# Approximate controllability and complete controllability of semilinear fractional functional differential systems with control 

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#### Abstract

This paper is concerned with the approximate controllability and complete controllability of semilinear fractional functional differential systems with control involving Caputo fractional derivative. By using the operator semigroup theory and the fixed point theorem, we establish sufficient conditions for each of these types of controllability. The results are obtained under the assumption that the corresponding linear system is approximately controllable and completely controllable, respectively. In the end, an example is presented to illustrate the obtained theory.


Keywords: Fractional functional differential system; Approximate controllability; Complete controllability; Mild solution

## 1 Introduction

In this paper, we assume that $X$ is a Hilbert space with the norm $\|\cdot\|$. Let $I=[0, T]$. Denote $C$ to be the Banach space of continuous functions from $[-h, 0]$ into $X$ with the norm $\|x\|=$ $\sup _{t \in[-h, 0]}|x(t)|, x \in C$. The aim of this paper is to study the controllability (approximate and complete controllability) of semilinear fractional functional differential system with control of the form:

$$
\left\{\begin{array}{l}
{ }^{C} D_{t}^{q} x(t)=A x(t)+B u(t)+f\left(t, x_{t}, u(t)\right), \quad t \in I=[0, T],  \tag{1}\\
x_{0}(\theta)=\phi(\theta), \quad-h \leq \theta \leq 0,
\end{array}\right.
$$

where ${ }^{C} D_{t}^{q}$ is the Caputo fractional derivative of order $0<q<1$, the state variable $x(\cdot)$ takes values in $X$, the control function $u(\cdot) \in L^{2}(I, U)$ takes values in a Hilbert space $U$. $A$ is the infinitesimal generator of an alytic semigroup $\{T(t)\}_{t \geq 0}$ of bounded operators on $X, B: U \rightarrow X$ is a bounded linear operator, and $f: I \times C \times U \rightarrow X$ is a given function satisfying certain assumptions. If $x:[-h, T] \rightarrow X$ is a continuous function, then $x_{t}$ is an element in $C$ defined by $x_{t}(\theta)=x(t+\theta), \theta \in[-h, 0]$, and $\phi \in C$.

The controllability theory plays an important role in abstract control systems. Many researchers investigated the approximate or complete controllability of control systems (see [1-12] and the references therein). In particular, many authors [12-19] studied the approximate or complete controllability for various semilinear fractional evolution systems. In particular, Sakthivel et al. [17] formulated a new set of sufficient conditions for
the approximate controllability of the semilinear fractional differential system

$$
\left\{\begin{array}{l}
{ }^{C} D_{t}^{q} x(t)=A x(t)+B u(t)+f(t, x(t)), \quad t \in J=[0, T]  \tag{2}\\
x(0)=x_{0},
\end{array}\right.
$$

where ${ }^{C} D_{t}^{q}$ is the Caputo fractional derivative of order $q \in(0,1), A$ is the infinitesimal generator of a $C_{0}$ semigroup $T(t)$ of bounded operators on the Hilbert space $X, B: U \rightarrow X$ is a bounded linear operator, $U$ is a Hilbert space, $f: J \times X \rightarrow X$ is a given function.

In $[20,21]$, the authors paid attention to the approximate controllability of semilinear delay control systems in which the nonlinear terms depend on both state function and control function under the assumption that the corresponding linear systems are approximately controllable. On the other hand, Sukavanam et al. [22] formulated some sufficient conditions for the approximate controllability of a semilinear fractional delay control system. Moreover, the approximate controllability of fractional order semilinear systems with bounded delay when the nonlinear term is independent of a control function was addressed in [23]. The approximate controllability for a class of Riemann-Liouville fractional differential inclusions via fractional calculus, multi-valued analysis, semigroup theory, and the fixed-point technique was investigated by Yang and Wang in [24]. Also, Sakthivel [25] dealt with the exact controllability for fractional neutral control systems by using a fixed point analysis approach.

On the other hand, in order to study the various fractional systems, introducing a suitable concept of mild solutions will be of great importance. Zhou and Jiao [26] studied the existence of mild solution of fractional neutral evolution equations by the fractional power of operators and some fixed point theorems. Also, Wang and Zhou [27] investigated a new mild solution for a class of fractional evolution equations, and then the existence of optimal pairs of the considered system was also obtained. Furthermore, Wang and Zhou [8] discussed the complete controllability of fractional evolution systems based on the fractional calculus, properties of characteristic solution operators, and fixed point theorems.
Motivated by the above work, in this paper we first use the above suitable concept of mild solution in [26,27] and then adopt a method similar to paper [20] which investigates the approximate controllability of integer semilinear functional differential equations. In the end, we establish different sufficient conditions for the approximate and complete controllability of the semilinear fractional functional differential system (1).
This paper is organized as follows. Section 2 is devoted to some preliminaries. In Sect. 3, we give the approximate controllability result of system (1). In Sect. 4, we formulate sufficient conditions for the complete controllability of system (1) by using the Banach fixed point theorem when the semigroup $\{T(t)\}_{t \geq 0}$ is not compact. An example is presented to demonstrate the approximate controllability result in Sect. 5.

## 2 Preliminaries

In this section, we introduce some notations, definitions and lemmas which will be used throughout this paper.

Definition 2.1 ([28]) The fractional integral of order $\alpha$ with the lower limit 0 for a function $g$ is defined as

$$
\begin{equation*}
I^{\alpha} g(t)=\frac{1}{\Gamma(\alpha)} \int_{0}^{t} \frac{g(s)}{(t-s)^{1-\alpha}} d s, \quad t>0, \quad \alpha>0 \tag{3}
\end{equation*}
$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where $\Gamma$ is the gamma function.

Definition 2.2 ([28]) The Caputo derivative of order $\alpha$ with the lower limit 0 for a function $g$ can be written as

$$
\begin{equation*}
{ }^{C} D^{\alpha} g(t)=\frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} \frac{g^{(n)}(s)}{(t-s)^{\alpha+1-n}} d s, \quad t>0, \quad 0 \leq n-1<\alpha<n . \tag{4}
\end{equation*}
$$

If $g$ is an abstract function with value in the Hilbert space $X$, then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

By comparing with the fractional differential equations given in [26], we give the following definition of a mild solution of system (1).

Definition 2.3 ([26]) A function $x \in C([-h, T] ; X)$ is called a mild solution of system (1) if on $[-h, T]$ it satisfies

$$
\left\{\begin{array}{l}
x(t)=S_{q}(t) \phi(0)+\int_{0}^{t}(t-s)^{q-1} T_{q}(t-s)\left(B u(s)+f\left(s, x_{s}, u(s)\right)\right) d s, \quad t \in[0, T]  \tag{5}\\
x_{0}(\theta)=\phi(\theta), \quad-h \leq \theta \leq 0
\end{array}\right.
$$

where

$$
\begin{aligned}
& S_{q}(t)=\int_{0}^{\infty} \phi_{q}(\theta) T\left(t^{q} \theta\right) d \theta \\
& T_{q}(t)=q \int_{0}^{\infty} \theta \phi_{q}(\theta) T\left(t^{q} \theta\right) d \theta \\
& \phi_{q}(\theta)=\frac{1}{q} \theta^{-1-\frac{1}{q}} \psi_{q}\left(\theta^{-\frac{1}{q}}\right) \\
& \psi_{q}(\theta)=\frac{1}{\pi} \sum_{n=1}^{\infty}(-1)^{n-1} \theta^{-q n-1} \frac{\Gamma(n q+1)}{n!} \sin (n \pi q), \quad \theta \in(0, \infty)
\end{aligned}
$$

is a probability density function. In addition, $\phi_{q}(\theta)$ is the probability density function defined as

$$
\phi_{q}(\theta) \geq 0, \quad \theta \in(0, \infty), \quad \text { and } \quad \int_{0}^{\infty} \phi_{q}(\theta) d \theta=1
$$

Remark 2.1 For $\xi \in[0,1], \int_{0}^{\infty} \theta^{\xi} \phi_{q}(\theta) d \theta=\frac{\Gamma(1+\xi)}{\Gamma(1+q \xi)}$.

