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Positive solutions for a singular fractional nonlocal boundary value problem

Luyao Zhang[1], Zhongmin Sun[2] and Xinan Hao[1*]

*Correspondence:
haoxinan2004@163.com
[1]School of Mathematical Sciences, Qufu Normal University, Qufu, P.R. China
Full list of author information is available at the end of the article

Abstract

We investigate a singular fractional differential equation with an infinite-point fractional boundary condition, where the nonlinearity $f(t,x)$ may be singular at $x = 0$, and $g(t)$ may also have singularities at $t = 0$ or $t = 1$. We establish the existence of positive solutions using the fixed point index theory in cones.

MSC: 26A33; 34B08; 34B10; 34B16; 34B18

Keywords: Positive solution; Singular; Infinite-point fractional boundary condition; Fixed point index

1 Introduction

We consider the existence of positive solutions for the following fractional nonlocal boundary value problem:

$$\begin{cases} D_{0+}^{\alpha}x(t)+\lambda g(t)f(t,x(t))=0, & t\in(0,1), \\ x(0)=x'(0)=\cdots=x^{(n-2)}(0)=0, & D_{0+}^{\beta}x(1)=\sum_{i=1}^{\infty}\alpha_i D_{0+}^{\gamma}x(\xi_i), \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, D_{0+}^{α}, D_{0+}^{β}, and D_{0+}^{γ} denote the Riemann–Liouville fractional derivatives, $2 \le n-1 < \alpha \le n$, $1 \le \beta \le n-2$, $0 \le \gamma \le \beta$, $\alpha_i \ge 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{i-1} < \xi_i < \cdots < 1$ $(i = 1, 2, \ldots)$, and $\Gamma(\alpha-\gamma) > \Gamma(\alpha-\beta)\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-\gamma-1}$. The function $f(t,x)$ may have singularity at $x = 0$, and $g(t)$ may be singular at $t = 0$ and/or $t = 1$.

Fractional differential equations describe many phenomena in various fields of science and engineering [1–4]. For the development of the fractional differential equations, see [5–23] and the references therein. Recently, the existence of positive solutions for fractional differential equation multipoint boundary value problems (BVPs) have been studied by many authors; see [24–33]. Using the compression expansion fixed point theorem due to Krasnosel'skii, Henderson and Luca [27] studied the fractional BVP

$$\begin{cases} D_{0+}^{\alpha}x(t)+\lambda f(t,x(t))=0, & 0<t<1, \\ x(0)=x'(0)=\cdots=x^{(n-2)}(0)=0, & D_{0+}^{\beta}x(1)=\sum_{i=1}^{m}\alpha_i D_{0+}^{\gamma}x(\xi_i), \end{cases} \tag{1.2}$$

where $\lambda > 0$, $2 \le n-1 < \alpha \le n$, $\alpha_i \ge 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$ $(i = 1, 2, \ldots, m)$, $1 \le \beta \le n-2$, $0 \le \gamma \le \beta$, and $f(t,x)$ may be singular at $t = 0, 1$ and may change sign. In [28], for

$\lambda = 1$, the authors investigated the existence and multiplicity of positive solutions for BVP (1.2). In [29, 30], the authors discussed the following infinite-point BVP:

$$\begin{cases} D_{0^+}^{\alpha}x(t) + f(t,x(t)) = 0, & 0 < t < 1, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, & x^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j x(\xi_j), \end{cases} \tag{1.3}$$

where $i \in \{1, 2, \ldots, n-2\}$, and $\sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-1} < (\alpha-1)\cdots(\alpha-i)$. The existence, uniqueness, and multiplicity of positive solutions for BVP (1.3) are established. Qiao and Zhou [31] discussed the singular BVP

$$\begin{cases} D_{0^+}^{\alpha}x(t) + g(t)f(t,x(t)) = 0, & 0 < t < 1, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, & D_{0^+}^{\beta}x(1) = \sum_{i=1}^{\infty} \alpha_i x(\xi_i), \end{cases} \tag{1.4}$$

where $\beta \in [1, \alpha-1]$, and $\Gamma(\alpha) > \Gamma(\alpha-\beta)\sum_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1}$. For more results on the fractional infinite-point BVPs, see [24, 25, 32, 33] and the references therein.

In the present paper, we investigate the existence of positive solutions for the singular fractional infinite-point BVP (1.1) using the fixed point index theory in cones. Note that $f(t,x)$ may be singular at $x = 0$ and $g(t)$ may be singular at $t = 0$ or $t = 1$.

