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Existence of solutions for a multi-point boundary value problem with a $p(r)$-Laplacian

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Abstract

In this paper, we consider the existence of solutions to the $p(r)$-Laplacian equation with multi-point boundary conditions. Under some new criteria and by utilizing degree methods and also the Leray–Schauder fixed point theorem, the new existence results of the solutions have been established. Some results in the literature can be generalized and improved. And as an application, two examples are provided to demonstrate the effectiveness of our theoretical results.

Keywords: $p(r)$-Laplacian; Boundary condition; Fixed point theorem; Leray–Schauder degree method

1 Introduction

In recent years, there has been extensive interest in boundary value problems (BVPs) with variable exponent in a Banach space, see [1–9]. Such problems usually arise in the study of image processing, elastic mechanics, electrorheological fluids dynamics, etc. (see [10–18]).

In the case when p is a constant and $f(r,u(r),u'(r)) = f(r,u(r))$, the first differential equation of Eq. (1.1) subjected to some other boundary conditions becomes the classical p-Laplacian problem, which has been extensively researched in [19–21]:

$$\begin{cases} (|u'(r)|^{p-2}u'(r))' + a(r)f(r,u(r)) = 0, & r \in (0,1), \\ u(0) = 0, \qquad u(1) = 1, \end{cases}$$

and we have obtained the existence of solutions for the addressed equations. For more information on the problems of differential equations with p-Laplacian operator, the readers may refer to [22–30].

This paper focuses on the following $p(r)$-Laplacian differential equations with multi-point boundary conditions:

$$\begin{cases} (|u'(r)|^{p(r)-2}u'(r))' + a(r)f(r,u(r),u'(r)) = 0, & r \in (0,1), \\ u(0) - \alpha u'(\xi) = 0, \qquad u(1) + \sum_{i=1}^{m-3} \beta_i u(\eta_i) = 0, \end{cases} \tag{1.1}$$

where the functions f, p, a and the constants α, β_i, ξ, η_i ($1 \le i \le m-3$) satisfy:

(H1) $f \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $p \in C([0,1], \mathbb{R})$, $p(r) > 1$, $a \in C((0,1), \mathbb{R})$ is possibly singular at $r = 0$ or $r = 1$ and satisfies $0 < \int_0^1 |a(r)|\,\mathrm{d}r < +\infty$;

(H2) $\alpha, \beta_i > 0$, $0 < \xi < \eta_1 < \eta_2 < \cdots < \eta_{m-3} < 1$.

Compared with some new achievements in the articles, such as [19–30], the major contributions of our research contain at least the following three:

(1) f is the nonlinear term and $a(r)$ is allowed to be singular at $r = 0$ or $r = 1$. Additionally, compared to two-point or three-point BVPs, which have been extensively studied, we discuss a multi-point BVP in this article.

(2) The model we are concerned with is more generalized, some ones in the articles [19–22] are the special cases of it. $p(r)$ is a general function, which is more complicated than the case when p is a fixed constant. That is to say, the comprehensive model is originally considered in the present paper.

(3) An innovative approach based on degree methods and the Leray–Schauder fixed point theorem are utilized to obtain the existence of solutions for the addressed equations (1.1). The results established are essentially new.

The following article is organized as follows: In Sect. 2, we introduce some necessary notations and important lemmas, while Sect. 3 is devoted to establishing the existence of solutions for problem (1.1) by a fixed point theorem and degree methods, and then we come up with the main theorems. To explain the results clearly, we finally give two examples in Sect. 4.

2 Preliminaries

In this section, we are going to present some basic notations and lemmas which are used throughout this paper.

Let $U = C^1[0,1]$. It is well known that U is a Banach space with the norm $\|\cdot\|_1$ defined by

$$\|u\|_1 = \|u\| + \|u'\|,$$

where

$$\|u\| = \max_{r\in[0,1]} |u(r)|, \qquad \|u'\| = \max_{r\in[0,1]} |u'(r)|.$$

Besides, we denote

$$p^- = \min_{r\in[0,1]} p(r), \qquad p^+ = \max_{r\in[0,1]} p(r).$$

Set

$$\varphi(r,x) = |x|^{p(r)-2}x \quad \text{for fixed } r \in [0,1], x \in \mathbb{R},$$

and set $\varphi^{-1}(r,\cdot)$ as

$$\varphi^{-1}(r,x) = |x|^{\frac{2-p(r)}{p(r)-1}}x \quad \text{for fixed } r \in [0,1], x \in \mathbb{R} \setminus \{0\},$$

where $\varphi^{-1}(r,0) = 0$.

