) \in [a, b] \times [0, T]$ . Spatial step size  $h = \frac{b-a}{M}$  and time step size  $\tau = \frac{T}{N}$ , respectively, where  $M$  and  $N$  are integers and assume that the mesh sizes  $h$  and  $\tau$  satisfy  $0 < h < 1$  and  $0 < \tau < 1$ . Grid points  $(x_j, t_n)$  are defined by  $x_j = a + jh$  ( $0 \leq j \leq M$ );  $t_n = n\tau$  ( $0 \leq n \leq N$ ). Let  $u_j^n$  denote the numerical approximation of  $u(x_j, t_n)$ . Sometimes, suppose that  $u^n = \{u_j^n | 0 \leq j \leq M\}$  is a grid function. We introduce the following notations and an L1 approximation formula of the  $\frac{1}{2}$ -order Caputo's fractional derivative [23]:

$$\begin{aligned} u_j^{n+\frac{1}{2}} &= \frac{u_j^n + u_j^{n+1}}{2}, & D_t u_j^n &= \frac{u_j^{n+1} - u_j^n}{\tau}, & D_+ u_j^n &= \frac{u_{j+1}^n - u_j^n}{h}, \\ D_- u_j^n &= \frac{u_j^n - u_{j-1}^n}{h}, & D^2 u_j^n &= D_+ D_- u_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}, \\ D_0 u_j^n &= \frac{u_{j+1}^n - u_{j-1}^n}{2h}, & (u^n, v^n) &= \sum_{j=0}^{M-1} h u_j^n v_j^n, & \|u^n\| &= \sqrt{(u^n, u^n)}, \\ \|u^n\|_\infty &= \max_{0 \leq j \leq M-1} |u_j^n|, & (u, v)_1 &= (u, v) + \beta(D_+ u, D_+ v), \end{aligned}$$

and

$$\frac{1}{\Gamma(\frac{1}{2})} \int_0^{t_n} \frac{u'(s)}{(t_n - s)^{\frac{1}{2}}} ds := \frac{\tau^{-\frac{1}{2}}}{\Gamma(\frac{3}{2})} \left( u(t_n) - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k})u(t_k) - a_{n-1}u(t_0) \right),$$

where  $a_j = (j + 1)^{1-\alpha} - j^{1-\alpha}$  and  $j \geq 0$ .

We use the L1 formula to approximate the Caputo fractional derivatives, use the central differences to approximate for both first and second space derivatives, use the linear differences to approximate the nonlinear term and use the Crank–Nicolson difference formula to approximate the first-order time derivative. Then, we get the C-N-L1 scheme for problems (2)–(4):

$$\begin{aligned} D_t u_j^n + D_0 u_j^{n+\frac{1}{2}} - \beta D_t D^2 u_j^n + \frac{\epsilon^{\frac{1}{2}} \tau^{-\frac{1}{2}}}{\Gamma(\frac{3}{2})} \left( u_j^{n+\frac{1}{2}} - \sum_{i=0}^{n-1} (a_i - a_{i+1}) u_j^{n-\frac{1}{2}-i} - a_n u_j^0 \right) \\ + \frac{\gamma}{3} (u_j^n D_0 u_j^{n+\frac{1}{2}} + D_0 (u_j^n u_j^{n+\frac{1}{2}})) - \alpha D^2 u_j^{n+\frac{1}{2}} = 0, \end{aligned} \tag{6}$$

$$u_j^0 = u_0(x_j), \quad 0 \leq j \leq M, \tag{7}$$

$$u(a, t) = u(b, t) = 0, \quad 0 \leq n \leq N. \tag{8}$$

In order to prove the stability and convergence of the above constructed C-N-L1 scheme, we collect useful facts on the truncation error of the L1 approximate operator for the Caputo fractional derivatives, the discrete Sobolev inequality, and the discrete Gronwall inequality.

**Lemma 2.1** ([24]) *Suppose  $f(t) \in C^2[0, t_n]$ . Let*

$$\begin{aligned} \bar{R}(f(t_n)) := & \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{f'(s)}{(t_n-s)^\alpha} ds \\ & - \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left( a_0 f(t_n) - \sum_{j=1}^{n-1} (a_{n-j-1} - a_{n-j}) f(t_j) - a_{n-1} f(t_0) \right), \end{aligned} \tag{9}$$

then

$$|\bar{R}(f(t_n))| \leq \frac{1}{\Gamma(2-\alpha)} \left[ \frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1+2^{-\alpha}) \right] \max_{0 \leq t \leq t_n} |f''(t)| \tau^{2-\alpha}, \tag{10}$$

where  $0 < \alpha \leq 1$  and  $a_j = (j+1)^{1-\alpha} - j^{1-\alpha}$ ,  $j \geq 0$ .