2 Preliminaries and lemmas

Definition 2.1 ([1–4]) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $h : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds,$$

provided that the right-hand side is defined pointwise on $(0, +\infty)$.

Definition 2.2 ([1–4]) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $h : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{h(s)}{(t-s)^{\alpha-n+1}}\, ds,$$

where n is the smallest integer not less than α, provided that the right-hand side is defined pointwise on $(0, +\infty)$.

By arguments similar to those in [30, 31], we have the following two lemmas.

Lemma 2.1 *Given $y \in C(0,1) \cap L^1(0,1)$, the solution of the BVP*

$$\begin{cases} D_{0^+}^{\alpha}x(t) + y(t) = 0, & t \in (0,1), \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, & D_{0^+}^{\beta}x(1) = \sum_{i=1}^{\infty} \alpha_i D_{0^+}^{\gamma}x(\xi_i), \end{cases}$$

is

$$x(t) = \int_0^1 G(t,s)y(s)\, ds,$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = \frac{1}{\Gamma(\alpha)q(0)} \begin{cases} q(s)(1-s)^{\alpha-\beta-1}t^{\alpha-1} - q(0)(t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ q(s)(1-s)^{\alpha-\beta-1}t^{\alpha-1}, & 0 \le t \le s \le 1, \end{cases}$$

and

$$q(s) = \frac{1}{\Gamma(\alpha-\beta)} - \frac{1}{\Gamma(\alpha-\gamma)} \sum_{s\le\xi_i} \alpha_i \left(\frac{\xi_i - s}{1-s}\right)^{\alpha-\gamma-1} (1-s)^{\beta-\gamma}.$$

Lemma 2.2 *The functions q and G given in Lemma 2.1 have the following properties:*
(i) $q \in C([0,1],(0,+\infty))$ *is nondecreasing;*
(ii) $G(t,s) \in C([0,1]\times[0,1],[0,+\infty))$;
(iii) $p(t)G(1,s) \le G(t,s) \le G(1,s), t,s \in [0,1]$, *where* $p(t) = t^{\alpha-1}$.

Set $E = C[0,1]$ and $\|x\| = \sup_{t\in[0,1]} |x(t)|$. We define the cones

$$P = \{x \in E : x(t) \ge 0, t \in [0,1]\} \quad \text{and} \quad K = \{x \in P : x(t) \ge p(t)\|x\|, t \in [0,1]\}.$$

For $0 < r < +\infty$, denote $K_r = \{x \in K : \|x\| < r\}$, $\partial K_r = \{x \in K : \|x\| = r\}$ and $\overline{K}_r = \{x \in K : \|x\| \le r\}$. Define the operators $A : \overline{K}_R \backslash K_r \to P$ and $L : E \to E$ by

$$Ax(t) = \lambda \int_0^1 G(t,s)g(s)f\big(s,x(s)\big)\,ds, \quad t \in [0,1],$$

$$Lx(t) = \int_0^1 G(t,s)g(s)x(s)\,ds, \quad t \in [0,1].$$

Clearly, $L : K \to K$ is a completely continuous linear operator. Moreover, if x is a fixed point of A, then x is a solution of BVP (1.1).

We further assume that:
(H_1) $g \in C((0,1),[0,\infty))$ and $0 < \int_0^1 G(1,s)g(s)\,ds < +\infty$.
(H_2) $f \in C([0,1]\times(0,\infty),[0,\infty))$, and for any $0 < r < R < +\infty$,

$$\lim_{m\to\infty} \sup_{u\in\overline{K}_R\backslash K_r} \int_{D(m)} g(s)f\big(s,x(s)\big)\,ds = 0,$$

where $D(m) = [0,\frac{1}{m}] \cup [\frac{m-1}{m},1]$.

We obtain the following lemma using proofs similar to those in [34, 35].

Lemma 2.3 *Suppose that (H_1) and (H_2) hold. Then $A : \overline{K}_R\backslash K_r \to K$ is completely continuous.*

By Lemma 2.2 we can show that the spectral radius $r(L) > 0$; see, for example, Lemma 2.5 of [36]. Using the Krein–Rutman theorem (see Theorem 19.2 on p. 226 of [37]), we have the following result.

Lemma 2.4 *Suppose that (H_1) and (H_2) are satisfied. Then the first eigenvalue of L is $\lambda_1 = (r(L))^{-1} > 0$, and there exists a positive eigenfunction φ_1 such that $\varphi_1 = \lambda_1 L\varphi_1$.*

The main tool in the paper is the following fixed point index theorem.

