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Equivalence of the mean square stability between the partially truncated Euler–Maruyama method and stochastic differential equations with super-linear growing coefficients

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Abstract

For stochastic differential equations (SDEs) whose drift and diffusion coefficients can grow super-linearly, the equivalence of the asymptotic mean square stability between the underlying SDEs and the partially truncated Euler–Maruyama method is studied. Using the finite time convergence as a bridge, a twofold result is proved. More precisely, the mean square stability of the SDEs implies that of the partially truncated Euler–Maruyama method, and the mean square stability of the partially truncated Euler–Maruyama method indicates that of the SDEs given the step size is carefully chosen.

Keywords: Partially truncated Euler–Maruyama method; Stochastic differential equations; Mean square exponential stability; Super-linear growing coefficients

1 Introduction

In [9], the authors initialed the study on the equivalence of the stability between the underlying equations and their numerical methods for stochastic differential equations (SDEs). More precisely, for the SDE of the Itô type

$$dy(t) = \mu\big(y(t)\big)\,dt + \sigma\big(y(t)\big)\,dB(t)$$

with the initial value $y(0) = y_0$ satisfying $\mathbb{E}|y_0|^2 < \infty$. One says that the solution is mean square stable if there exist positive constants K_s and λ_s such that

$$\mathbb{E}\big|y(t)\big|^2 \le K_s\mathbb{E}\big|y(0)\big|^2 e^{-\lambda_s t}. \tag{1.1}$$

Denote some numerical approximations to $y(t)$ by $x_\Delta(t)$ with $x_\Delta(0) = y(0)$. One claims that the numerical solution is mean square stable if there exist positive constants K_n and λ_n such that

$$\mathbb{E}\big|x_\Delta(t)\big|^2 \le K_n\mathbb{E}\big|x_\Delta(0)\big|^2 e^{-\lambda_n t}. \tag{1.2}$$

The authors in [9] proved a very general "if and only if" theorem between (1.1) and (1.2). The theorem states in twofold that

(A) the mean square stability of the SDE implies the mean square stability of the numerical method,

(B) the mean square stability of the numerical method implies the mean square stability of the SDE if the step size Δ of the numerical method is carefully chosen.

By saying the theorem is very general, one means that conditions required are the second moment boundedness of the numerical method and the finite time convergence of the numerical method to the SDE (see the details in Sect. 3), where the structure of the numerical method is not needed to be specified.

Since then, many works have been devoted to the study on the equivalence of the mean square stability between different types of SDEs and their numerical methods. The author in [18] investigated the stochastic differential delay equations (SDDEs) and the Euler–Maruyama (EM) method. The SDDEs with Poisson jump and Markov switching and the semi-implicit Euler method are studied in [28]. The authors in [17] analyze the neutral delayed stochastic differential equations and the EM method.

It should be pointed out that although the very general results are obtained in the papers above for different types of SDEs and numerical methods, the global Lipschitz condition is always imposed on the drift and diffusion coefficients when the theorems are applied to some specified numerical methods. One may notice that SDEs with the coefficients not obeying the global Lipschitz condition have been extensively applied more recently in many areas such as biology, finance, and epidemiology [1, 4, 19, 23]. Therefore, the first motivation of this paper is as follows.

(M1) For some SDEs whose coefficients do not satisfy the global Lipschitz condition, can we find some numerical method that shares the mean square stability with the underlying SDEs, i.e., both (A) and (B) hold?

Actually, many interesting works have been devoted to (A), i.e., given the SDE is stable under certain conditions, some numerical method can reproduce such a stability. We just mention some of the works here [2, 3, 7, 10, 11, 14, 22, 26, 27] and refer the readers to the references therein. It is not hard to observe that when the coefficients of SDE do not satisfy the global Lipschitz condition, the classical EM method is always abandoned and some implicit methods are adapted as alternatives. This phenomenon is explained in [12], where the authors proved that the classical EM diverges from the SDE if either drift or diffusion coefficient grows super-linearly.

However, due to the advantages such as simple structure and less computational cost in each iteration (not like implicit methods in which some non-linear equation system needs to be solved in each iteration) [8], the explicit Euler-type methods are still attracting lots of attention. Therefore, in the past several years some modified explicit Euler methods, such as the tamed Euler method [13, 24, 25, 29] and the truncated EM method [6, 15, 16, 20, 21], have been developed. The bloom of explicit methods brings the second motivation of this paper as follows.

(M2) Can we use some explicit method to answer (M1), i.e., can we find some explicit methods for some SDEs with super-linear growing coefficients that shares the mean square stability with the underlying SDEs?

Bearing (M1) and (M2) in mind, in this paper we investigate the partially truncated EM method to see if it could share the mean square stability with the SDEs when both the drift and diffusion coefficients can grow super-linear. By using the finite time convergence as the bridge, we prove the "if and only if" theorem for the partially truncated EM method. More precisely, we prove that

(a) the mean square stability of the SDE implies the mean square stability of the partially truncated EM method,

(b) the mean square stability of the partially truncated EM method implies the mean square stability of the SDE if the step size Δ of the numerical method is carefully chosen.

To our best knowledge, few works have dealt with (B) using explicit methods for SDEs with super-linear coefficients, though some works have studied (A) using explicit methods [6, 29]. As pointed out in [9], sometimes (B) is more interesting and important since if (B) is true for some numerical method, then by carefully conducting the numerical simulation one can know whether the SDE is stable or not without using the Lyapunov technique.

This paper is constructed as follows. In Sect. 2, a general theorem that guarantees the equivalence of the stability between the SDEs and their numerical methods is provided, which is a slight generalization of Theorem 2.6 in [9]. Our main results for the partially truncated EM method are presented in Sect. 3. Section 4 concludes this paper with some future research mentioned.