Obviously, $\varphi^{-1}(r,\cdot)$ is continuous and sends a bounded set into a boundary set. Aiming to obtain the existence of solutions to problem (1.1), we need the following lemmas. The proofs are standard, thus some details can be omitted.

Lemma 2.1 ([31]) *φ is a continuous function and satisfies that, for any $x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2$, for any $r \in [T_1, T_2]$,*

$$\big(\varphi(r,x_1) - \varphi(r,x_2), x_1 - x_2\big) > 0,$$

which implies it is monotone increasing.

Lemma 2.2 *Let U be a Banach space. Provided that the operator $T(u,\lambda) : U \times [0,1] \to U$ is a map satisfying the conditions as follows*:

(S1) *T is a compact map*;

(S2) *For any $u \in U$, $T(u,0) = 0$*;

(S3) *If one has $u = T(u,\lambda)$ for some $\lambda \in [0,1]$, then there exists $M > 0$ such that $\|u\|_1 \leq M$ for any $u \in U$. Then $T(u,1)$ has a fixed point in U.*

Lemma 2.3 *Suppose that $g \in L^1[0,1]$ and $g(r) \neq 0$ on any subinterval of $[0,1]$. Then the BVP*

$$\begin{cases} (\varphi(r,u'))' + g(r) = 0, & 0 < r < 1, \\ u(0) - \alpha u'(\xi) = 0, & u(1) + \sum_{i=1}^{m-3} \beta_i u'(\eta_i) = 0, \end{cases} \tag{2.1}$$

has a unique solution $u(r)$, which is

$$u(r) = \alpha\varphi^{-1}\left(\rho - \int_0^{\xi} g(s)\,\mathrm{d}s\right) + \int_0^r \varphi^{-1}\left(\rho - \int_0^s g(t)\,\mathrm{d}t\right)\mathrm{d}s$$

or

$$u(r) = \sum_{i=1}^{m-3} \beta_i\varphi^{-1}\left(\rho - \int_0^{\eta_i} g(s)\,\mathrm{d}s\right) - \int_r^1 \varphi^{-1}\left(\rho - \int_0^s g(t)\,\mathrm{d}t\right)\mathrm{d}s,$$

where $\rho = \varphi(0,u'(0))$ and ρ is dependent on g.

Now, for any $h \in C[0,1]$, we define

$$\begin{aligned} \Lambda_h(\rho) = {}& \alpha\varphi^{-1}\left(\rho - \int_0^{\xi} h(s)\,\mathrm{d}s\right) + \sum_{i=1}^{m-3} \beta_i\varphi^{-1}\left(\rho - \int_0^{\eta_i} h(s)\,\mathrm{d}s\right) \\ & - \int_0^1 \varphi^{-1}\left(\rho - \int_0^s h(r)\,\mathrm{d}r\right)\mathrm{d}s. \end{aligned}$$

The properties of the operator Λ_h are described in the following lemma.

Lemma 2.4 *For any $h \in C[0,1]$, the equation*

$$\Lambda_h(\rho) = 0 \tag{2.2}$$

has a unique solution $\overline{\rho}(h) \in \mathbb{R}$.

Proof From Lemma 2.1, it is apparent that

$$\big(\Lambda_h(a_1) - \Lambda_h(a_2), a_1 - a_2\big) > 0 \quad \text{for } a_1 \neq a_2.$$

Hence, if Eq. (2.2) has a solution, then it is unique.

Since $h \in C[0,1]$, and let $R_0 = 2\|h\|$. It is easy to see that if $|\rho| > R_0$, then for any $r \in [0,1]$ we have $(\rho - \int_0^r h(s)\,ds) \cdot \rho > 2\|h\|^2$.