**Lemma 2.2** ([24]) *Let  $0 < \alpha < 1$ ,  $a_j = (j+1)^{1-\alpha} - j^{1-\alpha}$ ,  $j = 0, 1, 2, \dots$ . Then*

- (1)  $1 = a_0 > a_1 > \dots > a_k \rightarrow 0$ , as  $k \rightarrow +\infty$ ;
- (2)  $a_{n-1} > \frac{1-\alpha}{j^2}$ .

**Lemma 2.3** (Discrete Sobolev’s inequality [19]) *There exist constants  $C_1$  and  $C_2$  such that*

$$\|u^n\|_\infty \leq C_1 \|u^n\| + C_2 \|D_+ u^n\|. \tag{11}$$

**Lemma 2.4** (Discrete Gronwall’s inequality [21]) *Suppose that  $\theta(k)$ ,  $\rho(k)$  are nonnegative mesh functions and  $\rho(k)$  is nondecreasing. For  $C > 0$ , if*

$$\theta(k) \leq \rho(k) + C\tau \sum_{l=0}^{k-1} \theta(l), \quad \forall k,$$

then

$$\theta(k) \leq \rho(k) e^{C\tau k}, \quad \forall k.$$

### 3 Stability

In this section, we study the stability of the proposed C-N-L1 scheme (6).

**Theorem 3.1** *The difference scheme (6) is unconditionally stable, and it holds*

$$\|u^n\|_1 \leq C, \quad \|u^n\|_\infty \leq C, \quad n = 0, 1, \dots, N.$$

*Proof* For  $n = 0, 1, \dots, N$ , computing the inner product of (6) with  $2u^{n+\frac{1}{2}}$  (i.e.,  $u^{n+1} + u^n$ ), we obtain

$$\begin{aligned} & \frac{1}{\tau} (\|u^{n+1}\|^2 - \|u^n\|^2) + (D_0 u^{n+\frac{1}{2}}, 2u^{n+\frac{1}{2}}) - \beta (D_t D^2 u^n, 2u^{n+\frac{1}{2}}) \\ & + \gamma (Z, 2u^{n+\frac{1}{2}}) - \mu \left( u^{n+\frac{1}{2}} - \sum_{i=0}^{n-1} (a_i - a_{i+1}) u_j^{n-\frac{1}{2}-i} - a_n u_j^0, 2u^{n+\frac{1}{2}} \right) \\ & - \alpha (D^2 u^{n+\frac{1}{2}}, 2u^{n+\frac{1}{2}}) = 0, \end{aligned} \tag{12}$$

where  $Z = \frac{1}{3}(u^n D_0 u^{n+\frac{1}{2}} + D_0(u^n u^{n+\frac{1}{2}}))$ ,  $\mu = \frac{\frac{1}{2}\tau^{-\frac{1}{2}}}{\Gamma(\frac{3}{2})}$ .

For the second and fourth terms in (12), we use the definition of inner product and note that [20]

$$(D_0 u^{n+\frac{1}{2}}, 2u^{n+\frac{1}{2}}) = \sum_{j=1}^M [u_{j+1}^{n+\frac{1}{2}} u_j^{n+\frac{1}{2}} - u_{j-1}^{n+\frac{1}{2}} u_j^{n+\frac{1}{2}}] = 0, \tag{13}$$