2 A general theorem

Throughout this paper, unless otherwise specified, the following notations are used. The transpose of a vector or matrix A is denoted by A^T. $|y|$ is the Euclidean norm if $y \in \mathbb{R}^d$. If A is a matrix, we let $|A| = \sqrt{\text{trace}(A^T A)}$ be its trace norm. For two real numbers α and β, we use $\alpha \vee \beta = \max(\alpha, \beta)$ and $\alpha \wedge \beta = \min(\alpha, \beta)$. If D is a set, its indicator function is denoted by I_D, namely $I_D(x) = 1$ if $x \in D$ and 0 otherwise. Moreover, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and let $\mathbb{E}$ denote the expectation corresponding to $\mathbb{P}$. Let $B(t)$ be an m-dimensional Brownian motion defined on the space.

In this paper, we study stochastic differential equations of the Itô type

$$dy(t) = \mu\bigl(y(t)\bigr)\,dt + \sigma\bigl(y(t)\bigr)\,dB(t) \tag{2.1}$$

with the initial value $y(0) = y_0$, where

$$\mu : \mathbb{R}^d \to \mathbb{R}^d \quad \text{and} \quad \sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}.$$

Theorem 2.1 *Assume that, for all sufficiently small step size Δ, a numerical method applied to (2.1) with initial value $x_\Delta(0) = y(0) = y_0$ satisfies:*

(I) *for any $T > 0$,*

$$\sup_{0\leq t\leq T} \mathbb{E}\bigl|x_\Delta(t)\bigr|^2 < C(y_0, T),$$

where $C(y_0, T)$ depends on y_0 and T, but not on Δ;

(II) *there exists a strictly increasing continuous function* $\alpha(\Delta)$ *with* $\alpha(0) = 0$ *such that*

$$\sup_{0\le t\le T} \mathbb{E}\big|x_\Delta(t) - y(t)\big|^2 \le \Big(\sup_{0\le t\le T} \mathbb{E}\big|x_\Delta(t)\big|^2\Big) C_T \alpha(\Delta),$$

where C_T *depends on* T *but not on* y_0 *and* Δ.

Then the SDE is mean square exponentially stable if and only if there exists $\Delta > 0$ *such that the numerical method is mean square exponentially stable with rate constant* λ_n, *growth constant* K_n, *step size* Δ, *and the global error constant* C_T *with* $T := 1 + (4 \log K_n)/\lambda_n$ *satisfying*

$$C_{2T}\big(\alpha(\Delta) + \sqrt{\alpha(\Delta)}\big)e^{\lambda_n T} + 1 + \sqrt{\alpha(\Delta)} \le e^{(1/4)\lambda_n T} \quad \text{and} \quad C_T \Delta \le 1.$$

Remark 2.2 Theorem 2.1 is a general theorem for any numerical method. Compared with Condition 2.3 for Theorem 2.6 in [9], one may notice that a more general function of Δ, i.e., $\alpha(\Delta)$, is used in Theorem 2.1. This change enables the new theorem to cover numerical methods with different convergence rates. For example, in this paper the partially truncated EM method needs $\alpha(\Delta) = \Delta^\epsilon$ with $\epsilon \in (0, 1)$, which is not covered by Theorem 2.6 in [9].

Remark 2.3 Although Theorem 2.1, to some extent, could be regarded as a generalization of Theorem 2.6 in [9], the proof follows a similar manner. Therefore, we refer the readers to [9] for the detailed proof. In this paper, we focus on how to fulfill (I) and (II) by using the partially truncated EM method for SDEs with super-linear growing coefficients.

3 Main results

We start this section by imposing some conditions on the coefficient and introducing the partially truncated EM method in Sect. 3.1. The main results and proofs are presented in Sect. 3.2

3.1 Partially truncated EM method

We assume that both the drift and diffusion coefficients in (2.1) could be separated into two parts as follows:

$$\mu(y) = \mu_1(y) + \mu_2(y) \quad \text{and} \quad \sigma(y) = \sigma_1(y) + \sigma_2(y).$$

We impose some assumptions on μ_i and σ_i for $i = 1, 2$.

Assumption 3.1 Assume that there exist constants $L_1 \ge 1$ and $\gamma \ge 0$ such that, for any $x, y \in \mathbb{R}^d$,

$$\big|\mu_1(x) - \mu_1(y)\big| \vee \big|\sigma_1(x) - \sigma_1(y)\big| \le L_1|x - y|$$

and

$$\big|\mu_2(x) - \mu_2(y)\big| \vee \big|\sigma_2(x) - \sigma_2(y)\big| \le L_1\big(1 + |x|^\gamma + |y|^\gamma\big)|x - y|.$$

Assumption 3.2 Assume that there is a pair of constants $\bar{r} > 2$ and L_2 such that

$$(x-y)^T\big(\mu_2(x)-\mu_2(y)\big)+\frac{\bar{r}-1}{2}\big|\sigma_2(x)-\sigma_2(y)\big|^2 \le L_2|x-y|^2$$

for all $x, y \in \mathbb{R}^d$.

For the purpose of the study on the stability, we require

$$\mu_1(0)=\mu_2(0)=\sigma_1(0)=\sigma_2(0)=0.$$

Assumption 3.3 Assume that there are constants $\bar{p} > \bar{r}$ and $K_2 > 0$ such that

$$x^T\mu_2(x)+\frac{\bar{p}-1}{2}\big|\sigma_2(x)\big|^2 \le K_2|x|^2$$

for all $x \in \mathbb{R}^d$.

