# Variational iterative method: an appropriate numerical scheme for solving system of linear Volterra fuzzy integro-differential equations 

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#### Abstract

In this research article, we focus on the system of linear Volterra fuzzy integro-differential equations and we propose a numerical scheme using the variational iteration method (VIM) to get a successive approximation under uncertainty aspects. We have $$
\begin{equation*} U^{j}(t)=f(t)+\int_{a}^{t} k(t, x) u(x) d x \tag{1} \end{equation*}
$$ where $j$ refers to the $j$ th order of the integro-differential equation and $j=1,2,3, \ldots, n$. $k(t, x)$ are integral kernel and a function of $t$ and $x$, which arise in mathematical biology, physics and more. The variational iteration technique gives the more accurate results at the very small cost of iterations leading to exact solutions quickly. The benefits of the proposal, an algorithmic form of the VIM, are also designed. To illustrate the potentiality of the scheme, two test problems are given and the approximate solutions are compared with the exact solution and also represented graphically.


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## 1 Introduction

The relationship between physical quantities and their rate of changes is named a system of differential equations. Integral and integro-differential equations with fuzzy primal conditions have led to many practical approaches and are essential tools for various realworld problems in science and engineering. These arise in mathematical biology models, chemical engineering, and fluid dynamics. Many others utilized a variety of approach for the fuzzy system of Volterra integro-differential equations based on a bloodline of differential inclusions. Fuzzy differential equations have been suggested as a way of modeling uncertain and incompletely specified systems and were studied by many researchers [7, $11,13,15,17,19,20]$. Nowadays, the examination of linear and nonlinear dynamical sys-
tems with uncertainty aspects is a rapid development and is concerned with world wide studied phenomena.
Now, we consider the system of linear Volterra integro-differential equation of such a form that Eq. (1) can be rewritten as

$$
\begin{align*}
U_{i}^{j}(t)= & f_{i}(t)+\sum_{h=1}^{j} F_{i, h}\left(t, u_{1}(t), \ldots, u_{n}^{j}(t)\right) \\
& +\sum_{h=1}^{j} \int_{0}^{t} k_{i, h}(t, x) G_{i, h}\left(u_{1}(t), \ldots, u_{n}^{j}(t)\right) d x, \quad 0 \leq t \leq 1, \tag{2}
\end{align*}
$$

where $k_{i, h}(t, x)$ is an arbitrary kernel function over $\{(t, x): 0 \leq t \leq x \leq 1\}, j \in Z^{+}, f_{i}(t)$ and $u_{i}(t)$ are known functions of $\{t: 0 \leq t \leq 1\}, F_{i, h}$ and $G_{i, h}$ are linear or nonlinear functions and $i=1,2, \ldots, n$. If $f_{i}(t)$ is a crisp function then the solutions of Eq. (2) are crisp as well. If $f(t, r)$ is a fuzzy function then the above equation may only possess a fuzzy solution, and this solution is a fuzzy function on the interval $r \in[0,1]$.

In general, many real-world experiments rarely can be expected to be close to an analytical solution, so an efficient approximate method has to be developed. An iterative technique can explore the output of the dynamical systems that were closer to the exact result. Recently, many researchers have utilized approximate techniques like He's variational iteration, Adomian's decomposition, Laplace transforms, homotopy perturbation techniques and others $[1,3,5,6,8,10,14,21]$. Here we mainly concentrate on the variational iterative technique by which we study the solution of a linear Volterra fuzzy integro-differential system. The variational iteration technique has been extensively applied in recent years by numerous researchers [9]. Starting from the pioneer ideas of the Inokuti Sekine Mura method [14], Ji Huan He [12] developed the variational iteration technique. In this technique the output comes out more accurately and closer to the exact results of the Volterra fuzzy integro-differential system.
Furthermore, we explained and successfully applied the variational iteration technique to evaluate a class of linear Volterra fuzzy integro-differential equations; this technique gives a better accuracy of the solution. This research article is organized as follows: in Sect. 2, we provide some basic concepts, definition, and background on fuzzy numbers and fuzzy differential equations. In Sect. 3, we explain the variational iterative technique and successfully demonstrate the linear Volterra fuzzy integro-differential system. In Sect. 4, we prepare two examples of the linear Volterra fuzzy integro-differential system, and for the technique we show the high accuracy of the results. Finally, we draw conclusions using the approximate results and exact solutions.

## 2 Basic concepts and definitions

In this section, we present the most basic ideas, definitions and useful results, which are used throughout this article $[2,4,16,22-24]$.

Definition 1 Let $U, V \in F(R)$. If there exists $W \in F(R)$, such that $U=V+W$, then $W$ is called the Hukuhara difference of $U$ and $V$ and it is denoted by $U \ominus V$.