and

$$\begin{aligned} (Z, 2u^{n+\frac{1}{2}}) &= \frac{2h}{3} \sum_{j=1}^M (u_j^n D_0 u_j^{n+\frac{1}{2}} + D_0(u^n u_j^{n+\frac{1}{2}})) u_j^{n+\frac{1}{2}} \\ &= \frac{1}{3} \sum_{j=1}^M (u_j^n (u_{j+1}^{n+\frac{1}{2}} - u_{j-1}^{n+\frac{1}{2}}) + u_{j+1}^n u_{j+1}^{n+\frac{1}{2}} - u_{j-1}^n u_{j-1}^{n+\frac{1}{2}}) u_j^{n+\frac{1}{2}} \\ &= \frac{1}{3} \sum_{j=1}^M [(u_j^n u_{j+1}^{n+\frac{1}{2}} u_j^{n+\frac{1}{2}} + u_{j+1}^n u_{j+1}^{n+\frac{1}{2}} u_j^{n+\frac{1}{2}}) - (u_j^n u_{j-1}^{n+\frac{1}{2}} u_j^{n+\frac{1}{2}} + u_{j-1}^n u_{j-1}^{n+\frac{1}{2}} u_j^{n+\frac{1}{2}})] \\ &= 0. \end{aligned} \tag{14}$$

From (13) and (14), (12) can be rewritten as

$$\begin{aligned} &\frac{1}{\tau} (\|u^{n+1}\|^2 - \|u^n\|^2) + \frac{\beta}{\tau} (\|D_+ u^{n+1}\|^2 - \|D_+ u^n\|^2) + 2\alpha \|D_+ u^{n+\frac{1}{2}}\|^2 + 2\mu \|u^{n+\frac{1}{2}}\|^2 \\ &= 2\mu \sum_{i=0}^{n-1} (a_i - a_{i+1}) (u^{n-\frac{1}{2}-i}, u^{n+\frac{1}{2}}) + 2\mu a_n (u^0, u^{n+\frac{1}{2}}) \\ &\leq \mu \sum_{i=0}^{n-1} (a_i - a_{i+1}) (\|u^{n-\frac{1}{2}-i}\|^2 + \|u^{n+\frac{1}{2}}\|^2) + \mu a_n (\|u^0\|^2 + \|u^{n+\frac{1}{2}}\|^2) \\ &= \mu \sum_{i=0}^{n-1} (a_i - a_{i+1}) \|u^{n-\frac{1}{2}-i}\|^2 + \mu ((1 - a_n) + a_n) \|u^{n+\frac{1}{2}}\|^2 + \mu a_n \|u^0\|^2. \end{aligned} \tag{15}$$

Multiplying (15) by  $\tau$ , we have

$$\begin{aligned} &\|u^{n+1}\|_1^2 + \tau \mu \sum_{i=0}^n a_{i+1} \|u^{n-\frac{1}{2}-i}\|^2 \\ &\leq \|u^n\|_1^2 + \tau \mu \sum_{i=0}^{n-1} a_{i+1} \|u^{n-\frac{1}{2}-i}\|^2 + \tau \mu a_n \|u^0\|^2. \end{aligned} \tag{16}$$

Let

$$B^{n+1} = \|u^{n+1}\|_1^2 + \tau \mu \sum_{i=0}^n a_{i+1} \|u^{n-\frac{1}{2}-i}\|^2, \tag{17}$$

(16) can be rewritten as follows:

$$B^{n+1} \leq B^n + \mu a_n \|u^0\|^2. \tag{18}$$

It follows that

$$B^{n+1} \leq B^0 + \mu\sqrt{n+1}\|u^0\|^2. \tag{19}$$

Then we get

$$\|u^n\|_1 \leq C. \tag{20}$$

And using Lemma 2.3, we have

$$\|u^n\|_\infty \leq C. \tag{21}$$

This completes the proof. □

### 4 Convergence

In this section, the rate of convergence of scheme (6) is guaranteed and explicitly proved. Firstly, we consider the truncation error of the difference scheme (6). Let  $v(x, t)$  be the solution of problems (2)–(4),  $v_j^n = v(x_j, t_n)$ , then the truncation error of scheme (6) is

$$\begin{aligned} r_j^n &= D_t v_j^n + D_0 v_j^{n+\frac{1}{2}} - \beta D_t D^2 v_j^n + \mu \left( v_j^{n+\frac{1}{2}} - \sum_{i=0}^{n-1} (a_i - a_{i+1}) v_j^{n-\frac{1}{2}-i} - a_n v_j^0 \right) \\ &\quad + \frac{\gamma}{3} (v_j^n D_0 v_j^{n+\frac{1}{2}} + D_0 (v_j^n v_j^{n+\frac{1}{2}})) - \alpha D^2 v_j^{n+\frac{1}{2}}, \end{aligned} \tag{22}$$

where  $\mu = \frac{\epsilon^{\frac{1}{2}} \tau^{-\frac{1}{2}}}{\Gamma(\frac{3}{2})}$ .