As pointed out in [6], Assumption 3.3 cannot be deduced from Assumption 3.2 as we need $\bar{p} > \bar{r}$.

In addition, it can be derived from Assumption 3.1 that μ_1 and σ_1 satisfy the linear growth condition and μ_2 and σ_2 satisfy the polynomial growth condition that there exists a constant $K_1 > 0$ such that, for all $x \in \mathbb{R}^d$,

$$\big|\mu_1(x)\big| \vee \big|\sigma_1(x)\big| \le K_1|x| \tag{3.1}$$

and

$$\big|\mu_2(x)\big| \vee \big|\sigma_2(x)\big| \le K_1\big(1+|x|^{\gamma+1}\big). \tag{3.2}$$

It is not hard to see that the combination of Assumption 3.3 and (3.1) can imply that, for any $p \in (2, \bar{p})$,

$$x^T\mu(x)+\frac{p-1}{2}\big|\sigma(x)\big|^2 \le K_3|x|^2 \tag{3.3}$$

for all $x \in \mathbb{R}^d$, where K_3 is a constant dependent on K_1, K_2, p, and $\bar{p}$.

It can also be seen that under Assumptions 3.1 and 3.2, for any $q \in (2, \bar{r})$,

$$(x-y)^T\big(\mu(x)-\mu(y)\big)+\frac{q-1}{2}\big|\sigma(x)-\sigma(y)\big|^2 \le L_3|x-y|^2, \tag{3.4}$$

where L_3 is a constant dependent on L_1, L_2, q, and $\bar{r}$.

To make the paper self-contained, we provide the definition of the partially truncated EM method here and refer for the original ideas to [5, 6]. Firstly, we choose a strictly increasing continuous function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\kappa(r) \to \infty \quad \text{as } r \to \infty \quad \text{and} \quad \sup_{|x|\le r}\big(\big|\mu_2(x)\big| \vee \big|\sigma_2(x)\big|\big) \le \kappa(r) \quad \text{for any } r \ge 1.$$

It is clear to see that the inverse function of κ, denoted by κ^{-1}, is strictly increasing continuous from $[\kappa(0), \infty)$ to $\mathbb{R}_+$.

Next, we choose a strictly decreasing continuous function $h : (0,1] \to (0,\infty)$ such that

$$\lim_{\Delta \to 0} h(\Delta) = \infty \quad \text{and} \quad \Delta^{1/4} h(\Delta) \le |y_0| \quad \text{for any } \Delta \in (0,1]. \tag{3.5}$$

Remark 3.4 Comparing (3.5) with (2.9) in [5], we need $\Delta^{1/4}h(\Delta)$ to be bounded by the initial value here. It should be noted that this requirement is not hard to satisfy as the initial value y_0 is always provided. In addition, such an upper bound is key in the proofs of our results.

For a given time step $\Delta \in (0,1]$, we define the truncating function $\pi_\Delta : \mathbb{R}_d \to \mathbb{R}_d$ by

$$\pi_\Delta(x) = \bigl(|x| \wedge \kappa^{-1}\bigl(h(\Delta)\bigr)\bigr)\frac{x}{|x|}.$$

Now, we embed the truncating function into μ_2 and σ_2 to get

$$\mu_{2,\Delta}(x) = \mu_2\bigl(\pi_\Delta(x)\bigr) \quad \text{and} \quad \sigma_{2,\Delta}(x) = \sigma_2\bigl(\pi_\Delta(x)\bigr) \quad \text{for } x \in \mathbb{R}^d.$$

It is easy to see that, for any $x \in \mathbb{R}^d$,

$$\bigl|\mu_{2,\Delta}(x)\bigr| \vee \bigl|\sigma_{2,\Delta}(x)\bigr| \le \kappa\bigl(\kappa^{-1}\bigl(h(\Delta)\bigr)\bigr) = h(\Delta). \tag{3.6}$$

Using the notations that

$$\mu_\Delta(x) = \mu_1(x) + \mu_{2,\Delta}(x) \quad \text{and} \quad \sigma_\Delta(x) = \sigma_1(x) + \sigma_{2,\Delta}(x),$$

the partially truncated EM method is defined as

$$x_{\Delta,i+1} = x_{\Delta,i} + \mu_\Delta(x_{\Delta,i})\Delta + \sigma_\Delta(x_{\Delta,i})\Delta B_i \quad \text{with } x_{\Delta,0} = y_0, \tag{3.7}$$

where $\Delta B_i = B((i+1)\Delta) - B(i\Delta)$ is the Brownian motion increment for $i = 1,2,3,\ldots$, and $x_{\Delta,i}$ is the numerical approximation to $y(i\Delta)$ for $i = 1,2,3,\ldots$.

In some cases, it is more convenient to work with the continuous version of the numerical method. Thus, we define the continuous version of (3.7) by

$$x_\Delta(t) = x_{\Delta,0} + \int_0^t \mu_\Delta\bigl(\bar{x}_\Delta(t)\bigr)\,dt + \int_0^t \sigma_\Delta\bigl(\bar{x}_\Delta(t)\bigr)\,dB(t), \tag{3.8}$$

where

$$\bar{x}_\Delta(t) = x_{\Delta,i}, \quad \text{when } t \in [i\Delta, (i+1)\Delta).$$

3.2 (I) and (II) for the partially truncated EM method

To show that the SDE is mean square exponentially stable if and only if the partially truncated EM method is mean square exponentially stable, we need to prove that (I) and (II) in Theorem 2.1 hold for the method.

