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On parametric Gevrey asymptotics for initial value problems with infinite order irregular singularity and linear fractional transforms

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Abstract

This paper is a continuation of the work (Lastra and Malek in J. Differ. Equ. 259(10):5220–5270, 2015) where singularly perturbed nonlinear PDEs have been studied from an asymptotic point of view. Here, the partial differential operators are combined with particular Moebius transforms in the time variable. As a result, the leading term of the main problem needs to be regularized by means of a singularly perturbed infinite order formal irregular operator that allows us to construct a set of genuine solutions in the form of a Laplace transform in time and an inverse Fourier transform in space. Furthermore, we obtain Gevrey asymptotic expansions for these solutions of some order $K > 1$ in the perturbation parameter.

MSC: 35R10; 35C10; 35C15; 35C20

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1 Introduction

In this work, we deal with the study of a family of nonlinear singularly perturbed equations which combine linear fractional transforms, partial derivatives, and differential operators of infinite order of the form

$$Q(\partial_z)u(t,z,\epsilon) = \exp\bigl(\alpha\epsilon^k t^{k+1}\partial_t\bigr)R(\partial_z)u(t,z,\epsilon) + P\bigl(t,\epsilon,\{m_{\kappa,t,\epsilon}\}_{\kappa\in I},\partial_t,\partial_z\bigr)u(t,z,\epsilon) + Q_1(\partial_z)u(t,z,\epsilon)Q_2(\partial_z)u(t,z,\epsilon) + f(t,z,\epsilon), \qquad (1)$$

where $\alpha, k > 0$ are real numbers, $Q(X)$, $R(X)$, $Q_1(X)$, $Q_2(X)$ stand for polynomials with complex coefficients and $P(t,\epsilon,\{U_\kappa\}_{\kappa\in I},V_1,V_2)$ represents a polynomial in t, V_1, V_2, of degree at most one with respect to U_κ, and holomorphic coefficients w.r.t. ϵ near the origin in $\mathbb{C}$, where the symbol $m_{\kappa,t,\epsilon}$ denotes a Moebius operator acting on the time variable.

More precisely, we have

$$m_{\kappa,t,\epsilon}u(t,z,\epsilon) = u\left(\frac{t}{1+\kappa\epsilon t},z,\epsilon\right),$$

where κ belongs to some finite subset I of the positive real numbers $\mathbb{R}_+^*$. The forcing term $f(t,z,\epsilon)$ turns out to be an analytic function in a vicinity of the origin with respect to (t,ϵ) and holomorphic w.r.t. z on a horizontal strip of the form $H_\beta = \{z \in \mathbb{C}/|\operatorname{Im}(z)| < \beta\}$ for some $\beta > 0$.

This work is a continuation of our previous study [15], where the following problem is considered:

$$Q(\partial_z)\partial_t y(t,z,\epsilon) = H(t,\epsilon,\partial_t,\partial_z)y(t,z,\epsilon) + Q_1(\partial_z)y(t,z,\epsilon)Q_2(\partial_z)y(t,z,\epsilon) + f(t,z,\epsilon), \tag{2}$$

for given vanishing initial data $y(0,z,\epsilon) \equiv 0$, where Q_1, Q_2, H stand for polynomials and $f(t,z,\epsilon)$ is of the same nature as above. Under suitable constraints on the components of (2), we make use of Laplace and inverse Fourier transforms in order to construct a set of genuine bounded holomorphic solutions $y_p(t,z,\epsilon)$, $0 \le p \le \varsigma - 1$, for some integer $\varsigma \ge 2$. Such solutions are defined on domains $\mathcal{T} \times H_\beta \times \mathcal{E}_p$ for some well-selected bounded sector $\mathcal{T}$ with vertex at 0 and $\underline{\mathcal{E}} = \{\mathcal{E}_p\}_{0\le p\le\varsigma-1}$ a set of bounded sectors whose union contains a full neighborhood of 0 in $\mathbb{C}^*$. Such solutions share a common asymptotic expansion with respect to the perturbation parameter, say $\hat{y}(t,z,\epsilon) = \sum_{n\ge0} y_n(t,z)\epsilon^n$. The formal power series $\hat{y}(t,z,\epsilon)$ presents bounded holomorphic coefficients $y_n(t,z)$ on $\mathcal{T} \times H_\beta$. Furthermore, this asymptotic expansion turns out to be (at most) of Gevrey order $1/k'$ for some integer $k' \ge 1$ (see Definition 7). The Gevrey asymptotic behavior comes from the highest order term of the differential operator L, which is of irregular type in the sense of [20] and has the form

$$L(t,\epsilon,\partial_t,\partial_z) = \epsilon^{(\delta_D-1)k'} t^{(\delta_D-1)(k'+1)} \partial_t^{\delta_D} R_D(\partial_z) \tag{3}$$

for some integer $\delta_D \ge 2$ and a polynomial $R_D(X)$. In the case that the aperture of $\mathcal{E}_p$ can be chosen to be larger than π/k', the function $\epsilon \mapsto y_p(t,z,\epsilon)$ represents the k'-sum of $\hat{y}$ on $\mathcal{E}_p$ as described in Definition 7.

The purpose of the present contribution is to come up with a comparable statement, namely the existence of sectorial holomorphic solutions and associated asymptotic expansions as ϵ tends to 0 with controlled Gevrey bounds. However, the appearance of the nonlocal Moebius operator $m_{\kappa,t,\epsilon}$ changes drastically the whole picture in comparison with our previous investigation [15]. More precisely, a leading term of finite order $\delta_D \ge 2$ in time as described in (3) is not satisfactory enough to ensure the construction of actual holomorphic solutions to our initial problem (1). In contrast, it is substituted by an exponential formal differential operator

$$\exp\left(\alpha\epsilon^k t^{k+1}\partial_t\right)R(\partial_z) = \sum_{p\ge0} \frac{(\alpha\epsilon^k)^p}{p!}\left(t^{k+1}\partial_t\right)^{(p)} R(\partial_z)$$

of infinite order w.r.t. t. Here, $(t^{k+1}\partial_t)^{(p)}$ represents the pth iterate of the irregular differential operator $t^{k+1}\partial t$. As a result, (1) becomes singularly perturbed of irregular type of infinite order in time. The reason for the choice of such a new leading term will be put into light later on in the introduction.

A similar regularization procedure has been introduced in a different context in the paper [4] in order to obtain entire solutions in space of hydrodynamical PDEs such as the 3D Navier–Stokes equations

$$\partial_t v(t,x) + v(t,x) \cdot \nabla v(t,x) = -\nabla p(t,x) - \mu \Delta v(t,x), \quad \nabla \cdot v(t,x) = 0,$$

for given 2π-periodic initial data $v(0,x) = v_0(x_1,x_2,x_3)$ on $\mathbb{R}^3$. In that study, the usual Laplacian $\Delta = \sum_{j=1}^{3} \partial_{x_j}^2$ is replaced by a (pseudo differential) operator $\exp(\lambda A^{1/2})$, where $\lambda > 0$ and A stands for the differential operator $-\nabla^2$, whose Fourier symbol is $\exp(\lambda|k|)$ for $k \in \mathbb{Z}^3 \setminus \{0\}$. The resulting problem admits a solution $v(t,x)$ that is analytic w.r.t. x in $\mathbb{C}^3$ for all $t > 0$, whereas the solutions of the initial problem are expected to exhibit singularities in space.

Under appropriate restrictions on the shape of equation (1) (see Theorem 1), we prove the existence of

1. A set $\underline{\mathcal{E}}$ of bounded sectors as mentioned above, which forms a so-called good covering in $\mathbb{C}^*$ (see Definition 5);
2. A bounded sector $\mathcal{T}$ with bisecting direction $d = 0$;
3. A set of directions $\mathfrak{d}_p \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $0 \leq p \leq \varsigma - 1$ such that the halflines $L_{\mathfrak{d}_p} = \mathbb{R}_+ \exp(\sqrt{-1}\mathfrak{d}_p)$ avoid the infinite set of zeros of the map $\tau \mapsto Q(im) - \exp(\alpha k \tau^k) R(im)$ for all $m \in \mathbb{R}$,

for which we can construct a family of bounded holomorphic solutions $u_p(t,z,\epsilon)$ on the products $\mathcal{T} \times H_\beta \times \mathcal{E}_p$. Each solution u_p can be expressed as a Laplace transform of some order k and Fourier inverse transform

$$u_p(t,z,\epsilon) = \frac{k}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\mathfrak{d}_p}} w^{\mathfrak{d}_p}(u,m,\epsilon) \exp\left(-\left(\frac{u}{\epsilon t}\right)^k\right) e^{izm} \frac{du}{u} \, dm, \tag{4}$$

where $w^{\mathfrak{d}_p}(u,m,\epsilon)$ stands for a function with (at most) exponential growth of order k on a sector containing $L_{\mathfrak{d}_p}$ w.r.t. u, owning exponential decay w.r.t. m on $\mathbb{R}$ and analytic on ϵ near 0 (see Theorem 1). Moreover, we show that the functions $\epsilon \mapsto u_p(t,z,\epsilon)$ admit a common asymptotic expansion $\hat{u}(t,z,\epsilon) = \sum_{m\geq 0} h_m(t,z)\epsilon^m$ on $\mathcal{E}_p$ that defines a formal power series with bounded holomorphic coefficients on $\mathcal{T} \times H_\beta$. Besides, it turns out that this asymptotic expansion is (at most) of Gevrey order $1/k$ and leads to k-summability on $\mathcal{E}_{p_0}$ provided that one sector $\mathcal{E}_{p_0}$ has opening larger than π/k (see Theorem 2).

Another substantial difference between problems (1) and (2) lies in the fact that the real number k is asked to be less than 1. The situation $k = 1$ is not covered by the techniques developed in this work and is postponed for future inspection. However, the special case $k = 1$ has already been considered by the authors at the time of studying some families of Cauchy problems, giving rise to double scale structures involving 1 and 1^+ Gevrey estimates (see [17, 19]). Observe that if one performs the change of variable $t = 1/s$, then equation (1) is transformed into a singularly perturbed PDE combined with small shifts. More precisely, let $u(t,z,\epsilon) = X(s,z,\epsilon)$. Then, small shifts of the form $T_{\kappa,\epsilon}X(s,z,\epsilon) = X(s+\kappa\epsilon,z,\epsilon)$ are found for $\kappa \in I$. This restriction concerning the Gevrey order of formal expansions of the analytic solutions is rather natural in the context of difference equations as observed by Braaksma and Faber in [5]. Namely, if $A(x)$ stands for an invertible matrix of dimension $n \geq 1$ with meromorphic coefficients at ∞, and $G(x,y)$ represents a holomorphic function in $1/x$ and y near $(\infty, 0)$, then it holds that, under suitable assumptions on the formal fundamental matrix $\hat{Y}(x)$ of the linear equation $y(x+1) = A(x)y(x)$, any formal solution $\hat{y}(x) \in \mathbb{C}^n[[1/x]]$ of the nonlinear difference equation

$$y(x+1) - A(x)y(x) = G\big(x, y(x)\big)$$

can be decomposed as a sum of formal series $\hat{y}(x) = \sum_{h=1}^{q} \hat{y}_h(x)$ where each $\hat{y}_h(x)$ turns out to be k_h-summable on suitable sectors for some real numbers $0 < k_h \leq 1$ for $1 \leq h \leq q$.

In order to construct the family of solutions $\{u_p\}_{0 \leq p \leq \varsigma-1}$ mentioned above, we follow an approach that has been successfully applied by Faber and van der Put in [8], in the study of formal aspects of differential-difference operators such as the construction of Newton polygons, factorizations, and the extraction of formal solutions. This consists in considering the shift $x \mapsto x + \kappa$ as a formal differential operator of infinite order via the Taylor expansion at x, see (25). In our framework, the action of the Moebius transform $T \mapsto \frac{T}{1+\kappa T}$ is seen as an irregular operator of infinite order that can be formally written in the exponential form

$$\exp\big(-\kappa T^2 \partial_T\big) = \sum_{p \geq 0} (-1)^p \frac{\kappa^p}{p!} \big(T^2 \partial_T\big)^{(p)}.$$

If one seeks for genuine solutions in the form (4), then $w^{\mathfrak{d}_p}(\tau, m, \epsilon)$ would have to solve a related convolution equation (31) that involves infinite order operators $\exp(-\kappa \mathcal{C}_k(\tau))$, where $\mathcal{C}_k(\tau)$ denotes the convolution map given by (28). It turns out that this operator $\exp(-\kappa \mathcal{C}_k(\tau))$ acts on spaces of analytic functions $f(\tau)$ with (at most) exponential growth of order k, i.e., bounded by $C \exp(\nu |\tau|^k)$ for some $C, \nu > 0$ with type ν depending on κ, k, and ν as shown in Proposition 2 (48). It is worth mentioning that the use of precise bounds at infinity on the so-called Wiman special function $E_{\alpha,\beta}(z) = \sum_{n \geq 0} z^n / \Gamma(\beta + \alpha n)$, for $\alpha, \beta > 0$, is crucial in the proof, so the order k is preserved under the action of $\exp(-\kappa \mathcal{C}_k(\tau))$. Observe that this function also played a central role at the moment of proving multisummability properties of formal solutions in a perturbation parameter to certain families of nonlinear PDEs as described in our previous work [16]. As a result, the presence of an exponential type term $\exp(\alpha k \tau^k)$ in front of equation (31), and therefore the infinite order operator $\exp(\alpha \epsilon^k t^{k+1} \partial_t)$ as a leading term of (1) is well motivated in our problem and seems unavoidable to us in order to compensate such exponential growth.

We mention that a similar strategy has been carried out by Ōuchi in [21] who considered functional equations

$$u(z) + \sum_{j=2}^{m} a_j u\big(z + z^p \varphi_j(z)\big) = f(z),$$

where $p \geq 1$ is an integer, $a_j \in \mathbb{C}^*$ and $\varphi_j(z), f(z)$ stand for holomorphic functions near $z = 0$. He established the existence of formal power series solutions $\hat{u}(z) \in \mathbb{C}[[z]]$ that are p-summable in suitable directions. This result is attained by solving an associated convolution equation of infinite order for the Borel transform of order p in analytic functional spaces with (at most) exponential growth of order p on convenient unbounded sectors. More recently, in the work in progress [10], Hirose, Yamazawa, and Tahara are extending the above statement to more general functional PDEs such as

$$u(t,x) = a_1(t,x)(t\partial_t)^2\big(u\big(t+t^2,x\big)\big) + a_2(t,x)\partial_x u\big(t+t^2,x\big) + f(t,x)$$

for analytic coefficients a_1, a_2, f near $0 \in \mathbb{C}^2$ for which the formal series solutions

$$\hat{u}(t,x) = \sum_{n\geq 1} u_n(x)t^n,$$

which can be built up, are shown to be multisummable following appropriate multidirections in the sense defined in [2].

In a wider framework, there exists a vast amount of literature dealing with infinite order PDEs/ODEs both in mathematics and in theoretical physics. We just quote some recent references somehow related to our research interests. In the paper [1], the authors study formal solutions and their Borel transform of singularly perturbed differential equations of infinite order

$$\sum_{j\geq 0} \epsilon^j P_j(x, \epsilon\partial_x)\psi(x,\epsilon) = 0,$$

where $P_j(x,\xi) = \sum_{k\geq 0} a_{j,k}(x)\xi^k$ represent entire functions with appropriate growth features. For a nice introduction on the point of view introduced by Sato, the algebraic microlocal analysis, we refer to [12]. Other important contributions on infinite order ODEs in this context of algebraic microlocal analysis are [13, 14].

In our work, we apply the classical parameter expanding method. More precisely, our solutions $u_p(t,z,\epsilon)$ can be approximated by its power series expansion in the small parameter ϵ:

$$u_p(t,z,\epsilon) \sim \sum_{n\geq 0} h_m(t,z)\epsilon^m$$

as ϵ tends to 0. However, it should be mentioned that there exist many other powerful recent alternative analytic asymptotic approaches that can handle singularly perturbed problems such as the one considered in this paper. For an excellent survey of these techniques (variational iteration methods, homotopy perturbation methods) illustrated by concrete examples, we refer the reader to the paper by Ji-Huan He [9].

The paper is arranged as follows.

In Sect. 2, we recall the definition of Laplace transform of order k and basic formulas on the Fourier inverse transform acting on exponentially flat functions.

In Sect. 3, we display our main problem (11) and describe the strategy used to solve it. We search for the potential candidates for solutions among the Laplace of order k and Fourier inverse transforms of certain *Borel maps* w with exponential growth on unbounded sectors and exponential decay on the real line. In the last step, the convolution problem (31) provides the solution for w.

In Sect. 4, we analyze bounds for linear/nonlinear convolution operators of finite/infinite orders acting on different spaces of analytic functions on sectors.

In Sect. 5, we solve the principal convolution problem (31) within the Banach spaces of functions described in Sects. 3 and 4 by means of a fixed point argument.

In Sect. 6, we provide a set of genuine holomorphic solutions (104) to our initial equation (11) following the argument described in Sect. 3. Furthermore, we show that the difference of any two neighboring solutions is of some exponential decay at 0 with respect to the perturbation parameter.

In Sect. 7, we prove the existence of a common Gevrey asymptotic expansion for the solutions mentioned above leaning on the estimates reached in Sect. 6. This last result is obtained by means of the classical Ramis–Sibuya theorem.

2 Laplace, Borel transforms of order k and Fourier inverse maps

We recall the definition of Laplace transform of order k as introduced in [15]. In contrast to that work, the order k is assumed to be a real number less than 1 and larger than 1/2. If $z \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, we set $z^k = \exp(k\log(z))$, where $\log(z)$ stands for the principal branch of the complex logarithm defined as $\log(z) = \log|z| + i\arg(z)$ with $-\pi < \arg(z) < \pi$.

Definition 1 Let $\frac{1}{2} < k < 1$. Let $S_{d,\delta} = \{\tau \in \mathbb{C}^* : |d - \arg(\tau)| < \delta\}$ be some unbounded sector with bisecting direction $d \in \mathbb{R}$ and aperture $2\delta > 0$.

Consider a holomorphic function $w : S_{d,\delta} \to \mathbb{C}$ which satisfies there exist $C > 0$ and $K > 0$ such that

$$\bigl|w(\tau)\bigr| \leq C|\tau|^k \exp\bigl(K|\tau|^k\bigr)$$

for all $\tau \in S_{d,\delta}$. We define the Laplace transform of w of order k in the direction d as the integral transform

$$\mathcal{L}_k^d(w)(T) = k\int_{L_\gamma} w(u)\exp\biggl(-\biggl(\frac{u}{T}\biggr)^k\biggr)\frac{du}{u}$$

along a half-line $L_\gamma = \mathbb{R}_+ e^{\sqrt{-1}\gamma} \subset S_{d,\delta} \cup \{0\}$, where γ depends on T and is chosen in such a way that $\cos(k(\gamma - \arg(T))) \geq \delta_1 > 0$ for some fixed δ_1. The function $\mathcal{L}_k^d(w)(T)$ is well-defined, holomorphic, and bounded on any sector

$$S_{d,\theta,R^{1/k}} = \bigl\{T \in \mathbb{C}^* : |T| < R^{1/k}, \bigl|d - \arg(T)\bigr| < \theta/2\bigr\},$$

where $\frac{\pi}{k} < \theta < \frac{\pi}{k} + 2\delta$ and $0 < R < \delta_1/K$.

We consider the family of Banach spaces introduced in [15].

Definition 2 Let $\beta, \mu \in \mathbb{R}$, and $E_{(\beta,\mu)}$ be the vector space of continuous functions $h : \mathbb{R} \to \mathbb{C}$ such that

$$\bigl\|h(m)\bigr\|_{(\beta,\mu)} = \sup_{m\in\mathbb{R}}\bigl(1 + |m|\bigr)^{\mu}\exp\bigl(\beta|m|\bigr)\bigl|h(m)\bigr|$$

is finite. The space $E_{(\beta,\mu)}$ endowed with the norm $\|\cdot\|_{(\beta,\mu)}$ becomes a Banach space.