Denote

$$s(r) = \varphi^{-1}\left(r, \rho - \int_0^r h(s)\,ds\right),$$

then

$$\rho - \int_0^r h(s)\,ds = \varphi\big(r, s(r)\big),$$

it follows that

$$\rho - \int_0^r h(s)\,ds = \big|s(r)\big|^{p(r)-1} s(r).$$

While $|s(r)|^{p(r)-1}s(r) \cdot \rho = (\rho - \int_0^r h(s)\,ds) \cdot \rho > 2\|h\|^2 > 0$, thus $s(r) \cdot \rho > 0$.

From $|\rho| > R_0$ we have $s(r) \neq 0$, then we also have

$$\int_0^1 \varphi^{-1}\left(r, \rho - \int_0^r h(s)\,ds\right) dr \neq 0.$$

So, when $|\rho| > R_0$, $\Lambda_h(\rho) \neq 0$.

Let us consider the following equation:

$$f(\lambda, \rho) \triangleq \lambda\Lambda_h(\rho) + (1-\lambda)a = 0, \quad \lambda \in [0,1]. \tag{2.3}$$

It is easy to prove that all the solutions of Eq. (2.3) belong to $b(R_0) \triangleq \{x \in \mathbb{R} : |x| < R_0\}$. From the homotopy invariance property on Leray–Schauder degree theory, we have

$$\begin{aligned}\deg\big(\Lambda_h(\rho), b(R_0), 0\big) &= \deg\big(f(0,\rho), b(R_0), 0\big) \\ &= \deg\big(f(1,\rho), b(R_0), 0\big) = \deg\big(I, b(R_0), 0\big) \neq 0,\end{aligned}$$

which implies the existence of solution of $\Lambda_h(\rho) = 0$. Consequently, $\Lambda_h(\rho) = 0$ has a solution $\overline{\rho}(h) \in \mathbb{R}$. □

Lemma 2.5 *Assume that u is the solution of problem* (2.1), *then it can also be rewritten in the following form*:

$$u(r) = \begin{cases} \alpha\varphi^{-1}(\int_\xi^\sigma g(s)\,ds) + \int_0^r \varphi^{-1}(\int_s^\sigma g(t)\,dt)\,ds, & 0 \leq r \leq \sigma, \\ \sum_{i=1}^{m-3} \beta_i \varphi^{-1}(\int_\sigma^{\eta_i} g(s)\,ds) + \int_r^1 \varphi^{-1}(\int_\sigma^s g(t)\,dt)\,ds, & \sigma \leq r \leq 1, \end{cases} \tag{2.4}$$

where $\sigma \in (0,1)$.

Proof Assume that $u(r)$ is the solution of problem (2.1), then there exists $\sigma \in (0,1)$ such that $u'(\sigma) = 0$. Otherwise, suppose that $u'(r) < 0$ for any $r \in (0,1)$, which means that $u(t)$ is nonincreasing. From the boundary value conditions, it follows that

$$u(0) = \alpha u'(\xi) < 0,$$

but

$$u(1) = -\sum_{i=1}^{m-3} \beta_i u'(\xi_i) > 0,$$

which is a contradiction. Similarly, if $u'(r) > 0$ for any $t \in (0,1)$, we know that $u(t)$ is nondecreasing, which with boundary conditions yields a contradiction. Then, through direct computations, (2.4) holds. □

3 Existence of solutions

In this section, we will show that under some suitable conditions solutions to problem (1.1) do exist.

Theorem 3.1 *Suppose that* (H1), (H2) *hold and* f *satisfies*

$$\lim_{|u|+|v|\to\infty} \frac{f(r,u,v)}{(|u|+|v|)^{q(r)-1}} = 0, \quad 1 < q^- \le q^+ < p^-.$$

Then problem (1.1) *has at least one solution.*

Proof To obtain the existence of solutions of problem (1.1), consider the BVP

$$\begin{cases} (|u'(r)|^{p(r)-2}u'(r))' + \lambda a(r) f(r,u(r),u'(r)) = 0, & r \in (0,1), \lambda \in [0,1], \\ u(0) - \alpha u'(\xi) = 0, \qquad u(1) + \sum_{i=1}^{m-3} \beta_i u'(\eta_i) = 0, \end{cases}$$

