Shen and Li *Advances in Difference Equations* (2018) 2018:396
https://doi.org/10.1186/s13662-018-1843-0

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

The z-transform method for the Ulam stability of linear difference equations with constant coefficients

Yonghong Shen[1*] and Yongjin Li[2]

*Correspondence:
shenyonghong2008@hotmail.com
[1] School of Mathematics and Statistics, Tianshui Normal University, Tianshui, P.R. China
Full list of author information is available at the end of the article

Abstract

Applying the z-transform method, we study the Ulam stability of linear difference equations with constant coefficients. To a certain extent, our results can be viewed as an important complement to the existing methods and results.

Keywords: Ulam stability; z-transform; Inverse z-transform; Linear difference equations

1 Introduction

The notion of the Ulam stability was originated from a question on group homomorphisms posed by Ulam [24] in 1940. The essential problem of this type of stability is summarized as follows: "Under what conditions can a solution of a perturbed equation be close to a solution of the original equation?" In the following year, Hyers [9] gave a first affirmative partial answer to the Cauchy equation in a Banach space. Afterward, this work was generalized by Rassias [19] for linear mappings by considering unbounded Cauchy differences. It is worth mentioning that Rassias' work has a great impact on the development of the Ulam stability of functional equations. Since then, almost all studies related to the Ulam stability have focused on different types of functional equations or abstract spaces. For more detailed results, we refer to the monographs [1, 7, 10, 12, 21] and references therein.

Let $(X, \|\cdot\|)$ be a normed space, n a positive integer, and $F : X^n \to X$ a mapping. The nth-order difference equation

$$F(y_k, y_{k+1}, \dots, y_{k+n}) = 0, \quad k \in \mathbb{N}, \tag{1}$$

is said to have Hyers–Ulam stability or to be stable in the Hyers–Ulam sense if whenever a given $\varepsilon > 0$ and a sequence $\{x_k\}$ satisfy the inequality

$$\left\| F(x_k, x_{k+1}, \dots, x_{k+n}) \right\| \le \varepsilon, \quad k \in \mathbb{N},$$

there exists a solution $\{y_k\}$ of (1) such that

$$\|x_k - y_k\| \le K(\varepsilon), \quad k \in \mathbb{N},$$

where $K(\varepsilon)$ depends only on ε, and $\lim_{\varepsilon\to 0} K(\varepsilon) = 0$. More generally, we say that the difference equation (1) is Hyers–Ulam–Rassias stable or generalized Hyers–Ulam stable if ε and $K(\varepsilon)$ are replaced by two control sequences φ_k and Φ_k, respectively.

In 2005, Popa [17] initiated the study of the Ulam stability of difference equations. Specifically, the author proved the Hyers–Ulam–Rassias stability of the linear difference equation of first order $x_{k+1} = a_k x_k + b_k$ in a Banach space. At the same time, Popa [18] also established the Hyers–Ulam stability of higher-order linear difference equations with constant coefficients. These results showed that the Hyers–Ulam stability of a linear difference equation with constant coefficients depends strongly on the roots of the characteristic equation. Several examples showed that the difference equation is not Hyers–Ulam stable if the characteristic equation admits a root with modulus equal to 1. For this reason, Brzdęk et al. [2] considered the nonstability of linear difference equations with constant coefficients. In 2007, Brzdęk et al. [3] investigated the Ulam stability of the nonlinear difference equation $x_{k+1} = a_k(x_k) + b_k$ in an Abelian group with invariant metric d. Afterward, Brzdęk et al. [4] presented the $(\varepsilon_k)_{k\geq 0}$-stability of difference equations that is weaker than the Hyers–Ulam stability, and they considered the $(\varepsilon_k)_{k\geq 0}$-stability of first-order linear difference equations. Based on the previous related works, they also gave a systematic answer to the Hyers–Ulam stability of higher-order linear difference equations with constants coefficients. In 2015, Xu and Brzdęk [26] studied the Hyers–Ulam stability of systems of first-order linear difference equations with constant coefficients in a Banach space. Furthermore, Brzdęk and Wójcik [6] established the Ulam stability of two kinds of difference equations in a metric space. As far as we know, this is the most general result associated with the Ulam stability of difference equations. In addition, Jung [13] also considered the Hyers–Ulam stability of first-order linear homogeneous matrix difference equations. Later, Jung and Nam [14] investigated the Hyers-stability of the Pielou logistic difference equation from a more general perspective. Recently, Onitsuka [16] established the Hyers-stability of first-order nonhomogeneous linear difference equations with constant stepsize, and the author further considered the best Hyers–Ulam stability constant of the corresponding difference equations.

As is well known, many different methods for solving differential equations have been used to study the Ulam stability of the corresponding equations, such as the integrating factor [22, 25], the power series method [11], the Laplace transform [20], the method of variation of constants [23], and so on. The Laplace transform method has a significant advantage in solving linear differential equations with constant coefficients, because it can turn a differential equation into an algebraic one. As a discrete case, the z-transform has a similar advantage in solving linear difference equations with constant coefficients. Inspired by the reference [20], the main purpose of this paper is to investigate the Ulam stability of linear difference equations with constant coefficients by using the z-transform method.