Finally, we remind the reader the definition of the inverse Fourier transform acting on the latter Banach spaces and some of its properties concerning derivation and convolution product. We refer to [15] for further details.

Definition 3 Let $f \in E_{(\beta,\mu)}$ with $\beta > 0$, $\mu > 1$. The inverse Fourier transform of f is given by

$$\mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} f(m)\exp(ixm)\,dm$$

for all $x \in \mathbb{R}$. The function $\mathcal{F}^{-1}(f)$ extends to an analytic bounded function on the strip

$$H_{\beta'} = \{z \in \mathbb{C} / |\mathrm{Im}(z)| < \beta'\} \tag{5}$$

for all given $0 < \beta' < \beta$.

(a) Let ϕ be the function defined by $m \mapsto \phi(m) = imf(m)$. It holds that $\phi \in E_{(\beta,\mu-1)}$. Moreover, one has that

$$\partial_z \mathcal{F}^{-1}(f)(z) = \mathcal{F}^{-1}(\phi)(z).$$

(b) Take $g \in E_{(\beta,\mu)}$ and let

$$\psi(m) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} f(m-m_1)g(m_1)\,dm_1$$

be the convolution product of f and g. Then ψ belongs to $E_{(\beta,\mu)}$, and one has

$$\mathcal{F}^{-1}(f)(z)\mathcal{F}^{-1}(g)(z) = \mathcal{F}^{-1}(\psi)(z)$$

for all $z \in H_\beta$.

3 Outline of the main initial value problem and related auxiliary problems

Let $k \in (\frac{1}{2}, 1)$. Let $D \geq 2$ be an integer, $\alpha_D > 0$ be a positive real number, and c_{12}, c_f be nonzero complex numbers. For $1 \leq l \leq D-1$, we consider $c_l \in \mathbb{C}^*$ and nonnegative integers d_l, δ_l, Δ_l together with positive real numbers $\kappa_l > 0$. We assume that

$$1 = \delta_1, \qquad \delta_l < \delta_{l+1} \tag{6}$$

for all $1 \leq l \leq D-2$. We also take for granted that

$$d_l > \delta_l(k+1), \qquad \Delta_l - d_l + \delta_l \geq 0 \tag{7}$$

whenever $1 \leq l \leq D-1$. Let $Q(X), Q_1(X), Q_2(X), R_l(X) \in \mathbb{C}[X]$, $1 \leq l \leq D$, such that

$$\begin{aligned} &\deg(Q) = \deg(R_D) \geq \deg(R_l), \qquad \deg(R_D) \geq \deg(Q_1), \qquad \deg(R_D) \geq \deg(Q_2), \\ &Q(im) \neq 0, \qquad R_D(im) \neq 0 \end{aligned} \tag{8}$$

for all $m \in \mathbb{R}$, all $1 \leq l \leq D-1$.

For all $n \geq 1$, we consider a function $m \mapsto F_n(m,\epsilon)$ that belongs to the Banach space $E_{(\beta,\mu)}$ for some $\beta > 0$ and $\mu > \max(\deg(Q_1)+1, \deg(Q_2)+1)$ and that depends analytically on $\epsilon \in D(0,\epsilon_0)$. Here, $D(0,\epsilon_0)$ stands for the open disc centered at 0 in $\mathbb{C}$ with radius $\epsilon_0 > 0$. We assume there exist constants $K_0, T_0 > 0$ such that

$$\sup_{\epsilon \in D(0,\epsilon_0)} \left\| F_n(m,\epsilon) \right\|_{(\beta,\mu)} \leq K_0 \left(\frac{1}{T_0} \right)^n, \quad n \geq 1. \tag{9}$$

We define

$$F(T,z,\epsilon) = \sum_{n \geq 1} \mathcal{F}^{-1}\big(m \mapsto F_n(m,\epsilon)\big)(z) T^n,$$

which represents a convergent series on $D(0,T_0/2)$ with holomorphic and bounded coefficients on $H_{\beta'}$ for any given $0 < \beta' < \beta$. For all $1 \leq l \leq D-1$, we set the polynomials $A_l(T,\epsilon) = \sum_{n \in I_l} A_{l,n}(\epsilon) T^n$, where I_l are finite subsets of $\mathbb{N}$ and $A_{l,n}(\epsilon)$ represent bounded holomorphic functions on the disc $D(0,\epsilon_0)$. We put

$$f(t,z,\epsilon) = F(\epsilon t,z,\epsilon), \qquad a_l(t,\epsilon) = A_l(\epsilon t,\epsilon)$$

for all $1 \leq l \leq D-1$. By construction, $f(t,z,\epsilon)$ (resp. $a_l(t,\epsilon)$) defines a bounded holomorphic function on $D(0,r) \times H_{\beta'} \times D(0,\epsilon_0)$ (resp. $D(0,r) \times D(0,\epsilon_0)$) for any given $0 < \beta' < \beta$ and radii $r, \epsilon_0 > 0$ with $r\epsilon_0 \leq T_0/2$.

Let us introduce the next differential operator of infinite order which is formally defined by

$$\exp\big(\alpha_D \epsilon^k t^{k+1} \partial_t\big) = \sum_{p \geq 0} \frac{(\alpha_D \epsilon^k)^p}{p!} \big(t^{k+1}\partial_t\big)^{(p)}, \tag{10}$$

where $(t^{k+1}\partial_t)^{(p)}$ stands for the pth iterate of the differential operator $t^{k+1}\partial_t$. We consider a family of nonlinear singularly perturbed initial value problems involving this latter operator as their leading term and linear fractional transforms:

$$\begin{aligned} Q(\partial_z)u(t,z,\epsilon) &= \exp\big(\alpha_D \epsilon^k t^{k+1}\partial_t\big) R_D(\partial_z) u(t,z,\epsilon) \\ &\quad + \sum_{l=1}^{D-1} \epsilon^{\Delta_l} c_l a_l(t,\epsilon) t^{d_l} R_l(\partial_z) \partial_t^{\delta_l} \left(u\left(\frac{t}{1+\kappa_l \epsilon t}, z, \epsilon \right) \right) \\ &\quad + c_{12} Q_1(\partial_z) u(t,z,\epsilon) Q_2(\partial_z) u(t,z,\epsilon) + c_f f(t,z,\epsilon) \end{aligned} \tag{11}$$

for vanishing initial data $u(0,z,\epsilon) = 0$.

In this work, we search for time rescaled solutions of (11) of the form

$$u(t,z,\epsilon) = U(\epsilon t,z,\epsilon). \tag{12}$$

After the change of variable $T = \epsilon t$, one has that $U(T,z,\epsilon)$ solves the next nonlinear singular problem involving fractional transforms:

$$
\begin{aligned}
&Q(\partial_z)U(T,z,\epsilon)\\
&\quad = \exp\bigl(\alpha_D T^{k+1}\partial_T\bigr)R_D(\partial_z)U(T,z,\epsilon)\\
&\qquad + \sum_{l=1}^{D-1} \epsilon^{\Delta_l - d_l + \delta_l} c_l A_l(T,\epsilon) T^{d_l} R_l(\partial_z)\partial_T^{\delta_l}\left(U\left(\frac{T}{1+\kappa_l T}, z, \epsilon\right)\right)\\
&\qquad + c_{12}Q_1(\partial_z)U(T,z,\epsilon)Q_2(\partial_z)U(T,z,\epsilon) + c_f F(T,z,\epsilon)
\end{aligned} \tag{13}
$$

for given initial data $U(0,z,\epsilon) = 0$. According to assumption (7), there exist real numbers $d_{l,k} > 0$ with

$$
d_l = \delta_l(k+1) + d_{l,k} \tag{14}
$$

for all $1 \le l \le D-1$. The application of formula (8.7) from [22] p. 3630 yields

$$
T^{\delta_l(k+1)}\partial_T^{\delta_l} = \bigl(T^{k+1}\partial_T\bigr)^{\delta_l} + \sum_{1\le p\le \delta_l - 1} A_{\delta_l,p} T^{k(\delta_l - p)}\bigl(T^{k+1}\partial_T\bigr)^p \tag{15}
$$

for some $A_{\delta_l,p} \in \mathbb{R}$, for $1 \le p \le \delta_l - 1$ and $1 \le l \le D-1$. Hence, according to (14) together with (15), equation (13) reads as follows:

$$
\begin{aligned}
&Q(\partial_z)U(T,z,\epsilon)\\
&\quad = \exp\bigl(\alpha_D T^{k+1}\partial_T\bigr)R_D(\partial_z)U(T,z,\epsilon) + \sum_{l=1}^{D-1} \epsilon^{\Delta_l - d_l + \delta_l} R_l(\partial_z) c_l A_l(T,\epsilon)\\
&\qquad \times T^{d_{l,k}}\left(\bigl(T^{k+1}\partial_T\bigr)^{\delta_l} + \sum_{1\le p\le\delta_l-1} A_{\delta_l,p}T^{k(\delta_l-p)}\bigl(T^{k+1}\partial_T\bigr)^p\right)\left(U\left(\frac{T}{1+\kappa_l T}, z, \epsilon\right)\right)\\
&\qquad + c_{12}Q_1(\partial_z)U(T,z,\epsilon)Q_2(\partial_z)U(T,z,\epsilon) + c_f F(T,z,\epsilon).
\end{aligned} \tag{16}
$$

We now provide the definition of a modified version of some Banach spaces introduced in [15, 16] that takes into account a ramified variable τ^k for k fixed above.

Definition 4 Let S_d be an unbounded sector centered at 0 with bisecting direction $d \in \mathbb{R}$. Let $\nu, \beta, \mu > 0$ and $\rho > 0$ be positive real numbers. Let $k \in (\frac{1}{2}, 1)$ defined as above. We write $F^d_{(\nu,\beta,\mu,k,\rho)}$ for the vector space of continuous functions $(\tau, m) \mapsto h(\tau, m)$ on $S_d \times \mathbb{R}$, which are holomorphic with respect to τ on S_d such that

(1) For all $m \in \mathbb{R}$, the function $\tau \mapsto h(\tau, m)$ extends analytically on $D(0,\rho) \setminus L_-$, where L_- denotes the segment $(-\rho, 0]$.

(2) The norm

$$
\bigl\|h(\tau,m)\bigr\|_{(\nu,\beta,\mu,k,\rho)} = \sup_{\tau\in S_d\cup D(0,\rho)\setminus L_-, m\in\mathbb{R}} \bigl(1+|m|\bigr)^{\mu}\frac{1+|\tau|^{2k}}{|\tau|^k}e^{\beta|m|-\nu|\tau|^k}\bigl|h(\tau,m)\bigr|
$$

is finite.

The space $F^d_{(\nu,\beta,\mu,k,\rho)}$ equipped with the norm $\|\cdot\|_{(\nu,\beta,\mu,k,\rho)}$ forms a Banach space.

Lemma 1 *For β, μ given in (9), there exists $\nu > 0$ such that the series*

$$\sum_{n\geq 1} F_n(m,\epsilon)\frac{\tau^n}{\Gamma(\frac{n}{k})}$$

defines a function $\psi(\tau,m,\epsilon)$ that belongs to the space $F^d_{(\nu,\beta,\mu,k,\rho)}$ for all $\epsilon \in D(0,\epsilon_0)$, any radius $\rho > 0$, and any sector S_d for $d \in \mathbb{R}$.

Proof By Definition of the norm $\|\cdot\|_{(\nu,\beta,\mu,k,\rho)}$, we get

$$\begin{aligned}
&\bigl\|\psi(\tau,m,\epsilon)\bigr\|_{(\nu,\beta,\mu,k,\rho)} \\
&\quad\leq \sum_{n\geq 1}\bigl\|F_n(m,\epsilon)\bigr\|_{(\beta,\mu)}\biggl(\sup_{\tau\in(D(0,\rho)\setminus L_-)\cup S_d}\frac{1+|\tau|^{2k}}{|\tau|^k}\exp\bigl(-\nu|\tau|^k\bigr)\frac{|\tau|^n}{\Gamma(\frac{n}{k})}\biggr).
\end{aligned} \tag{17}$$

Due to the classical estimates

$$\sup_{x\geq 0} x^{m_1}\exp(-m_2 x) = \left(\frac{m_1}{m_2}\right)^{m_1} e^{-m_1},$$

valid for any $m_1 \geq 0$, $m_2 > 0$, together with the Stirling formula (see [3], Appendix B.3)

$$\Gamma(n/k) \sim (2\pi)^{1/2}(n/k)^{\frac{n}{k}-\frac{1}{2}}e^{-n/k}, \quad n \to +\infty,$$

we guarantee the existence of two constants $A_1 > 0$ depending on k, ν and $A_2 > 0$ depending on k such that

$$\begin{aligned}
&\sup_{\tau\in(D(0,\rho)\setminus L_-)\cup S_d}\frac{1+|\tau|^{2k}}{|\tau|^k}\exp\bigl(-\nu|\tau|^k\bigr)\frac{|\tau|^n}{\Gamma(\frac{n}{k})} \\
&\quad\leq \sup_{x\geq 0}\bigl(1+x^2\bigr)x^{\frac{n}{k}-1}\frac{e^{-\nu x}}{\Gamma(\frac{n}{k})} \\
&\quad\leq \left(\left(\frac{\frac{n}{k}-1}{\nu}\right)^{\frac{n}{k}-1}e^{-(\frac{n}{k}-1)} + \left(\frac{\frac{n}{k}+1}{\nu}\right)^{\frac{n}{k}+1}e^{-(\frac{n}{k}+1)}\right)\Big/\Gamma(n/k) \\
&\quad\leq A_1\left(\frac{A_2}{\nu^{1/k}}\right)^n
\end{aligned} \tag{18}$$

for $n \geq 1$. Therefore, if $\nu^{1/k} > A_2/T_0$, then we obtain

$$\bigl\|\psi(\tau,m,\epsilon)\bigr\|_{(\nu,\beta,\mu,k,\rho)} \leq A_1K_0\sum_{n\geq 1}\left(\frac{A_2}{T_0\nu^{1/k}}\right)^n = \frac{A_1K_0A_2}{T_0\nu^{1/k}}\frac{1}{1-\frac{A_2}{T_0\nu^{1/k}}} \tag{19}$$

for all $\epsilon \in D(0,\epsilon_0)$. □

By construction, according to the definition of gamma function, the function $F(T,z,\epsilon)$ can be represented as a Laplace transform of order k following direction d and Fourier inverse transform

$$F(T,z,\epsilon)=\frac{k}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_\gamma}\psi(u,m,\epsilon)\exp\left(-\left(\frac{u}{T}\right)^k\right)e^{izm}\frac{du}{u}\,dm, \tag{20}$$

where the integration path $L_\gamma=\mathbb{R}_+e^{\sqrt{-1}\gamma}$ stands for a halfline of direction $\gamma\in\mathbb{R}$ which belongs to the set $S_d\cup\{0\}$, whenever T belongs to some sector $S_{d,\theta,\varrho}$ with bisecting direction d, aperture $\frac{\pi}{k}<\theta<\frac{\pi}{k}+\mathrm{Ap}(S_d)$, and radius ϱ; with $\mathrm{Ap}(S_d)$ the aperture of S_d, some $\varrho>0$ and z belonging to a strip $H_{\beta'}$ for any $0<\beta'<\beta$ together with $\epsilon\in D(0,\epsilon_0)$.

In the next step, we seek for solutions $U(T,z,\epsilon)$ of (16) defined on the same domains as above that can be expressed via an integral representation of Laplace of order k and Fourier inverse transforms

$$U_\gamma(T,z,\epsilon)=\frac{k}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_\gamma}w(u,m,\epsilon)\exp\left(-\left(\frac{u}{T}\right)^k\right)e^{izm}\frac{du}{u}\,dm. \tag{21}$$

We aim to describe a related problem fulfilled by the expression $w(\tau,m,\epsilon)$. This is considered in the next section. For this purpose, we make use of the Banach spaces introduced above in Definition 4. Through this section, we assume that the function $w(\tau,m,\epsilon)$ belongs to the Banach space $F^d_{(\nu,\beta,\mu,k,\rho)}$.

We first display some formulas related to the action of the differential operators of irregular type and multiplication by monomials. A similar statement has been given in Sect. 3 of [15] for formal series expansions.

Lemma 2

(1) *The action of the differential operator $T^{k+1}\partial_T$ on U_γ is given by*

$$\begin{aligned}&T^{k+1}\partial_T U_\gamma(T,z,\epsilon)\\&\quad=\frac{k}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_\gamma}ku^k w(u,m,\epsilon)\exp\left(-\left(\frac{u}{T}\right)^k\right)e^{izm}\frac{du}{u}\,dm.\end{aligned} \tag{22}$$

(2) *Let $m'>0$ be a real number. The action of the multiplication by $T^{m'}$ on U_γ is described by*

$$\begin{aligned}T^{m'}U_\gamma(T,z,\epsilon)&=\frac{k}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_\gamma}\left(\frac{u^k}{\Gamma(\frac{m'}{k})}\int_0^{u^k}\left(u^k-s\right)^{\frac{m'}{k}-1}w\left(s^{1/k},m,\epsilon\right)\frac{ds}{s}\right)\\&\quad\times\exp\left(-\left(\frac{u}{T}\right)^k\right)e^{izm}\frac{du}{u}\,dm.\end{aligned} \tag{23}$$

(3) *The action of polynomial differential operators and multiplication can be described by*

$$\begin{aligned}&Q_1(\partial_z)U_\gamma(T,z,\epsilon)Q_2(\partial_z)U_\gamma(T,z,\epsilon)\\&\quad=\frac{k}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\int_{L_\gamma}\left(\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}u^k\right.\end{aligned}$$

$$\times \int_0^{u^k} Q_1\big(i(m-m_1)\big)w\big((u^k-s)^{1/k},m-m_1,\epsilon\big)Q_2(im_1)w\big(s^{1/k},m_1,\epsilon\big)$$
$$\times \frac{1}{(u^k-s)s}\,ds\,dm_1\bigg)\exp\bigg(-\bigg(\frac{u}{T}\bigg)^k\bigg)e^{izm}\frac{du}{u}\,dm. \tag{24}$$

Proof We present direct analytic proofs which avoid the use of summability arguments through Watson's lemma. The first point (1) is obtained by a mere derivation under the integral symbol. We turn to the second point (2). By the application of Fubini's theorem, we get that

$$A = \int_{L_\gamma} u^{k-1}\int_0^{u^k}\big(u^k-s\big)^{\frac{m'}{k}-1}w\big(s^{1/k},m,\epsilon\big)\frac{ds}{s}\exp\bigg(-\bigg(\frac{u}{T}\bigg)^k\bigg)\,du$$
$$= \int_{L_{\gamma'}}\bigg(\int_{L_{s^{1/k},\gamma}} u^{k-1}\big(u^k-s\big)^{\frac{m'}{k}-1}\exp\bigg(-\bigg(\frac{u}{T}\bigg)^k\bigg)\,du\bigg)w\big(s^{1/k},m,\epsilon\big)\frac{ds}{s},$$

where $\gamma' = k\gamma$ and $L_{s^{1/k},\gamma} = [|s|^{1/k},+\infty)e^{\sqrt{-1}\gamma}$. On the other hand, by successive path deformations $u^k = v$ and $v - s = v'$, we get that

$$\int_{L_{s^{1/k},\gamma}} u^{k-1}\big(u^k-s\big)^{\frac{m'}{k}-1}\exp\bigg(-\bigg(\frac{u}{T}\bigg)^k\bigg)\,du = \int_{L_{s,\gamma'}}(v-s)^{\frac{m'}{k}-1}\exp\bigg(-\frac{v}{T^k}\bigg)\frac{1}{k}\,dv,$$

where $L_{s,\gamma'} = [|s|,+\infty)e^{\sqrt{-1}\gamma'}$ and

$$\int_{L_{s,\gamma'}}(v-s)^{\frac{m'}{k}-1}\exp\bigg(-\frac{v}{T^k}\bigg)\frac{1}{k}\,dv = \int_{L_{\gamma'}}\big(v'\big)^{\frac{m'}{k}-1}\exp\bigg(-\frac{v'}{T^k}\bigg)\frac{1}{k}\,dv'\exp\bigg(-\frac{s}{T^k}\bigg).$$

The definition of gamma function together with a path deformation yields

$$\int_{L_{\gamma'}}\big(v'\big)^{\frac{m'}{k}-1}\exp\bigg(-\frac{v'}{T^k}\bigg)\,dv' = \Gamma\bigg(\frac{m'}{k}\bigg)T^{m'}.$$

As a result, according to the path deformation $s = u^k$, we finally get

$$A = \int_{L_{\gamma'}}\frac{\Gamma(\frac{m'}{k})}{k}T^{m'}w\big(s^{1/k},m,\epsilon\big)\exp\bigg(-\frac{s}{T^k}\bigg)\frac{ds}{s}$$
$$= \int_{L_\gamma} w(u,m,\epsilon)\exp\bigg(-\bigg(\frac{u}{T}\bigg)^k\bigg)\frac{du}{u}\Gamma\bigg(\frac{m'}{k}\bigg)T^{m'},$$

which implies identity (23).