According to Taylor’s expansion,

$$\begin{aligned} &D_t v_j^n + D_0 v_j^{n+\frac{1}{2}} - \beta D_t D^2 v_j^n \\ &+ \left[ \frac{\mu}{2} \left( (v_j^{n+1} + v_j^n) - \sum_{i=0}^{n-1} (a_i - a_{i+1}) (v_j^{n-i} + v_j^{n-1-i}) - a_n v_j^0 \right) + O(\tau^{\frac{3}{2}}) \right] \\ &+ \left\{ \frac{\gamma}{12h} [u_j^n (u_{j+1}^{n+1} - u_{j-1}^{n+1} + u_{j+1}^n - u_{j-1}^n) \right. \\ &\quad \left. + (u_{j+1}^n u_{j+1}^{n+1} - u_{j-1}^n u_{j-1}^{n+1} + u_{j+1}^n u_{j+1}^n - u_{j-1}^n u_{j-1}^n)] + O(\tau^2 + h^2) \right\} \\ &- \alpha D^2 v_j^{n+\frac{1}{2}} = 0, \end{aligned} \tag{23}$$

it can be easily obtained that  $r_j^n = O(\tau^{\frac{3}{2}} + h^2)$  holds if  $h, \tau \rightarrow 0$ .

**Theorem 4.1** *Suppose  $u_0 \in H_0^1[0, L]$ , then the solution  $u_j^n$  of (6) converges to the solution  $v_j^n$  of (2) in norm  $\|\cdot\|_\infty$  and the rate of convergence is  $O(\tau^{\frac{3}{2}} + h^2)$ .*

*Proof* Let  $e_j^n = v_j^n - u_j^n$ , subtracting (6) from (22), we have

$$r_j^n = \frac{e_j^{n+1} - e_j^n}{\tau} + D_0 e_j^{n+\frac{1}{2}} - \beta D_t D^2 e_j^n$$

$$\begin{aligned}
 & + \mu \left( e_j^{n+\frac{1}{2}} - \sum_{i=0}^{n-1} (a_i - a_{i+1}) e_j^{n-\frac{1}{2}-i} - a_n e_j^0 \right) \\
 & + \frac{\gamma}{3} (v_j^n D_0 v_j^{n+\frac{1}{2}} - u_j^n D_0 u_j^{n+\frac{1}{2}}) \\
 & + \frac{\gamma}{3} (D_0 (v_j^n v_j^{n+\frac{1}{2}}) - D_0 (u_j^n u_j^{n+\frac{1}{2}})) - \alpha D^2 e_j^{n+\frac{1}{2}}.
 \end{aligned} \tag{24}$$

For convenience, we denote

$$P = \frac{\gamma}{3} (v_j^n D_0 v_j^{n+\frac{1}{2}} - u_j^n D_0 u_j^{n+\frac{1}{2}}), \quad Q = \frac{\gamma}{3} (D_0 (v_j^n v_j^{n+\frac{1}{2}}) - D_0 (u_j^n u_j^{n+\frac{1}{2}})).$$

Multiplying by  $2e^{n+\frac{1}{2}} (i.e. e_j^{n+1} + e_j^n)$  on both sides of (24), we have

$$\begin{aligned}
 & \|e^{n+1}\|^2 - \|e^n\|^2 + \beta \|D_+ e^{n+1}\|^2 - \beta \|D_+ e^n\|^2 \\
 & = \tau \left[ -(D_0 e^{n+\frac{1}{2}}, 2e^{n+\frac{1}{2}}) - (P, 2e^{n+\frac{1}{2}}) - (Q, 2e^{n+\frac{1}{2}}) + (r^n, 2e^{n+\frac{1}{2}}) \right. \\
 & \quad \left. - \mu \left( 2\|e^{n+\frac{1}{2}}\|^2 - \sum_{i=0}^{n-1} (a_i - a_{i+1}) (e^{n-\frac{1}{2}-i}, 2e^{n+\frac{1}{2}}) - a_n (e^0, 2e^{n+\frac{1}{2}}) \right) \right. \\
 & \quad \left. + \alpha (D^2 e^{n+\frac{1}{2}}, 2e^{n+\frac{1}{2}}) \right].
 \end{aligned} \tag{25}$$