Lemma $2.1([6,26])$ The operators $S_{q}(t)$ and $T_{q}(t)$ have the following properties:
(i) For any $t \geq 0$, the operators $S_{q}(t)$ and $T_{q}(t)$ are linear and bounded operators, that is, for any $x \in X$,

$$
\left\|S_{q}(t) x\right\| \leq M\|x\| \quad \text { and } \quad\left\|T_{q}(t) x\right\| \leq \frac{M q}{\Gamma(1+q)}\|x\|
$$

(ii) $\left\{S_{q}(t)\right\}_{t \geq 0}$ and $\left\{T_{q}(t)\right\}_{t \geq 0}$ are strongly continuous.
(iii) For every $t>0, S_{q}(t)$ and $T_{q}(t)$ are also compact operators if $T(t), t>0$ is compact.

Definition 2.4 System (1) is said to be approximately controllable (completely controllable) on the interval $I$ if $\overline{\mathcal{R}(T, \phi)}=X(\mathcal{R}(T, \phi)=X)$, where $\mathcal{R}(T, \phi)=\left\{x_{T}(\phi, u)(0): u(\cdot) \in\right.$ $L^{2}(I, U)$.

In order to deal with our problems, we introduce the following two operators:

$$
\begin{aligned}
& \Gamma_{0}^{T}=\int_{0}^{T}(T-s)^{q-1} T_{q}(T-s) B B^{*} T_{q}^{*}(T-s) d s \\
& R\left(\mu, \Gamma_{0}^{T}\right)=\left(\mu I+\Gamma_{0}^{T}\right)^{-1}, \quad \mu>0
\end{aligned}
$$

where $B^{*}, T_{q}^{*}(t)$ denote the adjoint of $B$ and $T_{q}(t)$, respectively.
By [17, 29], we know that the linear fractional control system

$$
\left\{\begin{array}{l}
{ }^{C} D_{t}^{q} x(t)=A x(t)+B u(t), \quad t \in I,  \tag{6}\\
x(0)=\phi(0)
\end{array}\right.
$$

is approximately controllable on $I$ if and only if
(H0) $\mu R\left(\mu, \Gamma_{0}^{T}\right) \rightarrow 0$ as $\mu \rightarrow 0^{+}$in the strong operator topology.
Lemma 2.2 (Hölder's inequality) Assume $p, q \geq 1$ and $\frac{1}{p}+\frac{1}{q}=1$. Iff $\in L^{p}(I, R), g \in L^{q}(I, R)$, then $f g \in L^{1}(I, R)$ and

$$
\|f g\|_{L^{1}(I)} \leq\|f\|_{L^{p}(I)}\|g\|_{L^{q}(I)} .
$$

Lemma 2.3 (Schauder's fixed point theorem) If $Q$ is a closed bounded and convex subset of a Banach space $E$ and $\Psi: Q \rightarrow Q$ is completely continuous, then $\Psi$ has a fixed point in $Q$.

## 3 Approximate controllability

In this section, we first prove that the operator $F^{\mu}, 0<\mu \leq 1$ (which is defined below) has a fixed point based on the Schauder's fixed point theorem. Then we consider the approximate controllability of system (1) under the condition that the linear fractional system (6) is approximately controllable.
In the Banach space $C(I, C) \times C(I, U)$, let

$$
Y_{r}=\left\{(x, u) \in C(I, C) \times C(I, U) \mid\|(x, u)\|=\left\|x_{t}\right\|+\|u(t)\| \leq r, t \in I\right\},
$$

where $r$ is a positive constant.
To prove the main results of this section, we impose the following hypotheses:
(H1) $T(t)$ is a compact operator for every $t>0$.
(H2) The function $f: I \times C \times U \rightarrow X$ is continuous, and there exist a constant $q_{1} \in(0, q)$, $\lambda_{i}(t) \in L^{\frac{1}{q_{1}}}\left(I, R^{+}\right)$and $g_{i} \in L^{1}\left(C \times U, R^{+}\right), i=1,2, \ldots, m$, such that

$$
\begin{aligned}
& \left\|f\left(t, x_{t}, u\right)\right\| \\
& \quad \leq \sum_{i=1}^{m} \lambda_{i}(t) \sup \left\{g_{i}\left(x_{t}, u\right):\left\|x_{t}\right\|+\|u(t)\| \leq r\right\}, \quad\left(t, x_{t}, u\right) \in I \times C \times U .
\end{aligned}
$$

(H3) For each $\mu>0$,

$$
\limsup _{r \rightarrow \infty}\left(r-\sum_{i=1}^{m} \frac{c_{i}}{\mu} \sup \left\{g_{i}\left(x_{t}, u\right):\left\|x_{t}\right\|+\|u(t)\| \leq r\right\}\right)=\infty
$$

(H4) The function $f: I \times C \times U \rightarrow X$ is continuous and uniformly bounded, and there is a constant $N>0$ such that

$$
\|f(t, \varphi, u)\| \leq N, \quad(t, \varphi, u) \in I \times C \times U
$$

For the sake of convenience, we also introduce the following notations.

$$
\begin{aligned}
& k=\max \left\{1, \frac{M M_{B} T^{q}}{\Gamma(1+q)}\right\}, \quad M_{B}=\|B\|, \quad a=\frac{q-1}{1-q_{1}} \in(-1,0), \\
& a_{i}=\frac{3 k M_{B} M q}{\Gamma(1+q)} \times \frac{M q T^{(a+1)\left(1-q_{1}\right)}}{\Gamma(1+q)(1+a)^{1-q_{1}}}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}}[0, T]}^{\prime} \\
& b_{i}=\frac{3 M q T^{(a+1)\left(1-q_{1}\right)}}{\Gamma(1+q)(1+a)^{1-q_{1}}}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}}[0, T]}, \\
& c_{i}=\max \left\{a_{i}, b_{i}\right\}, \\
& d_{1}=\frac{3 k M_{B} M q}{\Gamma(1+q)}\left(\left\|x_{T}\right\|+M\|\phi\|\right), \quad d_{2}=3(M+1)\|\phi\|, \quad d=\max \left\{d_{1}, d_{2}\right\} .
\end{aligned}
$$

For $\mu>0$, we define the operator $F^{\mu}$ on $C(I, C) \times C(I, U)$ as

$$
\begin{equation*}
F^{\mu}(x, u)=(z, v), \tag{7}
\end{equation*}
$$

where

$$
\begin{align*}
& v(t)=B^{*} T_{q}^{*}(T-t) R\left(\mu, \Gamma_{0}^{T}\right) p(x, u),  \tag{8}\\
& z(t)=S_{q}(t) \phi(0)+\int_{0}^{t}(t-s)^{q-1} T_{q}(t-s)\left(B v(s)+f\left(s, x_{s}, u(s)\right)\right) d s, \quad t>0,  \tag{9}\\
& z_{0}(\theta)=\phi(\theta), \quad-h \leq \theta \leq 0 \\
& p(x, u)=x_{T}-S_{q}(T) \phi(0)-\int_{0}^{T}(T-s)^{q-1} T_{q}(T-s) f\left(s, x_{s}, u(s)\right) d s
\end{align*}
$$

We next prove that the operator $F^{\mu}$ has a fixed point in $C(I, C) \times C(I, U)$.

Theorem 3.1 If the hypotheses (H0)-(H2) are satisfied, then for each $0<\mu \leq 1$, the operator $F^{\mu}$ has a fixed point in $C(I, C) \times C(I, U)$.

Proof The proof of Theorem 3.1 is divided into several steps.
Step 1. We first prove that for an arbitrary $0<\mu \leq 1$, there is a constant $r_{0}=r_{0}(\mu)>0$ such that $F^{\mu}: Y_{r_{0}} \rightarrow Y_{r_{0}}$. Let

$$
\begin{equation*}
\psi_{i}(r)=\sup \left\{g_{i}\left(x_{t}, u\right):\left\|x_{t}\right\|+\|u(t)\| \leq r,\left(x_{t}, u\right) \in C \times U\right\} \tag{10}
\end{equation*}
$$

It follows from (H3) and (10) that there exists $r_{0}$ such that

$$
\begin{equation*}
r_{0}-\sum_{i=1}^{m} \frac{c_{i}}{\mu} \psi_{i}\left(r_{0}\right) \geq \frac{d}{\mu} \tag{11}
\end{equation*}
$$

that is,

$$
\begin{equation*}
\frac{d}{\mu}+\sum_{i=1}^{m} \frac{c_{i}}{\mu} \psi_{i}\left(r_{0}\right) \leq r_{0} \tag{12}
\end{equation*}
$$