We aim our attention to point (3). Again, Fubini's theorem yields

$$B = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{L_\gamma}\int_0^{u^k} u^{k-1}Q_1\big(i(m-m_1)\big)w\big((u^k-s)^{1/k},m-m_1,\epsilon\big)$$
$$\times Q_2(im_1)w\big(s^{1/k},m_1,\epsilon\big)\frac{1}{(u^k-s)s}\exp\bigg(-\bigg(\frac{u}{T}\bigg)^k\bigg)e^{izm}\,ds\,du\,dm_1\,dm$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{L_{\gamma'}}\int_{L_{s^{1/k},\gamma}} u^{k-1}Q_1\bigl(i(m-m_1)\bigr)w\bigl(\bigl(u^k-s\bigr)^{1/k},m-m_1,\epsilon\bigr)$$
$$\times Q_2(im_1)w\bigl(s^{1/k},m_1,\epsilon\bigr)\frac{1}{(u^k-s)s}\exp\biggl(-\biggl(\frac{u}{T}\biggr)^k\biggr)e^{izm}\,du\,ds\,dm\,dm_1,$$

where $\gamma' = k\gamma$ and $L_{s^{1/k},\gamma} = [|s|^{1/k},+\infty)e^{\sqrt{-1}\gamma}$. By the path deformation $v = u^k$, and then $v - s = v'$, we have

$$C = \int_{L_{s^{1/k},\gamma}} u^{k-1}w\bigl(\bigl(u^k-s\bigr)^{1/k},m-m_1,\epsilon\bigr)\frac{1}{(u^k-s)}\exp\biggl(-\biggl(\frac{u}{T}\biggr)^k\biggr)\,du$$
$$= \int_{L_{s,\gamma'}}\frac{1}{k}w\bigl((v-s)^{1/k},m-m_1,\epsilon\bigr)\frac{1}{v-s}\exp\biggl(-\frac{v}{T^k}\biggr)\,dv$$
$$\times \int_{L_{\gamma'}}\frac{1}{k}w\bigl(\bigl(v'\bigr)^{1/k},m-m_1,\epsilon\bigr)\frac{1}{v'}\exp\biggl(-\frac{v'}{T^k}\biggr)\,dv'\exp\biggl(-\frac{s}{T^k}\biggr).$$

Therefore, we obtain

$$B = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{L_{\gamma'}}\int_{L_{\gamma'}} Q_1\bigl(i(m-m_1)\bigr)\frac{1}{k}w\bigl(\bigl(v'\bigr)^{1/k},m-m_1,\epsilon\bigr)$$
$$\times \frac{1}{v'}\exp\biggl(-\frac{v'}{T^k}\biggr)\exp\biggl(-\frac{s}{T^k}\biggr)Q_2(im_1)w\bigl(s^{1/k},m_1,\epsilon\bigr)\frac{1}{s}e^{izm}\,dv'\,ds\,dm\,dm_1.$$

Besides, by the change of variable, $m - m_1 = m'$ yields

$$\int_{-\infty}^{+\infty} Q_1\bigl(i(m-m_1)\bigr)w\bigl(\bigl(v'\bigr)^{1/k},m-m_1,\epsilon\bigr)e^{izm}\,dm$$
$$= \int_{-\infty}^{+\infty} Q_1\bigl(im'\bigr)w\bigl(\bigl(v'\bigr)^{1/k},m',\epsilon\bigr)e^{izm'}\,dm'\,e^{izm_1}.$$

As a result,

$$B = \frac{1}{k}\int_{-\infty}^{+\infty}\int_{L_{\gamma'}} Q_1\bigl(im'\bigr)w\bigl(\bigl(v'\bigr)^{1/k},m',\epsilon\bigr)\frac{1}{v'}\exp\biggl(-\frac{v'}{T^k}\biggr)e^{izm'}\,dv'\,dm'$$
$$\times \int_{-\infty}^{+\infty}\int_{L_{\gamma'}} Q_2(im_1)w\bigl(s^{1/k},m_1,\epsilon\bigr)\frac{1}{s}\exp\biggl(-\frac{s}{T^k}\biggr)e^{izm_1}\,ds\,dm_1,$$

and according to the paths deformations $s = u^k$ and $v' = u^k$, we finally arrive at

$$B = k\int_{-\infty}^{+\infty}\int_{L_{\gamma}} Q_1\bigl(im'\bigr)w\bigl(u,m',\epsilon\bigr)\exp\biggl(-\biggl(\frac{u}{T}\biggr)^k\biggr)e^{izm'}\frac{du}{u}\,dm'$$
$$\times \int_{-\infty}^{+\infty}\int_{L_{\gamma}} Q_2(im_1)w(u,m_1,\epsilon)\exp\biggl(-\biggl(\frac{u}{T}\biggr)^k\biggr)e^{izm_1}\frac{du}{u}\,dm_1,$$

from which identity (24) follows. □

At the next level, we describe the action of the Moebius transform $T \mapsto \frac{T}{1+\kappa_l T}$ on U_γ. We need some preliminaries before this consideration.

We depart, as in the work of Faber and van der Put [8], describing the shift $x \mapsto x + \kappa_l$ as a differential operator of infinite order through Taylor expansions. Namely, for any holomorphic function $f : U \mapsto \mathbb{C}$ defined on an open convex set $U \subset \mathbb{C}$ containing x and $x + \kappa_l$, the following Taylor formula holds:

$$f(x+\kappa_l) = \sum_{p\geq 0} \frac{f^{(p)}(x)}{p!}\kappa_l^p, \tag{25}$$

where $f^{(p)}(x)$ denotes the derivative of order $p \geq 0$ of f (where by convention $f^{(0)}(x) = f(x)$). If one performs the change of variable $f(x) = U(1/x)$, one obtains a corresponding formula for $U(T)$:

$$U\left(\frac{T}{1+\kappa_l T}\right) = \sum_{p\geq 0} \frac{(-1)^p (T^2\partial_T)^{(p)}}{p!}\kappa_l^p U(T), \tag{26}$$

where $(T^2\partial_T)^{(p)}$ represents the pth iterate of the irregular operator $T^2\partial_T$.

According to our hypothesis $k \in (1/2, 1)$, we can apply Lemma 2 in order to write

$$\begin{aligned} &T^2\partial_T U_\gamma(T,z,\epsilon) \\ &\quad = T^{1-k}T^{k+1}\partial_T U_\gamma(T,z,\epsilon) \\ &\quad = \frac{k}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty}\int_{L_\gamma} \left(\frac{u^k}{\Gamma(\frac{1}{k}-1)} \int_0^{u^k} \left(u^k - s\right)^{\frac{1}{k}-2} kw\left(s^{1/k},m,\epsilon\right) ds\right) \\ &\qquad \times \exp\left(-\left(\frac{u}{T}\right)^k\right) e^{izm}\frac{du}{u}\,dm. \end{aligned} \tag{27}$$

As a result, if one denotes $\mathcal{C}_k$ the operator defined as

$$\mathcal{C}_k\left(w(\tau,m,\epsilon)\right) := \frac{\tau^k}{\Gamma(\frac{1}{k}-1)} \int_0^{\tau^k} \left(\tau^k - s\right)^{\frac{1}{k}-2} kw\left(s^{1/k},m,\epsilon\right) ds, \tag{28}$$

then the expression $U_\gamma(\frac{T}{1+\kappa_l T},z,\epsilon)$ can be written as a Laplace transform of order k in direction d and the Fourier inverse transform

$$\begin{aligned} &U_\gamma\left(\frac{T}{1+\kappa_l T},z,\epsilon\right) \\ &\quad = \frac{k}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty}\int_{L_\gamma} \left(\exp(-\kappa_l \mathcal{C}_k)w\right)(u,m,\epsilon) \exp\left(-\left(\frac{u}{T}\right)^k\right) e^{izm}\frac{du}{u}\,dm, \end{aligned} \tag{29}$$

where the integrant is formally presented as a series of operators

$$\left(\exp(-\kappa_l \mathcal{C}_k)w\right)(\tau,m,\epsilon) := \sum_{p\geq 0} \frac{(-1)^p \kappa_l^p}{p!}\mathcal{C}_k^{(p)} w(\tau,m,\epsilon) \tag{30}$$

and $\mathcal{C}_k^{(p)}$ stands for the kth order iterate of the operator $\mathcal{C}_k$ described above.

In virtue of the identities (22), (23), and (24) in Lemma 2, and according to the integral representation for the Moebius map acting on U_γ as described above in (29), we are now in a position to state the main equation satisfied by $w(\tau,m,\epsilon)$, provided that $U_{\gamma}(T,z,\epsilon)$ solves the equation in prepared form (16). More precisely, we consider

$$\begin{aligned}
&Q(im)w(\tau,m,\epsilon)\\
&\quad= \exp\bigl(\alpha_D k\tau^k\bigr)R_D(im)w(\tau,z,\epsilon)\\
&\qquad+ \sum_{l=1}^{D-1}\epsilon^{\Delta_l-d_l+\delta_l}R_l(im)c_l\sum_{n\in I_l}A_{l,n}(\epsilon)\\
&\qquad\times\biggl(\frac{\tau^k}{\Gamma(\frac{n+d_{l,k}}{k})}\int_0^{\tau^k}\bigl(\tau^k-s\bigr)^{\frac{n+d_{l,k}}{k}-1}k^{\delta_l}s^{\delta_l}\bigl(\exp(-\kappa_l\mathcal{C}_k)w\bigr)\bigl(s^{1/k},m,\epsilon\bigr)\frac{ds}{s}\\
&\qquad+\sum_{1\le p\le\delta_l-1}A_{\delta_l,p}\frac{\tau^k}{\Gamma(\frac{n+d_{l,k}}{k}+\delta_l-p)}\\
&\qquad\times\int_0^{\tau^k}\bigl(\tau^k-s\bigr)^{\frac{n+d_{l,k}}{k}+\delta_l-p-1}k^p s^p\bigl(\exp(-\kappa_l\mathcal{C}_k)w\bigr)\bigl(s^{1/k},m,\epsilon\bigr)\frac{ds}{s}\biggr)\\
&\qquad+c_{12}\frac{\tau^k}{(2\pi)^{1/2}}\int_0^{\tau^k}\int_{-\infty}^{+\infty}Q_1\bigl(i(m-m_1)\bigr)w\bigl(\bigl(\tau^k-s\bigr)^{1/k},m-m_1,\epsilon\bigr)\\
&\qquad\times Q_2(im_1)w\bigl(s^{1/k},m_1,\epsilon\bigr)\frac{1}{(\tau^k-s)s}\,ds\,dm_1+c_f\psi(\tau,m,\epsilon).
\end{aligned}\tag{31}$$

4 Action of convolution operators on certain spaces of functions

The principal goal of this section is to present bounds for convolution maps acting on spaces of functions that are analytic on sectors in $\mathbb{C}$ and continuous on $\mathbb{R}$. In the whole section, S_d denotes an unbounded sector centered at 0 with bisecting direction d in $\mathbb{R}$ and $D(0,\rho)\setminus L_-$ stands for a cut disc centered at 0 where $L_-=(-\rho,0]$.

Proposition 1 *Let $k\in(\frac{1}{2},1)$ be a real number. We fix real numbers γ_2, γ_3 satisfying that*

$$\gamma_2>-1,\qquad\gamma_3\ge 0,\qquad k(\gamma_2+\gamma_3+2)\in\mathbb{N}.\tag{32}$$

Let $(\tau,m)\mapsto f(\tau,m)$ be a continuous function on $S_d\times\mathbb{R}$, holomorphic w.r.t. τ on S_d, for which there exist a constant $C_1>0$, a positive integer $N\in\mathbb{N}^*$, and real numbers $\sigma>0$, $\mu>1$, $\beta>0$ with

$$\bigl|f(\tau,m)\bigr|\le C_1|\tau|^{kN}\exp\bigl(\sigma|\tau|^k\bigr)\bigl(1+|m|\bigr)^{-\mu}\exp\bigl(-\beta|m|\bigr)\tag{33}$$

for all $\tau\in S_d$, all $m\in\mathbb{R}$. Assume, moreover, that for all $m\in\mathbb{R}$, the map $\tau\mapsto f(\tau,m)$ extends analytically on the cut disc $D(0,\rho)\setminus L_-$ and for which one can choose a constant $C_1'>0$ such that

$$\bigl|f(\tau,m)\bigr|\le C_1'|\tau|^k\bigl(1+|m|\bigr)^{-\mu}e^{-\beta|m|}\tag{34}$$

whenever $\tau\in D(0,\rho)\setminus L_-$ and $m\in\mathbb{R}$.

We set

$$\mathcal{C}_{k,\gamma_2,\gamma_3}(f)(\tau,m)=\tau^k\int_0^{\tau^k}\bigl(\tau^k-s\bigr)^{\gamma_2}s^{\gamma_3}f\bigl(s^{1/k},m\bigr)\,ds.\tag{35}$$

Then

(1) *The map* $(\tau,m) \mapsto \mathcal{C}_{k,\gamma_2,\gamma_3}(f)(\tau,m)$ *is a continuous function on* $S_d \times \mathbb{R}$*, holomorphic w.r.t.* τ *on* S_d *such that a constant* $K_1 > 0$ *(depending on* γ_2*,* σ*) exists, and*

$$\left|\mathcal{C}_{k,\gamma_2,\gamma_3}(f)(\tau,m)\right| \leq C_1 K_1 |\tau|^{k(N+1)} |\tau|^{k\gamma_3} \exp\left(\sigma|\tau|^k\right)\left(1+|m|\right)^{-\mu} e^{-\beta|m|} \tag{36}$$

for all $\tau \in S_d$*, all* $m \in \mathbb{R}$*.*

(2) *For all* $m \in \mathbb{R}$*, the function* $\tau \mapsto \mathcal{C}_{k,\gamma_2,\gamma_3}(f)(\tau,m)$ *extends analytically on* $D(0,\rho) \setminus L_-$*. Furthermore, it holds that*

$$\left|\mathcal{C}_{k,\gamma_2,\gamma_3}(f)(\tau,m)\right| \leq C_1' \frac{\Gamma(\gamma_2+1)\Gamma(\gamma_3+2)}{\Gamma(\gamma_2+\gamma_3+3)} \rho^{k(\gamma_2+\gamma_3+2)} |\tau|^k \left(1+|m|\right)^{-\mu} e^{-\beta|m|} \tag{37}$$

for all $\tau \in D(0,\rho) \setminus L_-$*, and* $m \in \mathbb{R}$*.*

Proof We first investigate the global behavior of the convolution operator $\mathcal{C}_{k,\gamma_2,\gamma_3}$ w.r.t. τ on the unbounded sector S_d. Owing to the bounds (33), we get

$$\left|\mathcal{C}_{k,\gamma_2,\gamma_3}(f)(\tau,m)\right| \leq C_1 |\tau|^k \int_0^{|\tau|^k} \left(|\tau|^k - h\right)^{\gamma_2} h^{\gamma_3} h^N \exp(\sigma h)\, dh \left(1+|m|\right)^{-\mu} e^{-\beta|m|}. \tag{38}$$

In the next part of the proof, we need to attain sharp upper bounds for the function

$$G(x) = \int_0^x \exp(\sigma h) h^{\gamma_3+N}(x-h)^{\gamma_2}\, dh.$$

It is worth mentioning that the proof is a sharp adaptation of that of Proposition 1 in [16]. In accordance with the uniform expansion $e^{\sigma h} = \sum_{n\geq 0}(\sigma h)^n/n!$ on every compact interval $[0,x]$, $x \geq 0$, we can write

$$G(x) = \sum_{n\geq 0} \frac{\sigma^n}{n!} \int_0^x h^{n+N+\gamma_3}(x-h)^{\gamma_2}\, dh.$$

According to a beta integral formula (see Appendix B in [3]), we recall that

$$\int_0^x (x-h)^{\alpha-1} h^{\beta-1}\, dh = x^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{39}$$

holds for any real numbers $x \geq 0$ and $\alpha > 0$, $\beta > 0$. Therefore, since $N + \gamma_3 \geq 1$ and $\gamma_2 > -1$, we can rewrite

$$G(x) = \sum_{n\geq 0} \frac{\sigma^n}{n!} \frac{\Gamma(\gamma_2+1)\Gamma(n+N+1+\gamma_3)}{\Gamma(n+N+2+\gamma_2+\gamma_3)} x^{n+1+\gamma_2+\gamma_3+N}$$

for all $x \geq 0$. On the other hand, as a consequence of Stirling formula $\Gamma(x) \sim (2\pi)^{1/2} x^x e^{-x} \times x^{-1/2}$ as $x \to +\infty$, given $a > 0$, there exist two constants $K_{1.1}, K_{1.2} > 0$ (depending on a) such that

$$\frac{K_{1.1}}{x^a} \leq \frac{\Gamma(x)}{\Gamma(x+a)} \leq \frac{K_{1.2}}{x^a} \tag{40}$$

for all $x \geq 1$. As a result, there exists a constant $K_{1.2} > 0$ (depending on γ_2) for which

$$\frac{\Gamma(n+N+1+\gamma_3)}{\Gamma(n+N+1+\gamma_3+\gamma_2+1)} \leq \frac{K_{1.2}}{(n+N+1+\gamma_3)^{\gamma_2+1}} \leq \frac{K_{1.2}}{(n+1)^{\gamma_2+1}}$$

for all $n \geq 0$. Hence, we guarantee the existence of a constant $K_{1.3} > 0$ (depending on γ_2) such that

$$G(x) \leq K_{1.3} x^{1+\gamma_2+\gamma_3+N} \sum_{n\geq 0} \frac{1}{(n+1)^{\gamma_2+1} n!} (\sigma x)^n$$

for all $x \geq 0$. The second application of (40) shows the existence of a constant $K_{1.1} > 0$ (depending in γ_2) for which

$$\frac{1}{(n+1)^{\gamma_2+1}} \leq \frac{\Gamma(n+1)}{K_{1.1}\Gamma(n+\gamma_2+2)}$$

for all $n \geq 0$. Subsequently, a constant $K_{1.4} > 0$ (depending on γ_2) exists such that

$$G(x) \leq K_{1.4} x^{1+\gamma_2+\gamma_3+N} \sum_{n\geq 0} \frac{(\sigma x)^n}{\Gamma(n+\gamma_2+2)}$$

for all $x \geq 0$.