2 The z-transform and inverse z-transform

In this section, we recall some basic notions and results related to the z-transform and inverse z-transform. We denote by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ the sets of nonnegative integers, real numbers, and complex numbers, respectively. Moreover, $\mathbb{F}$ denotes either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

Definition 2.1 (Kelley and Peterson [15]) Let $\{x_k\}$ be a sequence in $\mathbb{F}$. The z-transform of $\{x_k\}$ is the function $X(z)$ of a complex variable $z \in \mathbb{C}$ defined by

$$X(z) = Z(x_k) = \sum_{k=0}^{\infty} \frac{x_k}{z^k},$$

and we say that the z-transform of $\{x_k\}$ exists if there is a real number $r > 0$ such that the series $\sum_{k=0}^{\infty} \frac{x_k}{z^k}$ converges for $|z| > r$.

A sequence $\{x_k\}$ is said to be *exponentially bounded* if there exist $M > 0$ and $c > 1$ such that

$$|x_k| < Mc^k, \quad k \in \mathbb{N}.$$

The following result is a sufficient condition for the existence of the z-transform of a sequence.

Theorem 2.1 (Kelley and Peterson [15]) *If the sequence $\{x_k\}$ is exponentially bounded, then the z-transform $X(z)$ of $\{x_k\}$ exists.*

According to Definition 2.1, it is easy to show that the z-transform is linear, that is, if the two sequences $\{x_k\}$ and $\{y_k\}$ are exponentially bounded, then

$$Z(\alpha x_k + \beta y_k) = \alpha Z(x_k) + \beta Z(y_k), \quad \alpha, \beta \in \mathbb{F},$$

for all z in the common domain of $X(z)$ and $Y(z)$.

Remark 1 Suppose that the sequence $\{x_k\}$ is exponentially bounded. For a positive integer n, it is easy to verify that

$$Z(x_{k+n}) = z^n Z(x_k) - \sum_{m=0}^{n-1} x_m z^{n-m}.$$

Definition 2.2 (Gradshteyn and Ryzhik [8]) Suppose that $X(z)$ is the z-transform of the sequence $\{x_k\}$ with the domain of convergence $D = \{z \in \mathbb{C} : |z| > r\}$. The inverse z-transform $\{x_k\}$ of $X(z)$ is given by

$$x_k = Z^{-1}\bigl(X(z)\bigr) = \frac{1}{2\pi i} \oint_C X(z) z^{k-1}\, dz,$$

where C is a counterclockwise simple closed contour encircling all poles of $X(z)$ and lying entirely within the domain of convergence D.

Now we define the unit step sequence $\{u_k(n)\}$, $n \geq 1$, and the unit impulse sequence $\{\delta_k(m)\}$, $m \geq 0$, respectively, by

$$u_k(n) = \begin{cases} 0, & 0 \leq k \leq n-1, \\ 1, & n \leq k, \end{cases} \qquad \delta_k(m) = \begin{cases} 1, & k = m, \\ 0, & k \neq m. \end{cases}$$

It follows immediately from Definition 2.1 that

$$Z(\delta_k(m)) = \frac{1}{z^m}, \quad |z| > 0.$$

Notice that the convolution of two sequences $\{x_k\}$ and $\{y_k\}$ is defined by

$$x_k * y_k = \sum_{m=0}^{k} x_{k-m} y_m.$$

Theorem 2.2 (Kelley and Peterson [15]) *Suppose that $X(z)$ and $Y(z)$ exist for $|z| > r_1$ and $|z| > r_2$, respectively. Then*

$$Z(x_k * y_k) = X(z)Y(z)$$

for $|z| > \max\{r_1, r_2\}$.

Theorem 2.3 (Kelley and Peterson [15]) *Let $\{y_k\}$ be a sequence such that the z-transform $Y(z) = Z(y_k)$ exists for $|z| > r$. Then*

$$Z(k^{(n)} y_k) = (-1)^n z^n \frac{d^n Y}{dz^n}(z), \quad n \geq 1,$$

where $k^{(n)} = k(k+1)(k+2)\cdots(k+n-1)$.

Lemma 2.4 *Let $a \in \mathbb{C}$. Then*

$$Z\left(\frac{k^{(n)}}{n!} a^k\right) = \frac{a z^n}{(z-a)^{n+1}}$$

for $|z| > |a|$.

Proof Since $Z(a^k) = \frac{z}{z-a}$ for $|z| > |a|$, from Theorem 2.3 we can infer that

$$\begin{aligned} Z\left(\frac{k^{(n)}}{n!} a^k\right) &= (-1)^n \frac{z^n}{n!} \frac{d^n}{dz^n}\left(\frac{z}{z-a}\right) \\ &= (-1)^n \frac{z^n}{n!} \frac{(-1)^n n! a}{(z-a)^{n+1}} \\ &= \frac{a z^n}{(z-a)^{n+1}}. \end{aligned}$$

□

3 The Ulam stability of linear difference equations with constant coefficients

Theorem 3.1 (Kelley and Peterson [15]) *Suppose that the sequence $\{f_k\}$ is exponentially bounded. Then each solution of the nth-order linear difference equation*

$$y_{k+n} + p_1 y_{k+n-1} + \cdots + p_n y_k = f_k$$

is exponentially bounded, and hence its z-transform exists.

First of all, we consider the Ulam stability of first-order linear difference equations with constant coefficients.