Definition 2 Let $X$ be a non-empty set, a fuzzy set $\bar{A} \in X$ is characterized by its membership $\mu_{\bar{A}}: X \rightarrow[0,1]$ and $\mu_{\bar{A}(x)}$ is interpreted as the degree of membership of element
$x$ in fuzzy set $\bar{A}$ for each $x \in X$. It is clear that $\bar{A}$ is determined by the set of tuples $\bar{A}=\left(x, \mu_{\bar{A}(x)}\right) \mid x \in X$.

Definition 3 Let $g: R \rightarrow F$ be a fuzzy-valued function. If for arbitrary fixed $u_{0} \in R$ and $\epsilon>0, \delta>0$ such that

$$
\left|u-u_{0}\right|<\delta \geq D\left(g(u), g\left(u_{0}\right)\right)<\epsilon,
$$

$g$ is said to be continuous.
Definition 4 Given a fuzzy set $\bar{A}$ defined on $X$ and a number $\alpha \in[0,1]$, the $\alpha$-cut, $\bar{A}^{\alpha}$ and the strong $\alpha$-cut, $\bar{A}^{\alpha+}$, are the crisp sets

$$
\begin{aligned}
& \bar{A}^{\alpha}=\{x \mid A(x) \geq \alpha\} \\
& \bar{A}^{\alpha+}=\{x \mid A(x)>\alpha\}
\end{aligned}
$$

Unlike in the conventional set theory, the convexity of fuzzy sets refers to properties of the membership function rather than to the support of a fuzzy set.

Definition 5 An arbitrary fuzzy number in parametric form is represented by an ordered pair of functions $\left(u_{l}(r), u_{u}(r)\right), r \in[0,1]$, which satisfy the following requirements:

1. $u_{l}(r)$ is a bounded left continuous non-decreasing function over $[0,1]$,
2. $u_{u}(r)$ is a bounded left continuous non-increasing function over $[0,1]$,
3. $u_{l}(r) \leq u_{u}(r), r \in[0,1]$.

Definition 6 A function $f: R \rightarrow F$ is said to be fuzzy function. Suppose that $f: R \rightarrow F$ and let $u_{0} \in R$, the derivative $f^{\prime}\left(u_{0}\right)$ of $f$ at the point $u_{0}$ is defined by

$$
f^{\prime}\left(u_{0}\right)=\lim _{h \rightarrow 0} \frac{f\left(u_{0}+h\right)-f\left(u_{0}\right)}{h}
$$

Definition 7 A fuzzy set $\bar{A}$ is the triangular fuzzy number with peak $a$, left width $\alpha>0$ and right $\beta>0$, if its membership function has the following form:

$$
\mu(x)= \begin{cases}1-\frac{a-x}{\alpha} & \text { if } a-\alpha<x<\alpha \\ 1-\frac{x-a}{\beta} & \text { if } a<x<\alpha+\beta \\ 0 & \text { otherwise }\end{cases}
$$

## 3 System of Volterra fuzzy integro-differential equations

In this section, the Volterra fuzzy integro-differential equations are discussed and we use uncertain primal conditions (Definitions 3, 4 and 5). We have

$$
\begin{align*}
& U^{j}(x)=f(x)+\int_{a}^{x} k(x, t) U(t) d t  \tag{3}\\
& V^{j}(x)=g(x)+\int_{a}^{x} k(x, t) V(t) d t \tag{4}
\end{align*}
$$

where $j$ refers to the $j$ th order of the fuzzy integro-differential equations $j=1,2,3, \ldots, a$ refers to a constant and $x$ is a variable. If $f(x), g(x)$ both are a crisp function, then we have the solutions of Eqs. (3) and (4)

$$
\begin{aligned}
& \int_{a}^{x} \underline{f(x ; r)} d t=\int_{a}^{x} f(x ; r) d t, \\
& \int_{a}^{x} \overline{f(x ; r)} d t=\int_{a}^{x} \bar{f}(x ; r) d t, \\
& \int_{a}^{x} \underline{g(x ; r)} d t=\int_{a}^{x} \underline{g(x ; r) d t,} \\
& \int_{a}^{x} \overline{g(x ; r)} d t=\int_{a}^{x} \bar{g}(x ; r) d t,
\end{aligned}
$$

where $(\underset{\sim}{f}(x ; r), \bar{f}(x ; r)),(\underline{g}(x ; r), \bar{g}(x ; r))$ is the parametric form of $f(x), g(x)$. If $f(x), g(x)$ is a fuzzy function the above equations are processes leading to fuzzy solutions.
Let $U: I \rightarrow E^{1}$ be a fuzzy-valued function. Then the $\int_{I} U(t) d t, \int_{a}^{b} U(t) d t$ is defined by $\left[\int_{I} U(t) d t\right]_{r}=\int_{I} U_{r} d t=\left\{\int_{I} U(t) d t|u: \rightarrow R|\right.$ is measurable function for $\left.U_{r}\right\}$ for every $r=$ [0,1] (Definition 6).
Then the parametric form of the Volterra fuzzy integro-differential equations is as follows (Definitions 1 and 2).
Let $(\underline{f}(x ; r), \bar{f}(x ; r)),(\underline{g}(x ; r), \bar{g}(x ; r))$ and $(\underline{U}(t ; r), \bar{U}(t ; r)),(\underline{V}(t ; r), \bar{V}(t ; r))$ is the parametric form of $f(x), g(x)$ and $U(t), V(t)$, respectively, and $0 \leq r \leq 1$. Then the parametric form of the Volterra fuzzy integro-differential equations is denoted by