Owing to the asymptotic property at infinity of the Wiman function $E_{\alpha,\beta}(z) = \sum_{n\geq 0} z^n / \Gamma(\beta + \alpha n)$, for given $\alpha, \beta > 0$ (see [7] p. 210), we get a constant $K_{1.5} > 0$ (depending on γ_2, σ) with

$$G(x) \leq K_{1.5} x^{\gamma_3+N} e^{\sigma x} \tag{41}$$

for all $x \geq 0$. In accordance with this last inequality, we obtain the expected bounds stated in inequality (36), namely

$$\left|\mathcal{C}_{k,\gamma_2,\gamma_3}(f)(\tau,m)\right| \leq C_1 K_{1.5} |\tau|^{k(N+1)} |\tau|^{k\gamma_3} \exp\bigl(\sigma|\tau|^k\bigr)\bigl(1+|m|\bigr)^{-\mu} e^{-\beta|m|} \tag{42}$$

for all $\tau \in S_d$, all $m \in \mathbb{R}$.

In the second part of the proof, we study local properties near the origin w.r.t. τ. First, we rewrite $\mathcal{C}_{k,\gamma_2,\gamma_3}$ by means of the parametrization $s = \tau^k u$ for $0 \leq u \leq 1$ in the form

$$\mathcal{C}_{k,\gamma_2,\gamma_3}(f)(\tau,m) = \tau^{k(\gamma_2+\gamma_3+2)} \int_0^1 (1-u)^{\gamma_2} u^{\gamma_3} f\bigl(\tau u^{1/k}, m\bigr)\, du \tag{43}$$

for all $\tau \in D(0,\rho) \setminus L_-$ whenever $m \in \mathbb{R}$. The last assumption in (32) and in view of the construction of $f(\tau,m)$, representation (43) induces that, for all $m \in \mathbb{R}$, the function $\tau \mapsto \mathcal{C}_{k,\gamma_2,\gamma_3}(f)(\tau,m)$ extends analytically on $D(0,\rho) \setminus L_-$. Furthermore, one may apply (34) in order to deduce the bounds

$$\left|\mathcal{C}_{k,\gamma_2,\gamma_3}(f)(\tau,m)\right| \leq C_1' |\tau|^k \int_0^{|\tau|^k} \bigl(|\tau|^k - h\bigr)^{\gamma_2} h^{\gamma_3+1}\, dh \bigl(1+|m|\bigr)^{-\mu} e^{-\beta|m|}$$

valid for all $\tau \in D(0,\rho) \setminus L_-$, all $m \in \mathbb{R}$. With the help of (39), we deduce that

$$\begin{aligned} \left|\mathcal{C}_{k,\gamma_2,\gamma_3}(f)(\tau,m)\right| &\le C_1'|\tau|^k \frac{\Gamma(\gamma_2+1)\Gamma(\gamma_3+2)}{\Gamma(\gamma_2+\gamma_3+3)}|\tau|^{k(\gamma_2+\gamma_3+2)}\big(1+|m|\big)^{-\mu}e^{-\beta|m|} \\ &\le C_1'\frac{\Gamma(\gamma_2+1)\Gamma(\gamma_3+2)}{\Gamma(\gamma_2+\gamma_3+3)}\rho^{k(\gamma_2+\gamma_3+2)}|\tau|^k\big(1+|m|\big)^{-\mu}e^{-\beta|m|}, \end{aligned} \tag{44}$$

when $\tau \in D(0,\rho) \setminus L_-$, all $m \in \mathbb{R}$ which entails (37). □

Proposition 2 *Let $k \in (\frac{1}{2},1)$. Let $(\tau,m) \mapsto f(\tau,m)$ be a continuous function on $S_d \times \mathbb{R}$, holomorphic w.r.t. τ on S_d, for which there exist constants $C_2 > 0$, $\nu > 0$ and $\mu > 1$, $\beta > 0$ fulfilling*

$$\big|f(\tau,m)\big| \le C_2|\tau|^k \exp\big(\nu|\tau|^k\big)\big(1+|m|\big)^{-\mu}e^{-\beta|m|} \tag{45}$$

for all $\tau \in S_d$, all $m \in \mathbb{R}$. Take for granted that, for all $m \in \mathbb{R}$, the map $\tau \mapsto f(\tau,m)$ extends analytically on the cut disc $D(0,\rho) \setminus L_-$ under the next bounds : there exists a constant $C_2' > 0$ with

$$\big|f(\tau,m)\big| \le C_2'|\tau|^k\big(1+|m|\big)^{-\mu}e^{-\beta|m|} \tag{46}$$

for all $\tau \in D(0,\rho) \setminus L_-$, all $m \in \mathbb{R}$.

Let $\kappa_l > 0$. We consider the operator

$$\big(\exp(-\kappa_l\mathcal{C}_k)f\big)(\tau,m) := \sum_{p\ge 0}\frac{(-1)^p\kappa_l^p}{p!}\mathcal{C}_k^{(p)}(f)(\tau,m), \tag{47}$$

where $\mathcal{C}_k^{(p)}$ denotes the iterate of order $p \ge 0$ of the operator $\mathcal{C}_k$ defined by

$$\begin{aligned} \mathcal{C}_k(f)(\tau,m) &= \frac{k\tau^k}{\Gamma(\frac{1}{k}-1)}\int_0^{\tau^k}\big(\tau^k-s\big)^{\frac{1}{k}-2}f\big(s^{1/k},m\big)\,ds \\ &= \frac{k}{\Gamma(\frac{1}{k}-1)}\mathcal{C}_{k,\frac{1}{k}-2,0}(f)(\tau,m) \end{aligned}$$

with the convention that $\mathcal{C}_k^{(0)}(f)(\tau,m) = f(\tau,m)$. Then

(1) *The map $(\tau,m) \mapsto (\exp(-\kappa_l\mathcal{C}_k)f)(\tau,m)$ represents a continuous function on $S_d \times \mathbb{R}$, holomorphic w.r.t. τ on S_d, for which a constant $K_1 > 0$ (depending on k, ν) exists such that*

$$\begin{aligned} &\big|\big(\exp(-\kappa_l\mathcal{C}_k)f\big)(\tau,m)\big| \\ &\quad\le C_2|\tau|^k\exp\left(\kappa_l K_1\frac{k}{\Gamma(\frac{1}{k}-1)}|\tau|^k\right)\exp\big(\nu|\tau|^k\big)\big(1+|m|\big)^{-\mu}e^{-\beta|m|} \end{aligned} \tag{48}$$

for all $\tau \in S_d$, all $m \in \mathbb{R}$.

(2) *For all* $m \in \mathbb{R}$, *the function* $\tau \mapsto (\exp(-\kappa_l \mathcal{C}_k)f)(\tau,m)$ *extends analytically on* $D(0,\rho) \setminus L_-$. *Furthermore,*

$$\left|\left(\exp(-\kappa_l \mathcal{C}_k)f\right)(\tau,m)\right| \leq \exp\left(\frac{\kappa_l k \rho}{\Gamma(\frac{1}{k}+1)}\right) C_2' |\tau|^k \left(1+|m|\right)^{-\mu} e^{-\beta|m|} \tag{49}$$

for all $\tau \in D(0,\rho) \setminus L_-$, *all* $m \in \mathbb{R}$.

Proof Let us provide estimates for each iterate $\mathcal{C}_k^{(N)}(f)(\tau,m)$, $N \geq 1$. We first consider such bounds on the unbounded sector S_d. By induction on $N \geq 0$, with the help of the estimates (33) and (36) for $\gamma_2 = \frac{1}{k} - 2$ and $\gamma_3 = 0$, we obtain a constant $K_1 > 0$ (depending on k, ν) with

$$\left|\mathcal{C}_k^{(N)}(f)(\tau,m)\right| \leq C_2 \left(\frac{k}{\Gamma(\frac{1}{k}-1)}\right)^N K_1^N |\tau|^{k(N+1)} \exp\left(\nu|\tau|^k\right)\left(1+|m|\right)^{-\mu} e^{-\beta|m|} \tag{50}$$

for all $\tau \in S_d$, all $m \in \mathbb{R}$, all $N \geq 0$. Similarly, owing to (34) and (37), and for the same choice $\gamma_2 = \frac{1}{k} - 2$ and $\gamma_3 = 0$, we get that

$$\left|\mathcal{C}_k^{(N)}(f)(\tau,m)\right| \leq C_2' k^N \left(\frac{\Gamma(2)}{\Gamma(\frac{1}{k}+1)}\right)^N \rho^N |\tau|^k \left(1+|m|\right)^{-\mu} e^{-\beta|m|} \tag{51}$$

for all $\tau \in D(0,\rho) \setminus L_-$, all $m \in \mathbb{R}$, all $N \geq 0$.

Finally, by summing up inequalities (50) (resp. (51)) over $N \geq 0$, we get the forecast bounds (48) (resp. (49)). □

Proposition 3 *Let* $Q_1(X), Q_2(X), R(X) \in \mathbb{C}[X]$ *such that*

$$\deg(R) \geq \deg(Q_1), \qquad \deg(R) \geq \deg(Q_2), \qquad R(im) \neq 0 \tag{52}$$

for all $m \in \mathbb{R}$. *Take for granted that* $\mu > \max(\deg(Q_1)+1, \deg(Q_2)+1)$. *Then there exists a constant* $C_3 > 0$ (*depending on* Q_1, Q_2, R, μ, k, ν) *for which*

$$\begin{aligned} &\left\| \frac{1}{R(im)} \tau^k \int_0^{\tau^k} \int_{-\infty}^{+\infty} Q_1\left(i(m-m_1)\right) f\left(\left(\tau^k - s\right)^{1/k}, m-m_1\right) \right. \\ &\qquad \left. \times Q_2(im_1) g\left(s^{1/k}, m_1\right) \frac{1}{(\tau^k - s)s} \, ds \, dm_1 \right\|_{(\nu,\beta,\mu,k,\rho)} \\ &\quad \leq C_3 \left\|f(\tau,m)\right\|_{(\nu,\beta,\mu,k,\rho)} \left\|g(\tau,m)\right\|_{(\nu,\beta,\mu,k,\rho)} \end{aligned} \tag{53}$$

for all $f(\tau,m), g(\tau,m) \in F^d_{(\nu,\beta,\mu,k,\rho)}$.

Proof We proceed as in the proof of Proposition 3 of [15]. Namely, according to Definition 4, we rewrite

$$\begin{aligned} B := &\left\| \frac{1}{R(im)} \tau^k \int_0^{\tau^k} \int_{-\infty}^{+\infty} Q_1\left(i(m-m_1)\right) f\left(\left(\tau^k - s\right)^{1/k}, m-m_1\right) \right. \\ &\left. \times Q_2(im_1) g\left(s^{1/k}, m_1\right) \frac{1}{(\tau^k - s)s} \, ds \, dm_1 \right\|_{(\nu,\beta,\mu,k,\rho)} \end{aligned}$$

$$
\begin{aligned}
&= \sup_{\tau \in (D(0,\rho)\setminus L_-)\cup S_d, m\in\mathbb{R}} \bigl(1+|m|\bigr)^{\mu} e^{\beta|m|} \frac{1+|\tau|^{2k}}{|\tau|^k} \\
&\quad \times \exp\bigl(-\nu|\tau|^k\bigr) \\
&\quad \times \left| \tau^k \int_0^{\tau^k} \int_{-\infty}^{+\infty} \left\{ \bigl(1+|m-m_1|\bigr)^{\mu} e^{\beta|m-m_1|} \frac{1+|\tau^k-s|^2}{|\tau^k-s|} \exp\bigl(-\nu\bigl|\tau^k-s\bigr|\bigr) \right.\right. \\
&\quad \left. \times f\bigl(\bigl(\tau^k-s\bigr)^{1/k}, m-m_1\bigr) \right\} \times \left\{ \bigl(1+|m_1|\bigr)^{\mu} e^{\beta|m_1|} \frac{1+|s|^2}{|s|} \exp\bigl(-\nu|s|\bigr) g\bigl(s^{1/k}, m_1\bigr) \right\} \\
&\quad \left. \times \mathcal{B}(\tau,s,m,m_1)\, ds\, dm_1 \right|
\end{aligned}
\tag{54}
$$

with

$$
\begin{aligned}
&\mathcal{B}(\tau,s,m,m_1) \\
&\quad = \frac{e^{-\beta|m-m_1|}e^{-\beta|m_1|}}{(1+|m-m_1|)^{\mu}(1+|m_1|)^{\mu}} \frac{Q_1(i(m-m_1))Q_2(im_1)}{R(im)} \frac{|s||\tau^k-s|}{(1+|\tau^k-s|^2)(1+|s|^2)} \\
&\quad\quad \times \exp\bigl(\nu\bigl|\tau^k-s\bigr|\bigr) \exp\bigl(\nu|s|\bigr) \frac{1}{(\tau^k-s)s}.
\end{aligned}
$$

According to the triangular inequality $|m| \leq |m-m_1| + |m_1|$ and bearing in mind the definition of the norms of f and g, we deduce

$$
B \leq C_{3.1} \bigl\| f(\tau,m) \bigr\|_{(\nu,\beta,\mu,k,\rho)} \bigl\| g(\tau,m) \bigr\|_{(\nu,\beta,\mu,k,\rho)},
\tag{55}
$$

where

$$
\begin{aligned}
C_{3.1} &= \sup_{\tau \in (D(0,\rho)\setminus L_-)\cup S_d, m\in\mathbb{R}} \bigl(1+|m|\bigr)^{\mu} \frac{1+|\tau|^{2k}}{|\tau|^k} \exp\bigl(-\nu|\tau|^k\bigr)|\tau|^k \\
&\quad \times \int_0^{|\tau|^k} \int_{-\infty}^{+\infty} \frac{|Q_1(i(m-m_1))||Q_2(im_1)|}{R(im)(1+|m-m_1|)^{\mu}(1+|m_1|)^{\mu}} \\
&\quad \times \frac{\exp(\nu(|\tau|^k-h))\exp(\nu h)}{(1+(|\tau|^k-h)^2)(1+h^2)}\, dh\, dm_1.
\end{aligned}
\tag{56}
$$

Now, we get bounds from above for $C_{3.1}$ via the splitting $C_{3.1} = C_{3.2}C_{3.3}$, where

$$
C_{3.2} = \sup_{m\in\mathbb{R}} \bigl(1+|m|\bigr)^{\mu} \frac{1}{|R(im)|} \int_{-\infty}^{+\infty} \frac{|Q_1(i(m-m_1))||Q_2(im_1)|}{(1+|m-m_1|)^{\mu}(1+|m_1|)^{\mu}}\, dm_1
$$

and

$$
C_{3.3} = \sup_{\tau \in (D(0,\rho)\setminus L_-)\cup S_d} \bigl(1+|\tau|^{2k}\bigr) \int_0^{|\tau|^k} \frac{1}{(1+(|\tau|^k-h)^2)(1+h^2)}\, dh.
$$

In the last step of the proof, we show that $C_{3.2}$ and $C_{3.3}$ are finite. By construction, there exist positive constants $\mathfrak{Q}_1$, $\mathfrak{Q}_2$, and $\mathfrak{R}$ such that

$$
\bigl|Q_1\bigl(i(m-m_1)\bigr)\bigr| \leq \mathfrak{Q}_1\bigl(1+|m-m_1|\bigr)^{\deg(Q_1)},
$$

$$\big|Q_2(im_1)\big| \leq \mathfrak{Q}_2\big(1+|m_1|\big)^{\deg(Q_2)},$$
$$\big|R(im)\big| \geq \mathfrak{R}\big(1+|m|\big)^{\deg(R)}$$

for all $m, m_1 \in \mathbb{R}$. Hence,

$$C_{3.2} \leq \frac{\mathfrak{Q}_1\mathfrak{Q}_2}{\mathfrak{R}} \sup_{m\in\mathbb{R}} \big(1+|m|\big)^{\mu-\deg(R)} \times \int_{-\infty}^{+\infty} \frac{1}{(1+|m-m_1|)^{\mu-\deg(Q_1)}(1+|m_1|)^{\mu-\deg(Q_2)}}\, dm_1 \tag{57}$$

that is finite owing to $\mu > \max(\deg(Q_1)+1, \deg(Q_2)+1)$ submitted to constraints (52) as shown in Lemma 4 in [18]. On the other hand,

$$\begin{aligned} C_{3.3} &\leq \sup_{x\geq 0}\big(1+x^2\big) \int_0^x \frac{1}{(1+(x-h)^2)(1+h^2)}\, dh \\ &= \sup_{x\geq 0}\big(1+x^2\big) 2\frac{\log(1+x^2)+x\arctan(x)}{x(x^2+4)}, \end{aligned} \tag{58}$$

which is also finite. □

5 Solutions of an auxiliary integral equation depending on a complex parameter

The main objective of this section is the construction of a unique solution of equation (31) for vanishing initial data within the Banach spaces given in Definition 4.

We first describe further analytic assumptions on the leading polynomials $Q(X)$ and $R_D(X)$ in order to be able to transform our problem (31) into a fixed point equation as stated below, see (101).

Namely, we take for granted that there exists a bounded sectorial annulus

$$S_{Q,R_D} = \big\{z \in \mathbb{C}/r_{Q,R_D,1} \leq |z| \leq r_{Q,R_D,2}, \big|\arg(z) - d_{Q,R_D}\big| \leq \eta_{Q,R_D}\big\}$$

with direction $d_{Q,R_D} \in [-\pi, \pi)$, small aperture $\eta_{Q,R_D} > 0$ for some radii $r_{Q,R_D,2} > r_{Q,R_D,1} > 1$ such that

$$\frac{Q(im)}{R_D(im)} \in S_{Q,R_D} \tag{59}$$

for all $m \in \mathbb{R}$. For any integer $l \in \mathbb{Z}$, we set

$$a_l(m) = \log\left|\frac{Q(im)}{R_D(im)}\right| + \sqrt{-1}\arg\left(\frac{Q(im)}{R_D(im)}\right) + 2l\pi\sqrt{-1}. \tag{60}$$

Figure 1 illustrates a configuration of the points $a_l(m)$, $l \in \mathbb{Z}$, and the set S_{Q,R_D} related to their definition.

By construction, we see that

$$Q(im) - e^{a_l(m)}R_D(im) = 0 \tag{61}$$

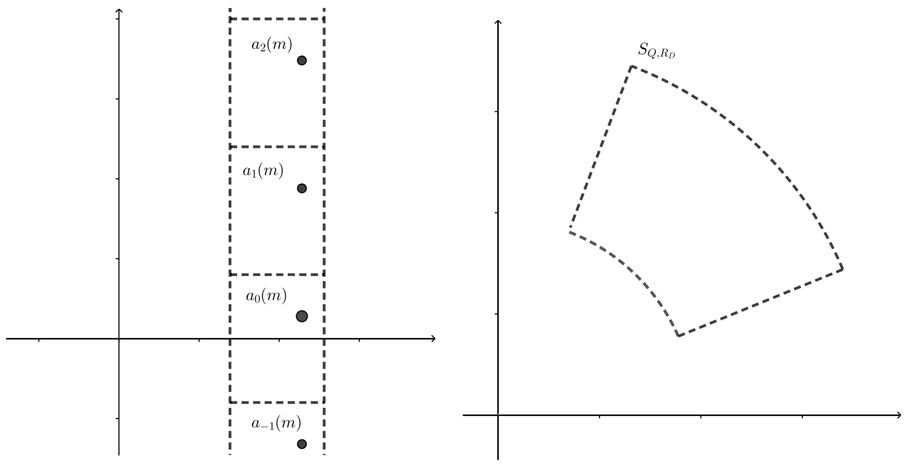


Figure 1 Configuration of $a_l(m)$ (left) related to S_{Q,R_D} (right)

for all $m \in \mathbb{R}$. Furthermore, for each $l \in \mathbb{Z}$, the equation

$$\alpha_D k \tau^k = a_l(m) \tag{62}$$

possesses one solution given by

$$\tau_l = \left| \frac{a_l(m)}{\alpha_D k} \right|^{1/k} \exp\left(\sqrt{-1} \frac{1}{k} \arg\bigl(a_l(m)\bigr) \right). \tag{63}$$

Indeed, by construction of $\tau^k = \exp(k \log(\tau))$, this equation is equivalent to

$$|\tau| = \left| \frac{a_l(m)}{\alpha_D k} \right|^{1/k}, \qquad \arg(\tau) = \frac{\arg(a_l(m))}{k} + \frac{2h\pi}{k} \tag{64}$$

for some $h \in \mathbb{Z}$. According to the hypothesis $r_{Q,R_D,1} > 1$, we have $|\arg(a_l(m))| < \pi/2$, and hence

$$\left| \frac{\arg(a_l(m))}{k} \right| < \frac{\pi}{2k} < \pi \tag{65}$$

since we assume that $\frac{1}{2} < k < 1$. Owing to the fact that $\arg(\tau)$ belongs to $(-\pi, \pi)$, we have $h = 0$ and hence $\arg(\tau) = \arg(a_l(m))/k$.