and define the integral operator $T : U \times [0,1] \to U$ by

$$T(u,\lambda)(r) = \begin{cases} \alpha \varphi^{-1}(\int_\xi^\sigma \lambda a(s) f(s,u(s),u'(s))\,\mathrm{d}s) \\ \quad + \int_0^r \varphi^{-1}(\int_s^\sigma \lambda a(t) f(t,u(t),u'(t))\,\mathrm{d}t)\,\mathrm{d}s, & 0 \le r \le \sigma, \\ \sum_{i=1}^{m-3} \beta_i \varphi^{-1}(\int_\sigma^{\eta_i} \lambda a(s) f(s,u(s),u'(s))\,\mathrm{d}s) \\ \quad + \int_r^1 \varphi^{-1}(\int_\sigma^s \lambda a(t) f(t,u(t),u'(t))\,\mathrm{d}t)\,\mathrm{d}s, & \sigma \le r \le 1. \end{cases} \tag{3.1}$$

From the continuity of f, φ^{-1} and also the definition of a, it is easy to see that u is a solution of problem (1.1) if and only if u is a fixed point of the integral operator T when $\lambda = 1$. In order to apply Lemma 2.2, the proof includes three steps:

(1) T is a compact map.

Let $D \subset U \times [0,1]$ be an arbitrary bounded subset, then there exists $M > 0$ such that

$$\|u\|_1 \le M.$$

And let $\{(u_n,\lambda_n)\}$ be a sequence in D. Firstly, we prove that $\{T(u_n,\lambda_n)\}$ has a convergent subsequence in $C[0,1]$. According to (H1), we find that there exists $N > 1$ such that

$$\bigl|f\bigl(r,u_n(r),u_n'(r)\bigr)\bigr| \le N, \quad r\in[0,1], \|u_n\|_1 \le M.$$

Thus, for any $(u_n,\lambda_n)\in D$, if $0\le r\le\sigma$, then

$$\begin{aligned}
&\bigl|T(u_n,\lambda_n)(r)\bigr| \\
&\quad\le \alpha\varphi^{-1}\biggl(\int_\xi^\sigma \lambda_n\bigl|a(s)f\bigl(s,u_n(s),u_n'(s)\bigr)\bigr|\,\mathrm{d}s\biggr) \\
&\qquad+\int_0^r \varphi^{-1}\biggl(\int_s^\sigma \lambda_n\bigl|a(t)f\bigl(t,u_n(t),u_n'(t)\bigr)\bigr|\,\mathrm{d}t\biggr)\,\mathrm{d}s \\
&\quad\le \alpha\varphi^{-1}\biggl(\int_\xi^\sigma \bigl|a(s)f\bigl(s,u_n(s),u_n'(s)\bigr)\bigr|\,\mathrm{d}s\biggr) \\
&\qquad+\int_0^r \varphi^{-1}\biggl(\int_s^\sigma \bigl|a(t)f\bigl(t,u_n(t),u_n'(t)\bigr)\bigr|\,\mathrm{d}t\biggr)\,\mathrm{d}s \\
&\quad\le (\alpha+1)N^{\frac{1}{p^--1}}\max\biggl\{\biggl(\int_0^1\bigl|a(s)\bigr|\,\mathrm{d}s\biggr)^{\frac{1}{p^--1}},\biggl(\int_0^1\bigl|a(s)\bigr|\,\mathrm{d}s\biggr)^{\frac{1}{p^+-1}}\biggr\}.
\end{aligned}$$