One may argue that the second moment boundedness and the L^2 strong convergence, i.e., requirements (I) and (II) in Theorem 2.1, have already been proven in [5, 6]. But it should be noted that (II) requires the L^2 strong error to be linearly related to $\sup_{0\le t\le T}\mathbb{E}|x_\Delta(t)|^2$ and this special structure of the error is important to the proof of Theorem 2.1. By looking into [5, 6], one notices that no such structure of error is provided.

Therefore, the main task of this part is to prove (II) for the partially truncated EM method by carefully tracking the related constants. We need several lemmas before we proceed to the main theorem.

Firstly, we cite two lemmas from [5]. Roughly speaking, Lemma 3.5 shows that $\mu_{2,\Delta}$ and $\sigma_{2,\Delta}$ inherit Assumption 3.3. And Lemma 3.6 shows that μ_Δ and σ_Δ inherit Assumption 3.1.

Lemma 3.5 *Let Assumption 3.3 hold. Then, for all $\Delta\in(0,1]$, we have*

$$x^T\mu_{2,\Delta}(x)+\frac{\bar{p}-1}{2}\left|\sigma_{2,\Delta}(x)\right|^2\le K_4|x|^2,\quad \forall x\in\mathbb{R}^d, \tag{3.9}$$

where $K_4=2K_2(1\vee[1/\kappa^{-1}(h(1))])$.

Lemma 3.6 *Let Assumption 3.1 hold. Then*

$$\left|\mu(x)-\mu(y)\right|\vee\left|\sigma(x)-\sigma(y)\right|\le 2L_1\left(1+|x|^\gamma+|y|^\gamma\right)|x-y| \tag{3.10}$$

for all $x,y\in\mathbb{R}^d$. Moreover, for any $\Delta\in(0,1]$,

$$\left|\mu_\Delta(x)-\mu_\Delta(y)\right|\vee\left|\sigma_\Delta(x)-\sigma_\Delta(y)\right|\le 3L_1\left(1+|x|^\gamma+|y|^\gamma\right)|x-y| \tag{3.11}$$

for all $x,y\in\mathbb{R}^d$.

Lemma 3.7 *Suppose that (3.5) and Assumptions 3.1, 3.2, and 3.3 hold, then the partially truncated EM method solution (3.8) satisfies*

$$\sup_{0<\Delta\le 1}\sup_{0\le t\le T}\mathbb{E}\left|x_\Delta(t)\right|^p<C_1\quad \textit{for } p\in(2,\bar{p}],$$

where

$$\begin{aligned}C_1&=\left(|y_0|+2^{p-1}y_0^{p/2}K_1\left(1+\left(p(p-2)/8\right)^{p/4}\right)T+2^py_0^pT\right)\\&\quad\times\exp\left(pK_4+2pK_1+(p-2)+2^{p-1}y_0^{p/2}K_1\left(1+\left(p(p-2)/8\right)^{p/4}\right)\right).\end{aligned}$$

Proof From (3.8) and by the Itô formula, we have

$$\begin{aligned}&\mathbb{E}\left|x_\Delta(t)\right|^p-\left|x_\Delta(0)\right|\\&\quad=\mathbb{E}\int_0^t p\left|x_\Delta(s)\right|^{p-2}\\&\qquad\times\left(x_\Delta^T(s)\left(\mu_1\left(\bar{x}_\Delta(s)\right)+\mu_{2,\Delta}\left(\bar{x}_\Delta(s)\right)\right)+\frac{p-2}{2}\left|\sigma_1\left(\bar{x}_\Delta(s)\right)+\sigma_{2,\Delta}\left(\bar{x}_\Delta(s)\right)\right|^2\right)ds\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\int_0^t p\left|x_\Delta(s)\right|^{p-2} \\
&\quad \times \left(\bar{x}_\Delta^T(s)\left(\mu_1\left(\bar{x}_\Delta(s)\right) + \mu_{2,\Delta}\left(\bar{x}_\Delta(s)\right)\right) + \frac{p-2}{2}\left|\sigma_1\left(\bar{x}_\Delta(s)\right) + \sigma_{2,\Delta}\left(\bar{x}_\Delta(s)\right)\right|^2\right) ds \\
&\quad + \mathbb{E}\int_0^t p\left|x_\Delta(s)\right|^{p-2}\left(x_\Delta(x) - \bar{x}_\Delta(s)\right)^T\left(\mu_1\left(\bar{x}_\Delta(s)\right) + \mu_{2,\Delta}\left(\bar{x}_\Delta(s)\right)\right) ds.
\end{aligned}
$$

By (3.9), we have

$$
\mathbb{E}\left|x_\Delta(t)\right|^p - \left|x_\Delta(0)\right| \le J_1 + J_2 + J_3,
$$

where

$$
\begin{aligned}
J_1 &\le \mathbb{E}\int_0^t pK_4\left|x_\Delta(s)\right|^{p-2}\left|\bar{x}_\Delta(s)\right|^2 ds, \\
J_2 &\le \mathbb{E}\int_0^t p\left|x_\Delta(s)\right|^{p-2}\left|x_\Delta(s) - \bar{x}_\Delta(s)\right|\left|\mu_1\left(\bar{x}_\Delta(s)\right)\right| ds,
\end{aligned}
$$

and

$$
J_3 \le \mathbb{E}\int_0^t p\left|x_\Delta(s)\right|^{p-2}\left|x_\Delta(s) - \bar{x}_\Delta(s)\right|\left|\mu_{2,\Delta}\left(\bar{x}_\Delta(s)\right)\right| ds.
$$