Direct calculation gives that $(t-s)^{q-1} \in L^{\frac{1}{1-q_{1}}}[0, t], t \in[0, T], q_{1} \in(0, q)$. By Lemma 2.1(i), (H2) and (10), using Hölder's inequality, we obtain that

$$
\begin{align*}
& \int_{0}^{t}\left\|(t-s)^{q-1} T_{q}(t-s) f\left(s, x_{s}, u(s)\right)\right\| d s \\
& \quad \leq \frac{M q}{\Gamma(1+q)} \int_{0}^{t}(t-s)^{q-1} \sum_{i=1}^{m} \lambda_{i}(s) \psi_{i}(r) d s \\
& \quad \leq \frac{M q}{\Gamma(1+q)}\left(\int_{0}^{t}\left((t-s)^{q-1}\right)^{\frac{1}{1-q_{1}}} d s\right)^{1-q_{1}} \sum_{i=1}^{m}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}[0, T]}} \psi_{i}(r) \\
& \quad \leq \frac{M q T^{(a+1)\left(1-q_{1}\right)}}{\Gamma(1+q)(1+a)^{1-q_{1}}} \sum_{i=1}^{m}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}}[0, T]} \psi_{i}(r) . \tag{13}
\end{align*}
$$

If $(x, u) \in Y_{r_{0}}$, then it follows from (8), (12) and (13) that

$$
\begin{align*}
\|v(t)\|= & \| B^{*} T_{q}^{*}(T-t) R\left(\mu, \Gamma_{0}^{T}\right)\left(x_{T}-S_{q}(T) \phi(0)\right. \\
& \left.-\int_{0}^{T}(T-s)^{q-1} T_{q}(T-s) f\left(s, x_{s}, u(s)\right) d s\right) \| \\
\leq & \frac{1}{\mu} \frac{M_{B} M q}{\Gamma(1+q)}\left(\left\|x_{T}\right\|+M\|\phi\|+\frac{M q T^{(a+1)\left(1-q_{1}\right)}}{\Gamma(1+q)(1+a)^{1-q_{1}}} \sum_{i=1}^{m}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}[0, T]}} \psi_{i}\left(r_{0}\right)\right) \\
\leq & \frac{1}{\mu}\left[\frac{d}{3 k}+\frac{1}{3 k} \sum_{i=1}^{m} c_{i} \psi_{i}\left(r_{0}\right)\right] \leq \frac{1}{3 k \mu}\left[d+\sum_{i=1}^{m} c_{i} \psi_{i}\left(r_{0}\right)\right] \\
\leq & \frac{r_{0}}{3 k} \tag{14}
\end{align*}
$$

and

$$
\begin{align*}
\left\|z_{t}(\theta)\right\| \leq & \left\|S_{q}(t+\theta) \phi(0)+\int_{0}^{t+\theta}(t+\theta-s)^{q-1} T_{q}(t+\theta-s)\left(B v(s)+f\left(s, x_{s}, u(s)\right)\right) d s\right\| \\
& +\|\phi\| \\
\leq & (M+1)\|\phi\|+\frac{M M_{B} T^{q}}{\Gamma(1+q)}\|v\|+\frac{M q T^{(a+1)\left(1-q_{1}\right)}}{\Gamma(1+q)(1+a)^{1-q_{1}}} \sum_{i=1}^{m}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}}[0, T]} \psi_{i}\left(r_{0}\right) \\
\leq & \frac{d}{3}+k\|v\|+\frac{1}{3} \sum_{i=1}^{m} c_{i} \psi_{i}\left(r_{0}\right) \\
\leq & \frac{1}{3}\left[d+\sum_{i=1}^{m} c_{i} \psi_{i}\left(r_{0}\right)\right]+k\|v\| \\
\leq & \frac{\mu r_{0}}{3}+\frac{r_{0}}{3} \leq \frac{2 r_{0}}{3} . \tag{15}
\end{align*}
$$

Thus,

$$
\begin{equation*}
\left\|\left(F^{\mu}(x, u)\right)(t)\right\|=\|(z, v)\|=\left\|z_{t}\right\|+\|v(t)\| \leq r_{0} \tag{16}
\end{equation*}
$$

Therefore, $F^{\mu}: Y_{r_{0}} \rightarrow Y_{r_{0}}$.
Step 2. We will prove that $F^{\mu}: Y_{r_{0}} \rightarrow Y_{r_{0}}$ is continuous. Let $\left\{\left(y^{n}, u^{n}\right)\right\} \subset Y_{r_{0}}$ and $\left(y^{n}, u^{n}\right) \rightarrow$ $(y, u),(y, u) \in Y_{r_{0}}$. By (H2), we have $f\left(s, y_{s}^{n}, u^{n}(s)\right) \rightarrow f\left(s, y_{s}, u(s)\right)$ as $n \rightarrow \infty, s \in I$ and

$$
\begin{equation*}
\left\|f\left(s, y_{s}^{n}, u^{n}(s)\right)-f\left(s, y_{s}, u(s)\right)\right\| \leq 2 \sum_{i=1}^{m} \lambda_{i}(s) \psi_{i}\left(r_{0}\right) \tag{17}
\end{equation*}
$$

It follows from the Lebesgue dominated convergence theorem, (8) and (9) that for each $t \in[0, T]$, there exists a constant $l$ such that

$$
\begin{align*}
\| & \left(F^{\mu}\left(y^{n}, u^{n}\right)\right)(t)-\left(F^{\mu}(y, u)\right)(t) \| \\
= & \left\|z_{t}^{n}-z_{t}\right\|+\left\|v^{n}(t)-v(t)\right\| \\
\leq & \left\|\int_{0}^{t+\theta}(t+\theta-s)^{q-1} T_{q}(t+\theta-s)\left[B\left(v^{n}(s)-v(s)\right)+f\left(s, y_{s}^{n}, u^{n}(s)\right)-f\left(s, y_{s}, u(s)\right)\right] d s\right\| \\
& +\left\|B^{*} T_{q}^{*}(T-t) R\left(\mu, \Gamma_{0}^{T}\right)\left[p\left(y^{n}, u^{n}\right)-p(y, u)\right]\right\| \\
\leq & l \int_{0}^{t+\theta}(t+\theta-s)^{q-1}\left\|f\left(s, y_{s}^{n}, u^{n}(s)\right)-f\left(s, y_{s}, u(s)\right)\right\| d s \rightarrow 0, \quad n \rightarrow \infty \tag{18}
\end{align*}
$$

which implies

$$
\begin{equation*}
\left\|F^{\mu}\left(y^{n}, u^{n}\right)-F^{\mu}(y, u)\right\| \rightarrow 0, \quad n \rightarrow \infty \tag{19}
\end{equation*}
$$

This means that $F^{\mu}$ is continuous.
Step 3. Next, we will show that

$$
V=\left\{\left(F^{\mu}(x, u)\right)(\cdot):(x, u) \in Y_{r_{0}}\right\}
$$

is equicontinuous on $[0, T]$. Indeed, for $0 \leq t_{1}+\theta<t_{2}+\theta \leq T$ and ( $\left.x, u\right) \in Y_{r_{0}}$, (8), (9) and (13) imply

$$
\begin{align*}
\left\|v\left(t_{2}\right)-v\left(t_{1}\right)\right\|= & \| B^{*}\left(T_{q}^{*}\left(T-t_{2}\right)-T_{q}^{*}\left(T-t_{1}\right)\right) R\left(\mu, \Gamma_{0}^{T}\right)\left(x_{T}-S_{q}(T) \phi(0)\right. \\
& \left.-\int_{0}^{T}(T-s)^{q-1} T_{q}(T-s) f\left(s, x_{s}, u(s)\right) d s\right) \| \\
\leq & \frac{M_{B}}{\mu} \|\left(T_{q}^{*}\left(T-t_{2}\right)-T_{q}^{*}\left(T-t_{1}\right)\right)\left(\left\|x_{T}\right\|+M\|\phi\|\right. \\
& \left.+\frac{M q T^{(a+1)\left(1-q_{1}\right)}}{\Gamma(1+q)(1+a)^{1-q_{1}}} \sum_{i=1}^{m}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}}{ }_{[0, T]}} \psi_{i}\left(r_{0}\right)\right) \| \\
= & I_{0} \tag{20}
\end{align*}
$$