Lemma 2.5 ([38]) *Let K be a cone in a Banach space E, and let $T:\overline{K}_r\to K$ be a completely continuous operator.*

(i) *If there exists $u_0\in K\backslash\{\theta\}$ such that $u-Tu\neq\mu u_0$ for any $u\in\partial K_r$ and $\mu\geq 0$, then $i(T,K_r,K)=0$.*

(ii) *If $Tu\neq\mu u$ for any $u\in\partial K_r$ and $\mu\geq 1$, then $i(T,K_r,K)=1$.*

3 Main results

Theorem 3.1 *Suppose that (H_1) and (H_2) are satisfied. If*

$$0\leq f^{\infty}:=\limsup_{x\to+\infty}\max_{t\in[0,1]}\frac{f(t,x)}{x}<\lambda_1<f_0:=\liminf_{x\to 0}\min_{t\in[0,1]}\frac{f(t,x)}{x}\leq+\infty,$$

then BVP (1.1) has at least one positive solution for any

$$\lambda\in\left(\frac{\lambda_1}{f_0},\frac{\lambda_1}{f^{\infty}}\right). \tag{3.1}$$

Proof By (3.1) we have $f_0>\frac{\lambda_1}{\lambda}$, and there exists $r_1>0$ such that $f(t,x)\geq\frac{\lambda_1}{\lambda}x$ for $0<x\leq r_1$ and $0\leq t\leq 1$. For any $x\in\partial K_{r_1}$, we obtain

$$(Ax)(t)=\lambda\int_0^1 G(t,s)g(s)f\bigl(s,x(s)\bigr)\,ds\geq\lambda_1(Lx)(t),\quad t\in[0,1].$$

Suppose that φ_1 is the positive eigenfunction corresponding to λ_1 and that A has no fixed points on ∂K_{r_1}. We claim that

$$x-Ax\neq\mu\varphi_1,\quad x\in\partial K_{r_1},\mu\geq 0. \tag{3.2}$$

Otherwise, there would exist $x_1\in\partial K_{r_1}$ and $\mu_1\geq 0$ such that $x_1-Ax_1=\mu_1\varphi_1$. Then $\mu_1>0$ and $x_1=Ax_1+\mu_1\varphi_1\geq\mu_1\varphi_1$. Denote $\overline{\mu}=\sup\{\mu\mid x_1\geq\mu\varphi_1\}$. Then $\overline{\mu}\geq\mu_1$, $x_1\geq\overline{\mu}\varphi_1$, and $Ax_1\geq\lambda_1\overline{\mu}L\varphi_1=\overline{\mu}\varphi_1$. Thus

$$x_1=Ax_1+\mu_1\varphi_1\geq\overline{\mu}\varphi_1+\mu_1\varphi_1=(\overline{\mu}+\mu_1)\varphi_1,$$

which contradicts to the definition of $\overline{\mu}$. It follows from (3.2) and Lemma 2.5(i) that

$$i(A,K_{r_1},K)=0. \tag{3.3}$$

On the other hand, by (3.1) we have $f^{\infty}<\frac{\lambda_1}{\lambda}$, and there exist $r_2>r_1$ and $0<\sigma<1$ such that $f(t,x)\leq\sigma\frac{\lambda_1}{\lambda}x$ for $x\geq r_2$ and $0\leq t\leq 1$. We define $L_1u=\sigma\lambda_1Lu$. Obviously, the linear operator $L_1:E\to E$ is bounded, and $L_1(K)\subset K$. From the definition of λ_1 and $0<\sigma<1$ it follows that

$$\bigl(r(L_1)\bigr)^{-1}=(\sigma\lambda_1)^{-1}\bigl(r(L)\bigr)^{-1}=\sigma^{-1}>1. \tag{3.4}$$

Choose $\varepsilon_0 = \frac{1}{2}(1 - r(L_1))$. Then by Gelfand's formula there exists a natural number $N \geq 1$ such that $\|L_1^k\| \leq [r(L_1) + \varepsilon_0]^k$ for $k \geq N$. We now define

$$\|x\|^* = \sum_{i=1}^{N} \bigl[r(L_1) + \varepsilon_0\bigr]^{N-i} \bigl\|L_1^{i-1}x\bigr\|, \quad x \in E,$$

where $L_1^0 = I$ is the identity operator. Since L_1 is linear, it is easy to verify that $\|x\|^*$ is a norm in E. Let $M_0 = \sup_{x\in\partial K_{r_2}} \lambda \int_0^1 G(1,s)g(s)f(s,x(s))\,ds$. Then $M_0 < +\infty$. We define $M_0^* = \|M_0\|^*$ and take $r_3 > \max\{r_2, 2M_0^*\varepsilon_0^{-1}\}$. Noting that $\|x\|^* > [r(L_1) + \varepsilon_0]^{N-1}\|x\|$, we can find $r_4 > r_3$ large enough such that $\|x\| \geq r_4$and thus $\|x\|^* > r_3$.