Now, we estimate the third and fourth terms of the right-hand side of (25), respectively [18, 20]:

$$\begin{aligned}
 (P, 2e^{n+\frac{1}{2}}) & = \frac{2h}{3} \sum_{j=1}^M (v_j^n D_0 v_j^{n+\frac{1}{2}} - u_j^n D_0 u_j^{n+\frac{1}{2}}) e_j^{n+\frac{1}{2}} \\
 & = \frac{2h}{3} \sum_{j=1}^M (v_j^n D_0 e_j^{n+\frac{1}{2}}) e_j^{n+\frac{1}{2}} + \frac{2h}{3} \sum_{j=1}^M (v_j^n D_0 u_j^{n+\frac{1}{2}} - u_j^n D_0 u_j^{n+\frac{1}{2}}) e_j^{n+\frac{1}{2}} \\
 & \leq Ch \sum_{j=1}^M (|D_0 e_j^{n+\frac{1}{2}}| + |e_j^n|) e_j^{n+\frac{1}{2}} \\
 & \leq Ch (\|D_+ e^{n+1}\|^2 + \|D_+ e^n\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2)
 \end{aligned} \tag{26}$$

and

$$\begin{aligned}
 (Q, 2e^{n+\frac{1}{2}}) & = \frac{2h}{3} \sum_{j=1}^M [D_0 (v_j^n v_j^{n+\frac{1}{2}}) - D_0 (u_j^n u_j^{n+\frac{1}{2}})] e_j^{n+\frac{1}{2}} \\
 & = -\frac{2h}{3} \sum_{j=1}^M (v_j^n e_j^{n+\frac{1}{2}}) D_0 e_j^{n+\frac{1}{2}} + \frac{2h}{3} \sum_{j=1}^M (v_j^n - u_j^n) u_j^{n+\frac{1}{2}} D_0 e_j^{n+\frac{1}{2}} \\
 & = -\frac{2h}{3} \sum_{j=1}^M (v_j^n e_j^{n+\frac{1}{2}}) D_0 e_j^{n+\frac{1}{2}} - \frac{2h}{3} \sum_{j=1}^M e_j^n u_j^{n+\frac{1}{2}} D_0 e_j^{n+\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 &\leq Ch \sum_{j=1}^M (|e_j^{n+\frac{1}{2}}| + |e_j^n|) D_0 e_j^{n+\frac{1}{2}} \\
 &\leq Ch (\|D_+ e^{n+1}\|^2 + \|D_+ e^n\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2).
 \end{aligned}
 \tag{27}$$

For the second term on the right-hand side of (25), we have

$$\begin{aligned}
 &\sum_{i=0}^{n-1} (a_{n-i-1} - a_{n-i})(e^{i+\frac{1}{2}}, 2e^{n+\frac{1}{2}}) + a_n(e^0, 2e^{n+\frac{1}{2}}) \\
 &\leq \sum_{i=0}^{n-1} (a_{n-i-1} - a_{n-i})(\|e^{i+\frac{1}{2}}\|^2 + \|e^{n+\frac{1}{2}}\|^2) + a_n(\|e^0\|^2 + \|e^{n+\frac{1}{2}}\|^2) \\
 &= \sum_{i=0}^{n-1} (a_{n-i-1} - a_{n-i})\|e^{i+\frac{1}{2}}\|^2 + ((1 - a_n) + a_n)\|e^{n+\frac{1}{2}}\|^2 + a_n\|e^0\|^2 \\
 &= \sum_{i=0}^{n-1} (a_{n-i-1} - a_{n-i})\|e^{i+\frac{1}{2}}\|^2 + \|e^{n+\frac{1}{2}}\|^2 + a_n\|e^0\|^2.
 \end{aligned}
 \tag{28}$$

Using Lemma 2.4, it is obvious that

$$\alpha(D_+ D_- e^{n+\frac{1}{2}}, 2e^{n+\frac{1}{2}}) = -2\alpha\|D_+ e^{n+\frac{1}{2}}\|^2
 \tag{29}$$

and

$$(r^n, 2e^{n+\frac{1}{2}}) \leq \|r^n\|^2 + \frac{1}{2}(\|e^{n+1}\|^2 + \|e^n\|^2).
 \tag{30}$$