By the Hölder inequality and the Young inequality $\alpha^b\beta^{1-b} \le b\alpha + (1-b)\beta$ for $\alpha, \beta \ge 0$ and $b \in (0,1)$, we have

$$
\begin{aligned}
\mathbb{E}\left|x_\Delta(s)\right|^{p-2}\left|\bar{x}_\Delta(s)\right|^2 &\le \left(\mathbb{E}\left|x_\Delta(s)\right|^{(p-2)\times\frac{p}{p-2}}\right)^{\frac{p-2}{p}}\left(\mathbb{E}\left|\bar{x}_\Delta(s)\right|^{2\times\frac{p}{2}}\right)^{\frac{2}{p}} \\
&\le \frac{p-2}{p}\mathbb{E}\left|x_\Delta(s)\right|^p + \frac{2}{p}\mathbb{E}\left|\bar{x}_\Delta(s)\right|^p.
\end{aligned}
$$

Then we obtain the estimate of J_1

$$
J_1 \le K_4\int_0^t \left((p-2)\mathbb{E}\left|x_\Delta(s)\right|^p + 2\mathbb{E}\left|\bar{x}_\Delta(s)\right|^p\right) ds.
$$

In a similar manner, by (3.1) we have

$$
\begin{aligned}
&\mathbb{E}\left(\left|x_\Delta(s)\right|^{p-2}\left|x_\Delta(s) - \bar{x}_\Delta(s)\right|\left|\mu_1\left(\bar{x}_\Delta(s)\right)\right|\right) \\
&\quad \le K_1\mathbb{E}\left(\left|x_\Delta(s)\right|^{p-2}\left(\left|x_\Delta(s)\right| + \left|\bar{x}_\Delta(s)\right|\right)\left|\bar{x}_\Delta(s)\right|\right) \\
&\quad \le K_1\left(\frac{2p-3}{p}\mathbb{E}\left|x_\Delta(s)\right|^p + \frac{3}{p}\mathbb{E}\left|\bar{x}_\Delta(s)\right|^p\right).
\end{aligned}
$$

Thus, we have the estimate for J_2

$$
J_2 \le pK_1\int_0^t \left(\frac{2p-3}{p}\mathbb{E}\left|x_\Delta(s)\right|^p + \frac{3}{p}\mathbb{E}\left|\bar{x}_\Delta(s)\right|^p\right) ds.
$$

Now we are going to estimate J_3. By the Hölder inequality, the Young inequality, and (3.6), we have

$$\begin{aligned}&\mathbb{E}\big(|x_\Delta(s)|^{p-2}|x_\Delta(s)-\bar{x}_\Delta(s)||\mu_{2,\Delta}(\bar{x}_\Delta(s))|\big)\\&\quad\le\frac{p-2}{p}\mathbb{E}|x_\Delta(s)|^p+\frac{2}{p}\big(h(\Delta)\big)^{p/2}\mathbb{E}|x_\Delta(s)-\bar{x}_\Delta(s)|^{p/2}.\end{aligned}$$

By the Hölder inequality, the elementary inequality, (3.1), (3.6), Theorem 7.1 on page 39 of [19], for $s\in[i\Delta,(i+1)\Delta)$, we have

$$\mathbb{E}|x_\Delta(s)-\bar{x}_\Delta(s)|^{p/2}\le 2^{p-2}\Delta^{p/4}\big(\big(1+\big(p(p-2)/8\big)^{p/4}\big)K_1\mathbb{E}|\bar{x}_\Delta(i\Delta)|^{p/2}+2\big(h(\Delta)\big)^{p/2}\big).$$

Combining the two inequalities above and using the fact that $|\bar{x}_\Delta(s)|^{p/2}\le 1+|\bar{x}_\Delta(s)|^p$, we have the estimate for J_3

$$\begin{aligned}J_3&\le\int_0^t\big((p-2)\mathbb{E}|x_\Delta(s)|^p+2^{p-1}\big(h(\Delta)\big)^{p/2}\Delta^{p/4}\\&\quad\times\big[K_1\big(1+\big(p(p-2)/8\big)^{p/4}\big)\big(\mathbb{E}|\bar{x}_\Delta(s)|^p+1\big)+2\big(h(\Delta)\big)^{p/2}\big]\big)\,ds\\&\le\int_0^t\big((p-2)\mathbb{E}|x_\Delta(s)|^p+2^{p-1}\Delta^{p/8}y_0^{p/2}K_1\big(1+\big(p(p-2)/8\big)^{p/4}\big)\mathbb{E}|\bar{x}_\Delta(s)|^p\big)\,ds\\&\quad+2^{p-1}\Delta^{p/8}y_0^{p/2}K_1\big(1+\big(p(p-2)/8\big)^{p/4}\big)t+2^py_0^pt,\end{aligned}$$

where (3.5) is used. Putting the estimates for J_1, J_2, and J_3 together, we have that

$$\begin{aligned}\mathbb{E}|x_\Delta(t)|^p&\le|y_0|+2^{p-1}y_0^{p/2}K_1\big(1+\big(p(p-2)/8\big)^{p/4}\big)t+2^py_0^pt\\&\quad+\int_0^t\Big(\big[(p-2)K_4+(2p-3)K_1+(p-2)\big]\sup_{0\le u\le s}\mathbb{E}|x_\Delta(u)|^p\\&\quad+\big[2K_4+3K_1+2^{p-1}y_0^{p/2}K_1\big(1+\big(p(p-2)/8\big)^{p/4}\big)\big]\sup_{0\le u\le s}\mathbb{E}|\bar{x}_\Delta(u)|^p\Big)\,ds,\end{aligned}$$

where $\Delta<1$ is used. Since $\sup_{0\le u\le s}\mathbb{E}|x_\Delta(u)|^p=\sup_{0\le u\le s}\mathbb{E}|\bar{x}_\Delta(u)|^p$ and the inequality above holds for any $t\in[0,T]$ and any $\Delta\in(0,1]$, we obtain that

$$\sup_{0<\Delta\le 1}\sup_{0\le t\le T}\mathbb{E}|x_\Delta(t)|^p\le C_1,$$

where

$$\begin{aligned}C_1&=\big(|y_0|+2^{p-1}y_0^{p/2}K_1\big(1+\big(p(p-2)/8\big)^{p/4}\big)T+2^py_0^pT\big)\\&\quad\times\exp\big(pK_4+2pK_1+(p-2)+2^{p-1}y_0^{p/2}K_1\big(1+\big(p(p-2)/8\big)^{p/4}\big)\big).\end{aligned}$$ □