We next prove that

$$Ax \neq \mu x, \quad x \in \partial K_{r_4}, \mu \geq 1. \tag{3.5}$$

Arguing indirectly, we get that there exist $x_2 \in \partial K_{r_4}$ and $\mu_2 \geq 1$ such that $Ax_2 = \mu_2 x_2$. We define $\widetilde{x}(t) = \min\{x_2(t), r_2\}$ for $t \in [0,1]$ and $H(x_2) = \{t \in [0,1] : x_2(t) > r_2\}$. It is easy to see that $\|\widetilde{x}\| = r_2$. We have $\widetilde{x} \in \partial K_{r_2}$ since $\widetilde{x}(t) = \min\{x_2(t), r_2\} \geq \min\{p(t)r_4, r_2\} \geq p(t)r_2$, $t \in [0,1]$. It follows that

$$\begin{aligned}
\mu_2 x_2(t) &= (Ax_2)(t) \\
&= \lambda \int_0^1 G(t,s)g(s)f\bigl(s, x_2(s)\bigr)\,ds \\
&\leq \lambda \int_{H(x_2)} G(t,s)g(s)f\bigl(s, x_2(s)\bigr)\,ds + \lambda \int_{[0,1]\setminus H(x_2)} G(1,s)g(s)f\bigl(s, x_2(s)\bigr)\,ds \\
&\leq \sigma\lambda_1 \int_0^1 G(t,s)g(s)x_2(s)\,ds + \lambda \int_0^1 G(1,s)g(s)f\bigl(s, \widetilde{x}(s)\bigr)\,ds \\
&\leq (L_1x_2)(t) + M_0, \quad t \in [0,1].
\end{aligned}$$

Since $L_1(K) \subset K$, we have $0 \leq (L_1^j(Ax_2)(t)) \leq (L_1^j(L_1x_2 + M_0)(t))$, $j = 0,1,2,\dots,N-1$. Then $\|L_1^j(Ax_2)\| \leq \|L_1^j(L_1x_2 + M_0)\|$, $j = 0,1,2,\dots,N-1$, and hence

$$\|Ax_2\|^* \leq \sum_{i=1}^{N} \bigl[r(L_1) + \varepsilon_0\bigr]^{N-i} \bigl\|L_1^{i-1}(L_1x_2 + M_0)\bigr\| = \|L_1x_2 + M_0\|^*.$$

Therefore we obtain

$$\begin{aligned}
\mu_2\|x_2\|^* &= \|Ax_2\|^* \\
&\leq \|L_1x_2\|^* + M_0^* \\
&= \sum_{i=1}^{N} \bigl[r(L_1) + \varepsilon_0\bigr]^{N-i} \bigl\|L_1^i x_2\bigr\| + M_0^* \\
&\leq \bigl[r(L_1) + \varepsilon_0\bigr] \sum_{i=1}^{N-1} \bigl[r(L_1) + \varepsilon_0\bigr]^{N-i-1} \bigl\|L_1^i x_2\bigr\| + \bigl[r(L_1) + \varepsilon_0\bigr]^N \|x_2\| + M_0^* \\
&= \bigl[r(L_1) + \varepsilon_0\bigr] \sum_{i=1}^{N} \bigl[r(L_1) + \varepsilon_0\bigr]^{N-i} \bigl\|L_1^{i-1} x_2\bigr\| + M_0^*
\end{aligned}$$

$$= \left[r(L_1) + \varepsilon_0\right]\|x_2\|^* + M_0^*$$
$$\leq \left[r(L_1) + \varepsilon_0\right]\|x_2\|^* + \frac{\varepsilon_0}{2} r_3$$
$$< \left[r(L_1) + \varepsilon_0\right]\|x_2\|^* + \frac{\varepsilon_0}{2}\|x_2\|^*$$
$$= \left[\frac{1}{4} r(L_1) + \frac{3}{4}\right]\|x_2\|^*.$$

Thus $\frac{1}{4}r(L_1) + \frac{3}{4} \geq 1$, that is, $r(L_1) \geq 1$, which contradicts (3.4). It follows from (3.5) and Lemma 2.5(ii) that

$$i(A, K_{r_4}, K) = 1. \tag{3.6}$$

By (3.3), (3.6), and the additivity of the fixed point index we have

$$i(A, K_{r_4} \backslash \overline{K}_{r_1}, K) = i(A, K_{r_4}, K) - i(A, K_{r_1}, K) = 1.$$