and

$$
\begin{align*}
\| z_{t_{2}}(\theta) & -z_{t_{1}}(\theta) \| \\
= & \left\|S_{q}\left(t_{2}+\theta\right) \phi(0)-S_{q}\left(t_{1}+\theta\right) \phi(0)\right\| \\
& +\| \int_{0}^{t_{2}+\theta}\left(t_{2}+\theta-s\right)^{q-1} T_{q}\left(t_{2}+\theta-s\right)\left(B v(s)+f\left(s, x_{s}, u(s)\right)\right) d s \\
& -\int_{0}^{t_{1}+\theta}\left(t_{1}+\theta-s\right)^{q-1} T_{q}\left(t_{1}+\theta-s\right)\left(B v(s)+f\left(s, x_{s}, u(s)\right)\right) d s \| \\
\leq & \left\|S_{q}\left(t_{2}+\theta\right) \phi(0)-S_{q}\left(t_{1}+\theta\right) \phi(0)\right\| \\
& +\left\|\int_{t_{1}+\theta}^{t_{2}+\theta}\left(t_{2}+\theta-s\right)^{q-1} T_{q}\left(t_{2}+\theta-s\right) f\left(s, x_{s}, u(s)\right) d s\right\| \\
& +\left\|\int_{0}^{t_{1}+\theta}\left[\left(t_{2}+\theta-s\right)^{q-1}-\left(t_{1}+\theta-s\right)^{q-1}\right] T_{q}\left(t_{2}+\theta-s\right) f\left(s, x_{s}, u(s)\right) d s\right\| \\
& +\left\|\int_{0}^{t_{1}+\theta}\left(t_{1}+\theta-s\right)^{q-1}\left[T_{q}\left(t_{2}+\theta-s\right)-T_{q}\left(t_{1}+\theta-s\right)\right] f\left(s, x_{s}, u(s)\right) d s\right\| \\
& +\left\|\int_{t_{1}+\theta}^{t_{2}+\theta}\left(t_{2}+\theta-s\right)^{q-1} T_{q}\left(t_{2}+\theta-s\right) B v(s) d s\right\| \\
& +\left\|\int_{0}^{t_{1}+\theta}\left[\left(t_{2}+\theta-s\right)^{q-1}-\left(t_{1}+\theta-s\right)^{q-1}\right] T_{q}\left(t_{2}+\theta-s\right) B v(s) d s\right\| \\
& +\left\|\int_{0}^{t_{1}+\theta}\left(t_{1}+\theta-s\right)^{q-1}\left[T_{q}\left(t_{2}+\theta-s\right)-T_{q}\left(t_{1}+\theta-s\right)\right] B v(s) d s\right\| \\
= & I_{1}+I_{2}+I_{3}+I_{4}+I_{5}+I_{6}+I_{7} . \tag{21}
\end{align*}
$$

We now need to check $I_{i} \rightarrow 0$ as $t_{2}-t_{1} \rightarrow 0, i=0,1,2, \ldots, 7$.
For $I_{0}$ and $I_{1}$, by Lemma 2.1(ii), $I_{0}, I_{1} \rightarrow 0$ as $t_{2}-t_{1} \rightarrow 0$.
For $I_{2}$ and $I_{3}$, similar to (13), by Lemma 2.1(i), (H2) and Lemma 2.2, we have

$$
I_{2}=\left\|\int_{t_{1}+\theta}^{t_{2}+\theta}\left(t_{2}+\theta-s\right)^{q-1} T_{q}\left(t_{2}+\theta-s\right) f\left(s, x_{s}, u(s)\right) d s\right\|
$$

$$
\begin{align*}
& \leq \frac{M q}{\Gamma(1+q)}\left(\int_{t_{1}+\theta}^{t_{2}+\theta}\left(\left(t_{2}+\theta-s\right)^{q-1}\right)^{\frac{1}{1-q_{1}}} d s\right)^{1-q_{1}} \sum_{i=1}^{m}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}}{ }_{[0, T]}} \psi_{i}\left(r_{0}\right) \\
& =\frac{M q\left(t_{2}-t_{1}\right)^{(1+a)\left(1-q_{1}\right)}}{\Gamma(1+q)(1+a)^{1-q_{1}}} \sum_{i=1}^{m}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}}[0, T]} \psi_{i}\left(r_{0}\right) \\
& \rightarrow 0 \quad \text { as } t_{2}-t_{1} \rightarrow 0 \tag{22}
\end{align*}
$$

and

$$
\begin{align*}
I_{3}= & \left\|\int_{0}^{t_{1}+\theta}\left[\left(t_{2}+\theta-s\right)^{q-1}-\left(t_{1}+\theta-s\right)^{q-1}\right] T_{q}\left(t_{2}+\theta-s\right) f\left(s, x_{s}, u(s)\right) d s\right\| \\
\leq & \frac{M q}{\Gamma(1+q)} \int_{0}^{t_{1}+\theta}\left[\left(t_{1}+\theta-s\right)^{q-1}-\left(t_{2}+\theta-s\right)^{q-1}\right] \sum_{i=1}^{m} \lambda_{i}(s) \psi_{i}\left(r_{0}\right) d s \\
\leq & \frac{M q}{\Gamma(1+q)}\left(\int_{0}^{t_{1}+\theta}\left(\left(\left(t_{1}+\theta-s\right)^{q-1}-\left(t_{2}+\theta-s\right)\right)^{q-1}\right)^{\frac{1}{1-q_{1}}} d s\right)^{1-q_{1}} \\
& \times \sum_{i=1}^{m}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}}[0, T]} \psi_{i}\left(r_{0}\right) \\
\leq & \frac{M q}{\Gamma(1+q)}\left(\int_{0}^{t_{1}+\theta}\left(t_{1}+\theta-s\right)^{a}-\left(t_{2}+\theta-s\right)^{a} d s\right)^{1-q_{1}} \sum_{i=1}^{m}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}}[0, T]} \psi_{i}\left(r_{0}\right) \\
= & \frac{M q}{\Gamma(1+q)(1+a)^{1-q_{1}}}\left(\left(t_{1}+\theta\right)^{a+1}-\left(t_{2}+\theta\right)^{a+1}+\left(t_{2}-t_{1}\right)^{a+1}\right)^{1-q_{1}} \\
& \times \sum_{i=1}^{m}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}}[0, T]} \psi_{i}\left(r_{0}\right) \\
\leq & \frac{M q\left(t_{2}-t_{1}\right)^{(1+a)\left(1-q_{1}\right)}}{\Gamma(1+q)(1+a)^{1-q_{1}}} \sum_{i=1}^{m}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}}[0, T]} \psi_{i}\left(r_{0}\right) \\
\rightarrow & 0 \text { as } t_{2}-t_{1} \rightarrow 0 . \tag{23}
\end{align*}
$$

For $t_{1}+\theta=0,0<t_{2}+\theta \leq T$, it can be easily seen that $I_{4}=0$. For $t_{1}+\theta>0$ and $\varepsilon>0$ small enough, by Lemma 2.1(i), (H2) and Lemma 2.2, we obtain

$$
\begin{aligned}
I_{4}= & \left\|\int_{0}^{t_{1}+\theta}\left(t_{1}+\theta-s\right)^{q-1}\left[T_{q}\left(t_{2}+\theta-s\right)-T_{q}\left(t_{1}+\theta-s\right)\right] f\left(s, x_{s}, u(s)\right) d s\right\| \\
\leq & \int_{0}^{t_{1}+\theta-\varepsilon}\left(t_{1}+\theta-s\right)^{q-1}\left\|T_{q}\left(t_{2}+\theta-s\right)-T_{q}\left(t_{1}+\theta-s\right)\right\|\left\|f\left(s, x_{s}, u(s)\right)\right\| d s \\
& +\int_{t_{1}+\theta-\varepsilon}^{t_{1}+\theta}\left(t_{1}+\theta-s\right)^{q-1}\left\|\left(T_{q}\left(t_{2}+\theta-s\right)-T_{q}\left(t_{1}+\theta-s\right)\right) f\left(s, x_{s}, u(s)\right)\right\| d s \\
\leq & \sup _{s \in\left[0, t_{1}+\theta-\varepsilon\right]}\left\|T_{q}\left(t_{2}+\theta-s\right)-T_{q}\left(t_{1}+\theta-s\right)\right\| \int_{0}^{t_{1}+\theta-\varepsilon}\left(t_{1}+\theta-s\right)^{q-1} \sum_{i=1}^{m} \lambda_{i}(s) \psi_{i}\left(r_{0}\right) d s \\
& +\frac{2 M q}{\Gamma(1+q)} \int_{t_{1}+\theta-\varepsilon}^{t_{1}+\theta}\left(t_{1}+\theta-s\right)^{q-1} \sum_{i=1}^{m} \lambda_{i}(s) \psi_{i}\left(r_{0}\right) d s \\
\leq & \sup _{s \in\left[0, t_{1}+\theta-\varepsilon\right]}\left\|T_{q}\left(t_{2}+\theta-s\right)-T_{q}\left(t_{1}+\theta-s\right)\right\|\left(\int_{0}^{t_{1}+\theta-\varepsilon}\left(\left(t_{1}+\theta-s\right)^{q-1}\right)^{\frac{1}{1-q_{1}}} d s\right)^{1-q_{1}}
\end{aligned}
$$