$$
\begin{align*}
& \underline{U^{j}}(x ; r)=\underline{f}(x ; r)+\int_{a}^{x} \underline{k(x, t) U(t ; r)} d t,  \tag{5}\\
& \overline{U^{j}}(x ; r)=\bar{f}(x ; r)+\int_{a}^{x} \overline{k(x, t) U(t ; r)} d t,  \tag{6}\\
& \underline{V^{j}}(x ; r)=\underline{g}(x ; r)+\int_{a}^{x} \underline{k(x, t) V(t ; r)} d t,  \tag{7}\\
& \overline{V^{j}}(x ; r)=\bar{g}(x ; r)+\int_{a}^{x} \overline{k(x, t) V(t ; r)} d t, \tag{8}
\end{align*}
$$

where

$$
\begin{aligned}
& \underline{k(x, t) U(t ; r)}= \begin{cases}k(x, t) \underline{U}(t ; r), & k(x, t) \geq 0, \\
k(x, t) \bar{U}(t ; r), & k(x, t)<0,\end{cases} \\
& \overline{k(x, t) U(t ; r)}= \begin{cases}k(x, t) \underline{U}(t ; r), & k(x, t) \geq 0, \\
k(x, t) \bar{U}(t ; r), & k(x, t)<0,\end{cases}
\end{aligned}
$$

and

$$
\underline{k(x, t) V(t ; r)}= \begin{cases}k(x, t) \underline{V}(t ; r), & k(x, t) \geq 0 \\ k(x, t) V(t ; r), & k(x, t)<0\end{cases}
$$

$$
\overline{k(x, t) V(t ; r)}= \begin{cases}k(x, t) \underline{V}(t ; r), & k(x, t) \geq 0 \\ k(x, t) V(t ; r), & k(x, t)<0\end{cases}
$$

for $0 \leq r \leq 1$. Suppose $k(x, t)$ is persisting in $a \leq t \leq b$ and for fixed $x$, and we take $f(x ; r)=$ $[f(x ; r), \bar{f}(x ; r)], \quad U(t ; r)=[G[\underline{u}(t ; r), \bar{u}(t ; r)], H[\underline{u}(t ; r), \bar{u}(t ; r)]], \quad V(t ; r)=[I[\underline{v}(t ; r), \bar{v}(t ; r)]$, $\bar{J}[\underline{v}(t ; r), \bar{v}(t ; r)]]$, where $r=[0,1]$ then we have the form

$$
\begin{align*}
& \underline{U^{j}}(x ; r)=\underline{f}(x ; r)+\int_{a}^{x} k(x, t) G[\underline{u}(t ; r), \bar{u}(t ; r)] d t,  \tag{9}\\
& \overline{U^{j}}(x ; r)=\bar{f}(x ; r)+\int_{a}^{x} k(x, t) H[\underline{u}(t ; r), \bar{u}(t ; r)] d t,  \tag{10}\\
& \underline{V^{j}}(x ; r)=\underline{g}(x ; r)+\int_{a}^{x} k(x, t) I[\underline{v}(t ; r), \bar{v}(t ; r)] d t,  \tag{11}\\
& \overline{V^{j}}(x ; r)=\bar{g}(x ; r)+\int_{a}^{x} k(x, t) J[\underline{v}(t ; r), \bar{v}(t ; r)] d t, \tag{12}
\end{align*}
$$

where $u(t ; r), v(t ; r)$ are fuzzy functions.

## 4 Proposed scheme for solving system of linear Volterra fuzzy integro-differential equations-variational iterative technique

In this section, we demonstrate the basic concept of the technique for evaluating a class of linear Volterra fuzzy integro-differential equations, we consider the following general differential equation:

$$
L u(t)+N u(t)=q(t),
$$

where $L, N$ are linear and nonlinear operators, respectively, and $q(t)$ is the source's inhomogeneous term. Assuming $u_{0}(t)$ is an appropriate solution of the linear, homogeneous equation

$$
L u_{0}(t)=0
$$

which depends on the conditions. According to the variational iteration technique $[3,18]$, we can construct a correct functional as follows:

$$
\begin{equation*}
u_{n+1}(t)=u_{n}(t)+\int_{0}^{t} \lambda\left(L u_{n}(\zeta)+N \tilde{u}_{n}(\zeta)-g(\zeta)\right) d \zeta, \quad n=0,1,2, \ldots \tag{13}
\end{equation*}
$$