Therefore A has at least one fixed point $x^* \in K_{r_4} \backslash \overline{K}_{r_1}$, which is a positive solution of BVP (1.1). □

4 An example

Let $\alpha = \frac{7}{2}, \beta = \frac{3}{2}, \gamma = \frac{1}{2}, \alpha_i = \frac{2}{i^2}, \xi_i = 1 - \frac{1}{i+1} (i = 1, 2, \ldots), g(t) = \frac{1}{\sqrt[4]{t(1-t)}}, f(t, x) = \sqrt{2 - t + |\ln x|}$. Consider the following fractional BVP:

$$\begin{cases} D_{0^+}^{\frac{7}{2}} x(t) + \lambda \frac{1}{\sqrt[4]{t(1-t)}} \sqrt{2 - t + |\ln x(t)|} = 0, & t \in (0,1), \\ x(0) = x'(0) = x''(0) = 0, & D_{0^+}^{\frac{3}{2}} x(1) = \sum_{i=1}^{\infty} \frac{2}{i^2} D_{0^+}^{\frac{1}{2}} x(1 - \frac{1}{i+1}). \end{cases} \tag{4.1}$$

Direct computation shows that $\Gamma(\alpha - \beta) = 1, \Gamma(\alpha - \gamma) = 2, \sum_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-\gamma-1} = 2(\frac{\pi^2}{6} - 1)$, and $\frac{1}{\Gamma(\alpha-\beta)} - \frac{1}{\Gamma(\alpha-\gamma)} \sum_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-\gamma-1} \approx 0.355 > 0$.

Let $K = \{x \in C[0,1] : x(t) \geq p(t)\|x\|, t \in [0,1]\}$, where $p(t) = t^{\frac{5}{2}}$. For $x \in \overline{K}_R \backslash K_r$, we obtain $|\ln x(t)| \leq |\ln rp(t)| + |\ln R|$. Due to $\int_0^1 |\ln p(t)| dt = \frac{5}{2}$, we have $\lim_{m\to\infty} \int_{D(m)} |\ln p(t)| \, dt = 0$. Since $0 \leq G(t,s) \leq G(1,s) \leq \frac{1}{\Gamma(\frac{7}{2})(2-\frac{\pi^2}{6})}$, it follows that $\int_0^1 G(1,s) g(s) \, ds \leq \frac{1}{\Gamma(\frac{7}{2})(2-\frac{\pi^2}{6})} \times \int_0^1 g(s) \, ds = \frac{2[\Gamma(\frac{3}{4})]^2}{\Gamma(\frac{7}{2})(2-\frac{\pi^2}{6})\sqrt{\pi}}$. For $x \in \overline{K}_R \backslash K_r$, we have

$$\int_0^1 f^2(s, x(s)) \, ds \leq \int_0^1 \left(2 - s + |\ln r| + |\ln R| + \left|\ln p(s)\right|\right) ds = 4 + |\ln r| + |\ln R|.$$

Therefore

$$\lim_{m\to\infty} \sup_{x \in \overline{K}_R \backslash K_r} \int_{D(m)} g(s) f(s, x(s)) \, ds$$
$$\leq \lim_{m\to\infty} \sup_{x \in \overline{K}_R \backslash K_r} \left(\int_{D(m)} g^2(s) \, ds\right)^{\frac{1}{2}} \left(\int_{D(m)} f^2(s, x(s)) \, ds\right)^{\frac{1}{2}}$$
$$\leq \lim_{m\to\infty} \sqrt{\pi} \left(\int_{D(m)} \left(2 - s + |\ln r| + |\ln R| + \left|\ln p(s)\right|\right) ds\right)^{\frac{1}{2}} = 0.$$

Direct computation yields $f^{\infty} = 0$ and $f_0 = +\infty$. Using Theorem 3.1, we can conclude that the BVP (4.1) has at least one positive solution for any $\lambda \in (0, +\infty)$.

5 Conclusions

We established the existence of positive solutions for the singular fractional differential equation infinite-point BVP (1.1) using the fixed point index theory in cones. Note that the nonlinearity may possess singularities, that is, $f(t,x)$ may have a singularity at $x = 0$, and $g(t)$ may be singular at $t = 0$ or $t = 1$.

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Author details
[1]School of Mathematical Sciences, Qufu Normal University, Qufu, P.R. China. [2]Weifang Engineer Vocational College, Shandong, P.R. China.

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