$$
\begin{align*}
& \quad \times \sum_{i=1}^{m}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}}{ }_{[0, T]} \psi_{i}\left(r_{0}\right)+\frac{2 M q}{\Gamma(1+q)}\left(\int_{t_{1}+\theta-\varepsilon}^{t_{1}+\theta}\left(\left(t_{1}+\theta-s\right)^{q-1}\right)^{\frac{1}{1-q_{1}}} d s\right)^{1-q_{1}}} \quad \times \sum_{i=1}^{m}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}}{ }_{[0, T]} \psi_{i}\left(r_{0}\right)}^{=\sup _{s \in\left[0, t_{1}+\theta-\varepsilon\right]}\left\|T_{q}\left(t_{2}+\theta-s\right)-T_{q}\left(t_{1}+\theta-s\right)\right\| \frac{\left(\left(t_{1}+\theta\right)^{1+b}-\varepsilon^{1+a}\right)^{1-q_{1}}}{(1+a)^{1-q_{1}}}} \begin{array}{l}
\quad \times \sum_{i=1}^{m}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}}[0, T]} \psi_{i}\left(r_{0}\right) \\
\quad+\frac{2 M q}{\Gamma(1+q)(a+1)^{1-q_{1}}} \varepsilon^{(a+1)\left(1-q_{1}\right)} \sum_{i=1}^{m}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}}}^{[0, T]} \psi_{i}\left(r_{0}\right)
\end{array} .
\end{align*}
$$

Assumption (H1) and Lemma 2.1(ii), (iii) imply the continuity of $T_{q}(t), t>0$ in the uniform operator topology, then it is easy to obtain that $I_{4}$ tends to zero as $t_{2}-t_{1} \rightarrow 0$ and $\varepsilon \rightarrow 0$.

On the other hand, by (14), we know that

$$
\begin{align*}
\|v\|= & \| B^{*} T_{q}^{*}(T-t) R\left(\mu, \Gamma_{0}^{T}\right)\left(x_{T}-S_{q}(T) \phi(0)\right. \\
& \left.-\int_{0}^{T}(T-s)^{q-1} T_{q}(T-s) f\left(s, x_{s}, u(s)\right) d s\right) \| \\
\leq & \frac{1}{\mu} \frac{M_{B} M q}{\Gamma(1+q)}\left(\left\|x_{T}\right\|+M\|\phi\|+\frac{M q T^{(a+1)\left(1-q_{1}\right)}}{\Gamma(1+q)(1+a)^{1-q_{1}}} \sum_{i=1}^{m}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}[0, T]}} \psi_{i}\left(r_{0}\right)\right) \tag{25}
\end{align*}
$$

is bounded. Then, by using similar methods as we did to $I_{2}, I_{3}$ and $I_{4}$, it follows

$$
\begin{align*}
I_{5} & =\left\|\int_{t_{1}+\theta}^{t_{2}+\theta}\left(t_{2}+\theta-s\right)^{q-1} T_{q}\left(t_{2}+\theta-s\right) B v(s) d s\right\| \\
& \leq \frac{M q M_{B}\|v\|}{\Gamma(1+q)} \int_{t_{1}+\theta}^{t_{2}+\theta}\left(t_{2}+\theta-s\right)^{q-1} d s \\
& \leq \frac{M M_{B}\|v\|}{\Gamma(1+q)}\left(t_{2}-t_{1}\right)^{q} \\
& \rightarrow 0 \quad \text { as } t_{2}-t_{1} \rightarrow 0,  \tag{26}\\
I_{6} & =\left\|\int_{0}^{t_{1}+\theta}\left[\left(t_{2}+\theta-s\right)^{q-1}-\left(t_{1}+\theta-s\right)^{q-1}\right] T_{q}\left(t_{2}+\theta-s\right) B v(s) d s\right\| \\
& \leq \frac{M q M_{B}\|v\|}{\Gamma(1+q)} \int_{0}^{t_{1}+\theta}\left[\left(\left(t_{1}+\theta-s\right)^{q-1}-\left(t_{2}+\theta-s\right)\right)^{q-1}\right] d s \\
& =\frac{M M_{B}\|v\|}{\Gamma(1+q)}\left[\left(t_{1}+\theta\right)^{q}-\left(t_{2}+\theta\right)^{q}+\left(t_{2}-t_{1}\right)^{q}\right] \\
& \leq \frac{M M_{B}\|v\|}{\Gamma(1+q)}\left(t_{2}-t_{1}\right)^{q} \\
& \rightarrow 0 \quad \text { as } t_{2}-t_{1} \rightarrow 0 \tag{27}
\end{align*}
$$

and

$$
\begin{align*}
I_{7}= & \left\|\int_{0}^{t_{1}+\theta}\left(t_{1}+\theta-s\right)^{q-1}\left[T_{q}\left(t_{2}+\theta-s\right)-T_{q}\left(t_{1}+\theta-s\right)\right] B v(s) d s\right\| \\
\leq & \int_{0}^{t_{1}+\theta-\varepsilon}\left(t_{1}+\theta-s\right)^{q-1}\left\|T_{q}\left(t_{2}+\theta-s\right)-T_{q}\left(t_{1}+\theta-s\right)\right\| M_{B}\|v\| d s \\
& +\int_{t_{1}+\theta-\varepsilon}^{t_{1}+\theta}\left(t_{1}+\theta-s\right)^{q-1}\left\|\left(T_{q}\left(t_{2}+\theta-s\right)-T_{q}\left(t_{1}+\theta-s\right)\right) B v(s)\right\| d s \\
\leq & \sup _{s \in\left[0, t_{1}+\theta-\varepsilon\right]}\left\|T_{q}\left(t_{2}+\theta-s\right)-T_{q}\left(t_{1}+\theta-s\right)\right\| M_{B}\|v\| \frac{\left(t_{1}+\theta\right)^{q}-\varepsilon^{q}}{q} \\
& +\frac{2 M M_{B}\|v\|}{\Gamma(1+q)} \varepsilon^{q} \\
\rightarrow & 0 \quad \text { as } t_{2}-t_{1} \rightarrow 0, \varepsilon \rightarrow 0 . \tag{28}
\end{align*}
$$

In consequence, $\left\|F^{\mu}(x, u)\left(t_{2}\right)-F^{\mu}(x, u)\left(t_{1}\right)\right\|=\left\|z_{t_{2}}-z_{t_{1}}\right\|+\left\|v\left(t_{2}\right)-v\left(t_{1}\right)\right\|$ tends to zero independently of $(x, u) \in Y_{r_{0}}$ as $t_{2}-t_{1} \rightarrow 0$. This means that $V=\left\{\left(F^{\mu}(x, u)\right)(\cdot):(x, u) \in Y_{r_{0}}\right\}$ is equicontinuous.
Since the other cases $t_{1}+\theta<t_{2}+\theta<0, t_{1}+\theta<0<t_{2}+\theta$ are very simple, we only consider the case $0 \leq t_{1}+\theta<t_{2}+\theta$. In conclusion, $F^{\mu}\left[Y_{r_{0}}\right]$ is equicontinuous and also bounded.
Step 4. It remains to prove that for any $t \in[0, T], V(t)=\left\{\left(F^{\mu}(x, u)\right)(t):(x, u) \in Y_{r_{0}}\right\}$ is relatively compact.