Similarly, if $\sigma\le r\le 1$, then

$$\begin{aligned}
&\bigl|T(u_n,\lambda_n)(r)\bigr| \\
&\quad\le \sum_{i=1}^{m-3}\beta_i\varphi^{-1}\biggl(\int_\sigma^{\eta_i} \bigl|a(s)f\bigl(s,u_n(s),u_n'(s)\bigr)\bigr|\,\mathrm{d}s\biggr) \\
&\qquad+\int_0^r \varphi^{-1}\biggl(\int_\sigma^s \bigl|a(t)f\bigl(t,u_n(t),u_n'(t)\bigr)\bigr|\,\mathrm{d}t\biggr)\,\mathrm{d}s \\
&\quad\le \Biggl(\sum_{i=1}^{m-3}\beta_i+1\Biggr)N^{\frac{1}{p^--1}}\max\biggl\{\biggl(\int_0^1\bigl|a(s)\bigr|\,\mathrm{d}s\biggr)^{\frac{1}{p^--1}},\biggl(\int_0^1\bigl|a(s)\bigr|\,\mathrm{d}s\biggr)^{\frac{1}{p^+-1}}\biggr\}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\bigl|T'(u_n,\lambda_n)(t)\bigr| \\
&\quad=\bigl|\varphi^{-1}\bigl|a(t)f\bigl(t,u_n(t),u_n'(t)\bigr)\bigr|\,\mathrm{d}r\bigr| \\
&\quad\le N^{\frac{1}{p^--1}}\max\biggl\{\biggl(\int_0^1\bigl|a(s)\bigr|\,\mathrm{d}s\biggr)^{\frac{1}{p^--1}},\biggl(\int_0^1\bigl|a(s)\bigr|\,\mathrm{d}s\biggr)^{\frac{1}{p^+-1}}\biggr\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\bigl|T(u_n,\lambda_n)(t)\bigr| &\le \max\Biggl\{\alpha+1,\sum_{i=1}^{m-3}\beta_i+1\Biggr\}N^{\frac{1}{p^--1}} \\
&\qquad\cdot\max\biggl\{\biggl(\int_0^1\bigl|a(s)\bigr|\,\mathrm{d}s\biggr)^{\frac{1}{p^--1}},\biggl(\int_0^1\bigl|a(s)\bigr|\,\mathrm{d}s\biggr)^{\frac{1}{p^+-1}}\biggr\},
\end{aligned}$$

and

$$\left|T'(u_n,\lambda_n)(t)\right| \le N^{\frac{1}{p^- -1}} \max\left\{\left(\int_0^1 \left|a(s)\right| \mathrm{d}s\right)^{\frac{1}{p^- -1}}, \left(\int_0^1 \left|a(s)\right| \mathrm{d}s\right)^{\frac{1}{p^+ -1}}\right\}.$$

Besides, we find that, for any $0 \le t_1 \le t_2 \le 1$,

$$\begin{aligned}
&\left|T(u_n,\lambda_n)(t_1) - T(u_n,\lambda_n)(t_2)\right| \\
&\quad = \left|\int_{t_1}^{t_2} T'(u_n,\lambda_n)(t)\,\mathrm{d}t\right| \\
&\quad \le N^{\frac{1}{p^- -1}} \max\left\{\left(\int_0^1 \left|a(s)\right| \mathrm{d}s\right)^{\frac{1}{p^- -1}}, \left(\int_0^1 \left|a(s)\right| \mathrm{d}s\right)^{\frac{1}{p^+ -1}}\right\}|t_1 - t_2|.
\end{aligned}$$

Hence, $\{T(u_n,\lambda_n)\}$ is equi-continuous and uniformly bounded.

Applying the Ascoli–Arzelà theorem, there exists a convergent subsequence of $\{T(u_n,\lambda_n)\}$ in $C[0,1]$. Without loss of generality, we denote the convergent subsequence again by $\{T(u_n,\lambda_n)\}$.

Next, we should show that $\{T'(u_n,\lambda_n)\}$ also has a convergent subsequence in $C[0,1]$. Denote

$$F_n(t) = \int_\sigma^t \lambda_n a(s) f\left(s, u_n(s), u_n'(s)\right) \mathrm{d}s.$$

Similar to the proof above, we can find that $\{F_n(t)\}$ has a convergent subsequence in $C[0,1]$, which we still denote by $\{F_n(t)\}$. From the continuity of φ^{-1}, we can easily get that $\{T'(u_n,\lambda_n)\}$ is convergent in $C[0,1]$.

From the above, we know that T is a compact operator, which implies that condition (S1) in Lemma 2.2 holds.

(2) Evidently, $T(u,0) = 0$ for $u \in U$, so condition (S2) is satisfied.

(3) Now, we verify condition (S3) in Lemma 2.2.