The next lemma can be proved by following the typical way, we refer the readers to, for example, [19] for details.

Lemma 3.8 *Suppose that* (3.3) *holds, the solution to* (2.1) *satisfies*

$$\sup_{0\le t\le T}\mathbb{E}|y(t)|^p\le C_2,$$

where $C_2=y_0\exp(pK_3T)$.

Lemma 3.9 *Suppose that Assumption 3.1 holds, then*

$$\big(\mathbb{E}\big|x_\Delta(t)-\bar{x}_\Delta(t)\big|^4\big)^{1/2}\le C_3\Delta^{1/2}\Big(\sup_{0\le t\le T}\mathbb{E}\big|x_\Delta(t)\big|^2\Big),$$

where

$$C_3=97\big(K_1^4C_1+1\big)^{1/2}.$$

Proof We see from (3.8) that, for any $t\in[i\Delta,(i+1)\Delta)$,

$$\begin{aligned}
&\mathbb{E}\big|x_\Delta(t)-\bar{x}_\Delta(t)\big|^4\\
&\quad\le 8\mathbb{E}\bigg|\int_{i\Delta}^t\big(\mu_1\big(\bar{x}_\Delta(s)\big)+\mu_{2,\Delta}\big(\bar{x}_\Delta(s)\big)\big)\,ds\bigg|^4+8\mathbb{E}\bigg|\int_{i\Delta}^t\big(\sigma_1\big(\bar{x}_\Delta(s)\big)+\sigma_{2,\Delta}\big(\bar{x}_\Delta(s)\big)\big)\,dB(s)\bigg|^4\\
&\quad\le 8\Delta^3\mathbb{E}\int_{i\Delta}^t\big|\mu_1\big(\bar{x}_\Delta(s)\big)+\mu_{2,\Delta}\big(\bar{x}_\Delta(s)\big)\big|^4\,ds\\
&\qquad+288\Delta\mathbb{E}\int_{i\Delta}^t\big|\sigma_1\big(\bar{x}_\Delta(s)\big)+\sigma_{2,\Delta}\big(\bar{x}_\Delta(s)\big)\big|^4\,ds\\
&\quad\le 64\Delta\big(\Delta^2+36\big)\\
&\qquad\times\mathbb{E}\int_{i\Delta}^t\big(\big|\mu_1\big(\bar{x}_\Delta(s)\big)\big|^4+\big|\mu_{2,\Delta}\big(\bar{x}_\Delta(s)\big)\big|^4+\big|\sigma_1\big(\bar{x}_\Delta(s)\big)\big|^4+\big|\sigma_{2,\Delta}\big(\bar{x}_\Delta(s)\big)\big|^4\big)\,ds,
\end{aligned}$$

where the Hölder inequality, Theorem 7.1 on page 39 of [19] and the elementary inequality $|a+b|^4\le 8(|a|^4+|b|^4)$ have been used.

Now, using (3.1) and (3.6), we have

$$\begin{aligned}
\mathbb{E}\big|x_\Delta(t)-\bar{x}_\Delta(t)\big|^4&\le 128\Delta\big(\Delta^2+36\big)\int_{i\Delta}^t\big(K_1^4\mathbb{E}\big|\bar{x}_\Delta(s)\big|^4+\big(h(\Delta)\big)^4\big)\,ds\\
&\le 128\Delta^2\big(\Delta^2+36\big)\big(K_1^4\mathbb{E}\big|\bar{x}_\Delta(i\Delta)\big|^4+\big(h(\Delta)\big)^4\big)\\
&\le 128\Delta^2\big(\Delta^2+36\big)\big(K_1^4C_1+\big(h(\Delta)\big)^4\big),
\end{aligned}$$

where Lemma 3.7 is used for the last inequality.

From (3.5), we derive $\Delta(h(\Delta))^4\le|y_0|^4$ and $\Delta\le|y_0|^4$. Then

$$\mathbb{E}\big|x_\Delta(t)-\bar{x}_\Delta(t)\big|^4\le 256\Delta\big(\Delta^2+36\big)\big(K_1^4C_1+1\big)|y_0|^4.$$

Taking square root on both sides, we have

$$\big(\mathbb{E}\big|x_\Delta(t)-\bar{x}_\Delta(t)\big|^4\big)^{1/2}\le 16\Delta^{1/2}(1+36)^{1/2}\big(K_1^4C_1+1\big)^{1/2}\Big(\sup_{0\le t\le T}\mathbb{E}\big|x_\Delta(t)\big|^2\Big),$$

where the fact $|y_0|^2=|x_\Delta(0)|^2\le\sup_{0\le t\le T}\mathbb{E}|x_\Delta(t)|^2$ is used. Since the analysis above holds for any $t\in[0,T]$, the assertion is obtained. □