We consider the set

$$\Theta_{Q,R_D} = \left\{ \frac{\arg(a_l(m))}{k} / m \in \mathbb{R}, l \in \mathbb{Z} \right\}$$

of the so-called forbidden directions. We choose the aperture $\eta_{Q,R_D} > 0$ small enough in a way that, for all directions $d \in (-\pi/2, \pi/2) \setminus \Theta_{Q,R_D}$, we can find some unbounded sector S_d centered at 0 with small aperture $\delta_{S_d} > 0$ and bisecting direction d such that $\tau_l \notin S_d \cup D(0, \rho)$ for some fixed $\rho > 0$ small enough and for all $l \in \mathbb{Z}$.

For all $\tau \in \mathbb{C} \setminus \mathbb{R}_-$, all $m \in \mathbb{R}$, we consider the function

$$H(\tau,m) = Q(im) - \exp\bigl(\alpha_D k \tau^k\bigr) R_D(im). \tag{66}$$

Let $d \in (-\pi/2, \pi/2) \setminus \Theta_{Q,R_D}$ and fix a sector S_d and a disc $D(0,\rho)$ as above.

(1) Our first goal is to provide lower bounds for the function $|H(\tau,m)|$ when $\tau \in S_d$ and $m \in \mathbb{R}$. Let $\tau \in S_d$. Then we can write

$$\tau = \tau_l r e^{\sqrt{-1}\theta} \tag{67}$$

for some well-chosen $l \in \mathbb{Z}$, where $r \geq 0$ and θ belongs to some small interval I_{S_d} which is close to 0 but such that $0 \notin I_{S_d}$. In particular, we choose I_{S_d} in a way that $\arg(\tau_l) + \theta$ belongs to $(-\pi, \pi)$ for all $\theta \in I_{S_d}$.

Hence, owing to the fact that τ_l solves (62), we can rewrite

$$\alpha_D k \tau^k - a_l(m) = \alpha_D k \tau_l^k r^k e^{\sqrt{-1}k\theta} - a_l(m) = a_l(m)\bigl(r^k e^{\sqrt{-1}k\theta} - 1\bigr).$$

In particular, if the radius $r_{Q,R_D,2}$ is chosen close enough to $r_{Q,R_D,1}$, we get a constant $\eta_{1,l} > 0$ (depending on l) for which

$$\bigl|\alpha_D k \tau^k - a_l(m) - \sqrt{-1}h2\pi\bigr| \geq \eta_{1,l} \tag{68}$$

for all $h \in \mathbb{Z}$, all $\tau \in S_d$, all $m \in \mathbb{R}$. More precisely, for each $m \in \mathbb{R}$, the set

$$\mathcal{L}_{l,m} = \bigl\{a_l(m)\bigl(xe^{\sqrt{-1}k\theta} - 1\bigr)/x \geq 0\bigr\}$$

represents a halfline passing through the point $-a_l(m)$ and close to the origin in $\mathbb{C}$. Consequently, it avoids the set of points $\{\sqrt{-1}h2\pi/h \in \mathbb{Z}\}$.

Figure 2 illustrates a configuration of some of the halflines described in the construction.

Now, owing to equality (61), we can rewrite

$$\begin{aligned} H(\tau,m) &= Q(im) - \exp\bigl(\alpha_D k \tau^k - a_l(m)\bigr)\exp\bigl(a_l(m)\bigr)R_D(im) \\ &= Q(im)\bigl(1 - \exp\bigl(\alpha_D k \tau^k - a_l(m)\bigr)\bigr). \end{aligned} \tag{69}$$

According to (68), we obtain a constant $\eta_{2,l} > 0$ (depending on l) for which

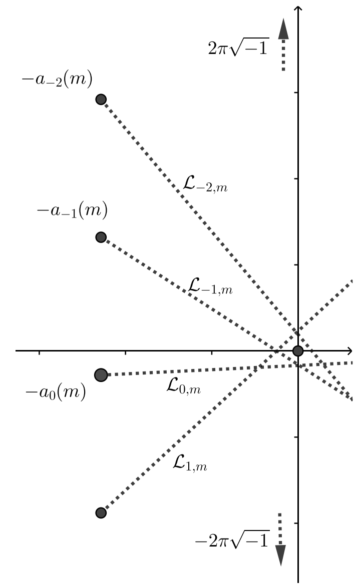
$$\bigl|H(\tau,m)\bigr| \geq \bigl|Q(im)\bigr|\eta_{2,l} \tag{70}$$

for all $\tau \in S_d$, all $m \in \mathbb{R}$.

In the second step, we aim attention at lower bounds for large values of $|\tau|$ on S_d. We first carry out some preliminary computations. For this purpose, we expand

$$\begin{aligned} &\mathrm{Re}\bigl(a_l(m)\bigl(r^k e^{\sqrt{-1}k\theta} - 1\bigr)\bigr) \\ &\quad = r^k\left(\log\left|\frac{Q(im)}{R_D(im)}\right|\cos(k\theta) - \left(\arg\left(\frac{Q(im)}{R_D(im)}\right) + 2l\pi\right)\sin(k\theta)\right) - \log\left|\frac{Q(im)}{R_D(im)}\right|. \end{aligned} \tag{71}$$

Figure 2 Some of the halflines $\mathcal{L}_{l,m}$



We assume that the segment I_{S_d} is close enough to 0 in such a way that a constant $\Delta_1 > 0$ exists submitted to the next inequality

$$\log\left|\frac{Q(im)}{R_D(im)}\right|\cos(k\theta) - \left(\arg\left(\frac{Q(im)}{R_D(im)}\right) + 2l\pi\right)\sin(k\theta) \geq \Delta_1 \tag{72}$$

for all $m \in \mathbb{R}$, all $\theta \in I_{S_d}$. Besides, according to (59), we notice that

$$0 < \log(r_{Q,R_D,1}) \leq \log\left|\frac{Q(im)}{R_D(im)}\right| \leq \log(r_{Q,R_D,2}) \tag{73}$$

for all $m \in \mathbb{R}$. As a result, collecting (71), (72), and (73), we arrive at the lower bounds

$$\operatorname{Re}\bigl(a_l(m)\bigl(r^k e^{\sqrt{-1}k\theta} - 1\bigr)\bigr) \geq \Delta_1 r^k - \log(r_{Q,R_D,2}) \tag{74}$$

for all $r \geq 0$, all $\theta \in I_{S_d}$, all $m \in \mathbb{R}$.

Departing from factorization (69), we get the next estimates from below

$$\begin{aligned}
\bigl|H(\tau,m)\bigr| &\geq \bigl|Q(im)\bigr|\bigl|1 - \bigl|\exp\bigl(a_l(m)\bigl(r^k e^{\sqrt{-1}k\theta} - 1\bigr)\bigr)\bigr|\bigr| \\
&= \bigl|Q(im)\bigr|\bigl|1 - \exp\bigl(\operatorname{Re}\bigl(a_l(m)\bigl(r^k e^{\sqrt{-1}k\theta} - 1\bigr)\bigr)\bigr)\bigr| \\
&= \bigl|Q(im)\bigr| \exp\bigl(\operatorname{Re}\bigl(a_l(m)\bigl(r^k e^{\sqrt{-1}k\theta} - 1\bigr)\bigr)\bigr) \\
&\quad \times \bigl|1 - \exp\bigl(-\operatorname{Re}\bigl(a_l(m)\bigl(r^k e^{\sqrt{-1}k\theta} - 1\bigr)\bigr)\bigr)\bigr|
\end{aligned} \tag{75}$$

for all $r \geq 0$, all $\theta \in I_{S_d}$, all $m \in \mathbb{R}$. We select a real number $r_1 > 0$ large enough such that

$$\exp\bigl(-\bigl(\Delta_1 r^k - \log(r_{Q,R_D,2})\bigr)\bigr) \leq 1/2 \tag{76}$$

for all $r \geq r_1$. Under (76), we deduce from (74) and (75) that

$$\left|H(\tau,m)\right| \geq \frac{1}{2}\left|Q(im)\right| \exp\bigl(\Delta_1 r^k - \log(r_{Q,R_D,2})\bigr)$$

for all $r \geq r_1$, all $\theta \in I_{S_d}$, all $m \in \mathbb{R}$. Now, in view of decomposition (67), we get in particular that $|\tau| = r|\tau_l|$. Consequently, we see that

$$\left|H(\tau,m)\right| \geq \frac{1}{2}\left|Q(im)\right| \exp\left(\frac{\Delta_1}{|\tau_l|^k}|\tau|^k - \log(r_{Q,R_D,2})\right) \tag{77}$$

for all $\tau \in S_d$ with $|\tau| \geq r_1|\tau_l|$.

As a result, gathering (70) and (77), together with the shape of $a_l(m)$ and τ_l given in (60), (63), we obtain two constants $A_{H,d}, B_{H,d} > 0$ (depending on k, S_{Q,R_D}, S_d) for which

$$\left|H(\tau,m)\right| \geq A_{H,d}\left|Q(im)\right| \exp\bigl(B_{H,d}\alpha_D|\tau|^k\bigr) \tag{78}$$

for all $\tau \in S_d$, all $m \in \mathbb{R}$.

(2) In the second step, we display lower bounds when τ belongs to the cut disc $D(0,\rho) \setminus L_-$, where $L_- = (-\rho, 0]$. Let $\tau = re^{\sqrt{-1}\theta}$ for some $\theta \in (-\pi,\pi)$ and $0 < r < \rho$. Let $l \in \mathbb{Z}$. We first compute the real part

$$\operatorname{Re}\bigl(\alpha_D k r^k e^{\sqrt{-1}k\theta} - a_l(m)\bigr) = \alpha_D k r^k \cos(k\theta) - \log\left|\frac{Q(im)}{R_D(im)}\right|$$

for all $m \in \mathbb{R}$. Therefore, owing to (73), we may select $r_{Q,R_D,1} > 0$ large enough such that

$$\exp\bigl(\operatorname{Re}\bigl(\alpha_D k r^k e^{\sqrt{-1}k\theta} - a_l(m)\bigr)\bigr) \leq \exp\bigl(\alpha_D k \rho^k - \log(r_{Q,R_D,1})\bigr) \leq \frac{1}{2} \tag{79}$$

for all $\theta \in (-\pi,\pi)$, $0 < r < \rho$ and $m \in \mathbb{Z}$. Hence, in view of factorization (69), it follows that

$$\begin{aligned} \left|H(\tau,m)\right| &\geq \left|Q(im)\right|\left|1 - \left|\exp\bigl(\alpha_D k \tau^k - a_l(m)\bigr)\right|\right| \\ &= \left|Q(im)\right|\left|1 - \exp\bigl(\operatorname{Re}\bigl(\alpha_D k r^k e^{\sqrt{-1}k\theta} - a_l(m)\bigr)\bigr)\right| \\ &\geq \frac{1}{2}\left|Q(im)\right| \end{aligned} \tag{80}$$

for all $\tau \in D(0,\rho) \setminus L_-$, all $m \in \mathbb{R}$.

In the next proposition we provide sufficient conditions for which equation (31) possesses a solution $w^d(\tau,m,\epsilon)$ within the Banach space $F^d_{(\nu,\beta,\mu,k,\rho)}$.

Proposition 4 *We make an additional assumption that*

$$B_{H,d}\alpha_D > \kappa_l K_1 \frac{k}{\Gamma(\frac{1}{k} - 1)} \tag{81}$$

for all $1 \leq l \leq D-1$, *where* K_1 *is the constant defined in Proposition* 2(1) *and* $B_{H,d}$ *is in* (78). *Under the condition that the moduli* $|c_{12}|$, $|c_f|$, *and* $|c_l|$ *for* $1 \leq l \leq D-1$ *are chosen small enough, we can find a constant* $\varpi > 0$ *for which equation* (31) *has a unique solution*

$w^{d}(\tau,m,\epsilon)$ in $F^{d}_{(\nu,\beta,\mu,k,\rho)}$, with $\|w^{d}(\tau,m,\epsilon)\|_{(\nu,\beta,\mu,k,\rho)} \leq \varpi$ for all $\epsilon \in D(0,\epsilon_0)$, where $\beta,\mu > 0$ are chosen as in (9), $\nu > 1$ is taken as in Lemma 1, the sector S_d and the disc $D(0,\rho)$ are suitably selected in a way that $\tau_l \notin S_d \cup D(0,\rho)$ for all $l \in \mathbb{Z}$ and where τ_l is described in (63).

Proof We initiate the proof with the lemma that introduces a map related to (31) and describes some of its properties that allow us to apply a fixed point theorem.

Lemma 3 *One can sort the moduli $|c_{12}|$, $|c_f|$, and $|c_l|$ for $1 \leq l \leq D-1$ in such a way that a constant $\varpi > 0$ can be chosen so that the map $\mathcal{H}_\epsilon$ defined as*

$$
\begin{aligned}
&\mathcal{H}_\epsilon\bigl(w(\tau,m)\bigr)\\
&\quad := \sum_{l=1}^{D-1} \epsilon^{\Delta_l-d_l+\delta_l} R_l(im) c_l \sum_{n\in I_l} A_{l,n}(\epsilon)\\
&\qquad \times \biggl(\frac{\tau^k}{H(\tau,m)\Gamma(\frac{n+d_{l,k}}{k})}\int_0^{\tau^k}\bigl(\tau^k-s\bigr)^{\frac{n+d_{l,k}}{k}-1} k^{\delta_l} s^{\delta_l}\bigl(\exp(-\kappa_l \mathcal{C}_k)w\bigr)\bigl(s^{1/k},m\bigr)\frac{ds}{s}\\
&\qquad + \sum_{1\leq p\leq \delta_l-1} A_{\delta_l,p}\frac{\tau^k}{H(\tau,m)\Gamma(\frac{n+d_{l,k}}{k}+\delta_l-p)}\int_0^{\tau^k}\bigl(\tau^k-s\bigr)^{\frac{n+d_{l,k}}{k}+\delta_l-p-1}\\
&\qquad \times k^p s^p\bigl(\exp(-\kappa_l \mathcal{C}_k)w\bigr)\bigl(s^{1/k},m\bigr)\frac{ds}{s}\biggr)\\
&\qquad + c_{12}\frac{\tau^k}{(2\pi)^{1/2}H(\tau,m)}\int_0^{\tau^k}\int_{-\infty}^{+\infty} Q_1\bigl(i(m-m_1)\bigr)w\bigl(\bigl(\tau^k-s\bigr)^{1/k},m-m_1\bigr)\\
&\qquad \times Q_2(im_1)w\bigl(s^{1/k},m_1\bigr)\frac{1}{(\tau^k-s)s}\,ds\,dm_1 + c_f\frac{\psi(\tau,m,\epsilon)}{H(\tau,m)}
\end{aligned}
\tag{82}
$$

fulfills the following statements:

(1) *The inclusion*

$$\mathcal{H}_\epsilon\bigl(\bar{B}(0,\varpi)\bigr) \subset \bar{B}(0,\varpi) \tag{83}$$

holds where $\bar{B}(0,\varpi)$ represents the closed ball of radius $\varpi > 0$ centered at 0 in $F^{d}_{(\nu,\beta,\mu,k,\rho)}$ for all $\epsilon \in D(0,\epsilon_0)$.

(2) *The shrinking condition*

$$\bigl\|\mathcal{H}_\epsilon(w_2)-\mathcal{H}_\epsilon(w_1)\bigr\|_{(\nu,\beta,\mu,k,\rho)} \leq \frac{1}{2}\|w_2-w_1\|_{(\nu,\beta,\mu,k,\rho)} \tag{84}$$

occurs whenever $w_1, w_2 \in \bar{B}(0,\varpi)$ for all $\epsilon \in D(0,\epsilon_0)$.

Proof We focus on the first property (83). Let $w(\tau,m) \in F^{d}_{(\nu,\beta,\mu,k,\rho)}$. We take $\epsilon \in D(0,\epsilon_0)$ and set $\varpi > 0$ such that $\|w(\tau,m)\|_{(\nu,\beta,\mu,k,\rho)} \leq \varpi$. In particular, we notice that the estimate

$$\bigl|w(\tau,m)\bigr| \leq \varpi|\tau|^k e^{\nu|\tau|^k} e^{-\beta|m|}\bigl(1+|m|\bigr)^{-\mu} \tag{85}$$

holds for each $\tau \in S_d \cup (D(0,\rho) \setminus L_-)$. As a consequence of Proposition 2, we get that $(\tau,m) \mapsto (\exp(-\kappa_l \mathcal{C}_k)w)(\tau,m)$ defines a continuous function on $S_d \times \mathbb{R}$, holomorphic w.r.t. τ on S_d. A constant $K_1 > 0$ (depending on k, ν) can be found such that

$$\begin{aligned}&\bigl|\bigl(\exp(-\kappa_l \mathcal{C}_k)w\bigr)(\tau,m)\bigr|\\&\quad\leq \varpi|\tau|^k \exp\biggl(\kappa_l K_1 \frac{k}{\Gamma(\frac{1}{k}-1)}|\tau|^k\biggr)\exp\bigl(\nu|\tau|^k\bigr)\bigl(1+|m|\bigr)^{-\mu}e^{-\beta|m|}\end{aligned} \tag{86}$$

for all $\tau \in S_d$, all $m \in \mathbb{R}$. Furthermore, the application of Proposition 1 for $\gamma_2 = \frac{n+d_{l,k}}{k} - 1$, $\gamma_3 = \delta_l - 1$ with $n \in I_l$ guarantees the existence of a constant $C_4 > 0$ (depending on I_l, k, κ_l, d_l, δ_l ,ν) with

$$\begin{aligned}&\biggl|\tau^k \int_0^{\tau^k} \bigl(\tau^k - s\bigr)^{\frac{n+d_{l,k}}{k}-1} s^{\delta_l}\bigl(\exp(-\kappa_l \mathcal{C}_k)w\bigr)\bigl(s^{1/k},m\bigr)\frac{ds}{s}\biggr|\\&\quad\leq \varpi C_4|\tau|^{2k+k(\delta_l-1)}\exp\biggl(\kappa_l K_1 \frac{k}{\Gamma(\frac{1}{k}-1)}|\tau|^k\biggr)\exp\bigl(\nu|\tau|^k\bigr)\bigl(1+|m|\bigr)^{-\mu}e^{-\beta|m|},\end{aligned}$$

provided that $\tau \in S_d$, $m \in \mathbb{R}$. From the lower bounds (78) and (81) we get

$$\begin{aligned}&\biggl|\frac{R_l(im)\tau^k}{H(\tau,m)}\int_0^{\tau^k} \bigl(\tau^k - s\bigr)^{\frac{n+d_{l,k}}{k}-1} s^{\delta_l}\bigl(\exp(-\kappa_l \mathcal{C}_k)w\bigr)\bigl(s^{1/k},m\bigr)\frac{ds}{s}\biggr|\\&\quad\leq \varpi \frac{C_4}{A_{H,d}}\biggl|\frac{R_l(im)}{Q(im)}\biggr||\tau|^{k(\delta_l+1)}\exp\biggl(\biggl(\kappa_l K_1 \frac{k}{\Gamma(\frac{1}{k}-1)} - B_{H,d}\alpha_D\biggr)|\tau|^k\biggr)\\&\qquad\times \exp\bigl(\nu|\tau|^k\bigr)\bigl(1+|m|\bigr)^{-\mu}e^{-\beta|m|}\\&\quad\leq \varpi \frac{C_4}{A_{H,d}}\sup_{m\in\mathbb{R}}\biggl|\frac{R_l(im)}{Q(im)}\biggr|\biggl(\sup_{|\tau|\geq 0}|\tau|^{k\delta_l}\bigl(1+|\tau|^{2k}\bigr)\\&\qquad\times \exp\biggl(\biggl(\kappa_l K_1 \frac{k}{\Gamma(\frac{1}{k}-1)} - B_{H,d}\alpha_D\biggr)|\tau|^k\biggr)\biggr)\\&\qquad\times \frac{|\tau|^k}{1+|\tau|^{2k}}\exp\bigl(\nu|\tau|^k\bigr)\bigl(1+|m|\bigr)^{-\mu}\exp\bigl(-\beta|m|\bigr)\end{aligned} \tag{87}$$

whenever $\tau \in S_d$, $m \in \mathbb{R}$.