Theorem 3.2 *Let* $\{f_k\}$ *and* $\{\varphi_k\}$ *be two exponentially bounded sequences, and let* $\{\varphi_k\}$ *be a positive sequence. Assume that* $p \in \mathbb{F} \setminus \{0\}$. *If a sequence* $\{x_k\}$ *satisfies the inequality*

$$|x_{k+1} - px_k - f_k| \le \varphi_k \tag{2}$$

for all $k \in \mathbb{N}$, *then there exists a solution* $\{y_k\}$ *of the first-order linear difference equation*

$$y_{k+1} = py_k + f_k \tag{3}$$

such that

$$|x_k - y_k| \le \sum_{m=1}^{k} \varphi_{k-m}|p|^{m-1} \tag{4}$$

for all $k \in \mathbb{N}$.

Proof Set $v_k = x_{k+1} - px_k - f_k$, $k \in \mathbb{N}$. Obviously, v_k is exponentially bounded. Taking into account Theorem 2.1, Theorem 3.1, and Remark 1 and applying the z-transform to v_k, we have

$$V(z) = zX(z) - zx_0 - pX(z) - F(z),$$

where $V(z)$ and $F(z)$ denote the z-transforms of the sequences $\{v_k\}$ and $\{f_k\}$, respectively. Then, we obtain

$$X(z) = \frac{z}{z-p}x_0 + \frac{V(z) + F(z)}{z-p}. \tag{5}$$

Set

$$y_k = p^k x_0 + f_k * \bigl(p^{k-1}u_k(1)\bigr).$$

It is easily seen that $y_0 = x_0$. By Theorem 2.2, using the identity $p^{k-1}u_k(1) = p^k * \delta_k(1)$, we get

$$Y(z) = \frac{z}{z-p}x_0 + \frac{F(z)}{z-p}. \tag{6}$$

Then we obtain

$$Z(y_{k+1} - py_k) = zY(z) - zy_0 - pY(z) = (z-p)Y(z) - zx_0 = F(z) = Z(f_k).$$

Notice that the inverse z-transform gives $y_{k+1} = py_k + f_k$. This shows that y_k is a solution of the difference equation (3).

From (5) and (6) it follows that

$$Z(x_k) - Z(y_k) = \frac{Z(v_k)}{z-p}.$$

Again using the inverse z-transform, we obtain

$$x_k - y_k = v_k * \bigl(p^{k-1}u_k(1)\bigr).$$

From (2) we can infer that

$$\begin{aligned} |x_k - y_k| &= \bigl|v_k * \bigl(p^{k-1}u_k(1)\bigr)\bigr| \\ &= \left|\sum_{m=0}^{k} v_{k-m}p^{m-1}u_m(1)\right| \\ &\le \sum_{m=0}^{k}\bigl|v_{k-m}p^{m-1}u_m(1)\bigr| \\ &\le \sum_{m=0}^{k}\varphi_{k-m}\bigl|p^{m-1}u_m(1)\bigr| \\ &= \sum_{m=1}^{k}\varphi_{k-m}|p|^{m-1}, \end{aligned}$$

which completes the proof. □

As a direct consequence of Theorem 3.2, for $|p| < 1$, we obtain the Hyers–Ulam stability of the linear difference equation (3).

Corollary 3.3 *Let $\{f_k\}$ be given as in Theorem 3.2, and let $0 < |p| < 1$. For a given $\varepsilon > 0$, if a sequence $\{x_k\}$ satisfies the inequality*

$$|x_{k+1} - px_k - f_k| \le \varepsilon$$

for all $k \in \mathbb{N}$, then there exists a solution $\{y_k\}$ of the first-order linear difference equation (3) such that

$$|x_k - y_k| \le \frac{\varepsilon}{1-|p|}$$

for all $k \in \mathbb{N}$.

Corollary 3.4 *Let $\{f_k\}$ be given as in Theorem 3.2, and let $|p| = 1$. Assume that $\lambda > 1$. For a given $\varepsilon > 0$, if a sequence $\{x_k\}$ satisfies the inequality*

$$|x_{k+1} - px_k - f_k| \le \frac{\varepsilon}{\lambda^k}$$

for all $k \in \mathbb{N}$, then there exists a solution y_k of the first-order linear difference equation (3) *such that*

$$|x_k - y_k| \le \frac{\lambda\varepsilon}{\lambda - 1}$$

for all $k \in \mathbb{N}$.

Lemma 3.5 *Let*

$$P(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_n z^n$$

and

$$Q(z) = \beta_0 + \beta_1 z + \beta_2 z^2 + \cdots + \beta_m z^m,$$

where $m, n \in \mathbb{N}$ with $m < n$, and α_j and β_j are scalars. Then there exists a sequence $\{g_k\}$ such that

$$Z(g_k) = G(z) = \frac{Q(z)}{P(z)}$$

for $|z| > r_p$, where $r_p = \max\{|z_1|, |z_2|, \dots, |z_l|\}$, and z_i, $i = 1, 2, \dots, l$, are mutually different roots of the polynomial equation $P(z) = 0$.

Proof Using the polynomial decomposition theorem, the polynomial $P(z)$ can be decomposed as

$$P(z) = \alpha_n (z - z_1)^{n_1} (z - z_2)^{n_2} \cdots (z - z_l)^{n_l},$$

for some complex numbers z_i, $n_i \in \mathbb{N}$, $i = 1, 2, \dots, l$, such that $n_1 + n_2 + \cdots + n_l = n$.