For any real number $R>|y_0|$, define the stopping times

$$\tau_R=\inf\big\{t\ge 0:\big|y(t)\big|\ge R\big\}\quad\text{and}\quad\rho_R=\inf\big\{t\ge 0:\big|x_\Delta(t)\big|\ge R\big\}.$$

Set

$$\theta_R = \tau_R \wedge \rho_R \quad \text{and} \quad e(t) = x_\Delta(t) - y(t).$$

Lemma 3.10 *Suppose that Assumptions 3.1, 3.2, and 3.3 hold, then for any* $t \in [0,T]$

$$\mathbb{E}\big|y(t\wedge\theta_R) - x_\Delta(t\wedge\theta_R)\big|^2 \le C_4\Delta^{1/2}\Big(\sup_{0\le t\le T}\mathbb{E}\big|x_\Delta(t)\big|^2\Big),$$

where

$$C_4 = \frac{3^{3/2}(8q-12)L_1^2(1+2C_1)^{1/2}C_3e^{(2L_3+1)T}}{q-2}.$$

Proof For any $t \in [0,T]$, it is clear that $t\wedge\theta_R \le \theta_R$ a.s. Therefore, we observe from the definitions of μ_Δ and σ_Δ that $\mu_\Delta(\bar{x}_\Delta(t)) = \mu(\bar{x}_\Delta(t))$ and $\sigma_\Delta(\bar{x}_\Delta(t)) = \sigma(x_\Delta(t))$.

By the Itô formula, we have

$$\mathbb{E}\big|e(t\wedge\theta_R)\big|^2 = 2\mathbb{E}\int_0^{t\wedge\theta_R}\Big(e^T(s)\big(\mu\big(y(s)\big) - \mu\big(\bar{x}_\Delta(s)\big)\big) + \frac{1}{2}\big|\sigma\big(y(s)\big) - \sigma\big(\bar{x}_\Delta(s)\big)\big|^2\Big)\,ds.$$

Now, using the elementary inequality, we can see that

$$\begin{aligned}&\big|\sigma\big(y(s)\big) - \sigma\big(\bar{x}_\Delta(s)\big)\big|^2\\ &\quad\le \big(1+(q-2)\big)\big|\sigma\big(y(s)\big) - \sigma\big(x_\Delta(s)\big)\big|^2 + \big(1+1/(q-2)\big)\big|\sigma\big(x_\Delta(s)\big) - \sigma\big(\bar{x}_\Delta(s)\big)\big|^2,\end{aligned}$$

where $q > 2$ is used. Thus we obtain

$$\mathbb{E}\big|e(t\wedge\theta_R)\big|^2 \le J_4 + J_5,$$

where

$$J_4 = 2\mathbb{E}\int_0^{t\wedge\theta_R}\Big(e^T(s)\big(\mu\big(y(s)\big) - \mu\big(x_\Delta(s)\big)\big) + \frac{q-1}{2}\big|\sigma\big(y(s)\big) - \sigma\big(x_\Delta(s)\big)\big|^2\Big)\,ds$$

and

$$J_5 = 2\mathbb{E}\int_0^{t\wedge\theta_R}\Big(e^T(s)\big(\mu\big(x_\Delta(s)\big) - \mu\big(\bar{x}_\Delta(s)\big)\big) + \frac{q-1}{2(q-2)}\big|\sigma\big(x_\Delta(s)\big) - \sigma\big(\bar{x}_\Delta(s)\big)\big|^2\Big)\,ds.$$

Applying (3.4) yields

$$J_4 \le 2L_3\mathbb{E}\int_0^t \mathbb{E}\big|e(s\wedge\theta_R)\big|^2\,ds.$$

Applying the elementary inequality and (3.10) gives

$$J_5 \le \mathbb{E}\int_0^{t\wedge\theta_R}\left(\left|e(s)\right|^2 + \left|\mu\left(x_\Delta(s)\right) - \mu\left(\bar{x}_\Delta(s)\right)\right|^2 + \frac{q-1}{q-2}\left|\sigma\left(x_\Delta(s)\right) - \sigma\left(\bar{x}_\Delta(s)\right)\right|^2\right)ds$$
$$\le \int_0^t \mathbb{E}\left|e(s\wedge\theta_R)\right|^2 ds$$
$$+ 4\left(1+\frac{q-1}{q-2}\right)L_1^2\mathbb{E}\int_0^{t\wedge\theta_R}\left(1+\left|x_\Delta(s)\right|^\gamma + \left|\bar{x}_\Delta(s)\right|^\gamma\right)^2\left|x_\Delta(s)-\bar{x}_\Delta(s)\right|^2 ds$$
$$\le \int_0^t \mathbb{E}\left|e(s\wedge\theta_R)\right|^2 ds$$
$$+ \frac{8q-12}{q-2}L_1^2\int_0^T\left(\mathbb{E}\left(1+\left|x_\Delta(s)\right|^\gamma + \left|\bar{x}_\Delta(s)\right|^\gamma\right)^4\right)^{1/2}\left(\mathbb{E}\left|x_\Delta(s)-\bar{x}_\Delta(s)\right|^4\right)^{1/2} ds,$$

where $t\wedge\theta_R \le T$ a.s. for $t\in[0,T]$ is used for the last integral. Using Lemmas 3.7 and 3.9, we have

$$J_5 \le \int_0^t \mathbb{E}\left|e(s\wedge\theta_R)\right|^2 ds + \frac{3^{3/2}(8q-12)L_1^2(1+2C_1)^{1/2}C_3}{q-2}\Delta^{1/2}\left(\sup_{0\le t\le T}\mathbb{E}\left|x_\Delta(t)\right|^2\right),$$

where the elementary inequality $|a+b+c|^4 \le 3^3(|a|^4+|b|^4+|c|^4)$ is used.