If condition (S3) does not hold, then we would find that there exists a subsequence $\{(u_n,\lambda_n)\}$ such that $\|u_n\|_1 \to \infty$ as $n \to \infty$ and $\|u_n\|_1 > 1$. According to Lemma 2.5, we have

$$\begin{aligned}
\left|u_n'(r)\right|^{p(r)-2} u_n'(r) &= \int_\sigma^r \left(\left(\left|u_n'(s)\right|^{p(s)-2}\right) u_n'(s)\right)' \mathrm{d}s \\
&= -\int_\sigma^r \lambda_n a(s) f\left(s, u_n(s), u_n'(s)\right) \mathrm{d}s.
\end{aligned}$$

Note that

$$\lim_{|u|+|u'|\to\infty} \frac{f(r,u,u')}{(|u| + |u'|)^{q(r)-1}} = 0,$$

then we get that there exist $M_1 > 0, c_1 > 0$ such that

$$\left|f\left(r,u,u'\right)\right| \le c_1\left(|u| + \left|u'\right|\right)^{q(r)-1}, \quad r \in [0,1], |u| + \left|u'\right| \in [M, +\infty).$$

Thus, for $|u_n| + |u_n'| \in [M_1, +\infty)$ and $r \in [0,1]$, we have

$$\begin{aligned} \bigl||u_n'(r)|^{p(r)-2}u_n'(r)\bigr| &\le \lambda_n \int_\sigma^r \bigl|a(s)f\bigl(s,u_n(s),u_n'(s)\bigr)\bigr|\,\mathrm{d}s \\ &\le c_1 \int_0^1 \bigl|a(s)\bigr|\bigl(\bigl|u(s)\bigr| + \bigl|u_n'(s)\bigr|\bigr)^{q(s)-1}\,\mathrm{d}s \\ &\le c_1\, |u_n\, |^{q^+-1} \int_0^1 \bigl|a(s)\bigr|\,\mathrm{d}s, \end{aligned}$$

it follows that

$$\bigl|u_n'(r)\bigr|^{p(r)-1} \le c_1 \|u_n\|^{q^+-1} \int_0^1 \bigl|a(s)\bigr|\,\mathrm{d}s.$$

Hence,

$$\bigl|u_n'(r)\bigr| \le C\|u_n\|^{\frac{q^+-1}{p^--1}} \max\biggl\{\biggl(\int_0^1 \bigl|a(s)\bigr|\,\mathrm{d}s\biggr)^{\frac{1}{p^--1}}, \biggl(\int_0^1 \bigl|a(s)\bigr|\,\mathrm{d}s\biggr)^{\frac{1}{p^+-1}}\biggr\},$$

and

$$\begin{aligned} \bigl|u_n(r)\bigr| &= \biggl|\int_\sigma^r u_n'(s)\,\mathrm{d}s\biggr| \\ &\le C\|u_n\|^{\frac{q^+-1}{p^--1}} \max\biggl\{\biggl(\int_0^1 \bigl|a(s)\bigr|\,\mathrm{d}s\biggr)^{\frac{1}{p^--1}}, \biggl(\int_0^1 \bigl|a(s)\bigr|\,\mathrm{d}s\biggr)^{\frac{1}{p^+-1}}\biggr\}, \end{aligned}$$

where C is a constant.

We can conclude that $\{(u_n, \lambda_n)\}$ is bounded, which leads to a contradiction. Therefore, condition (S3) in Lemma 2.2 holds.

Applying Lemma 2.2, we can obtain that $T(u,1)$ has a fixed point in U, that is to say, problem (1.1) has at least one solution. This completes the proof. □

Furthermore, we prove the existence of solutions to problem (1.1) under other innovative conditions.

Theorem 3.2 *Suppose that* $\Omega_t = \{u \in C^1[0,1] : \|u\|_1 < t\}$ *is a bounded open set in* U *and* (H1), (H2) *hold. If there exists* $t > 0$ *such that*

$$\bigl|f\bigl(r,u,u'\bigr)\bigr| \le \min\biggl\{\biggl(\frac{t}{3}\biggr)^{p^--1}, \biggl(\frac{t}{3}\biggr)^{p^+-1}\biggr\} \frac{1}{\int_0^1 |a(r)|\,\mathrm{d}r}, \tag{3.2}$$

where $u \in \overline{\Omega_t}$, $r \in [0,1]$, *then problem* (1.1) *has at least one solution.*