Furthermore, applying the partial fraction decomposition, we obtain

$$\frac{Q(z)}{P(z)} = \sum_{i=1}^{l} \sum_{j=1}^{n_i} \frac{\lambda_{ij}}{(z - z_i)^j},$$

where λ_{ij} is a scalar for all $i = 1, 2, \dots, l$, $j = 1, 2, \dots, n_i$.

Now we set

$$h_k^{ij} = \begin{cases} z_i^{k-1} u_k(j), & z_i \ne 0, j = 1, \\ \frac{(k-j+1)^{(j-1)}}{(j-1)!} z_i^{k-j} u_k(j-1), & z_i \ne 0, j = 2, 3, \dots, n_i, \\ \delta_k(j), & z_i = 0, \end{cases}$$

where $i = 1, 2, \dots, l$ and $j = 1, 2, \dots, n_i$.

Define

$$g_k = \sum_{i=1}^{l} \sum_{j=1}^{n_i} \lambda_{ij} h_k^{ij}.$$

From Lemma 2.4 and the linearity of the z-transform it follows that

$$\begin{aligned} G(z) = Z(g_k) &= Z\left(\sum_{i=1}^{l}\sum_{j=1}^{n_i}\lambda_{ij}h_k^{ij}\right) \\ &= \sum_{i=1}^{l}\sum_{j=1}^{n_i}\lambda_{ij}Z\left(h_k^{ij}\right) \\ &= \sum_{i=1}^{l}\sum_{j=1}^{n_i}\frac{\lambda_{ij}}{(z-z_i)^j} = \frac{Q(z)}{P(z)} \end{aligned} \tag{7}$$

for $|z| > r_p$, where $r_p = \max\{|z_1|, |z_2|, \ldots, |z_l|\}$. □

Lemma 3.6 *Let $\{f_k\}$ be an exponentially bounded sequence, and let $P(z)$ be a complex polynomial of degree $n \geq 1$. Then there exists a sequence $\{h_k\}$ such that*

$$Z(h_k) = \frac{Z(f_k)}{P(z)} \tag{8}$$

for $|z| > \max\{r_f, r_p\}$, where $r_p = \max\{|z^*| : P(z^*) = 0\}$, and r_f is a nonnegative real number such that the z-transform of $\{f_k\}$ exists for $|z| > r_f$.

Proof Set $Q(z) = 1$. Note that $P(z)$ is a complex polynomial of degree $n \geq 1$. By Lemma 3.5 there exists a sequence $\{g_k\}$ such that

$$Z(g_k) = \frac{1}{P(z)}$$

for $|z| > r_p$. Define $h_k = g_k * f_k$. For any $|z| > \max\{r_f, r_p\}$, by Theorem 2.2 we have

$$Z(h_k) = Z(g_k * f_k) = Z(g_k)Z(f_k) = \frac{Z(f_k)}{P(z)}.$$ □

Now we discuss the Ulam stability of linear difference equations of nth order ($n > 1$).

Theorem 3.7 *Let $\{f_k\}$ and $\{\varphi_k\}$ be two exponentially bounded sequences, and let $\{\varphi_k\}$ be a positive sequence. Assume that $p_0, p_1, \ldots, p_{n-1}$ are scalars, $n \in \mathbb{N} \setminus \{0, 1\}$. If a sequence $\{x_k\}$ satisfies the inequality*

$$\left| x_{k+n} + \sum_{m=0}^{n-1} p_m x_{k+m} - f_k \right| \leq \varphi_k \tag{9}$$

for all $k \in \mathbb{N}$, then there exists a solution $\{y_k\}$ of the nth-order linear difference equation

$$y_{k+n} + \sum_{m=0}^{n-1} p_m y_{k+m} = f_k \tag{10}$$

such that

$$|x_k - y_k| \leq \sum_{m=0}^{k} \varphi_{k-m} |\widetilde{p_m}| \tag{11}$$

for all $k \in \mathbb{N}$, where $\widetilde{p_k} = Z^{-1}(\frac{1}{P_{n,0}(z)})$, and $P_{n,0}(z)$ is the characteristic polynomial of the corresponding homogeneous equation of (10).

Proof By Remark 1, for any $n \in \mathbb{N} \setminus \{0,1\}$, we have

$$Z(y_{k+n}) = z^n Z(y_k) - \sum_{m=0}^{n-1} y_m z^{n-m}.$$

For notational convenience, we let $p_n = 1$. Then, we note that a sequence $\{\widetilde{y}_k\}$ is a solution of (10) if and only if

$$\begin{aligned} Z(f_k) &= \sum_{m=0}^{n} p_m z^m Z(\widetilde{y}_k) - \sum_{m=1}^{n} p_m \sum_{j=0}^{m-1} \widetilde{y}_j z^{m-j} \\ &= \sum_{m=0}^{n} p_m z^m Z(\widetilde{y}_k) - \sum_{m=1}^{n} p_m \sum_{j=1}^{m} \widetilde{y}_{j-1} z^{m-j+1} \\ &= \sum_{m=0}^{n} p_m z^m Z(\widetilde{y}_k) - \sum_{j=1}^{n} \sum_{m=j}^{n} p_m \widetilde{y}_{j-1} z^{m-j+1} \\ &= P_{n,0}(z) Z(\widetilde{y}_k) - \sum_{j=1}^{n} z P_{n,j}(z) \widetilde{y}_{j-1}, \end{aligned} \tag{12}$$

where the polynomial $P_{n,j}(z)$ is given by

$$P_{n,j}(z) = \sum_{m=j}^{n} p_m z^{m-j} = p_j + p_{j+1} z + \cdots + p_n z^{n-j}, \quad j = 0,1,2,\ldots,n.$$