Here $u_{0}$ is an initial approximation that satisfies the primal conditions, from the variational theory of $\lambda$ can be identified optimally and it is a general Lagrange multiplier [14]. The subscript $n$ refers to the $n$th approximation, $\tilde{u}_{n}$ is considered as a restricted variation i.e., $\delta u_{n}=0$. One is required first to determine the Lagrange multiplier $\lambda$ optimally. The successive approximation $u_{n+1}, n \geq 0$, of the solution $u$ will be readily obtained upon using the determined Lagrange multiplier and any selective function $u_{0}$, consequently, the solution is given by

$$
u=\lim _{n \rightarrow \infty} u_{n} .
$$

Then we have the existence of a unique solution to Eqs. (3) and (4).

Theorem 1 Let $U(x), U_{n}(x) \in[0,1], n=0,1,2, \ldots$. The sequence defined by Eq. (13) with $U_{(0)}(x)=U_{0}$ converges to $U(x)$, the exact solution of Eqs. (3) and (4).

Proof Obviously from Eqs. (3) and (4) we have

$$
U_{n+1}(x)=U_{n}(x)+\int_{0}^{x} \lambda\left(U_{n}^{\prime}(\zeta)-B U(\zeta)\right) d \zeta
$$

where $\lambda=\frac{(-1)^{m}}{(m-1)!}(\zeta-t)^{m-1}=(-1) e^{-(\zeta-t)}, n=0,1,2, \ldots$, by using the fact that $U_{n}(0), n=$ $0,1,2, \ldots$,

$$
\begin{equation*}
U_{n+1}(x)=U_{n}(x)-\int_{0}^{x} e^{-(\zeta-t)}\left(U_{n}^{\prime}(\zeta)-B U(\zeta)\right) d \zeta \tag{14}
\end{equation*}
$$

and using integration by parts and we conclude that

$$
\begin{aligned}
& =U_{n}(x)-\int_{0}^{x} e^{(\zeta B)+t} \frac{d}{d \zeta} e^{-(\zeta B)} U_{n}(\zeta) d \zeta \\
& =A \int_{0}^{x} e^{-(\zeta-t)} U_{n}(\zeta) d \zeta
\end{aligned}
$$

Therefore

$$
\left\|U_{n+1}(x)\right\| \leq\|A\| \int_{0}^{x}\left\|e^{-(\zeta-t)}\right\|\left\|U_{n}(\zeta)\right\| d \zeta
$$

Let us consider $C=\|A\|\left\|e^{-(\zeta-t)}\right\|$; then we have

$$
\left\|U_{n+1}(x)\right\| \leq C \int_{0}^{x}\left\|U_{n}(\zeta)\right\| d \zeta
$$

Now we proceed as follows:

$$
\begin{aligned}
&\left\|U_{1}(x)\right\| \leq C \int_{0}^{x}\left\|U_{0}(\zeta)\right\| d \zeta=C \operatorname{Max}\left\|U_{0}(\zeta)\right\| \int_{0}^{x} d \zeta=C \operatorname{Max}\left\|U_{0}(\zeta)\right\| x \\
&\left\|U_{2}(x)\right\| \leq C \int_{0}^{x}\left\|U_{1}(\zeta)\right\| d \zeta=C \int_{0}^{x} \operatorname{Max}\left\|U_{0}(\zeta)\right\| \zeta d \zeta=C^{2} \operatorname{Max}\left\|U_{0}(\zeta)\right\| \frac{x^{2}}{2!} \\
&\left\|U_{3}(x)\right\| \leq C \int_{0}^{x}\left\|U_{2}(\zeta)\right\| d \zeta=C \int_{0}^{x} \operatorname{Max}\left\|U_{0}(\zeta)\right\| \zeta d \zeta=C^{3} \operatorname{Max}\left\|U_{0}(\zeta)\right\| \frac{x^{3}}{3!} \\
& \vdots \\
&\left\|U_{n}(x)\right\| \leq C \int_{0}^{x}\left\|U_{n-1}(\zeta)\right\| d \zeta=C^{n} \int_{0}^{x} \operatorname{Max}\left\|U_{0}(\zeta)\right\| \frac{(\zeta)^{m-1}}{(m-1)!} d \zeta \\
&=\operatorname{Max}\left\|U_{0}(\zeta)\right\| \frac{(C x)^{n}}{n!} \rightarrow 0
\end{aligned}
$$

and hence as $n \rightarrow \infty$ we have $\operatorname{Max}\left\|U_{0}(\zeta)\right\| \frac{(C x)^{n}}{n!} \rightarrow 0$, then $U_{0}(x)=U_{0}$ converges.