Proof Let us consider the following BVP:

$$\begin{cases} (|u'(r)|^{p(r)-2}u'(r)) + \lambda a(r)f(r,u(r),u'(r)) = 0, & r \in (0,1), \lambda \in [0,1], \\ u(0) - \alpha u'(\xi) = 0, & u(1) + \sum_{I=1}^{m-3} \beta_i u'(\eta_i) = 0, \end{cases} \tag{3.3}$$

and define an integral operator $T : U \times [0,1] \to U$ by

$$T(u,\lambda)(r) = \begin{cases} \alpha\varphi^{-1}(\int_\xi^\sigma \lambda a(s)f(s,u(s),u'(s))\,\mathrm{d}s) \\ \quad + \int_0^r \varphi^{-1}(\int_s^\sigma \lambda a(t)f(t,u(t),u'(t))\,\mathrm{d}t)\,\mathrm{d}s, & 0 \le r \le \sigma, \\ \sum_{i=1}^{m-3} \beta_i\varphi^{-1}(\int_\sigma^{\eta_i} \lambda a(s)f(s,u(s),u'(s))\,\mathrm{d}s) \\ \quad + \int_r^1 \varphi^{-1}(\int_\sigma^s \lambda a(t)f(t,u(t),u'(t))\,\mathrm{d}t)\,\mathrm{d}s, & \sigma \le r \le 1, \end{cases}$$

where $\sigma \in (0,1)$. Similar to the above proof, we know that T is compact. Moreover, u is a fixed point of $u = T(u,1)$ if and only if u is a solution of problem (1.1). To achieve the result by Leray–Schauder degree theory, we just need to prove that

(i) for any $\lambda \in [0,1)$, $u = T(u,\lambda)$ has no solution on $\partial\Omega_t$;

(ii) $\deg(I - T(u,0),\Omega_t,0) \neq 0$.

Firstly, we verify that (i) holds. Without loss of generality, there exist $\lambda \in [0,1)$ and $u \in \partial\Omega_t$ such that $u = T(u,\lambda)$, then we have

$$\left|u'(r)\right|^{p(r)-2}u'(r) = -\lambda \int_\sigma^1 a(s)f\bigl(s,u(s),u'(s)\bigr)\,\mathrm{d}s, \quad r \in (0,1).$$

Since $u \in \partial\Omega_t$, it is easy to see that

$$\|u\| + \left\|u'\right\| = t.$$

If $\|u\| \ge 2t/3$, then $\|u'\| \le t/3$, but

$$\left|u(r)\right| = \left|\int_\sigma^r u'(s)\right|\mathrm{d}s \le \int_0^1 \left|u'(s)\right|\mathrm{d}s \le \frac{t}{3},$$

which is a contradiction.

Similarly, if $\|u\| \le 2t/3$, then $\|u'\| > t/3$. Hence, there exists $r_0 \in [0,1]$ such that

$$\left|u'(r_0)\right|^{p(r_0)-1} > \left(\frac{t}{3}\right)^{p(r_0)-1}.$$

According to condition (3.2), we get that

$$\begin{aligned} \left|u'(r_0)\right|^{p(r_0)-1} &= \left|\int_\sigma^{r_0} \lambda a(s)f\bigl(s,u(s),u'(s)\bigr)\,\mathrm{d}s\right| \\ &\le \int_0^1 \left|a(s)f\bigl(s,u(s),u'(s)\bigr)\right|\mathrm{d}s \\ &\le \min\left\{\left(\frac{t}{3}\right)^{p^- -1}, \left(\frac{t}{3}\right)^{p^+ -1}\right\}, \end{aligned}$$

which together with $\|u'\| \le t/3$ leads to a contradiction. So problem (3.3) has no solution on $\partial\Omega_t$.

Secondly, when $\lambda = 0$, problem (3.3) becomes the following one:

$$\begin{cases} (|u'(r)|^{p(r)-2})u'(r) = 0, & r \in (0,1), \\ u(0) - \alpha u'(\xi) = 0, & u(1) + \sum_{i=1}^{m-3} \beta_i u'(\eta_i) = 0. \end{cases} \tag{3.4}$$

We can easily find that problem (3.4) has a solution on Ω_t. From the solvability on Leray–Schauder degree theory, we get that

$$\deg\bigl(I - T(u,0), \Omega_t, 0\bigr) \neq 0.$$

Thus condition (ii) is satisfied.

Therefore, upon an application of Leray–Schauder degree method, we obtain that problem (1.1) has at least one solution. This completes the proof. □

4 Example

In the section, we will present the following two examples to illustrate our main results.