Substituting (26)–(30) into (25) and using the fact that  $\|e^0\| = 0$ , we get

$$\begin{aligned}
 &\|e^{n+1}\|^2 - \|e^n\|^2 + \beta\|D_+ e^{n+1}\|^2 - \beta\|D_+ e^n\|^2 + 2\tau\mu\|e^{n+\frac{1}{2}}\|^2 \\
 &\leq \tau C(\|D_+ e^{n+1}\|^2 + \|D_+ e^n\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2) \\
 &\quad + \tau\mu \left[ \sum_{i=0}^{n-1} (a_{n-i-1} - a_{n-i})\|e^{i+\frac{1}{2}}\|^2 + \|e^{n+\frac{1}{2}}\|^2 \right] \\
 &\quad + \tau\|r^n\|^2 + \frac{1}{2}\tau(\|e^{n+1}\|^2 + \|e^n\|^2), \\
 &\leq \tau C(\|D_+ e^{n+1}\|^2 + \|D_+ e^n\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2) \\
 &\quad + \tau\mu \left[ \sum_{i=0}^{n-1} (\|e^i\|^2 + \|e^{i+1}\|^2) + \|e^{n+\frac{1}{2}}\|^2 \right] \\
 &\quad + \tau\|r^n\|^2 + \frac{1}{2}\tau(\|e^{n+1}\|^2 + \|e^n\|^2).
 \end{aligned}
 \tag{31}$$

Using Lemma 2.4, we have

$$(1 - \tau C)(\|e^{n+1}\|_1^2 - \|e^n\|_1^2) \leq 2\tau C\|e^n\|_1^2 + \tau\|r^n\|^2,
 \tag{32}$$



where  $\|e^n\|_1^2 = \|e^n\|^2 + \beta \|D_+ e^n\|^2$ . Moreover, let  $\tau$  be suitably small, then

$$\|e^{n+1}\|_1^2 - \|e^n\|_1^2 \leq C\tau \|e^n\|_1^2 + C\tau \|r^n\|^2. \tag{33}$$

Summing up (33) from 0 to  $n$ , we get

$$\|e^{n+1}\|_1^2 \leq \|e^0\|_1^2 + C\tau \sum_{k=1}^n \|e^k\|_1^2 + C\tau \sum_{k=1}^n \|r^k\|^2. \tag{34}$$

Using Lemma 2.4 and the boundedness of  $r^k$ , we have

$$\|e^{n+1}\|_1^2 \leq O(\tau^{\frac{3}{2}} + h^2)^2. \tag{35}$$

By Lemma 2.3 and (35), we obtain

$$\|e^n\| \leq O(\tau^{\frac{3}{2}} + h^2), \quad \|D_+ e^n\| \leq O(\tau^{\frac{3}{2}} + h^2), \quad \|e^n\|_\infty \leq O(\tau^{\frac{3}{2}} + h^2). \tag{36}$$

This completes the proof of Theorem 4.1. □

### 5 Numerical experiments

In this section, we provide two numerical experiments to show the performance of the proposed numerical scheme (6).

*Example 1* We check the accuracy and convergence order of the numerical scheme (6). Since we do not obtain the exact solution of (2), we consider the nonhomogeneous problem

$$\begin{aligned} u_t + u_x - \beta u_{xxt} + \epsilon \frac{1}{2} {}^C D_t^{\frac{1}{2}} u + \gamma uu_x - \alpha u_{xx} &= s(x, t), \quad x \in (0, 1), t \in (0, 1], \\ u(x, 0) &= u_0(x), \quad x \in (0, 1), \\ u(0, t) = u(1, t) &= 0, \quad t \in (0, 1], \end{aligned} \tag{37}$$

where the source term  $s(x, t)$  is determined by the exact  $u(x, t) = x^4(x - 1)t^{\frac{3}{2}}$ , which does not belong to  $C^2[0, T]$ .

In Tables 1–3, we list the error and convergence order for scheme (6) with the same parameters.

In Table 3, we can observe the convergence order is  $\frac{3}{2}$  for  $\epsilon = 1$  and 2 for  $\epsilon = 0$ , respectively. That is a good agreement with the theoretical prediction of Theorem 4.1.