Obviously, for $t=0, V(0)$ is compact. Let $0<t \leq T$ be fixed and $\eta$ be a given real number satisfying $0<\eta<t$. For $\eta \in(0, t)$ and $\forall \delta>0$, define an operator $F_{\eta, \delta}^{\mu}$ as

$$
\begin{align*}
&\left(F_{\eta, \delta}^{\mu}(x, u)\right)(t) \\
&=\left(\int_{\delta}^{\infty} \phi_{q}(\theta) T\left(t^{q} \theta\right) \phi(0) d \theta+q \int_{0}^{t-\eta} \int_{\delta}^{\infty} \theta(t-s)^{q-1} \phi_{q}(\theta) T\left((t-s)^{q} \theta\right)\right. \\
&\left.\times\left(B v(s)+f\left(s, x_{s}, u(s)\right)\right) d s, v(t)\right) \\
&=\left(T ( \eta ^ { q } \delta ) \left[\int_{\delta}^{\infty} \phi_{q}(\theta) T\left(t^{q} \theta-\eta^{q} \delta\right) \phi(0) d \theta+q \int_{0}^{t-\eta} \int_{\delta}^{\infty} \theta(t-s)^{q-1} \phi_{q}(\theta)\right.\right. \\
&\left.\left.\times T\left((t-s)^{q} \theta-\eta^{q} \delta\right)\left(B v(s)+f\left(s, x_{s}, u(s)\right)\right) d s\right], v(t)\right) \\
&=\left(T\left(\eta^{q} \delta\right) z(t, \eta), v(t)\right), \tag{29}
\end{align*}
$$

where $(x, u) \in Y_{r_{0}}$. Since $T\left(\eta^{q} \delta\right), \eta^{q} \delta>0$ is compact, $z(t, \eta)$ and $v(t)$ are bounded on $Y_{r_{0}}$, the set

$$
V_{\eta, \delta}(t)=\left\{\left(F_{\eta, \delta}^{\mu}(x, u)\right)(t):(x, u) \in Y_{r_{0}}\right\}
$$

is relatively compact in $C \times U$ for $\eta \in(0, t)$ and $\forall \delta>0$. Furthermore, for $(x, u) \in Y_{r_{0}}$,

$$
\begin{aligned}
& \left\|\left(F^{\mu}(x, u)\right)(t)-\left(F_{\eta, \delta}^{\mu}(x, u)\right)(t)\right\| \\
& \quad=\| \int_{0}^{\delta} \phi_{q}(\theta) T\left(t^{q} \theta\right) d \theta \phi(0)+q \int_{0}^{t} \int_{0}^{\delta} \theta(t-s)^{q-1} \phi_{q}(\theta) T\left((t-s)^{q} \theta\right) B v(s) d s
\end{aligned}
$$

$$
\begin{align*}
&+q \int_{0}^{t} \int_{0}^{\delta} \theta(t-s)^{q-1} \phi_{q}(\theta) T\left((t-s)^{q} \theta\right) f\left(s, x_{s}, u(s)\right) d s \\
&+\int_{t-\eta}^{t} \int_{\delta}^{\infty} \theta(t-s)^{q-1} \phi_{q}(\theta) T\left((t-s)^{q} \theta\right) B v(s) d s \\
&+\int_{t-\eta}^{t} \int_{\delta}^{\infty} \theta(t-s)^{q-1} \phi_{q}(\theta) T\left((t-s)^{q} \theta\right) f\left(s, x_{s}, u(s)\right) d s \| \\
& \leq M_{1}\|\phi\| \int_{0}^{\delta} \phi_{q}(\theta) d \theta+q M_{1} M_{B}\|v\| \int_{0}^{t}(t-s)^{q-1} d s \int_{0}^{\delta} \theta \phi_{q}(\theta) d \theta \\
&+q M_{1}\left(\int_{0}^{t}\left((t-s)^{q-1}\right)^{\frac{1}{1-q_{1}}} d s\right)^{1-q_{1}} \cdot \sum_{i=1}^{m}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}}[0, T]} \psi_{i}\left(r_{0}\right) \int_{0}^{\delta} \theta \phi_{q}(\theta) d \theta \\
&+q M_{1} M_{B}\|v\| \int_{t-\eta}^{t}(t-s)^{q-1} d s \int_{0}^{\infty} \theta \phi_{q}(\theta) d \theta \\
&+q M_{1}\left(\int_{t-\eta}^{t}\left((t-s)^{q-1}\right)^{\frac{1}{1-q_{1}}} d s\right)^{1-q_{1}} \cdot \sum_{i=1}^{m}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}}[0, T]} \psi_{i}\left(r_{0}\right) \int_{0}^{\infty} \theta \phi_{q}(\theta) d \theta \\
& \leq {\left[M_{1}\|\phi\|+M_{1} M_{B}\|v\| T^{q}\right.} \\
&\left.+q M_{1} \frac{T^{(a+1)\left(1-q_{1}\right)}}{(a+1)^{1-q_{1}}} \cdot \sum_{i=1}^{m}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}}[0, T]} \psi_{i}\left(r_{0}\right)\right] \int_{0}^{\delta} \theta \phi_{q}(\theta) d \theta \\
& \rightarrow+\frac{M_{1} M_{B}\|v\| \eta^{q}}{\Gamma(1+q)}+\frac{q M_{1}}{\Gamma(1+q)(a+1)^{1-q_{1}} \eta^{(a+1)\left(1-q_{1}\right)} \sum_{i=1}^{m}\left\|\lambda_{i}\right\|_{L^{\frac{1}{q_{1}}}[0, T]} \psi_{i}\left(r_{0}\right)} \\
& \rightarrow \quad \text { as } \delta, \eta \rightarrow 0 . \tag{30}
\end{align*}
$$

Here the last inequality is based on Remark 2.1. This means that there are relatively compact sets arbitrarily close to the set $V(t), t \in(0, T]$. Thus, for each $t \in(0, T], V(t)$ is relatively compact in $C \times U$. By the Arzelá-Ascoli theorem, $F^{\mu}\left[Y_{r_{0}}\right]$ is relatively compact in $C(I, C) \times C(I, U)$. We obtain that $F^{\mu}$ is a completely continuous operator. Hence, Lemma 2.3 shows that $F^{\mu}$ has a fixed point. The proof is complete.

Theorem 3.2 Assume that the hypotheses of Theorem 3.1 and (H4) are satisfied and that the linear system (6) is approximately controllable on I. Then system (1) is approximately controllable on I.

Proof Let $\left(\bar{x}^{\mu}(\cdot), \bar{u}^{\mu}\right)$ be a fixed point in $Y_{r_{0}}$, i.e.,

$$
F^{\mu}\left(\bar{x}^{\mu}(\cdot), \bar{u}^{\mu}\right)=\left(\bar{x}^{\mu}(\cdot), \bar{u}^{\mu}\right),
$$

where

$$
\begin{align*}
& \bar{x}^{\mu}(t)=S_{q}(t) \phi(0)+\int_{0}^{t}(t-s)^{q-1} T_{q}(t-s)\left(B \bar{u}^{\mu}(s)+f\left(s, \bar{x}_{s}^{\mu}, \bar{u}^{\mu}(s)\right)\right) d s, \quad t>0  \tag{31}\\
& \bar{x}_{0}^{\mu}(\theta)=\bar{\phi}(\theta), \quad-h \leq \theta \leq 0
\end{align*}
$$

and the control function

$$
\begin{equation*}
\bar{u}^{\mu}(t)=B^{*} T_{q}^{*}(T-t) R\left(\mu, \Gamma_{0}^{T}\right) p\left(\bar{x}^{\mu}, \bar{u}^{\mu}\right), \tag{32}
\end{equation*}
$$

$p\left(\bar{x}^{\mu}, \bar{u}^{\mu}\right)=x_{T}-S_{q}(T) \phi(0)-\int_{0}^{T}(T-s)^{q-1} T_{q}(T-s) f\left(s, \bar{x}_{s}^{\mu}, \bar{u}^{\mu}(s)\right) d s$. By Definition 2.3, any fixed point of $F^{\mu}$ is a mild solution of (1) under the control $\bar{u}^{\mu}(t)$. Then, by (31), (32) and the fact $I-\Gamma_{0}^{T} R\left(\mu, \Gamma_{0}^{T}\right)=\mu R\left(\mu, \Gamma_{0}^{T}\right)$, we have

$$
\begin{align*}
\bar{x}^{\mu}(T) & =x_{T}-p\left(\bar{x}^{\mu}, \bar{u}^{\mu}\right)+\int_{0}^{T}(T-s)^{q-1} T_{q}(T-s) B \bar{u}^{\mu}(s) d s \\
& =x_{T}-p\left(\bar{x}^{\mu}, \bar{u}^{\mu}\right)+\int_{0}^{T}(T-s)^{q-1} T_{q}(T-s) B B^{*} T_{q}^{*}(T-s) R\left(\mu, \Gamma_{0}^{T}\right) p\left(\bar{x}^{\mu}, \bar{u}^{\mu}\right) d s \\
& =x_{T}-p\left(\bar{x}^{\mu}, \bar{u}^{\mu}\right)+\Gamma_{0}^{T} R\left(\mu, \Gamma_{0}^{T}\right) p\left(\bar{x}^{\mu}, \bar{u}^{\mu}\right) \\
& =x_{T}+\left(\Gamma_{0}^{T} R\left(\mu, \Gamma_{0}^{T}\right)-I\right) p\left(\bar{x}^{\mu}, \bar{u}^{\mu}\right) \\
& =x_{T}-\mu R\left(\mu, \Gamma_{0}^{T}\right) p\left(\bar{x}^{\mu}, \bar{u}^{\mu}\right) . \tag{33}
\end{align*}
$$