Set

$$h_k = x_{k+n} - \sum_{m=0}^{n-1} p_m x_{k+m} - f_k, \quad k \in \mathbb{N}. \tag{13}$$

Similarly, applying the z-transform to both sides of (13), we obtain

$$Z(h_k) = P_{n,0}(z) Z(x_k) - \sum_{j=1}^{n} z P_{n,j}(z) x_{j-1} - Z(f_k).$$

Then we have

$$Z(x_k) - \frac{1}{P_{n,0}(z)} \left(\sum_{j=1}^{n} z P_{n,j}(z) x_{j-1} + Z(f_k) \right) = \frac{Z(h_k)}{P_{n,0}(z)}. \tag{14}$$

For convenience, we assume that the z-transform of $\{f_k\}$ exists for $|z| > r_f$. Let $z_1, z_2, \ldots, z_n$ be the roots of the polynomial $P_{n,0}(z)$, and let $r_p = \max\{|z_i| : i = 1,2,\ldots n\}$.

For $|z| > \max\{r_f, r_p\}$, we let

$$G(z) = \frac{1}{P_{n,0}(z)} \left(\sum_{j=1}^{n} z P_{n,j}(z) x_{j-1} + Z(f_k) \right). \tag{15}$$

By Lemma 3.6 there exists a sequence $\{\widetilde{f_k}\}$ such that

$$Z(\widetilde{f_k}) = \frac{Z(f_k)}{P_{n,0}(z)} \tag{16}$$

for $|z| > \max\{r_f, r_p\}$. Moreover, we also notice that

$$\begin{aligned} \frac{zP_{n,j}(z)}{P_{n,0}(z)} &= \frac{p_j z + p_{j+1} z^2 + \cdots + p_n z^{n-j+1}}{P_{n,0}(z)} \\ &= \frac{p_j z^j + p_{j+1} z^{j+1} + \cdots + p_n z^n}{z^{j-1} P_{n,0}(z)} \\ &= \frac{P_{n,0}(z) - (p_0 + p_1 z + \cdots + p_{j-1} z^{j-1})}{z^{j-1} P_{n,0}(z)} \\ &= \frac{1}{z^{j-1}} - \frac{p_0 + p_1 z + \cdots + p_{j-1} z^{j-1}}{z^{j-1} P_{n,0}(z)} \end{aligned} \tag{17}$$

for $j = 1, 2, \ldots, n$ and $|z| > r_p$.

Let $Q(z) = p_0 + p_1 z + \cdots + p_{j-1} z^{j-1}$ and $P(z) = z^{j-1} P_{n,0}(z)$. By Lemma 3.5 there exists a sequence $\{g_k\}$ such that

$$Z(g_k) = \frac{Q(z)}{P(z)} = \frac{p_0 + p_1 z + \cdots + p_{j-1} z^{j-1}}{z^{j-1} P_{n,0}(z)}. \tag{18}$$

Let us define

$$\widehat{f_k^j} = \delta_k(j-1) - g_k, \quad j = 1, 2, \ldots, n. \tag{19}$$

Furthermore, we define

$$y_k = \sum_{j=1}^{n} \widehat{f_k^j} x_{j-1} + \widetilde{f_k}, \quad k \in \mathbb{N}. \tag{20}$$

Then it follows from (15)–(20) that

$$\begin{aligned} Z(y_k) &= \sum_{j=1}^{n} x_{j-1} Z(\widehat{f_k^j}) + Z(\widetilde{f_k}) \\ &= \frac{1}{P_{n,0}(z)} \left(\sum_{j=1}^{n} z P_{n,j}(z) x_{j-1} + Z(f_k) \right) \end{aligned} \tag{21}$$

for $|z| > \max\{r_f, r_p\}$. From (14) and (21) we know that the sequence $\{y_k\}$ given in (20) is a solution of (10). Meantime, we can infer from (14) and (20) that

$$Z(x_k) - Z(y_k) = \frac{Z(h_k)}{P_{n,0}(z)}.$$

Therefore, using the inverse z-transform, we get

$$\begin{aligned}|x_k - y_k| &= |h_k * \widetilde{p_k}| \\ &= \left|\sum_{m=0}^{k} h_{k-m}\widetilde{p_m}\right| \\ &\le \sum_{m=0}^{k} |\varphi_{k-m}\widetilde{p_m}| = \sum_{m=0}^{k} \varphi_{k-m}|\widetilde{p_m}|\end{aligned}$$

for each $k \in \mathbb{N}$, where $\widetilde{p_k} = Z^{-1}(\frac{1}{P_{n,0}(z)})$. □

Remark 2 In fact, $P_{n,0}(z)$ is the characteristic polynomial of the corresponding homogeneous equation of (10). Assume that $z_1, z_2, \dots, z_l$ are distinct roots of the polynomial equation $P_{n,0}(z) = 0$ with multiplicities $n_1, n_2, \dots, n_l$, respectively, such that $n_1 + n_2 + \cdots + n_l = n$. Then we obtain

$$P_{n,0}(z) = (z - z_1)^{n_1}(z - z_2)^{n_2} \cdots (z - z_l)^{n_l}.$$

Using the partial fraction decomposition, we have

$$\frac{1}{P_{n,0}(z)} = \sum_{i=1}^{l}\sum_{j=1}^{n_i} \frac{\mu_{ij}}{(z - z_i)^j},$$

where μ_{ij} are scalars, $i = 1, 2, \dots, l$, $j = 1, 2, \dots, n_i$.