To determine Eqs. (3) and (4) by utilizing the variational iteration method, we build the correction function as follows:

$$
\begin{aligned}
& \underline{u}_{n+1}(x, r)=\underline{u}_{n}(x, r)+\int_{0}^{x} \lambda\left[\underline{U}_{n}^{j}(\tau ; r)-\underline{f}(\tau ; r)-\int_{a}^{\tau} k(\tau, t) G\left[\underline{\underline{u}}_{n}(t ; r), \tilde{\bar{u}}_{n}(t ; r)\right] d t\right] d \tau, \\
& \bar{u}_{n+1}(x, r)=\bar{u}_{n}(x, r)+\int_{0}^{x} \lambda\left[\bar{U}_{n}^{j}(\tau ; r)-\bar{f}(\tau ; r)-\int_{a}^{\tau} k(\tau, t) H\left[\underline{\tilde{u}}_{n}(t ; r), \tilde{\bar{u}}_{n}(t ; r)\right] d t\right] d \tau,
\end{aligned}
$$

and

$$
\begin{aligned}
& \underline{v}_{n+1}(x, r)=\underline{v}_{n}(x, r)+\int_{0}^{x} \lambda\left[\underline{V}_{n}^{j}(\tau ; r)-\underline{g}(\tau ; r)-\int_{a}^{\tau} k(\tau, t) I\left[\underline{\tilde{v}}_{n}(t ; r), \tilde{\bar{v}}_{n}(t ; r)\right] d t\right] d \tau, \\
& \bar{v}_{n+1}(x, r)=\bar{v}_{n}(x, r)+\int_{0}^{x} \lambda\left[\bar{V}_{n}^{j}(\tau ; r)-\bar{g}(\tau ; r)-\int_{a}^{\tau} k(\tau, t) J\left[\underline{\tilde{v}}_{n}(t ; r), \tilde{\bar{v}}_{n}(t ; r)\right] d t\right] d \tau,
\end{aligned}
$$

where $\lambda$ is for a Lagrange multiplier, which can be optimally defined via variational theory,

$$
\begin{aligned}
& \int_{a}^{\tau} k(\tau, t) G\left[\underline{\tilde{u}}_{n}(t ; r), \tilde{\bar{u}}_{n}(t ; r)\right] d t, \\
& \int_{a}^{\tau} k(\tau, t) H\left[\underline{\tilde{u}}_{n}(t ; r), \tilde{\bar{u}}_{n}(t ; r)\right] d t, \\
& \int_{a}^{\tau} k(\tau, t) I\left[\underline{\tilde{v}}_{n}(t ; r), \tilde{\bar{v}}_{n}(t ; r)\right] d t, \\
& \int_{a}^{\tau} k(\tau, t) J\left[\underline{\tilde{v}}_{n}(t ; r), \tilde{\bar{v}}_{n}(t ; r)\right] d t
\end{aligned}
$$

are restricted variations, i.e., $\delta \tilde{u}=0, \delta \tilde{v}=0$;

$$
\begin{align*}
\delta \underline{u}_{n+1}(x, r)= & \delta \underline{u}_{n}(x, r)+\delta \int_{0}^{x} \lambda\left[\underline{U}_{n}^{j}(\tau ; r)-\underset{-}{ }(\tau ; r)\right. \\
& \left.-\int_{a}^{\tau} k(\tau, t) G\left[\underline{\tilde{u}}_{n}(t ; r), \tilde{\bar{u}}_{n}(t ; r)\right] d t\right] d \tau, \\
= & \delta \underline{u}_{n}(x, r)+\delta \int_{0}^{x} \lambda\left[\underline{U}_{n}^{j}(\tau ; r)\right] d \tau  \tag{15}\\
\delta \underline{v}_{n+1}(x, r)= & \delta \underline{v}_{n}(x, r)+\delta \int_{0}^{x} \lambda\left[\underline{V}_{n}^{j}(\tau ; r)-\underline{g}(\tau ; r)\right. \\
& \left.-\int_{a}^{\tau} k(\tau, t) I\left[\underline{\tilde{v}}_{n}(t ; r), \tilde{\bar{v}}_{n}(t ; r)\right] d t\right] d \tau \\
= & \delta \underline{v}_{n}(x, r)+\delta \int_{0}^{x} \lambda\left[\underline{V}_{n}^{j}(\tau ; r)\right] d \tau . \tag{16}
\end{align*}
$$