Furthermore, by condition (H4), it follows

$$
\begin{equation*}
\left(\int_{0}^{T}\left\|f\left(s, \bar{x}_{s}^{\mu}, \bar{u}^{\mu}(s)\right)\right\|^{2} d s\right)^{\frac{1}{2}} \leq N T^{\frac{1}{2}} \tag{34}
\end{equation*}
$$

Consequently, $\left\{f\left(s, \bar{x}_{s}^{\mu}, \bar{u}^{\mu}(s)\right)\right\}$ is bounded in $L^{2}(I, X)$. Then there exists a subsequence, still denoted by $\left\{f\left(s, \bar{x}_{s}^{\mu}, \bar{u}^{\mu}(s)\right)\right\}$, which weakly converges to $f(s)$ in $L^{2}(I, X)$. Define

$$
\begin{equation*}
w=x_{T}-S_{q}(T) \phi(0)+\int_{0}^{T}(T-s)^{q-1} T_{q}(T-s) f(s) d s . \tag{35}
\end{equation*}
$$

It follows from (35) that

$$
\begin{align*}
\| p & \left(\bar{x}^{\mu}, \bar{u}^{\mu}\right)-w \| \\
& =\left\|\int_{0}^{T}(T-s)^{q-1} T_{q}(T-s)\left(f\left(s, \bar{x}_{s}^{\mu}, \bar{u}^{\mu}(s)\right)-f(s)\right) d s\right\| \\
& =\sup _{0 \leq t \leq T}\left\|\int_{0}^{t}(t-s)^{q-1} T_{q}(t-s)\left(f\left(s, \bar{x}_{s}^{\mu}, \bar{u}^{\mu}(s)\right)-f(s)\right) d s\right\| . \tag{36}
\end{align*}
$$

As Step 3, Step 4 in the proof of Theorem 3.1, using the Arzelá-Ascoli theorem one can obtain that an operator $g(\cdot) \rightarrow \int_{0}^{\cdot}(\cdot-s)^{q-1} T_{q}(\cdot-s) g(s) d s$ is compact. Thus, the right-hand side of (36) tends to zero as $\mu \rightarrow 0^{+}$. Then (H0), (33) and (36) imply

$$
\begin{align*}
\left\|\bar{x}^{\mu}(T)-x_{T}\right\| & =\left\|\mu R\left(\mu, \Gamma_{0}^{T}\right) p\left(\bar{x}^{\mu}, \bar{u}^{\mu}\right)\right\| \\
& =\left\|\mu R\left(\mu, \Gamma_{0}^{T}\right)(w)+\mu R\left(\mu, \Gamma_{0}^{T}\right)\left(p\left(\bar{x}^{\mu}, \bar{u}^{\mu}\right)-w\right)\right\| \\
& \leq\left\|\mu R\left(\mu, \Gamma_{0}^{T}\right)(w)\right\|+\left\|\mu R\left(\mu, \Gamma_{0}^{T}\right)\right\|\left\|p\left(\bar{x}^{\mu}, \bar{u}^{\mu}\right)-w\right\| \\
& \leq\left\|\mu R\left(\mu, \Gamma_{0}^{T}\right)(w)\right\|+\left\|p\left(\bar{x}^{\mu}, \bar{u}^{\mu}\right)-w\right\| \rightarrow 0 \tag{37}
\end{align*}
$$

as $\mu \rightarrow 0^{+}$. This proves that system (1) is approximately controllable on $I$. The proof is complete.

## 4 Complete controllability

In this section, we formulate sufficient conditions for complete controllability of the semilinear fractional system (1) under the assumption that the linear system (6) is completely controllable. In this case, the condition of compactness of $T(t)$ is not made. To prove our results, let us assume that
(H5) The function $f: I \times C \times U \rightarrow X$ is continuous, and there exists a constant $L>0$ such that

$$
\|f(t, \varphi, u)\| \leq L\left(1+\|\varphi\|_{C}+\|u\|\right), \quad(t, \varphi, u) \in I \times C \times U .
$$

(H6) There exists a constant $L^{\prime}$ such that

$$
\begin{aligned}
& \left\|f\left(t, \varphi_{1}, u_{1}\right)-f\left(t, \varphi_{2}, u_{2}\right)\right\| \\
& \quad \leq L^{\prime}\left(\left\|\varphi_{1}-\varphi_{2}\right\|_{C}+\left\|u_{1}-u_{2}\right\|\right) \quad \text { for }\left(\varphi_{1}, u_{1}\right),\left(\varphi_{2}, u_{2}\right) \in C \times U .
\end{aligned}
$$

(H7) The linear system (6) is completely controllable.

Lemma $4.1([25,30])$ The linear fractional control system (6) is completely controllable on $I$ if and only if there exists $\gamma>0$ such that

$$
\left\langle\Gamma_{0}^{T} x, x\right\rangle \geq \gamma\|x\|^{2} \quad \text { in the Hilbert space } X, \quad \text { i.e., } \quad\left\|\left(\Gamma_{0}^{T}\right)^{-1}\right\| \leq \frac{1}{\gamma} .
$$

Define the operator $F^{0}$ on $C(I, C) \times C(I, U)$ as

$$
\begin{equation*}
F^{0}(x, u)=(z, v), \tag{38}
\end{equation*}
$$

where

$$
\begin{align*}
& v(t)=B^{*} T_{q}^{*}(T-t)\left(\Gamma_{0}^{T}\right)^{-1} p(x, u),  \tag{39}\\
& z(t)=S_{q}(t) \phi(0)+\int_{0}^{t}(t-s)^{q-1} T_{q}(t-s)\left(B v(s)+f\left(s, x_{s}, u(s)\right)\right) d s  \tag{40}\\
& z_{0}(\theta)=\phi(\theta), \quad-h \leq \theta \leq 0 \\
& p(x, u)=x_{T}-S_{q}(T) \phi(0)-\int_{0}^{T}(T-s)^{q-1} T_{q}(T-s) f\left(s, x_{s}, u(s)\right) d s .
\end{align*}
$$

Theorem 4.1 Assume that the hypotheses (H5), (H6) and (H7) are satisfied. If

$$
\begin{equation*}
\left(\frac{M_{B} M^{2} q T^{q}}{\gamma \Gamma^{2}(1+q)}+\frac{M_{B}^{2} M^{3} q T^{2 q}}{\gamma \Gamma^{3}(1+q)}+\frac{M T^{q}}{\Gamma(1+q)}\right) L^{\prime}<1, \tag{41}
\end{equation*}
$$

then the operator $F^{0}$ has a fixed point in $C(I, C) \times C(I, U)$.