On the other hand, Proposition 2 guarantees that, for all $m \in \mathbb{R}$, the function $\tau \mapsto (\exp(-\kappa_l \mathcal{C}_k)w)(\tau,m)$ extends analytically on $D(0,\rho) \setminus L_-$, with

$$\bigl|\bigl(\exp(-\kappa_l \mathcal{C}_k)w\bigr)(\tau,m)\bigr| \leq \exp\biggl(\frac{\kappa_l k\rho}{\Gamma(\frac{1}{k}+1)}\biggr)\varpi \exp\bigl(\nu\rho^k\bigr)|\tau|^k\bigl(1+|m|\bigr)^{-\mu}e^{-\beta|m|} \tag{88}$$

for all $\tau \in D(0,\rho) \setminus L_-$, all $m \in \mathbb{R}$. As a consequence, Proposition 1 specialized for $\gamma_2 = \frac{n+d_{l,k}}{k} - 1$, $\gamma_3 = \delta_l - 1$ with $n \in I_l$ gives rise to a constant $C_4' > 0$ (depending on I_l, k, κ_l, d_l, δ_l, ν, ρ) for which

$$\biggl|\tau^k \int_0^{\tau^k} \bigl(\tau^k - s\bigr)^{\frac{n+d_{l,k}}{k}-1} s^{\delta_l}\bigl(\exp(-\kappa_l \mathcal{C}_k)w\bigr)\bigl(s^{1/k},m\bigr)\frac{ds}{s}\biggr| \leq \varpi C_4'|\tau|^k\bigl(1+|m|\bigr)^{-\mu}e^{-\beta|m|}$$

provided that $\tau \in D(0,\rho) \setminus L_{-}$, $m \in \mathbb{R}$. Regarding (80) we notice that

$$\begin{aligned} &\left| \frac{R_l(im)\tau^k}{H(\tau,m)} \int_0^{\tau^k} (\tau^k - s)^{\frac{n+d_{l,k}}{k}-1} s^{\delta_l} \big(\exp(-\kappa_l \mathcal{C}_k) w\big)\big(s^{1/k},m\big) \frac{ds}{s} \right| \\ &\quad \leq 2\varpi C_4' \left| \frac{R_l(im)}{Q(im)} \right| |\tau|^k \big(1+|m|\big)^{-\mu} e^{-\beta|m|} \\ &\quad \leq 2\varpi C_4' \sup_{m\in\mathbb{R}} \left| \frac{R_l(im)}{Q(im)} \right| \big(1+\rho^{2k}\big) \frac{|\tau|^k}{1+|\tau|^{2k}} \exp\big(\nu|\tau|^k\big) \\ &\qquad \times \big(1+|m|\big)^{-\mu} \exp\big(-\beta|m|\big) \end{aligned} \tag{89}$$

for all $\tau \in D(0,\rho) \setminus L_{-}$, all $m \in \mathbb{R}$.

By clustering (87) and (89), we conclude there exists a constant $C_5 > 0$ (depending on I_l, k, κ_l, d_l, δ_l, ν, ρ, S_{Q,R_D}, S_d, R_l, Q) with

$$\left\| \frac{R_l(im)\tau^k}{H(\tau,m)} \int_0^{\tau^k} (\tau^k - s)^{\frac{n+d_{l,k}}{k}-1} s^{\delta_l} \big(\exp(-\kappa_l \mathcal{C}_k) w\big)\big(s^{1/k},m\big) \frac{ds}{s} \right\|_{(\nu,\beta,\mu,k,\rho)} \leq C_5 \varpi. \tag{90}$$

In view of bounds (86), the application of Proposition 1 for

$$\gamma_2 = \frac{n+d_{l,k}}{k} + \delta_l - p - 1, \qquad \gamma_3 = p-1,$$

where $n \in I_l$ with $1 \leq p \leq \delta_l - 1$, yields the existence of a constant $C_6 > 0$ (depending on I_l, k, κ_l, d_l, δ_l, ν) with

$$\begin{aligned} &\left| \tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{n+d_{l,k}}{k}+\delta_l-p-1} s^{p} \big(\exp(-\kappa_l \mathcal{C}_k) w\big)\big(s^{1/k},m\big) \frac{ds}{s} \right| \\ &\quad \leq \varpi C_6 |\tau|^{2k+k(p-1)} \exp\left(\kappa_l K_1 \frac{k}{\Gamma(\frac{1}{k}-1)} |\tau|^k \right) \exp\big(\nu|\tau|^k\big)\big(1+|m|\big)^{-\mu} e^{-\beta|m|} \end{aligned}$$

for all $\tau \in S_d$, $m \in \mathbb{R}$, and $1 \leq p \leq \delta_l - 1$.

Owing to the lower bounds (78) under restriction (81), we deduce that

$$\begin{aligned} &\left| \frac{R_l(im)\tau^k}{H(\tau,m)} \int_0^{\tau^k} (\tau^k - s)^{\frac{n+d_{l,k}}{k}+\delta_l-p-1} s^{p} \big(\exp(-\kappa_l \mathcal{C}_k) w\big)\big(s^{1/k},m\big) \frac{ds}{s} \right| \\ &\quad \leq \varpi \frac{C_6}{A_{H,d}} \left| \frac{R_l(im)}{Q(im)} \right| |\tau|^{k(p+1)} \exp\left(\left(\kappa_l K_1 \frac{k}{\Gamma(\frac{1}{k}-1)} - B_{H,d}\alpha_D \right) |\tau|^k \right) \\ &\qquad \times \exp\big(\nu|\tau|^k\big)\big(1+|m|\big)^{-\mu} e^{-\beta|m|} \\ &\quad \leq \varpi \frac{C_6}{A_{H,d}} \sup_{m\in\mathbb{R}} \left| \frac{R_l(im)}{Q(im)} \right| \left(\sup_{|\tau|\geq 0} |\tau|^{kp}\big(1+|\tau|^{2k}\big) \right. \\ &\qquad \left. \times \exp\left(\left(\kappa_l K_1 \frac{k}{\Gamma(\frac{1}{k}-1)} - B_{H,d}\alpha_D \right) |\tau|^k \right) \right) \\ &\qquad \times \frac{|\tau|^k}{1+|\tau|^{2k}} \exp\big(\nu|\tau|^k\big)\big(1+|m|\big)^{-\mu} \exp\big(-\beta|m|\big) \end{aligned} \tag{91}$$

whenever $\tau \in S_d$, $m \in \mathbb{R}$ with $1 \leq p \leq \delta_l - 1$.

Using bounds (88) and from Proposition 1 applied to

$$\gamma_2 = \frac{n + d_{l,k}}{k} + \delta_l - p - 1, \qquad \gamma_3 = p - 1,$$

where $n \in I_l$ with $1 \le p \le \delta_l - 1$, we obtain a constant $C_6' > 0$ (depending on I_l, k, κ_l, d_l, δ_l, ν, ρ) for which

$$\left| \tau^k \int_0^{\tau^k} (\tau^k - s)^{\frac{n+d_{l,k}}{k} + \delta_l - p - 1} s^p \bigl(\exp(-\kappa_l \mathcal{C}_k) w\bigr)(s^{1/k}, m) \frac{ds}{s} \right| \le \varpi C_6' |\tau|^k (1 + |m|)^{-\mu} e^{-\beta|m|}$$

for $\tau \in D(0,\rho) \setminus L_-$, $m \in \mathbb{R}$. With the help of the lower bounds (80), we deduce

$$\begin{aligned}
&\left| \frac{R_l(im)\tau^k}{H(\tau,m)} \int_0^{\tau^k} (\tau^k - s)^{\frac{n+d_{l,k}}{k} + \delta_l - p - 1} s^p \bigl(\exp(-\kappa_l \mathcal{C}_k) w\bigr)(s^{1/k}, m) \frac{ds}{s} \right| \\
&\quad \le 2\varpi C_6' \left| \frac{R_l(im)}{Q(im)} \right| |\tau|^k (1 + |m|)^{-\mu} e^{-\beta|m|} \\
&\quad \le 2\varpi C_6' \sup_{m \in \mathbb{R}} \left| \frac{R_l(im)}{Q(im)} \right| (1 + \rho^{2k}) \frac{|\tau|^k}{1 + |\tau|^{2k}} \exp(\nu|\tau|^k) \\
&\qquad \times (1 + |m|)^{-\mu} \exp(-\beta|m|)
\end{aligned} \tag{92}$$

provided that $\tau \in D(0,\rho) \setminus L_-$ and $m \in \mathbb{R}$.

By means of (91) and (92), we deduce the existence of a constant $C_7 > 0$ (depending on I_l, k, κ_l, d_l, δ_l, ν, ρ, S_{Q,R_D}, S_d, R_l, Q) with

$$\left\| \frac{R_l(im)\tau^k}{H(\tau,m)} \int_0^{\tau^k} (\tau^k - s)^{\frac{n+d_{l,k}}{k} + \delta_l - p - 1} s^p \bigl(\exp(-\kappa_l \mathcal{C}_k) w\bigr)(s^{1/k}, m) \frac{ds}{s} \right\|_{(\nu,\beta,\mu,k,\rho)} \le C_7 \varpi. \tag{93}$$

On the other hand, taking into account assumption (8) and the lower bounds (70) together with (80), the application of Proposition 3 induces a constant $C_3 > 0$ (depending on Q_1, Q_2, Q, μ, k, ν) and a constant $\eta_2 > 0$ (equals to $\eta_{2,l}$ from (70)) for which

$$\begin{aligned}
&\left\| \frac{\tau^k}{H(\tau,m)} \int_0^{\tau^k} \int_{-\infty}^{+\infty} Q_1\bigl(i(m - m_1)\bigr) w\bigl((\tau^k - s)^{1/k}, m - m_1\bigr) \right. \\
&\qquad \left. \times Q_2(im_1) w(s^{1/k}, m_1) \frac{1}{(\tau^k - s)s} \, ds \, dm_1 \right\|_{(\nu,\beta,\mu,k,\rho)} \\
&\quad \le \sup_{\tau \in S_d \cup (D(0,\rho) \setminus L_-), m \in \mathbb{R}} \left| \frac{Q(im)}{H(\tau,m)} \right| \\
&\qquad \times \left\| \frac{\tau^k}{Q(im)} \int_0^{\tau^k} \int_{-\infty}^{+\infty} Q_1\bigl(i(m - m_1)\bigr) w\bigl((\tau^k - s)^{1/k}, m - m_1\bigr) \right. \\
&\qquad \left. \times Q_2(im_1) w(s^{1/k}, m_1) \frac{1}{(\tau^k - s)s} \, ds \, dm_1 \right\|_{(\nu,\beta,\mu,k,\rho)} \\
&\quad \le \frac{C_3}{\min(\eta_2, 1/2)} \bigl\| w(\tau,m) \bigr\|^2_{(\nu,\beta,\mu,k,\rho)} \\
&\quad \le \frac{C_3 \varpi^2}{\min(\eta_2, 1/2)}.
\end{aligned} \tag{94}$$

Furthermore, owing to Lemma 1 and in view of the lower estimates (70), (80), we obtain a constant $K_f > 0$ (depending on k, ν and K_0, T_0 from (9)) and $\eta_2 > 0$ such that

$$\left\| \frac{\psi(\tau,m,\epsilon)}{H(\tau,m)} \right\|_{(\nu,\beta,\mu,k,\rho)} \leq \frac{K_f}{\min(\eta_2,1/2)\min_{m\in\mathbb{R}}|Q(im)|} \tag{95}$$

for all $\epsilon \in D(0,\epsilon_0)$.

Now, we select $|c_{12}|$, $|c_f|$ with $|c_l|$, $1 \leq l \leq D-1$ such that the existence of a constant $\varpi > 0$ is guaranteed such that

$$\begin{aligned} &\sum_{l=1}^{D-1} \epsilon_0^{\Delta_l - d_l + \delta_l} |c_l| \sum_{n\in I_l} \sup_{\epsilon\in D(0,\epsilon_0)} \bigl|A_{l,n}(\epsilon)\bigr| \left(\frac{C_5 \varpi k^{\delta_l}}{\Gamma(\frac{n+d_{l,k}}{k})} + \sum_{1\leq p\leq \delta_l -1} |A_{\delta_l,p}| \frac{C_7 \varpi k^p}{\Gamma(\frac{n+d_{l,k}}{k} + \delta_l - p)} \right) \\ &\quad + |c_{12}| \frac{C_3 \varpi^2}{(2\pi)^{1/2}\min(\eta_2,1/2)} + |c_f| \frac{K_f}{\min(\eta_2,1/2)\min_{m\in\mathbb{R}}|Q(im)|} \leq \varpi. \end{aligned} \tag{96}$$

Finally, by collecting the norm estimates (90), (93) together with (94) and (95) under restriction (96), one gets (83).

In the next part of the proof, we focus on statement (84). Namely, let $w_1(\tau,m), w_2(\tau,m)$ belonging to $\bar{B}(0,\varpi) \subseteq F^d_{(\nu,\beta,\mu,k,\rho)}$. We get in particular that the next bound

$$\bigl|w_2(\tau,m) - w_1(\tau,m)\bigr| \leq \bigl\|w_2(\tau,m) - w_1(\tau,m)\bigr\|_{(\nu,\beta,\mu,k,\rho)} |\tau|^k \exp\bigl(\nu|\tau|^k\bigr) e^{-\beta|m|}\bigl(1+|m|\bigr)^{-\mu}$$

holds for all $\tau \in S_d \cup (D(0,\rho) \setminus L_-)$. Following analogous steps as for the sequence of inequalities (85), (86), (87), (88), (89), and (90), we observe that

$$\begin{aligned} &\left\| \frac{R_l(im)\tau^k}{H(\tau,m)} \int_0^{\tau^k} \bigl(\tau^k - s\bigr)^{\frac{n+d_{l,k}}{k}-1} s^{\delta_l} \bigl(\exp(-\kappa_l \mathcal{C})(w_2 - w_1)\bigr)\bigl(s^{1/k},m\bigr) \frac{ds}{s} \right\|_{(\nu,\beta,\mu,k,\rho)} \\ &\quad \leq C_5 \bigl\|w_2(\tau,m) - w_1(\tau,m)\bigr\|_{(\nu,\beta,\mu,k,\rho)} \end{aligned} \tag{97}$$

for the constant $C_5 > 0$ appearing in (90).

Similarly, tracking the progression (85), (86), (88), (91), (92), and (93) yields the next upper bound

$$\begin{aligned} &\left\| \frac{R_l(im)\tau^k}{H(\tau,m)} \int_0^{\tau^k} \bigl(\tau^k - s\bigr)^{\frac{n+d_{l,k}}{k}+\delta_l - p - 1} s^{p} \bigl(\exp(-\kappa_l \mathcal{C}_k)(w_2 - w_1)\bigr)\bigl(s^{1/k},m\bigr) \frac{ds}{s} \right\|_{(\nu,\beta,\mu,k,\rho)} \\ &\quad \leq C_7 \bigl\|w_2(\tau,m) - w_1(\tau,m)\bigr\|_{(\nu,\beta,\mu,k,\rho)} \end{aligned} \tag{98}$$

for all $1 \leq p \leq \delta_l - 1$, where the constant $C_7 > 0$ is given in (93).

In order to handle the nonlinear term, we write

$$\begin{aligned} &Q_1\bigl(i(m-m_1)\bigr) w_2\bigl(\bigl(\tau^k - s\bigr)^{1/k}, m - m_1\bigr) Q_2(im_1) w_2\bigl(s^{1/k}, m_1\bigr) \\ &\quad - Q_1\bigl(i(m-m_1)\bigr) w_1\bigl(\bigl(\tau^k - s\bigr)^{1/k}, m - m_1\bigr) Q_2(im_1) w_1\bigl(s^{1/k}, m_1\bigr) \\ &\quad = Q_1\bigl(i(m-m_1)\bigr) \bigl(w_2\bigl(\bigl(\tau^k - s\bigr)^{1/k}, m - m_1\bigr) \\ &\quad - w_1\bigl(\bigl(\tau^k - s\bigr)^{1/k}, m - m_1\bigr)\bigr) Q_2(im_1) w_2\bigl(s^{1/k}, m_1\bigr) \end{aligned}$$

$$+ Q_1\big(i(m-m_1)\big)w_1\big(\big(\tau^k-s\big)^{1/k},m-m_1\big)Q_2(im_1)\big(w_2\big(s^{1/k},m_1\big)-w_1\big(s^{1/k},m_1\big)\big)$$

for all $\tau \in S_d \cup (D(0,\rho)\setminus L_-)$, all $m, m_1 \in \mathbb{R}$.

Then, in view of assumption (8) and the lower bounds (70), (80), Proposition 3 gives rise to constants $C_3 > 0$ and $\eta_2 > 0$ for which

$$\begin{aligned}
&\bigg\| \frac{\tau^k}{H(\tau,m)} \int_0^{\tau^k} \int_{-\infty}^{+\infty} \big\{ Q_1\big(i(m-m_1)\big)w_2\big(\big(\tau^k-s\big)^{1/k},m-m_1\big)Q_2(im)w_2\big(s^{1/k},m_1\big) \\
&\quad - Q_1\big(i(m-m_1)\big)w_1\big(\big(\tau^k-s\big)^{1/k},m-m_1\big)Q_2(im_1)w_1\big(s^{1/k},m_1\big)\big\} \\
&\quad \times \frac{1}{(\tau^k-s)s}\,ds\,dm_1 \bigg\|_{(\nu,\beta,\mu,k,\rho)} \\
&\quad \le \frac{C_3}{\min(\eta_2,1/2)}\big(\big\|w_2(\tau,m)\big\|_{(\nu,\beta,\mu,k,\rho)} \\
&\quad + \big\|w_1(\tau,m)\big\|_{(\nu,\beta,\mu,k,\rho)}\big)\big\|w_2(\tau,m)-w_1(\tau,m)\big\|_{(\nu,\beta,\mu,k,\rho)} \\
&\quad \le \frac{2\varpi C_3}{\min(\eta_2,1/2)}\big\|w_2(\tau,m)-w_1(\tau,m)\big\|_{(\nu,\beta,\mu,k,\rho)}.
\end{aligned} \tag{99}$$

Now, we consider that $|c_{12}|$ and $|c_l|$, $1 \le l \le D-1$, are such that

$$\begin{aligned}
&\sum_{l=1}^{D-1} \epsilon_0^{\Delta_l-d_l+\delta_l}|c_l| \sum_{n\in I_l} \sup_{\epsilon\in D(0,\epsilon_0)} \big|A_{l,n}(\epsilon)\big| \bigg(\frac{C_5 k^{\delta_l}}{\Gamma(\frac{n+d_{l,k}}{k})} + \sum_{1\le p\le \delta_l-1} |A_{\delta_l,p}| \frac{C_7 k^p}{\Gamma(\frac{n+d_{l,k}}{k}+\delta_l-p)} \bigg) \\
&\quad + |c_{12}| \frac{2\varpi C_3}{(2\pi)^{1/2}\min(\eta_2,1/2)} \le \frac{1}{2}.
\end{aligned} \tag{100}$$

Submitting estimates (97), (98) with (99) to constraints (100), one achieves (84).