This implies that the stationary conditions of the correction function can be determined as

$$
\lambda=(-1)^{j} \frac{(\tau-x)^{(k-1)}}{(k-1)!}
$$

and from the result, we obtain the following iteration formula:

$$
\begin{align*}
\underline{u}_{n+1}(x, r)= & \underline{u}_{n}(x, r) \\
& +\int_{0}^{x}(-1)^{j} \frac{(\tau-x)^{(k-1)}}{(k-1)!}\left[\underline{U}_{n}(\tau ; r)-\underline{f}_{\underline{\prime}}(\tau ; r)\right. \\
& \left.-\int_{a}^{\tau} k(\tau, t) G[\underline{\tilde{u}}(t ; r), \tilde{\bar{u}}(t ; r)] d t\right] d \tau  \tag{17}\\
\bar{u}_{n+1}(x, r)= & \bar{u}_{n}(x, r) \\
& +\int_{0}^{x}(-1)^{j} \frac{(\tau-x)^{(k-1)}}{(k-1)!}\left[\bar{U}_{n}(\tau ; r)-\bar{f}(\tau ; r)\right. \\
& \left.-\int_{a}^{\tau} k(\tau, t) H[\underline{\tilde{u}}(t ; r), \tilde{\bar{u}}(t ; r)] d t\right] d \tau \tag{18}
\end{align*}
$$

and

$$
\begin{align*}
\underline{v}_{n+1}(x, r)= & \underline{v}_{n}(x, r) \\
& +\int_{0}^{x}(-1)^{j} \frac{(\tau-x)^{(k-1)}}{(k-1)!}\left[\underline{V}_{n}(\tau ; r)-\underline{g}(\tau ; r)\right. \\
& \left.-\int_{a}^{\tau} k(\tau, t) I[\underline{\tilde{v}}(t ; r), \tilde{\bar{v}}(t ; r)] d t\right] d \tau,  \tag{19}\\
\bar{v}_{n+1}(x, r)= & \bar{v}_{n}(x, r) \\
& +\int_{0}^{x}(-1)^{j} \frac{(\tau-x)^{(k-1)}}{(k-1)!}\left[\bar{V}_{n}(\tau ; r)-\bar{g}(\tau ; r)\right. \\
& \left.-\int_{a}^{\tau} k(\tau, t) J[\underline{\tilde{v}}(t ; r), \tilde{\bar{v}}(t ; r)] d t\right] d \tau, \tag{20}
\end{align*}
$$

and we assume that the iteration formula starts with an initial approximation, $\left[\underline{u}_{0}(x ; r)\right.$, $\left.\bar{u}_{0}(x ; r)\right],\left[\underline{v}_{0}(x ; r), \bar{v}_{0}(x ; r)\right]$.

## 5 Illustrative examples

Now, we apply the proposed approximation technique by evaluating the system of linear Volterra fuzzy integro-differential equations, our solutions of the variational iteration method and the demonstration are given in Sect. 4, convert to the numerical variational iteration algorithm is drawn in this section. Its application of the linear Volterra fuzzy integro-differential system is demonstrated.

## Algorithm

Step 1: Choose and initiate the initial conditions $u(x ; r), v(x ; r)$, i.e., $u_{0}=u(x=0)$.
Step 2: Compute Eqs. (17) to (20) by using the calculated values of $u_{0}$.
Step 3: If $\left(u_{n+1}(t ; r)-u_{n}(t ; r)\right)\left(v_{n+1}(t ; r)-v_{n}(t ; r)\right)$ is an approximate solution, and go to Step 4, else go to Step 2.
Step 4: Print the successive solutions of $u, v$.

In the forthcoming two illustrations, we utilize our variational iteration algorithm to get more efficiency and accuracy of the linear Volterra fuzzy integro-differential system.

Example 1 Now, we consider the linear Volterra fuzzy integro-differential system, with the functions $k(x ; t)=x-t, a=0, f(x ; r)=2 x^{2}, g(x ; r)=-3 x^{2}-\frac{1}{10} x^{5}$,

$$
\begin{aligned}
& G[\underline{u}(t ; r), \bar{u}(t ; r)]=\underline{u}(t ; r)-\underline{v}(t ; r), \quad H[\underline{u}(t ; r), \bar{u}(t ; r)]=\bar{u}(t ; r)-\bar{v}(t ; r), \\
& I[\underline{v}(t ; r), \bar{v}(t ; r)]=\underline{u}(t ; r)+\underline{v}(t ; r), \quad \text { and } \quad J[\underline{v}(t ; r), \bar{v}(t ; r)]=\bar{u}(t ; r)+\bar{v}(t ; r) .
\end{aligned}
$$

Then Eqs. (9) to (12) can be written in the form

$$
\begin{aligned}
& \left\{\begin{array}{l}
\underline{u^{\prime}}(x ; r)=2 x^{2}+\int_{0}^{x}(x-t)(\underline{u}(t ; r)-\underline{v}(t ; r)) d t, \\
\bar{u}^{\prime}(x ; r)=2 x^{2}+\int_{0}^{x}(x-t)(\bar{u}(t ; r)-\bar{v}(t ; r)) d t,
\end{array}\right. \\
& \left\{\begin{array}{l}
\underline{v}^{\prime}(x ; r)=-3 x^{2}-\frac{1}{10} x^{5}+\int_{0}^{x}(x-t)(\underline{u}(t ; r)+\underline{v}(t ; r)) d t, \\
\bar{v}^{\prime}(x ; r)=-3 x^{2}-\frac{1}{10} x^{5}+\int_{0}^{x}(x-t)(\bar{u}(t ; r)+\bar{v}(t ; r)) d t,
\end{array}\right.
\end{aligned}
$$