Similar to the proof of Lemma 3.5, we define

$$q_k^{ij} = \begin{cases} z_i^{k-1} u_k(j), & z_i \neq 0, j = 1, \\ \frac{(k-j+1)^{(j-1)}}{(j-1)!} z_i^{k-j} u_k(j-1), & z_i \neq 0, j = 2, 3, \dots, n_i, \\ \delta_k(j), & z_i = 0, \end{cases}$$

where $i = 1, 2, \dots, l$ and $j = 1, 2, \dots, n_i$. Furthermore, we set

$$\widetilde{p_k} = \sum_{i=1}^{l}\sum_{j=1}^{n_i} \mu_{ij} q_k^{ij}, \quad k \in \mathbb{N}. \tag{22}$$

Hence we can infer that

$$Z(\widetilde{p_k}) = \sum_{i=1}^{l}\sum_{j=1}^{n_i} \mu_{ij} Z\bigl(q_k^{ij}\bigr) = \sum_{i=1}^{l}\sum_{j=1}^{n_i} \frac{\mu_{ij}}{(z - z_i)^j} = \frac{1}{P_{n,0}(z)}.$$

Then the inverse z-transform implies that

$$\widetilde{p_k} = Z^{-1}\left(\frac{1}{P_{n,0}(z)}\right).$$

From the previous statement we can obtain the following results associated with the Ulam stability of the linear difference equation (10).

Corollary 3.8 *Let $\{f_k\}$, $\{\varphi_k\}$ be given as in Theorem 3.7, and let $p_0, p_1, \ldots, p_n$ be scalars with $p_n = 1$, $n \in \mathbb{N} \setminus \{0, 1\}$. Assume that $z_1, z_2, \ldots, z_l$ are distinct roots of the polynomial equation $P_{n,0}(z) = 0$ with multiplicities $n_1, n_2, \ldots, n_l$, respectively, such that $n_1 + n_2 + \cdots + n_l = n$. If a sequence $\{x_k\}$ satisfies inequality (9) for all $k \in \mathbb{N}$, then there exists a solution $\{y_k\}$ of the nth-order linear difference equation (10) such that*

$$|x_k - y_k| \leq \sum_{m=0}^{k} \varphi_{k-m} |\widetilde{p_m}|$$

for each $k \in \mathbb{N}$, where $\widetilde{p_m}$ is defined by (22).

Corollary 3.9 *Let $\{f_k\}$ be an exponentially bounded sequence, and let $p_0, p_1, \ldots, p_n$ be scalars with $p_n = 1$, $n \in \mathbb{N} \setminus \{0, 1\}$. Assume that $z_1, z_2, \ldots, z_n$ are n distinct roots of the polynomial equation $P_{n,0}(z) = 0$ with $\max\{|z_i| : i = 1, 2, \ldots, n\} < 1$. For a given $\varepsilon > 0$, if a sequence $\{x_k\}$ satisfies the inequality*

$$\left| x_{k+n} - \sum_{m=0}^{n-1} p_m x_{k+m} - f_k \right| \leq \varepsilon \tag{23}$$

for all $k \in \mathbb{N}$, then there exists a solution $\{y_k\}$ of the nth-order linear difference equation (10) such that

$$|x_k - y_k| \leq \sum_{i=1}^{n} \frac{\varepsilon |\mu_i|}{1 - |z_i|}$$

for all $k \in \mathbb{N}$, where μ_i can be determined by the partial fraction decomposition of $\frac{1}{P_{n,0}(z)}$.

Proof By Corollary 3.8 there exists a solution $\{y_k\}$ of (10) such that

$$|x_k - y_k| \leq \varepsilon \sum_{m=0}^{k} |\widetilde{p_m}|, \quad k \in \mathbb{N}.$$

However, if $z_i \neq 0$ for each i, then by (22) we obtain

$$\begin{aligned} \sum_{m=0}^{k} |\widetilde{p_m}| &= \sum_{m=0}^{k} \left| \sum_{i=1}^{n} \mu_i z_i^{m-1} u_m(1) \right| \\ &\leq \sum_{m=0}^{k} \sum_{i=1}^{n} \left| \mu_i z_i^{m-1} u_m(1) \right| \\ &= \sum_{i=1}^{n} |\mu_i| \sum_{m=0}^{k} \left| z_i^{m-1} u_m(1) \right| \\ &= \sum_{i=1}^{n} |\mu_i| \sum_{m=1}^{k} \left| z_i^{m-1} \right| \end{aligned}$$

$$\le \sum_{i=1}^{n} |\mu_i| \sum_{m=1}^{\infty} |z_i|^{m-1}$$
$$= \sum_{i=1}^{n} \frac{|\mu_i|}{1-|z_i|},$$

which implies the desired result. Moreover, if there exists $i_0 \in \{1,2,\dots,n\}$ such that $z_{i_0} = 0$, then the foregoing result remains true. □