Finally, we select $|c_{12}|$, $|c_f|$, and $|c_l|$, $1 \le l \le D-1$, small enough in a way that (96) and (100) are simultaneously fulfilled. □

We turn back again to the proof of Proposition 4. For $\varpi > 0$ chosen as in Lemma 3, we set the closed ball $\bar{B}(0,\varpi) \subset F^d_{(\nu,\beta,\mu,k,\rho)}$ which turns out to be a complete metric space for the distance $d(x,y) = \|x-y\|_{(\nu,\beta,\mu,k,\rho)}$. Owing to the previous lemma, we observe that $\mathcal{H}_\epsilon$ induces a contractive application from $(\bar{B}(0,\varpi),d)$ into itself. Then, according to the classical contractive mapping theorem, the map $\mathcal{H}_\epsilon$ possesses a unique fixed point that we denote by $w^d(\tau,m,\epsilon)$, i.e.,

$$\mathcal{H}_\epsilon\big(w^d(\tau,m,\epsilon)\big) = w^d(\tau,m,\epsilon) \tag{101}$$

that belongs to the ball $\bar{B}(0,\varpi)$ for all $\epsilon \in D(0,\epsilon_0)$. Besides, the function $w^d(\tau,m,\epsilon)$ depends holomorphically on ϵ in $D(0,\epsilon_0)$. Direct transformations on (31) turn such expression into (101). As a result, the unique fixed point $w^d(\tau,m,\epsilon)$ of $\mathcal{H}_\epsilon$ obtained solves equation (31). □

6 Analytic solutions on sectors of the main initial value problem

We turn back to the formal constructions obtained in Sect. 3 by taking into consideration the solution of the related problem (31) built up in Sect. 5 within the Banach spaces described in Definition 4.

We first recall the definition of a good covering in $\mathbb{C}^*$, and we disclose a modified version of the so-called associated sets of sectors as proposed in our previous work [15].

Definition 5 Let $\varsigma \geq 2$ be an integer. For all $0 \leq p \leq \varsigma - 1$, we fix an open sector $\mathcal{E}_p$, centered at 0, with radius $\epsilon_0 > 0$ such that $\mathcal{E}_p \cap \mathcal{E}_{p+1} \neq \emptyset$ for all $0 \leq p \leq \varsigma - 1$ (with the convention that $\mathcal{E}_\varsigma = \mathcal{E}_0$). Furthermore, we take for granted that the intersection of any three different elements of $\{\mathcal{E}_p\}_{0\leq p\leq\varsigma-1}$ is empty and that $\bigcup_{p=0}^{\varsigma-1} \mathcal{E}_p = \mathcal{U} \setminus \{0\}$, where $\mathcal{U}$ stands for some neighborhood of 0 in $\mathbb{C}$. A family of sectors $\{\mathcal{E}_p\}_{0\leq p\leq\varsigma-1}$ with the above properties is called a good covering in $\mathbb{C}^*$.

Definition 6 We consider a good covering $\underline{\mathcal{E}} = \{\mathcal{E}_p\}_{0\leq p\leq\varsigma-1}$ in $\mathbb{C}^*$. We fix a real number $\rho > 0$ and an open sector $\mathcal{T}$ centered at 0 with bisecting direction $d = 0$ and radius $r_{\mathcal{T}} > 0$, and we set a family of open sectors

$$S_{\mathfrak{d}_p,\theta,\epsilon_0 r_{\mathcal{T}}} = \left\{T \in \mathbb{C}^*/|T| < \epsilon_0 r_{\mathcal{T}}, \left|\mathfrak{d}_p - \arg(T)\right| < \theta/2\right\}$$

with aperture $\theta > \pi/k$ and $\mathfrak{d}_p \in [-\pi,\pi)$, $0 \leq p \leq \varsigma - 1$. We say that the set $\{\{S_{\mathfrak{d}_p,\theta,\epsilon_0 r_{\mathcal{T}}}\}_{0\leq p\leq\varsigma-1}, \mathcal{T}, \rho\}$ is associated to $\underline{\mathcal{E}}$ if the next two constraints hold:

(1) There exists a set of unbounded sectors $S_{\mathfrak{d}_p}$, $0 \leq p \leq \varsigma - 1$ centered at 0 with suitably chosen bisecting direction $\mathfrak{d}_p \in (-\pi/2, \pi/2)$ and small aperture satisfying that

$$\tau_l \notin S_{\mathfrak{d}_p} \cup D(0,\rho)$$

for some $\rho > 0$ and all $l \in \mathbb{Z}$ where τ_l stand for the complex numbers defined in (63).

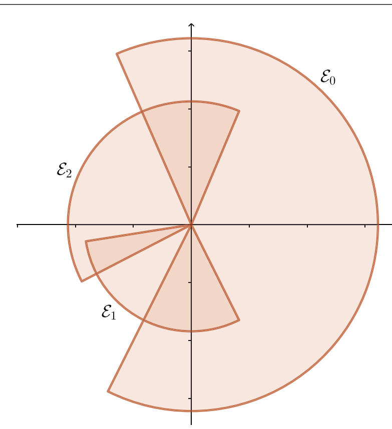
(2) For all $\epsilon \in \mathcal{E}_p$, all $t \in \mathcal{T}$,

$$\epsilon t \in S_{\mathfrak{d}_p,\theta,\epsilon_0 r_{\mathcal{T}}} \tag{102}$$

for all $0 \leq p \leq \varsigma - 1$.

Figure 3 shows a configuration of a good covering of three sectors, one of them of opening larger than π/k for some k close to 1. We illustrate in Fig. 4 a configuration of associated sectors.

Figure 3 Good covering in $\mathbb{C}^*$



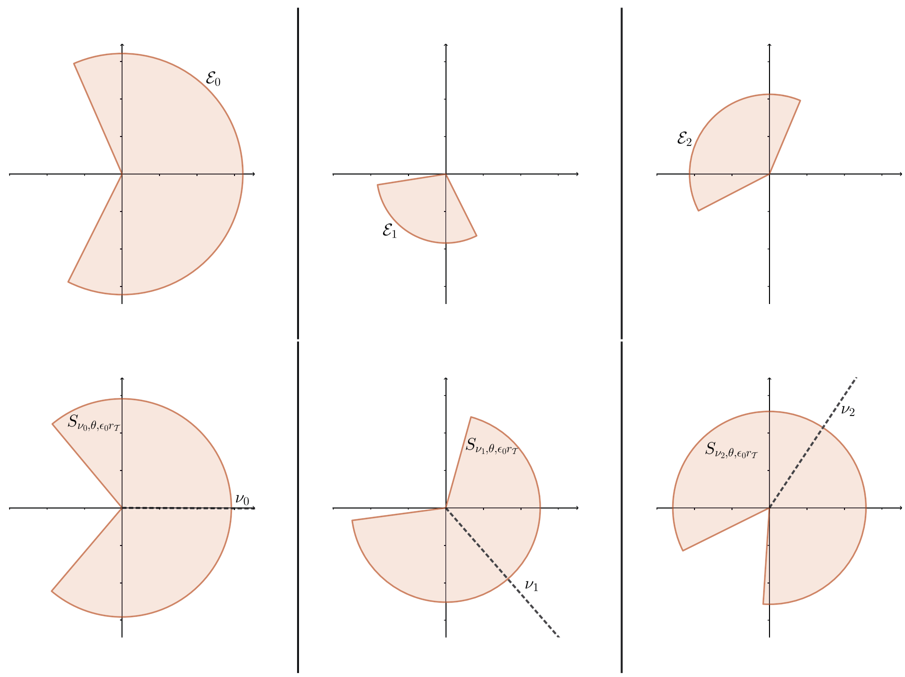


Figure 4 A configuration associated to the good covering in Fig. 3

In the following first principal result of the work, we build up a set of actual holomorphic solutions to the main initial value problem (11) defined on the sectors $\mathcal{E}_p$ w.r.t. ϵ. We also provide an upper control for the difference between any two neighboring solutions on $\mathcal{E}_p \cap \mathcal{E}_{p+1}$ that turn out to be at most exponentially flat of order k.

Theorem 1 *Let us assume that constraints* (6), (7), (8), (9), *and* (59) *hold. We consider a good covering* $\underline{\mathcal{E}} = \{\mathcal{E}_p\}_{0\leq p\leq \varsigma-1}$ *for which a set of data* $\{\{S_{\mathfrak{d}_p,\theta,\epsilon_0 r_{\mathcal{T}}}\}_{0\leq p\leq\varsigma-1}, \mathcal{T}, \rho\}$ *associated to* $\underline{\mathcal{E}}$ *can be singled out. We take for granted that the constants* α_D *and* κ_l, $1 \leq l \leq D-1$, *appearing in problem* (11) *are submitted to the inequality*

$$B_{H,\mathfrak{d}_p}\alpha_D > \kappa_l K_1 \frac{k}{\Gamma(\frac{1}{k}-1)} \tag{103}$$

for all $0 \leq p \leq \varsigma - 1$, *where* $B_{H,\mathfrak{d}_p}$ *is framed in construction* (78) *and depends on* k, S_{Q,R_D}, $S_{\mathfrak{d}_p}$ *and* $K_1 > 0$ *is a constant relying on* k, ν *defined in Proposition* 2(1).

Then, whenever the moduli $|c_{12}|$, $|c_f|$, *and* $|c_l|$, $1 \leq l \leq D-1$, *are taken sufficiently small, a family* $\{u_p(t,z,\epsilon)\}_{0\leq p\leq\varsigma-1}$ *of genuine solutions of* (11) *can be established. More precisely, each function* $u_p(t,z,\epsilon)$ *defines a bounded holomorphic function on the product* $(\mathcal{T} \cap D(0,\sigma)) \times H_{\beta'} \times \mathcal{E}_p$ *for any given* $0 < \beta' < \beta$ *and suitably tiny* $\sigma > 0$ (*where* β *comes out in* (9)) *and can be expressed as a Laplace transform of order* k *and Fourier inverse transform*

$$u_p(t,z,\epsilon) = \frac{k}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\gamma_p}} w^{\mathfrak{d}_p}(u,m,\epsilon) \exp\left(-\left(\frac{u}{\epsilon t}\right)^k\right) e^{izm} \frac{du}{u}\, dm \tag{104}$$

along a halfline $L_{\gamma_p} = \mathbb{R}_+ e^{\sqrt{-1}\gamma_p} \subset S_{\mathfrak{d}_p} \cup \{0\}$ and where $w^{\mathfrak{d}_p}(\tau,m,\epsilon)$ stands for a function that belongs to the Banach space $F^{\mathfrak{d}_p}_{(\nu,\beta,\mu,k,\rho)}$ for all $\epsilon \in D(0,\epsilon_0)$. Furthermore, one can choose constants $K_p, M_p > 0$ and $0 < \sigma' < \sigma$ (independent of ϵ) with

$$\sup_{t\in\mathcal{T}\cap D(0,\sigma'),z\in H_{\beta'}} \bigl|u_{p+1}(t,z,\epsilon) - u_p(t,z,\epsilon)\bigr| \le K_p \exp\biggl(-\frac{M_p}{|\epsilon|^k}\biggr) \tag{105}$$

for all $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$, all $0 \le p \le \varsigma - 1$ (with $u_\varsigma = u_0$).

Proof We select a good covering $\{\mathcal{E}_p\}_{0\le p\le\varsigma-1}$ in $\mathbb{C}^*$ with associated set $\{\{S_{\mathfrak{d}_p,\theta,\epsilon_0 r_{\mathcal{T}}}\}_{0\le p\le\varsigma-1}, \mathcal{T}, \rho\}$. According to Proposition 4, under the assumptions stated in Theorem 1, for suitable $|c_{12}|$, $|c_f|$, and $|c_l|$, $1 \le l \le D-1$, we observe that for each direction $\mathfrak{d}_p$, one can build a solution $w^{\mathfrak{d}_p}(\tau,m,\epsilon)$ satisfying (31) within the space $F^{\mathfrak{d}_p}_{(\nu,\beta,\mu,k,\rho)}$

$$\bigl|w^{\mathfrak{d}_p}(\tau,m,\epsilon)\bigr| \le \varpi_{\mathfrak{d}_p}\bigl(1+|m|\bigr)^{-\mu} e^{-\beta|m|} \frac{|\tau|^k}{1+|\tau|^{2k}} \exp\bigl(\nu|\tau|^k\bigr) \tag{106}$$

for all $\tau \in S_{\mathfrak{d}_p} \cup (D(0,\rho) \setminus L_-)$, all $m \in \mathbb{R}$, all $\epsilon \in D(0,\epsilon_0)$, for an adequate $\varpi_{\mathfrak{d}_p} > 0$. In particular, $w^{\mathfrak{d}_p}(\tau,m,\epsilon)$ conform the analytic continuation w.r.t. τ of a common holomorphic function that we call $\tau \mapsto w(\tau,m,\epsilon)$ whenever $\tau \in D(0,\rho) \setminus L_-$ which satisfies likewise the bounds above (106) provided that $m \in \mathbb{R}$, $\epsilon \in D(0,\epsilon_0)$.

As a consequence, the Laplace transform of order k and the Fourier inverse transform

$$U_{\gamma_p}(T,z,\epsilon) = \frac{k}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\gamma_p}} w^{\mathfrak{d}_p}(u,m,\epsilon) \exp\biggl(-\biggl(\frac{u}{T}\biggr)^k\biggr) e^{izm} \frac{du}{u}\, dm$$

along a halfline $L_{\gamma_p} \subset S_{\mathfrak{d}_p} \cup \{0\}$ represent

(1) A holomorphic bounded function w.r.t. T on a sector $S_{\mathfrak{d}_p,\theta,\varrho}$ with bisecting direction $\mathfrak{d}_p$, aperture $\frac{\pi}{k} < \theta < \frac{\pi}{k} + \mathrm{Ap}(S_{\mathfrak{d}_p})$, radius ϱ, where $\mathrm{Ap}(S_{\mathfrak{d}_p})$ stands for the aperture of $S_{\mathfrak{d}_p}$, for some real number $\varrho > 0$.
(2) A holomorphic bounded application w.r.t. z on $H_{\beta'}$ for any given $0 < \beta' < \beta$.
(3) A holomorphic bounded map w.r.t. ϵ on $D(0,\epsilon_0)$.

Furthermore, the integral representation (29) and bounds (48), (49) show that $U_{\gamma_p}(\frac{T}{1+\kappa_l T}, z,\epsilon)$ defines a holomorphic bounded function w.r.t. T on a sector $S_{\mathfrak{d}_p,\theta,\varrho_1}$ for some $0 < \varrho_1 < \varrho$ and the same θ as above in (1). Besides, a direct computation yields that

$$\begin{aligned} &\exp\bigl(\alpha_D T^{k+1}\partial_T\bigr) U_{\gamma_p}(T,z,\epsilon) \\ &\quad= \frac{k}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\gamma_p}} \exp\bigl(\alpha_D k u^k\bigr) w^{\mathfrak{d}_p}(u,m,\epsilon) \exp\biggl(-\biggl(\frac{u}{T}\biggr)^k\biggr) e^{izm} \frac{du}{u}\, dm \end{aligned} \tag{107}$$

defines a holomorphic bounded function w.r.t. T on a sector $S_{\mathfrak{d}_p,\theta,\varrho_2}$ for some $0 < \varrho_2 < \varrho$.

Finally, by combining these integral representations (29), (107) together with (22), (23), and (24) displayed in Lemma 2, from the fact that $w^{\mathfrak{d}_p}$ solves (31), we obtain that $U_{\gamma_p}(T,z,\epsilon)$ solves equation (16) and hence (13), whenever T belongs to some sector $S_{\mathfrak{d}_p,\theta,\varrho_3}$ where $0 < \varrho_3 < \varrho$, if z lies within $H_{\beta'}$ and $\epsilon \in D(0,\epsilon_0)$.

As a result, the function

$$u_p(t,z,\epsilon) = U_{\gamma_p}(\epsilon t,z,\epsilon)$$

defines a bounded holomorphic function w.r.t. t on $\mathcal{T} \cap D(0,\sigma)$ for some $\sigma > 0$, $\epsilon \in \mathcal{E}_p$, $z \in H_{\beta'}$ for any given $0 < \beta' < \beta$, owing to the fact that the sectors $\mathcal{E}_p$ and $\mathcal{T}$ fulfill (102). Moreover, $u_p(t,z,\epsilon)$ solves the main initial value problem (11) on the domain $(\mathcal{T} \cap D(0,\sigma)) \times H_{\beta'} \times \mathcal{E}_p$ for all $0 \leq p \leq \varsigma - 1$.

In the final part of the proof, we are concerned with bounds (105). The steps of verification are comparable to the arguments displayed in Theorem 1 of [15], but we still decide to present the details for the sake of clarity.

By construction, the map $u \mapsto w(u,m,\epsilon)\exp(-(\frac{u}{\epsilon t})^k)/u$ represents a holomorphic function on $D(0,\rho) \setminus L_-$ for all $(m,\epsilon) \in \mathbb{R} \times D(0,\epsilon_0)$. Therefore, its integral along the union of a segment joining 0 to $(\rho/2)e^{\sqrt{-1}\gamma_{p+1}}$ followed by an arc of circle with radius $\rho/2$ which relies on $(\rho/2)e^{\sqrt{-1}\gamma_{p+1}}$ and $(\rho/2)e^{\sqrt{-1}\gamma_p}$ and ending with a segment starting from $(\rho/2)e^{\sqrt{-1}\gamma_p}$ to 0 vanishes. The Cauchy formula allows us to write the difference $u_{p+1} - u_p$ as a sum of three integrals:

$$\begin{aligned} u_{p+1}(t,z,\epsilon) - u_p(t,z,\epsilon) &= \frac{k}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\rho/2,\gamma_{p+1}}} w^{\mathfrak{d}_{p+1}}(u,m,\epsilon) e^{-(\frac{u}{\epsilon t})^k} e^{izm} \frac{du}{u}\, dm \\ &\quad - \frac{k}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\rho/2,\gamma_p}} w^{\mathfrak{d}_p}(u,m,\epsilon) e^{-(\frac{u}{\epsilon t})^k} e^{izm} \frac{du}{u}\, dm \\ &\quad + \frac{k}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{C_{\rho/2,\gamma_p,\gamma_{p+1}}} w(u,m,\epsilon) e^{-(\frac{u}{\epsilon t})^k} e^{izm} \frac{du}{u}\, dm, \end{aligned} \tag{108}$$

where $L_{\rho/2,\gamma_{p+1}} = [\rho/2,+\infty)e^{\sqrt{-1}\gamma_{p+1}}$, $L_{\rho/2,\gamma_p} = [\rho/2,+\infty)e^{\sqrt{-1}\gamma_p}$, and $C_{\rho/2,\gamma_p,\gamma_{p+1}}$ stands for an arc of circle with radius connecting $(\rho/2)e^{\sqrt{-1}\gamma_p}$ and $(\rho/2)e^{\sqrt{-1}\gamma_{p+1}}$ with a well-chosen orientation.