Proof First we show that $F^{0}$ maps $C(I, C) \times C(I, U)$ into itself. By (H5), Lemma 2.1(i) and Lemma 4.1, there exist two constants $l_{1}, l_{2}>0$ such that

$$
\begin{align*}
\|v(t)\| & \leq \frac{M_{B} M q}{\gamma \Gamma(1+q)}\left(\left\|x_{T}\right\|+M\|\phi\|+\frac{M q}{\Gamma(1+q)} \int_{0}^{T}(T-s)^{q-1}\left\|f\left(s, x_{s}, u(s)\right)\right\| d s\right) \\
& \leq \frac{M_{B} M q}{\gamma \Gamma(1+q)}\left(\left\|x_{T}\right\|+M\|\phi\|+\frac{M T^{q}}{\Gamma(1+q)} L\left(1+\|x\|_{C}+\|u\|\right)\right) \\
& \leq l_{1}\left(1+\|x\|_{C}+\|u\|\right) \tag{42}
\end{align*}
$$

and

$$
\begin{align*}
\|z(t)\| \leq & M\|\phi\|+\frac{M q}{\Gamma(1+q)} \int_{0}^{t}(t-s)^{q-1}\|B v(s)\| d s \\
& +\frac{M q}{\Gamma(1+q)} \int_{0}^{t}(t-s)^{q-1}\left\|f\left(s, x_{s}, u(s)\right)\right\| d s \\
\leq & M\|\phi\|+\frac{M M_{B} T^{q}}{\Gamma(1+q)} C_{1}\left(1+\|x\|_{C}+\|u\|\right)+\frac{M T^{q}}{\Gamma(1+q)} L\left(1+\|x\|_{C}+\|u\|\right) \\
\leq & l_{2}\left(1+\|x\|_{C}+\|u\|\right) \tag{43}
\end{align*}
$$

respectively. It follows from (42) and (43) that there exists a constant $C_{3}$ such that

$$
\begin{equation*}
\left\|F^{0}(x, u)\right\|=\|v\|+\|z\| \leq l_{3}\left(1+\|x\|_{C}+\|u\|\right) \tag{44}
\end{equation*}
$$

This means $F^{0}$ maps $C(I, C) \times C(I, U)$ into itself.
We next prove that $F^{0}$ is a contraction mapping. For any $(x, u),(y, w) \in C(I, C) \times C(I, U)$, it holds

$$
\begin{align*}
&\left\|F^{0}(x, u)-F^{0}(y, w)\right\| \\
& \quad \leq\left\|v_{1}-v_{2}\right\|+\left\|z_{1}-z_{2}\right\| \\
& \leq\left\|v_{1}-v_{2}\right\|+\left\|\int_{0}^{t}(t-s)^{q-1} T_{q}(t-s) B\left(v_{1}(s)-v_{2}(s)\right) d s\right\| \\
& \quad+\left\|\int_{0}^{t}(t-s)^{q-1} T_{q}(t-s)\left(f\left(s, x_{s}, u(s)\right)-f\left(s, y_{s}, w(s)\right)\right) d s\right\| \\
& \quad I_{1}+I_{2}+I_{3} . \tag{45}
\end{align*}
$$

It follows from Lemma 2.1(i), Lemma 4.1 and (H6) that

$$
\begin{align*}
I_{1} & =\left\|v_{1}-v_{2}\right\| \\
& =\left\|B^{*} T_{q}^{*}(T-t)\left(\Gamma_{0}^{T}\right)^{-1} \int_{0}^{T}(T-s)^{q-1} T_{q}(T-s)\left(f\left(s, x_{s}, u(s)\right)-f\left(s, y_{s}, w(s)\right)\right) d s\right\| \\
& \leq \frac{M_{B}}{\gamma} \frac{M^{2} q^{2}}{\Gamma^{2}(1+q)} \int_{0}^{T}(T-s)^{q-1}\left\|f\left(s, x_{s}, u(s)\right)-f\left(s, y_{s}, w(s)\right)\right\| d s \\
& \leq \frac{M_{B} M^{2} q T^{q}}{\gamma \Gamma^{2}(1+q)} L^{\prime}(\|x-y\|+\|u-w\|), \tag{46}
\end{align*}
$$

$$
\begin{align*}
I_{2} & =\left\|\int_{0}^{t}(t-s)^{q-1} T_{q}(t-s) B\left(v_{1}(s)-v_{2}(s)\right) d s\right\| \\
& \leq \frac{M T^{q} M_{B}}{\Gamma(1+q)}\left\|v_{1}-v_{2}\right\| \\
& \leq \frac{M_{B}^{2} M^{3} q T^{2 q}}{\gamma \Gamma^{3}(1+q)} L^{\prime}(\|x-y\|+\|u-w\|) \tag{47}
\end{align*}
$$

and

$$
\begin{align*}
I_{3} & =\left\|\int_{0}^{t}(t-s)^{q-1} T_{q}(t-s)\left(f\left(s, x_{s}, u(s)\right)-f\left(s, y_{s}, w(s)\right)\right) d s\right\| \\
& \leq \frac{M q}{\Gamma(1+q)} \int_{0}^{t}(t-s)^{q-1}\left\|f\left(s, x_{s}, u(s)\right)-f\left(s, y_{s}, w(s)\right)\right\| d s \\
& \leq \frac{M T^{q}}{\Gamma(1+q)} L^{\prime}(\|x-y\|+\|u-w\|) \tag{48}
\end{align*}
$$

Then (45)-(48) imply

$$
\begin{align*}
\left\|F^{0}(x, u)-F^{0}(y, w)\right\| \leq & \left(\frac{M_{B} M^{2} q T^{q}}{\gamma \Gamma^{2}(1+q)}+\frac{M_{B}^{2} M^{3} q T^{2 q}}{\gamma \Gamma^{3}(1+q)}+\frac{M T^{q}}{\Gamma(1+q)}\right) \\
& \times L^{\prime}(\|x-y\|+\|u-w\|) \tag{49}
\end{align*}
$$

Combining (41) with (49) yields that $F^{0}$ is a contraction mapping. Consequently, $F^{0}$ has a fixed point in $C(I, C) \times C(I, U)$ by the Banach fixed point theorem, which is a mild solution of system (1). This completes the proof.

Remark 4.1 It can be easily seen that inequality (41) is satisfied if the constant $L^{\prime}$ is small enough.

Theorem 4.2 Assume that the assumptions (H5), (H6) and (H7) are satisfied. Then system (1) is completely controllable on I.

Proof If ( $\bar{x}^{0}, \bar{u}^{0}$ ) is a fixed point of the operator $F^{0}$, then (33) holds with $\mu=0$, i.e., $\bar{x}^{0}(T)=$ $x_{T}$ for any $x_{T} \in X$. Hence, system (1) is completely controllable on $I$. We omit the details here.

## 5 An example

As an application of our results, we consider the fractional partial differential equation of the form

$$
\left\{\begin{array}{l}
\partial_{t}^{q} x(t, z)=\partial_{z}^{2} x(t, z)+b(z) u(t)+f(t, x(t-h, z), u(t)), \quad t \in[0,1], \quad z \in[0, \pi]  \tag{50}\\
x(t, 0)=x(t, \pi)=0, \quad t>0 \\
x(t, z)=\phi(t, z), \quad-h \leq t \leq 0
\end{array}\right.
$$

here $\partial_{t}^{q}$ is the Caputo fractional derivative of order $0<q<1$ with respect to $t, \phi$ is continuous, $u \in U=L^{2}[0,1]$ is continuous, $X=L^{2}[0, \pi], b \in X$ and $f$ is a given continuous and uniformly bounded function.

Put $x(t)=x(t$. $\cdot$ ), i.e., $x(t)(z)=x(t, z)$. Define the function $f:[0,1] \times X \times U \rightarrow X$ by $f\left(t, x_{t}, u\right)(z)=f(t, x(t-h, z), u(t))$. Let $B: U \rightarrow X$ be a bounded linear operator defined by

$$
(B u)(z)=b(z) u, \quad z \in[0, \pi], \quad u \in U, \quad b(z) \in X
$$

and define an operator $A: X \rightarrow X$ by $A v=v^{\prime \prime}$ with the domain

$$
\begin{aligned}
D(A)= & \left\{v \in X \mid v(\cdot) \in L^{2}[0, \pi], v, v^{\prime}\right. \text { are absolutly continuous, } \\
& \left.v^{\prime \prime} \in X \text { and } v(0)=v(\pi)=0\right\} .
\end{aligned}
$$

It follows

$$
A v=-\sum_{n=1}^{\infty} n^{2}\left(v, e_{n}\right) e_{n}, \quad v \in D(A)
$$

where $e_{n}(z)=\sqrt{\frac{2}{\pi}} \sin n z, 0 \leq z \leq \pi, n=1,2, \ldots$. It is well known that the operator $A$ generates a compact semigroup $T(t), t>0$, which is given by

$$
T(t) v=\sum_{n=1}^{\infty} e^{-n^{2} t}\left(v, e_{n}\right) e_{n}, \quad v \in X
$$

For more details, please refer to [31]. Therefore, system (50) can be written to the abstract form (1) and assumption (H1) is satisfied. Then under the condition imposed on $f$ and the linear system corresponding to (50) is approximately controllable on [0, 1], see [17], system (50) is approximately controllable on $[0,1]$ by Theorem 3.2.

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## Competing interests

The authors declare that they have no competing interests.

## Authors' contributions

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