Corollary 3.10 *Let $\{f_k\}$ be an exponentially bounded sequence, and let $p_0,p_1,\dots,p_n$ be scalars with $p_n = 1$, $n \in \mathbb{N} \setminus \{0,1\}$. Assume that $z_1,z_2,\dots,z_l$ are roots of the polynomial equation $P_{n,0}(z) = 0$ with multiplicities $n_1,n_2,\dots,n_l$, respectively, such that $n_1 + n_2 + \cdots + n_l = n$, $\max\{|z_i| : i = 1,2,\dots,l\} < 1$. For a given $\varepsilon > 0$, if a sequence $\{x_k\}$ satisfies inequality (23) for all $k \in \mathbb{N}$, then there exists a solution $\{y_k\}$ of the nth-order linear difference equation (10) such that*

$$|x_k - y_k| \le \sum_{i=1}^{l} \sum_{j=1}^{n_i} \frac{\varepsilon|\mu_{ij}|}{(1-|z_i|)^j}, \quad k \in \mathbb{N},$$

where μ_{ij} can be determined by the partial fraction decomposition of $\frac{1}{P_{n,0}(z)}$.

Proof Without loss of generality, we assume that $z_1 = 0$ with multiplicity $n_1 < l$, and $z_2,z_3,\dots,z_t$ are simple roots, that is, $n_2 = n_3 = \cdots = n_t = 1$, $2 \le t < l$. Just notice that

$$\begin{aligned}
\sum_{m=0}^{k} |\widetilde{p_m}| &= \sum_{m=0}^{k} \left| \sum_{i=1}^{l} \sum_{j=1}^{n_i} \mu_{ij} q_m^{ij} \right| \\
&\le \sum_{m=0}^{k} \sum_{i=1}^{l} \sum_{j=1}^{n_i} \left| \mu_{ij} q_m^{ij} \right| \\
&= \sum_{m=0}^{k} \sum_{j=1}^{n_1} \left| \mu_{1j} q_m^{1j} \right| + \sum_{m=0}^{k} \sum_{i=2}^{t} \sum_{j=1}^{n_i} \left| \mu_{ij} q_m^{ij} \right| + \sum_{m=0}^{k} \sum_{i=t+1}^{l} \sum_{j=1}^{n_i} \left| \mu_{ij} q_m^{i,j} \right| \\
&= \sum_{m=0}^{k} \sum_{j=1}^{n_1} \left| \mu_{1j} \delta_m(j) \right| + \sum_{m=0}^{k} \sum_{i=2}^{t} \left| \mu_{i1} z_i^{m-1} u_m(1) \right| \\
&\quad + \sum_{m=0}^{k} \sum_{i=t+1}^{l} \sum_{j=1}^{n_i} \left| \mu_{ij} \frac{(m-j+1)^{(j-1)}}{(j-1)!} z_i^{m-j} u_m(j-1) \right| \\
&= \sum_{j=1}^{n_1} |\mu_{1j}| \sum_{m=0}^{k} \left| \delta_m(j) \right| + \sum_{i=2}^{t} |\mu_{i1}| \sum_{m=0}^{k} \left| z_i^{m-1} u_m(1) \right| \\
&\quad + \sum_{i=t+1}^{l} \sum_{j=1}^{n_i} \frac{|\mu_{ij}|}{(j-1)!} \sum_{m=0}^{k} \left| (m-j+1)^{(j-1)} z_i^{m-j} u_m(j-1) \right| \\
&\le \sum_{j=1}^{n_1} |\mu_{1j}| \sum_{m=0}^{\infty} \left| \delta_m(j) \right| + \sum_{i=2}^{t} |\mu_{i1}| \sum_{m=1}^{\infty} |z_i|^{m-1}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{i=t+1}^{l} \sum_{j=1}^{n_i} |\mu_{ij}| + \sum_{i=t+1}^{l} \sum_{j=1}^{n_i} \frac{|\mu_{ij}|}{(j-1)!} \sum_{m=j+1}^{\infty} (m-j+1)^{(j-1)} |z_i|^{m-j} \\
&= \sum_{j=1}^{n_1} |\mu_{1j}| + \sum_{i=2}^{t} \frac{|\mu_{i1}|}{1-|z_i|} + \sum_{i=t+1}^{l} \sum_{j=1}^{n_i} |\mu_{ij}| \\
&\quad + \sum_{i=t+1}^{l} \sum_{j=1}^{n_i} \frac{|\mu_{ij}|}{(j-1)!} \left(\frac{(j-1)!}{(1-|z_i|)^j} - \frac{1}{j} \right) \\
&= \sum_{j=1}^{n_1} |\mu_{1j}| + \sum_{i=2}^{t} \frac{|\mu_{i1}|}{1-|z_i|} + \sum_{i=t+1}^{l} \sum_{j=1}^{n_i} |\mu_{ij}| \\
&\quad + \sum_{i=t+1}^{l} \sum_{j=1}^{n_i} \frac{|\mu_{ij}|}{(1-|z_i|)^j} - \sum_{i=t+1}^{l} \sum_{j=1}^{n_i} \frac{|\mu_{ij}|}{j!} \\
&\le \sum_{i=1}^{l} \sum_{j=1}^{n_i} \frac{|\mu_{ij}|}{(1-|z_i|)^j}. \qquad \square
\end{aligned}
$$

4 Conclusion

By using the z-transform method we have established the Ulam stability of linear difference equations with constant coefficients. In fact, the results obtained in this paper can be regarded as a discrete analogue of the stability results for linear differential equations in [20]. To a certain extent, our results constitute an important complement to the stability results obtained in [4, 5]. Additionally, this paper also provides another method to study the Ulam stability of difference equations. Obviously, this paper shows that the z-transform method is more convenient to study the Ulam stability of linear difference equations with constant coefficients.