Combining the estimates of J_4 and J_5 and applying the Gronwall inequality prove the assertion. □

Theorem 3.11 *Suppose that Assumptions 3.1, 3.2, and 3.3 hold, then for any* $t\in[0,T]$

$$\mathbb{E}\left|e(t)\right|^2 \le C_5\Delta^{1/2}\left(\sup_{0\le t\le T}\mathbb{E}\left|x_\Delta(t)\right|^2\right),$$

where

$$C_5 = C_4 + \frac{(C_1+C_2)(2^p+p-2)}{p}.$$

Proof For any $t\in[0,T]$, we have

$$\mathbb{E}\left|e(t)\right|^2 = \mathbb{E}\left(\left|e(t)\right|^2 I_{\{\theta_R>t\}}\right) + \mathbb{E}\left(\left|e(t)\right|^2 I_{\{\theta_R\le t\}}\right).$$

For any $\delta>0$, the Young inequality yields

$$\mathbb{E}\left(\left|e(t)\right|^2 I_{\{\theta_R\le t\}}\right) \le \frac{2\delta}{p}\mathbb{E}\left|e(t)\right|^p + \frac{p-2}{p\delta^{2/(p-2)}}\mathbb{P}(\theta_R\le t).$$

Due to Lemmas 3.7 and 3.8, we obtain

$$\mathbb{E}\left|e(t)\right|^p \le 2^{p-1}\mathbb{E}\left|y(t)\right|^p + 2^{p-1}\mathbb{E}\left|x_\Delta(t)\right|^p \le 2^{p-1}(C_1+C_2).$$

By the definition of τ_R, it is not hard to see

$$\mathbb{P}(\tau_R\le t) = \mathbb{E}\left(I_{\{\tau_R\le t\}}\frac{|y(\tau_R)|^p}{R^p}\right) \le \frac{1}{R^p}\left(\sup_{0\le t\le T}\mathbb{E}\left|y(t)\right|^p\right) \le \frac{C_2}{R^p}.$$

In a similar way, we have

$$\mathbb{P}(\rho_R \le t) \le \frac{C_1}{R^p}.$$

Thus, we see that

$$\mathbb{P}(\theta_R \le t) \le \mathbb{P}(\tau_R \le t) + \mathbb{P}(\rho_R \le t) \le \frac{C_1 + C_2}{R^p}.$$

Now choosing $\delta = |y_0|^2 \Delta^{1/2}$ and $R = (|y_0|^2 \Delta^{1/2})^{-1/(p-2)}$ gives

$$\begin{aligned}\mathbb{E}\bigl(\bigl|e(t)\bigr|^2 I_{\{\theta_R \le t\}}\bigr) &\le \frac{(C_1 + C_2)(2^p + p - 2)}{p} |y_0|^2 \Delta^{1/2} \\ &\le \Bigl(\sup_{0\le t\le T} \mathbb{E}\bigl|x_\Delta(t)\bigr|^2\Bigr) \frac{(C_1 + C_2)(2^p + p - 2)}{p} \Delta^{1/2},\end{aligned}$$

where the fact $|y_0|^2 = |x_\Delta(0)|^2 \le (\sup_{0\le t\le T} \mathbb{E}|x_\Delta(t)|^2)$ is used. Using Lemma 3.10, we have

$$\begin{aligned}\mathbb{E}\bigl|e(t)\bigr|^2 &= \mathbb{E}\bigl(\bigl|e(t)\bigr|^2 I_{\{\theta_R > t\}}\bigr) + \mathbb{E}\bigl(\bigl|e(t)\bigr|^2 I_{\{\theta_R \le t\}}\bigr) \\ &\le \mathbb{E}\bigl|e(t \wedge \theta_R)\bigr|^2 + \frac{(C_1 + C_2)(2^p + p - 2)}{p} \Delta^{1/2} \Bigl(\sup_{0\le t\le T} \mathbb{E}\bigl|x_\Delta(t)\bigr|^2\Bigr) \\ &\le C_4 \Delta^{1/2} \Bigl(\sup_{0\le t\le T} \mathbb{E}\bigl|x_\Delta(t)\bigr|^2\Bigr) + \frac{(C_1 + C_2)(2^p + p - 2)}{p} \Delta^{1/2} \Bigl(\sup_{0\le t\le T} \mathbb{E}\bigl|x_\Delta(t)\bigr|^2\Bigr) \\ &\le C_5 \Delta^{1/2} \Bigl(\sup_{0\le t\le T} \mathbb{E}\bigl|x_\Delta(t)\bigr|^2\Bigr).\end{aligned}$$

Hence, the proof is completed. □

We finish this section by providing the theorem of equivalence for the partially truncated Euler–Maruyama method. The proof of the theorem is straightforward following Theorem 2.1, Lemma 3.7, and Theorem 3.11.

Theorem 3.12 *Suppose that Assumptions 3.1, 3.2, and 3.3 hold, then the SDE is mean square exponentially stable if and only if the partially truncated Euler–Maruyama method solution is mean square exponentially stable providing the step size is small enough and satisfies* (3.5).

4 Conclusion and future research

For stochastic differential equations with super-linear growing coefficients, this paper studies equivalence of the mean square stability between the partially truncated Euler–Maruyama method and the underlying SDEs. By carefully tracking the constant term in the finite time convergence error, the "if and only if" result is obtained.

For the equivalence of other types of stabilities, such as pth moment stability or almost sure stability, we will report in the future works.

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Competing interests
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Authors' contributions
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