then the system is subject to the triangular fuzzy initial conditions (Definition 7), $u_{l, u}(x ; r)=[0,1,2], v_{l, u}(x ; r)=[0,1,2], 0 \leq r \leq 1$. The exact solutions of this illustration are given by

$$
U(x ; r)=1+x^{3}, \quad V(x ; r)=1-x^{3} .
$$

Now, we begin with the primal approximation

$$
\begin{aligned}
& \underline{u}_{0}=r+\frac{2 x^{3}}{3}+\cdots, \quad \bar{u}_{0}=2-r+\frac{2 x^{3}}{3}+\cdots \quad \text { and } \\
& \underline{v}_{0}=r-x^{3}+\cdots, \quad \bar{v}_{0}=2-r-x^{3}+\cdots,
\end{aligned}
$$

and using the above iteration formula, we obtain the successive iterations by using Mathematica Package 10.0.

Error analysis The absolute errors are computed as

$$
\begin{aligned}
& \underline{E}(x ; r)=|\underline{U}(x ; r)-\underline{u}(x ; r)|, \\
& \bar{E}(x ; r)=|\bar{U}(x ; r)-\bar{u}(x ; r)| \quad \text { and } \\
& \underline{E}(x ; r)=|\underline{V}(x ; r)-\underline{v}(x ; r)|, \\
& \bar{E}(x ; r)=|\bar{V}(x ; r)-\bar{v}(x ; r)| .
\end{aligned}
$$

The numerical results of the obtained approximate solutions are compared with the exact solutions for different $r$-values and errors are presented in Table 1. Moreover, exact and approximate solutions are shown graphically in Figs. 1 and 2, the $x$-variation is also displayed in Figs. 3 and 4.

Example 2 Now, we concentrate on another linear Volterra fuzzy integro-differential system, with the functions $k(x ; t)=1, a=0, f(x ; r)=1+x^{2}+e^{x}, g(x ; r)=3-3 e^{x}$,

$$
\begin{aligned}
& G[\underline{u}(t ; r), \bar{u}(t ; r)]=\underline{u}(t ; r)+\underline{v}(t ; r), \quad H[\underline{u}(t ; r), \bar{u}(t ; r)]=\bar{u}(t ; r)+\bar{v}(t ; r), \\
& I[\underline{v}(t ; r), \bar{v}(t ; r)]=\underline{u}(t ; r)-\underline{v}(t ; r), \quad \text { and } \quad J[\underline{v}(t ; r), \bar{v}(t ; r)]=\bar{u}(t ; r)-\bar{v}(t ; r)
\end{aligned}
$$

Table 1 The error analysis of $U, V$ at $x=0.5$

| $r$ | $U(x ; r)$ at $t=0.5$ |  |  | $V(x ; r)$ at $t=0.5$ |  |
| :--- | :--- | :--- | :--- | :--- | :--- |
|  | $\underline{E}(x ; r)$ | $\bar{E}(x ; r)$ |  | $\underline{E}(x ; r)$ | $\bar{E}(x ; r)$ |
| 0 | 0.0000 | 0.0147 |  | 0.0000 | 0.0153 |
| 0.1 | 0.0007 | 0.0140 |  | 0.0008 | 0.0145 |
| 0.2 | 0.0015 | 0.0132 |  | 0.0015 | 0.0138 |
| 0.3 | 0.0022 | 0.0125 |  | 0.0023 | 0.0130 |
| 0.4 | 0.0029 | 0.0118 |  | 0.0031 | 0.0122 |
| 0.5 | 0.0037 | 0.0110 |  | 0.0038 | 0.0115 |
| 0.6 | 0.0044 | 0.0103 |  | 0.0046 | 0.0107 |
| 0.7 | 0.0051 | 0.0096 |  | 0.0054 | 0.0099 |
| 0.8 | 0.0059 | 0.0088 |  | 0.0061 | 0.0092 |
| 0.9 | 0.0066 | 0.0081 |  | 0.0069 | 0.0084 |
| 1 | 0.0073 | 0.0073 |  | 0.0077 | 0.0077 |