**Table 1** Error and convergence order in time when  $h = 1/1000, \epsilon = \alpha = \beta = \gamma = 1$

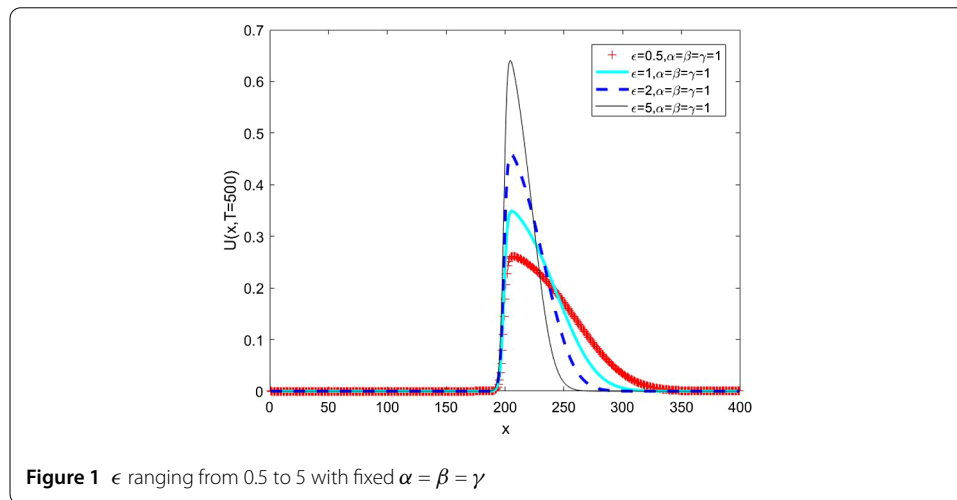
$\tau$	$\ e\ $	Order	$\ e\ _\infty$	Order
1/10	4.953935e-04	–	9.350106e-04	–
1/20	1.595958e-04	1.634152	3.029161e-04	1.626064
1/40	5.380815e-05	1.568526	1.026424e-04	1.561290
1/80	1.847254e-05	1.542441	3.545068e-05	1.533742
1/160	6.326795e-06	1.545836	1.225924e-05	1.531942

**Table 2** Error and convergence order in space when  $\tau = 1/1000, \epsilon = \alpha = \beta = \gamma = 1$

$h$	$\ e\ $	Order	$\ e\ _\infty$	Order
1/10	1.841489e-03	–	2.653046e-03	–
1/20	5.047981e-04	1.867094	7.279309e-04	1.865776
1/40	1.321991e-04	1.932994	1.905037e-04	1.933982
1/80	3.365747e-05	1.973713	4.854140e-05	1.972531
1/160	8.309556e-06	2.018083	1.199653e-05	2.016598

**Table 3** Error and convergence order in time and space when  $\epsilon = 0, \alpha = \beta = \gamma = 1$

$\tau = h$	$\ e\ $	Order	$\ e\ _\infty$	Order
1/10	1.588457e-03	–	2.350020e-03	–
1/20	4.183324e-04	1.924904	6.274669e-04	1.905061
1/40	1.030145e-04	2.021802	1.585501e-04	1.984600
1/80	2.440490e-05	2.077604	3.894461e-05	2.025443
1/160	5.878778e-06	2.053582	9.517392e-06	2.032785



*Example 2* We consider the following problem:

$$u_t + u_x - \beta u_{xxt} + \epsilon \frac{1}{2} {}^C D_t^{\frac{1}{2}} u + \gamma uu_x - \alpha u_{xx} = 0, \quad x \in (0, 400), t \in (0, T), \tag{38}$$