Example 4.1 Consider the following $p(r)$-Laplacian differential equations with six-point boundary conditions:

$$\begin{cases} (|u'(r)|^{e^r+2}u'(r))' + \sin r(r + u(r) + u'(r)) = 0, & r \in (0,1), \\ u(0) - 5u'(\frac{1}{6}) = 0, & u(1) + u(\frac{1}{5}) + 2u(\frac{1}{4}) + 3(\frac{1}{3}) = 0. \end{cases} \tag{4.1}$$

Conclusion Problem (4.1) has at least one solution.

Proof Corresponding to Eq. (1.1), we have

$$p(r) = e^{r+4}, \qquad a(r) = \sin r, \qquad f\bigl(r, u(r), u'(r)\bigr) = r + u(r) + u'(r), \quad r \in (0,1),$$
$$m = 6, \qquad \alpha = 5, \qquad \beta_1 = 1, \qquad \beta_2 = 2, \qquad \beta_3 = 3,$$
$$\xi = \frac{1}{6}, \qquad \eta_1 = \frac{1}{5}, \qquad \eta_2 = \frac{1}{4}, \qquad \eta_3 = \frac{1}{3}.$$

Thus, conditions (H1), (H2) are satisfied.

Choose $q(r) = e^r + 2 > 2$, we can easily get that $1 < q^- < q^+ < p^-$ and, when $|u| + |u'| \to \infty$, $r \in (0,1)$, we also get that

$$\frac{f(r,u,u')}{(|u| + |u'|)^{q(r)-1}} = \frac{r + u + u'}{(|u| + |u'|)^{q(r)-1}} \le \frac{r}{(|u| + |u'|)^{q(r)-1}} + \frac{|u| + |u'|}{(|u| + |u'|)^{q(r)-1}} \to 0.$$

Hence, by applying Theorem 3.1, we can see that Eq. (4.1) has at least one solution. □

Example 4.2 Consider the following $p(r)$-Laplacian differential equations with four-point boundary conditions:

$$\begin{cases} (|u'(r)|^{e^r-2}u'(r))' + r^2\cos(r + u(r) + u'(r)) = 0, & r \in (0,1), \\ u(0) - \frac{3}{2}u'(\frac{2}{5}) = 0, & u(1) + \frac{2}{7}u(\frac{4}{5}) = 0. \end{cases} \tag{4.2}$$

Conclusion Problem (4.2) has at least one solution.

Proof Corresponding to Eq. (1.1), we have

$$f\bigl(r, u(r), u'(r)\bigr) = \cos\bigl(r + u(r) + u'(r)\bigr),$$

$$p(r) = e^r, \qquad p^- = e^0 = 1, \qquad p^+ = e^1 = e, \qquad a(r) = r^2, \quad r \in (0,1),$$
$$m = 4, \qquad \alpha = \frac{3}{2}, \qquad \beta_1 = \frac{7}{2}, \qquad \xi = \frac{2}{5}, \qquad \eta_1 = \frac{4}{5}.$$

So conditions (H1), (H2) are satisfied.

Choose $t = 3$, then, when $u \in \Omega_t = \{u \in U : \|u\| < 3\}$, we have

$$\begin{aligned} \left|f\left(r,u,u'\right)\right| &= \left|\cos\left(r + u + u'\right)\right| \\ &\le \min\left\{\left(\frac{t}{3}\right)^{p^- -1}, \left(\frac{t}{3}\right)^{p^+ -1}\right\} \frac{1}{\int_0^1 |a(r)|\,\mathrm{d}r} \\ &= 3 \cdot \min\left\{1, \left(\frac{t}{3}\right)^{e-1}\right\} = 3, \end{aligned}$$

and inequality (3.2) holds, which implies that all the conditions in Theorem 3.1 are satisfied. Thus, we can see that Eq. (4.2) has at least one solution. □

5 Conclusions

In this paper, we are concerned with a class of differential equations involving a $p(r)$-Laplacian operator. The addressed equation with the multi-point boundary value is quite different from the related references discussed in the literature [26–28, 32]. The nonlinear differential system studied in the present paper is more generalized and more practical. By applying the degree methods (see Lemma 2.4, Theorem 3.2) and the fixed point theorem (see Theorems 3.1, 3.2), we employ innovative arguments, and easily verifiable sufficient conditions have been provided to determine the existence of the solutions to the considered equation. Consequently, this paper shows theoretically that some related references known in the literature can be enriched and complemented.

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