Acknowledgements
The authors would like to express their sincere thanks to the editor and three anonymous referees for their useful comments and valuable suggestions, which helped to improve the quality of the paper.

Funding
This work was supported by the National Natural Science Foundation of China (No. 11701425). The second author acknowledges the support from the National Natural Science Foundation of China (No. 11571378).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YS drafted the manuscript and completed all proofs of the results in this paper. Both authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Statistics, Tianshui Normal University, Tianshui, P.R. China. [2]Department of Mathematics, Sun Yat-Sen University, Guangzhou, P.R. China.

Publisher's Note

Received: 22 June 2018 Accepted: 9 October 2018 Published online: 25 October 2018

References

1. Brzdęk, J., Popa, D., Raşa, I., Xu, B.: Ulam Stability of Operators. Elsevier, San Diego (2018)
2. Brzdęk, J., Popa, D., Xu, B.: Note on nonstability of the linear recurrence. Abh. Math. Semin. Univ. Hamb. **76**, 183–189 (2006)

3. Brzdęk, J., Popa, D., Xu, B.: The Hyers–Ulam stability of nonlinear recurrences. J. Math. Anal. Appl. **335**, 443–449 (2007)
4. Brzdęk, J., Popa, D., Xu, B.: On nonstability of the linear recurrence of order one. J. Math. Anal. Appl. **367**, 146–153 (2010)
5. Brzdęk, J., Popa, D., Xu, B.: Remarks on stability of linear recurrence of higher order. Appl. Math. Lett. **23**, 1459–1463 (2010)
6. Brzdęk, J., Wójcik, P.: On approximate solutions of some difference equations. Bull. Aust. Math. Soc. **95**, 476–481 (2017)
7. Czerwik, S.: Functional Equations and Inequalities in Several Variables. World Scientific, Singapore (2002)
8. Gradshteyn, I.S., Ryzhik, I.M.: Table of Integrals, Series, and Products, 7th edn. Elsevier, San Diego (2007)
9. Hyers, D.H.: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA **27**, 222–224 (1941)
10. Hyers, D.H., Isac, G., Rassias, T.M.: Stability of Functional Equations in Several Variables. Birkhäuser, Boston (1998)
11. Jung, S.M.: Legendre's differential equation and its Hyers–Ulam stability. Abstr. Appl. Anal. **2007**, Article ID 56419 (2007)
12. Jung, S.M.: Hyers–Ulam–Rassias Stability of Functional Equations in Nonlinear Analysis. Springer, New York (2011)
13. Jung, S.M.: Hyers–Ulam stability of the first-order matrix difference equations. Adv. Differ. Equ. **2015**, 170 (2015)
14. Jung, S.M., Nam, Y.W.: Hyers–Ulam stability of Pielou logistic difference equation. J. Nonlinear Sci. Appl. **10**, 3115–3122 (2017)
15. Kelley, W.G., Peterson, A.C.: Difference Equations: An Introduction with Applications, 2nd edn. Academic Press, San Diego (2001)
16. Onitsuka, M.: Hyers–Ulam stability of first-order nonhomogeneous linear difference equations with a constant stepsize. Appl. Math. Comput. **330**, 143–151 (2018)
17. Popa, D.: Heyers–Ulam–Rassias stability of a linear recurrence. J. Math. Anal. Appl. **309**, 591–597 (2005)
18. Popa, D.: Hyers–Ulam stability of the linear recurrence with constant coefficients. Adv. Differ. Equ. **2005**, 101–107 (2005)
19. Rassias, T.M.: On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc. **72**, 297–300 (1978)
20. Rezaei, H., Jung, S.M., Rassias, T.M.: Laplace transform and Hyers–Ulam stability of linear differential equations. J. Math. Anal. Appl. **403**, 244–251 (2013)
21. Sahoo, P.K., Kannappan, P.: Introduction to Functional Equations. CRC Press, Boca Raton (2011)
22. Shen, Y.H.: An integrating factor approach to the Hyers–Ulam stability of a class of exact differential equations of second order. J. Nonlinear Sci. Appl. **9**, 2520–2526 (2016)
23. Shen, Y.H., Li, Y.J.: A general method for the Ulam stability of linear differential equations. Bull. Malays. Math. Sci. Soc. (2018). https://doi.org/10.1007/s40840-018-0653-6
24. Ulam, S.M.: Problems in Modern Mathematics. Wiley, New York (1960)
25. Wang, G.W., Zhou, M.R., Sun, L.: Hyers–Ulam stability of linear differential equations of first order. Appl. Math. Lett. **21**, 1024–1028 (2008)
26. Xu, B., Brzdęk, J.: Hyers–Ulam stability of a system of first order linear recurrences with constant coefficients. Discrete Dyn. Nat. Soc. **2015**, Article ID 269356 (2015)