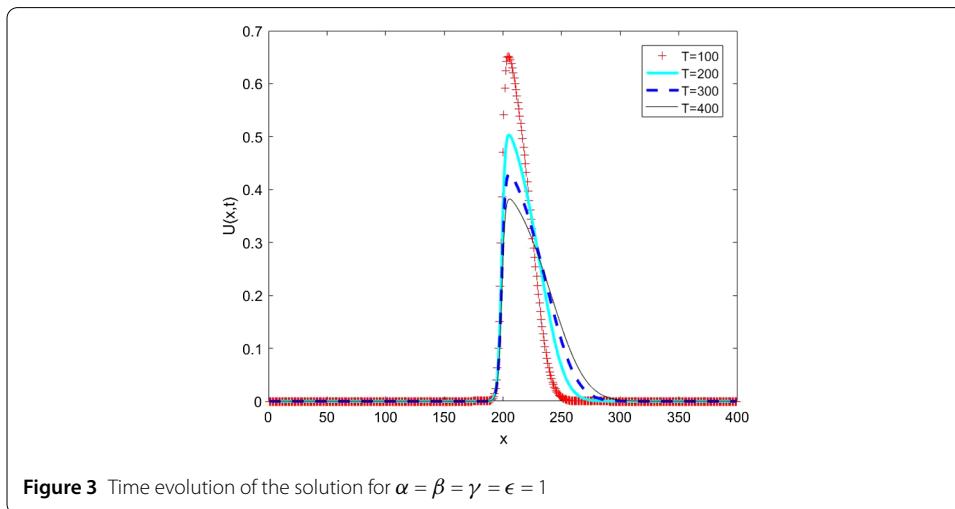
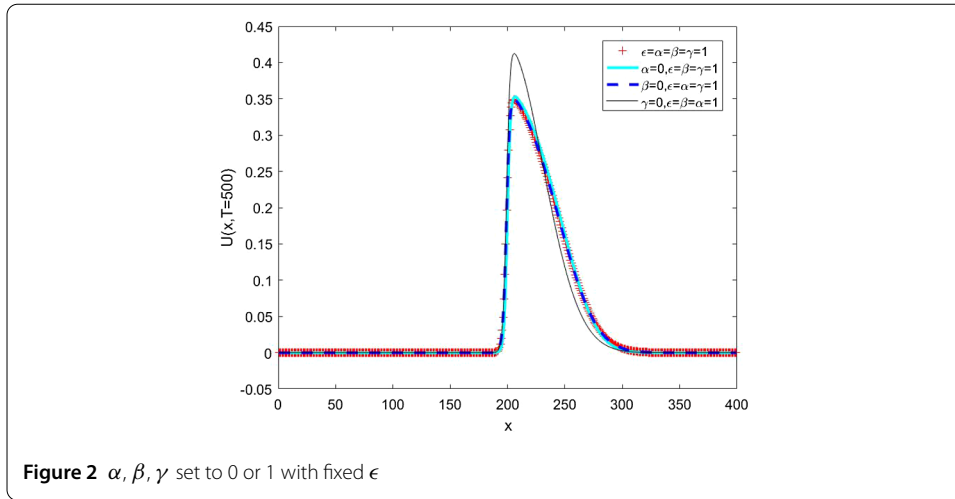
$$u(x, 0) = 3 \operatorname{sech}^2(\sqrt{0.5^3}(x - x_0)), \tag{39}$$

where  $x_0$  is the middle point of the interval  $(0, 400)$  [10]. Since the initial value converges to 0 exponentially as  $|x| \rightarrow +\infty$ , we can approximate the initial value problem (38) by an initial-boundary value for  $x \in (0, 400)$  as long as the solution does not reach the boundaries. The numerical simulations are plotted in Figs. 1–3.

In Fig. 1, we plot the solutions for several different values of  $\epsilon$  with  $\alpha, \beta$ , and  $\gamma$  fixed, and  $M = 500, \tau = 1$ . We observe that the nonlocal viscous term damps the wave down significantly.

Figure 2 shows the solutions for  $\alpha, \beta, \gamma$  set to 0 or 1 with fixed  $\epsilon$ , and  $M = 500, \tau = 1$ . We can observe that the parameters  $\alpha$  and  $\beta$  have no significant impact on the wave evolution, whereas the parameter  $\gamma$  plays an important role in wave shape.

Figure 3 shows the wave evolution in different time with  $\alpha = \beta = \gamma = \epsilon = 1$  and  $\tau = 1$ .



### 6 Conclusions

In this paper, we propose a C-N-L1 linearized difference scheme for a nonlocal viscous wave equation in a BBM form. The convergence and stability of the scheme are discussed in detail. Numerical experiments show that our scheme is efficient and accurate. In addition, we also verify the influence of different parameters such as the diffusion term, nonlocal viscous term, dispersion term, and nonlinear term. We would extend the scope of our work in the future by investigating the finite element Galerkin scheme and compact finite difference scheme for the nonlocal viscous wave equation in a BBM form.

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#### Competing interests

The authors declare that they have no competing interests.

**Authors' contributions**

The authors read and approved the final manuscript.

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