Figure 1 Comparing the exact and approximate solutions of $U$ and $u$ at $x=0.5$


Figure 2 Comparing the exact and approximate solutions of $V$ and $v$ at $x=0.5$


Figure $3 \times$ variations of $U$ and $u$ at $r=0.5$


Figure $4 \times$ variations of $V$ and $v$ at $r=0.5$
then Eqs. (9) to (12) can be written in the form

$$
\begin{aligned}
& \left\{\begin{array}{l}
\underline{u^{\prime}}(x ; r)=1+x^{2}+e^{x}+\int_{0}^{x}(x-t)(\underline{u}(t ; r)+\underline{v}(t ; r)) d t, \\
\bar{u}^{\prime}(x ; r)=1+x^{2}+e^{x}+\int_{0}^{x}(x-t)(\bar{u}(t ; r)+\bar{v}(t ; r)) d t,
\end{array}\right. \\
& \left\{\begin{array}{l}
\underline{v}^{\prime}(x ; r)=3-3 e^{x}+\int_{0}^{x}(x-t)(\underline{u}(t ; r)-\underline{v}(t ; r)) d t, \\
\bar{v}^{\prime}(x ; r)=3-3 e^{x}+\int_{0}^{x}(x-t)(\bar{u}(t ; r)-\bar{v}(t ; r)) d t,
\end{array}\right.
\end{aligned}
$$

then the system is subject to the triangular fuzzy initial conditions (Definition 7), $u_{l, u}(x ; r)=[0,1,2], v_{l, u}(x ; r)=[0,1,2], 0 \leq r \leq 1$. The exact solutions of this illustration is given by

$$
U(x ; r)=1+e^{x}, \quad V(x ; r)=1-e^{x} .
$$

Now, we begin with the primal approximation

$$
\begin{array}{ll}
\underline{u}_{0}=-1+e^{x}+r+\cdots, & \bar{u}_{0}=1+e^{x}-r+\cdots \quad \text { and } \\
\underline{v}_{0}=3-3 e^{x}-r+\cdots, & \bar{v}_{0}=2-r-x^{3}+\cdots,
\end{array}
$$

and using the above iteration formula, we obtain the successive iterations by using Mathematica Package 10.0.

Error analysis The absolute errors are computed as

$$
\begin{aligned}
& \underline{E}(x ; r)=|\underline{U}(x ; r)-\underline{u}(x ; r)|, \\
& \bar{E}(x ; r)=|\bar{U}(x ; r)-\bar{u}(x ; r)| \quad \text { and } \\
& \underline{E}(x ; r)=|\underline{V}(x ; r)-\underline{v}(x ; r)|, \\
& \bar{E}(x ; r)=|\bar{V}(x ; r)-\bar{v}(x ; r)| .
\end{aligned}
$$

Table 2 The error analysis of $U, V$ at $x=0.5$

| $r$ | $U(x ; r)$ at $t=0.5$ |  | $V(x ; r)$ at $t=0.5$ |  |
| :---: | :---: | :---: | :---: | :---: |
|  | $\underline{E}(x ; r)$ | $\bar{E}(x ; r)$ | $\underline{E}(x ; r)$ | $\bar{E}(x ; r)$ |
| 0 | 0.0000 | 0.0217 | 0.0000 | 0.0053 |
| 0.1 | 0.0011 | 0.0206 | 0.0003 | 0.0050 |
| 0.2 | 0.0022 | 0.0195 | 0.0005 | 0.0048 |
| 0.3 | 0.0033 | 0.0184 | 0.0008 | 0.0045 |
| 0.4 | 0.0043 | 0.0174 | 0.0011 | 0.0042 |
| 0.5 | 0.0054 | 0.0163 | 0.0013 | 0.0040 |
| 0.6 | 0.0065 | 0.0152 | 0.0016 | 0.0037 |
| 0.7 | 0.0076 | 0.0141 | 0.0019 | 0.0034 |
| 0.8 | 0.0087 | 0.0130 | 0.0021 | 0.0032 |
| 0.9 | 0.0098 | 0.0119 | 0.0024 | 0.0029 |
| 1 | 0.0108 | 0.0108 | 0.0026 | 0.0026 |



Figure 5 Comparing the exact and approximate solutions of $U$ and $u$ at $x=0.5$


Figure 6 Comparing the exact and approximate solutions of $V$ and $v$ at $x=0.5$


Figure $7 \times$ variations of $U$ and $u$ at $r=0.5$


Figure $8 \times$ variations of $V$ and $v$ at $r=0.5$

The numerical results of the obtained approximate solutions are compared with the exact solutions for different $r$-values and errors are presented in Table 2. Moreover, exact and approximate solutions are shown graphically in Figs. 5 and 6, the $x$-variation is also displayed in Figs. 7 and 8. It is clear that we obtain the minimum rate of computation and also get the high accuracy of the result.

## 6 Conclusion

Recently, many computer programs and techniques have been highly developed for these types of problems, but their scientific discipline basis is for a great deal insufficiently appreciated, and the proposed technique is well known as regards the accurate effect of the tomography results. In this research, He's variational iteration technique is successfully applied on demonstrating results of the Volterra fuzzy integro-differential systems. Utilizing this technique is to quickly lead to the exact result within the minimum rate of iterations and is a very effective tool for evaluating the solutions. The illustrative approaches are tested by the variational iteration technique (by using Mathematica Package 10.0).

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The authors declare that they have no competing interests.

## Authors' contributions

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