We first provide bounds for the first term in decomposition (108), namely

$$I_1 = \left| \frac{k}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\rho/2,\gamma_{p+1}}} w^{\mathfrak{d}_{p+1}}(u,m,\epsilon) e^{-(\frac{u}{\epsilon t})^k} e^{izm} \frac{du}{u}\, dm \right|.$$

By construction, γ_{p+1} (which may depend on ϵt) is chosen in such a way that $\cos(k(\gamma_{p+1} - \arg(\epsilon t))) \geq \delta_1$ for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, all $t \in \mathcal{T} \cap D(0,\sigma)$, for some fixed $\delta_1 > 0$. From estimates (106), we get that

$$\begin{aligned} I_1 &\leq \frac{k}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{\rho/2}^{+\infty} \varpi_{\mathfrak{d}_{p+1}} \bigl(1+|m|\bigr)^{-\mu} e^{-\beta|m|} \frac{r^k}{1+r^{2k}} \\ &\quad \times \exp\bigl(\nu r^k\bigr) \exp\biggl(-\frac{\cos(k(\gamma_{p+1} - \arg(\epsilon t)))}{|\epsilon t|^k} r^k\biggr) e^{-m\operatorname{Im}(z)} \frac{dr}{r}\, dm \\ &\leq \frac{k\varpi_{\mathfrak{d}_{p+1}}}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|}\, dm \int_{\rho/2}^{+\infty} r^{k-1} \exp\biggl(-\biggl(\frac{\delta_1}{|t|^k} - \nu|\epsilon|^k\biggr)\biggl(\frac{r}{|\epsilon|}\biggr)^k\biggr) dr \\ &\leq \frac{2k\varpi_{\mathfrak{d}_{p+1}}}{(2\pi)^{1/2}} \int_0^{+\infty} e^{-(\beta-\beta')m}\, dm \int_{\rho/2}^{+\infty} \frac{|\epsilon|^k}{(\frac{\delta_1}{|t|^k} - \nu|\epsilon|^k)k} \\ &\quad \times \biggl\{ \frac{(\frac{\delta_1}{|t|^k} - \nu|\epsilon|^k)}{|\epsilon|^k} k r^{k-1} \exp\biggl(-\biggl(\frac{\delta_1}{|t|^k} - \nu|\epsilon|^k\biggr)\biggl(\frac{r}{|\epsilon|}\biggr)^k\biggr)\biggr\} dr \\ &\leq \frac{2k\varpi_{\mathfrak{d}_{p+1}}}{(2\pi)^{1/2}} \frac{|\epsilon|^k}{(\beta-\beta')(\frac{\delta_1}{|t|^k} - \nu|\epsilon|^k)k} \end{aligned}$$

$$\times \exp\left(-\left(\frac{\delta_1}{|t|^k} - \nu|\epsilon|^k\right)\left(\frac{\rho/2}{|\epsilon|}\right)^k\right)$$
$$\leq \frac{2k\varpi_{\mathfrak{d}_{p+1}}}{(2\pi)^{1/2}} \frac{|\epsilon|^k}{(\beta-\beta')\delta_2 k} \exp\left(-\delta_2\left(\frac{\rho/2}{|\epsilon|}\right)^k\right) \tag{109}$$

for all $t \in \mathcal{T} \cap D(0,\sigma)$ and $z \in H_{\beta'}$ with $|t| < (\frac{\delta_1}{\delta_2+\nu\epsilon_0^k})^{1/k}$, for some $\delta_2 > 0$, whenever $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$.

The estimates for the second term in the sum of (108) are obtained in the same manner:

$$I_2 = \left|\frac{k}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{L_{\rho/2,\gamma_p}} w^{\mathfrak{d}_p}(u,m,\epsilon) e^{-(\frac{u}{\epsilon t})^k} e^{izm} \frac{du}{u}\, dm\right|.$$

As above, the direction γ_p (which relies on ϵt) is taken in a way that $\cos(k(\gamma_p - \arg(\epsilon t))) \geq \delta_1$ for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, all $t \in \mathcal{T} \cap D(0,\sigma)$, for some fixed $\delta_1 > 0$. Again, according to (106), we obtain

$$I_2 \leq \frac{2k\varpi_{\mathfrak{d}_p}}{(2\pi)^{1/2}} \frac{|\epsilon|^k}{(\beta-\beta')\delta_2 k} \exp\left(-\delta_2 \frac{(\rho/2)^k}{|\epsilon|^k}\right) \tag{110}$$

for all $t \in \mathcal{T} \cap D(0,\sigma)$ and $z \in H_{\beta'}$ with $|t| < (\frac{\delta_1}{\delta_2+\nu\epsilon_0^k})^{1/k}$, for some $\delta_2 > 0$, for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$.

Lastly, we deal with the remaining term in the sum (108), that is,

$$I_3 = \left|\frac{k}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_{C_{\rho/2,\gamma_p,\gamma_{p+1}}} w(u,m,\epsilon) e^{-(\frac{u}{\epsilon t})^k} e^{izm} \frac{du}{u}\, dm\right|.$$

By construction, the arc of circle $C_{\rho/2,\gamma_p,\gamma_{p+1}}$ is chosen in order that $\cos(k(\theta - \arg(\epsilon t))) \geq \delta_1$ for all $\theta \in [\gamma_p, \gamma_{p+1}]$ (if $\gamma_p < \gamma_{p+1}$), $\theta \in [\gamma_{p+1}, \gamma_p]$ (if $\gamma_{p+1} < \gamma_p$), for all $t \in \mathcal{T}$, all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, for some fixed $\delta_1 > 0$.

Owing to (106) we notice that

$$I_3 \leq \frac{k}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \left|\int_{\gamma_p}^{\gamma_{p+1}} \max_{0\leq p\leq\varsigma-1} \varpi_{\mathfrak{d}_p} \big(1+|m|\big)^{-\mu} e^{-\beta|m|} \frac{(\rho/2)^k}{1+(\rho/2)^{2k}} \exp\big(\nu(\rho/2)^k\big)\right.$$
$$\left.\times \exp\left(-\frac{\cos(k(\theta-\arg(\epsilon t)))}{|\epsilon t|^k}\left(\frac{\rho}{2}\right)^k\right) e^{-m\operatorname{Im}(z)}\, d\theta\right|\, dm$$
$$\leq \frac{k\max_{0\leq p\leq\varsigma-1}\varpi_{\mathfrak{d}_p}}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|}\, dm$$
$$\times |\gamma_p - \gamma_{p+1}|(\rho/2)^k \exp\left(-\left(\frac{\delta_1}{|t|^k} - \nu|\epsilon|^k\right)\left(\frac{\rho/2}{|\epsilon|}\right)^k\right)$$
$$\leq \frac{2k\max_{0\leq p\leq\varsigma-1}\varpi_{\mathfrak{d}_p}}{(2\pi)^{1/2}} \frac{|\gamma_p-\gamma_{p+1}|(\rho/2)^k}{\beta-\beta'} \exp\left(-\delta_2 \frac{(\rho/2)^k}{|\epsilon|^k}\right) \tag{111}$$

for all $t \in \mathcal{T} \cap D(0,\sigma)$ and $z \in H_{\beta'}$ whenever $|t| < (\frac{\delta_1}{\delta_2+\nu\epsilon_0^k})^{1/k}$, for some $\delta_2 > 0$, for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$.

By collecting (109), (110), and (111), we derive

$$\big|u_{p+1}(t,z,\epsilon) - u_p(t,z,\epsilon)\big|$$

$$\leq \left(\frac{2k(\varpi_{\mathfrak{d}_p} + \varpi_{\mathfrak{d}_{p+1}})}{(2\pi)^{1/2}} \frac{|\epsilon|^k}{(\beta - \beta')\delta_2 k} + \frac{2k \max_{0 \leq p \leq \varsigma-1} \varpi_{\mathfrak{d}_p}}{(2\pi)^{1/2}} \frac{|\gamma_p - \gamma_{p+1}|(\rho/2)^k}{\beta - \beta'} \right) \times \exp\left(-\delta_2 \frac{(\rho/2)^k}{|\epsilon|^k} \right)$$

for all $t \in \mathcal{T} \cap D(0,\sigma)$ and $z \in H_{\beta'}$ with $|t| < (\frac{\delta_1}{\delta_2 + \nu\epsilon_0^k})^{1/k}$, for some $\delta_2 > 0$, for all $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$. The result follows directly from here. □

7 Parametric Gevrey asymptotic expansions of order 1/*k* of the solutions

7.1 Gevrey asymptotic expansions of order 1/*k* and *k*-summable formal series

We first remind the reader the concept of k-summability of formal series with coefficients in a Banach space as defined in classical textbooks such as [2].

Definition 7 Let $(\mathbb{F}, \|\cdot\|_{\mathbb{F}})$ be a complex Banach space. Let $k \in (\frac{1}{2}, 1)$. A formal series

$$\hat{a}(\epsilon) = \sum_{j=0}^{\infty} a_j \epsilon^j \in \mathbb{F}[[\epsilon]]$$

with coefficients belonging to $(\mathbb{F}, \|\cdot\|_{\mathbb{F}})$ is said to be k-summable with respect to ϵ in the direction $d \in \mathbb{R}$ if

(i) A positive number $\rho \in \mathbb{R}_+$ exists in such a way that the formal series, called formal Borel transform of order k of $\hat{a}$,

$$B_k(\hat{a})(\tau) = \sum_{j=0}^{\infty} \frac{a_j \tau^j}{\Gamma(1 + \frac{j}{k})} \in \mathbb{F}[[\tau]]$$

converges absolutely for $|\tau| < \rho$.

(ii) One can find an aperture $2\delta > 0$ such that the series $B_k(\hat{a})(\tau)$ can be analytically continued with respect to τ on the unbounded sector $S_{d,\delta} = \{\tau \in \mathbb{C}^* : |d - \arg(\tau)| < \delta\}$. Moreover, there exist $C > 0$ and $K > 0$ with

$$\|B_k(\hat{a})(\tau)\|_{\mathbb{F}} \leq Ce^{K|\tau|^k}$$

for all $\tau \in S_{d,\delta}$.

If the conditions above are fulfilled, the vector-valued Laplace transform of order k of $B_k(\hat{a})(\tau)$ in the direction d is defined by

$$L_k^d\big(B_k(\hat{a})\big)(\epsilon) = \epsilon^{-k} \int_{L_\gamma} B_k(\hat{a})(u) e^{-(u/\epsilon)^k} k u^{k-1}\, du$$

along a half-line $L_\gamma = \mathbb{R}_+ e^{\sqrt{-1}\gamma} \subset S_{d,\delta} \cup \{0\}$, where γ depends on ϵ and is chosen in such a way that $\cos(k(\gamma - \arg(\epsilon))) \geq \delta_1 > 0$ for some fixed δ_1, for all ϵ in a sector

$$S_{d,\theta,R^{1/k}} = \{\epsilon \in \mathbb{C}^* : |\epsilon| < R^{1/k}, |d - \arg(\epsilon)| < \theta/2\},$$

where the angle θ and radius R satisfy that $\frac{\pi}{k} < \theta < \frac{\pi}{k} + 2\delta$ and $0 < R < \delta_1/K$.

Notice that this version of Laplace transform of order k slightly differs from the one introduced in Definition 1 which turns out to be more suitable for the problems under study in this work.

The function $L_k^d(B_k(\hat{a}))(\epsilon)$ is called the k-sum of the formal series $\hat{a}(\epsilon)$ in the direction d. It represents a bounded and holomorphic function on the sector $S_{d,\theta,R^{1/k}}$ and is the *unique* such function that admits the formal series $\hat{a}(\epsilon)$ as Gevrey asymptotic expansion of order $1/k$ with respect to ϵ on $S_{d,\theta,R^{1/k}}$. This means that, for all $\frac{\pi}{k} < \theta_1 < \theta$, there exist $C, M > 0$ such that

$$\left\| L_k^d\big(B_k(\hat{a})\big)(\epsilon) - \sum_{p=0}^{n-1} a_p \epsilon^p \right\|_{\mathbb{F}} \leq CM^n \Gamma\left(1 + \frac{n}{k}\right) |\epsilon|^n$$

for all $n \geq 1$, all $\epsilon \in S_{d,\theta_1,R^{1/k}}$.

In the sequel, we present a cohomological criterion for the existence of Gevrey asymptotics of order $1/k$ for suitable families of sectorial holomorphic functions and k-summability of formal series with coefficients in Banach spaces (see [3], p. 121 or [11], Lemma XI-2-6) which is known as a Ramis–Sibuya theorem in the literature. This result is an essential tool in the proof of our second main statement (Theorem 2).

Theorem (RS) *Let $(\mathbb{F}, \|\cdot\|_{\mathbb{F}})$ be a Banach space over $\mathbb{C}$ and $\{\mathcal{E}_p\}_{0 \leq p \leq \varsigma-1}$ be a good covering in $\mathbb{C}^*$. For all $0 \leq p \leq \varsigma - 1$, let G_p be a holomorphic function from $\mathcal{E}_p$ into the Banach space $(\mathbb{F}, \|\cdot\|_{\mathbb{F}})$, and let the cocycle $\Theta_p(\epsilon) = G_{p+1}(\epsilon) - G_p(\epsilon)$ be a holomorphic function from the sector $Z_p = \mathcal{E}_{p+1} \cap \mathcal{E}_p$ into $\mathbb{E}$ (with the convention that $\mathcal{E}_\varsigma = \mathcal{E}_0$ and $G_\varsigma = G_0$). We make the following assumptions*:

1. *The functions $G_p(\epsilon)$ are bounded as $\epsilon \in \mathcal{E}_p$ tends to the origin in $\mathbb{C}$ for all $0 \leq p \leq \varsigma - 1$.*
2. *The functions $\Theta_p(\epsilon)$ are exponentially flat of order k on Z_p for all $0 \leq p \leq \varsigma - 1$. This means that there exist constants $C_p, A_p > 0$ such that*

$$\left\| \Theta_p(\epsilon) \right\|_{\mathbb{F}} \leq C_p e^{-A_p/|\epsilon|^k}$$

for all $\epsilon \in Z_p$, all $0 \leq p \leq \varsigma - 1$.

Then, for all $0 \leq p \leq \varsigma - 1$, the functions $G_p(\epsilon)$ admit a common formal power series $\hat{G}(\epsilon) \in \mathbb{F}[[\epsilon]]$ as Gevrey asymptotic expansion of order $1/k$ on $\mathcal{E}_p$. Moreover, if the aperture of one sector $\mathcal{E}_{p_0}$ is slightly larger than π/k, then $G_{p_0}(\epsilon)$ represents the k-sum of $\hat{G}(\epsilon)$ on $\mathcal{E}_{p_0}$.

7.2 Gevrey asymptotic expansion in the complex parameter for the analytic solutions to the initial value problem

In this subsection, we show the second central result of our work, namely we establish the existence of a formal power series in the parameter ϵ whose coefficients are bounded holomorphic functions on the product of a sector $\mathcal{T}$ with small radius centered at 0 and a strip $H_{\beta'}$ in $\mathbb{C}^2$, which represent the common Gevrey asymptotic expansion of order $1/k$ of the actual solutions $u_p(t, z, \epsilon)$ of (11) constructed in Theorem 1.

The second main result of this work can be stated as follows.

Theorem 2 *We set $\mathbb{F}$ as the Banach space of complex valued bounded holomorphic functions on the product $(\mathcal{T} \cap D(0, \sigma')) \times H_{\beta'}$ endowed with the supremum norm where the sector $\mathcal{T}$, radius $\sigma' > 0$, and width $\beta' > 0$ are determined in Theorem 1. For all $0 \leq p \leq \varsigma - 1$,* *the holomorphic and bounded functions $\epsilon \mapsto u_p(t,z,\epsilon)$ from $\mathcal{E}_p$ into $\mathbb{F}$ built up in Theorem 1 admit a formal power series*

$$\hat{u}(t,z,\epsilon) = \sum_{m\geq 0} h_m(t,z)\epsilon^m \in \mathbb{F}[[\epsilon]]$$

as Gevrey asymptotic expansion of order $1/k$. More precisely, for all $0 \leq p \leq \varsigma - 1$, we can pick up two constants $C_p, M_p > 0$ with

$$\sup_{t\in\mathcal{T}\cap D(0,\sigma'),z\in H_{\beta'}} \left| u_p(t,z,\epsilon) - \sum_{m=0}^{n-1} h_m(t,z)\epsilon^m \right| \leq C_p M_p^n \Gamma\left(1+\frac{n}{k}\right)|\epsilon|^n$$

for all $n \geq 1$, whenever $\epsilon \in \mathcal{E}_p$. Furthermore, if the aperture of one sector $\mathcal{E}_{p_0}$ can be taken slightly larger than π/k, then the map $\epsilon \mapsto u_{p_0}(t,z,\epsilon)$ is the k-sum of $\hat{u}(t,z,\epsilon)$ on $\mathcal{E}_{p_0}$.

Proof We focus on the family of functions $u_p(t,z,\epsilon)$, $0 \leq p \leq \varsigma - 1$ constructed in Theorem 1. For all $0 \leq p \leq \varsigma - 1$, we define $G_p(\epsilon) := (t,z) \mapsto u_p(t,z,\epsilon)$, which represents a holomorphic and bounded function from $\mathcal{E}_p$ into the Banach space $\mathbb{F}$ of bounded holomorphic functions on $(\mathcal{T} \cap D(0,\sigma')) \times H_{\beta'}$ equipped with the supremum norm, where $\mathcal{T}$ is a bounded sector selected in Theorem 1, the radius $\sigma' > 0$ is taken small enough, and $H_{\beta'}$ is a horizontal strip of width $0 < \beta' < \beta$. In accordance with (105), we deduce that the cocycle $\Theta_p(\epsilon) = G_{p+1}(\epsilon) - G_p(\epsilon)$ is exponentially flat of order k on $Z_p = \mathcal{E}_p \cap \mathcal{E}_{p+1}$ for any $0 \leq p \leq \varsigma - 1$.

Owing to Theorem (RS), we obtain a formal power series $\hat{G}(\epsilon) \in \mathbb{F}[[\epsilon]]$ which represents the Gevrey asymptotic expansion of order $1/k$ of each $G_p(\epsilon)$ on $\mathcal{E}_p$ for $0 \leq p \leq \varsigma - 1$. Besides, when the aperture of one sector $\mathcal{E}_{p_0}$ is slightly larger than π/k, the function $G_{p_0}(\epsilon)$ defines the k-sum of $\hat{G}(\epsilon)$ on $\mathcal{E}_{p_0}$ as described in Definition 7. □

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References

1. Aoki, T., Kawai, T., Koike, T., Takei, Y.: On the exact WKB analysis of operators admitting infinitely many phases. Adv. Math. **181**(1), 165–189 (2004)
2. Balser, W.: From Divergent Power Series to Analytic Functions. Theory and Application of Multisummable Power Series. Lecture Notes in Mathematics, vol. 1582. Springer, Berlin (1994)
3. Balser, W.: Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations. Universitext. Springer, New York (2000)
4. Bardos, C., Frisch, U., Pauls, W., Ray, S., Titi, E.: Entire solutions of hydrodynamical equations with exponential dissipation. Commun. Math. Phys. **293**, 519–543 (2010)

5. Braaksma, B., Faber, B.: Multisummability for some classes of difference equations. Ann. Inst. Fourier (Grenoble) **46**(1), 183–217 (1996)
6. Costin, O., Tanveer, S.: Existence and uniqueness for a class of nonlinear higher-order partial differential equations in the complex plane. Commun. Pure Appl. Math. **53**(9), 1092–1117 (2000)
7. Erdelyi, A.: Higher Transcendental Functions, vol. III. McGraw-Hill, New York (1953)
8. Faber, B., Van der Put, M.: Formal theory for differential-difference operators. J. Abstr. Differ. Equ. Appl. **7**, 63–104 (2001)
9. He, J.H.: Some asymptotic methods for strongly nonlinear equations. Int. J. Mod. Phys. B **20**(10), 1141–1199 (2006)
10. Hirose, S., Tahara, H., Yamazawa, H.: Multisummability of formal solutions of some infinite order PDEs, Talk at the workshop FASdiff17 in Alcalá de Henares, Spain, 2017. http://www3.uah.es/fasdiff17/
11. Hsieh, P., Sibuya, Y.: Basic Theory of Ordinary Differential Equations. Universitext. Springer, New York (1999)
12. Kato, G., Struppa, D.: Fundamentals of Algebraic Microlocal Analysis. Monographs and Textbooks in Pure and Applied Mathematics, vol. 217. Dekker, New York (1999)
13. Kawai, T., Struppa, D.: On the existence of holomorphic solutions of systems of linear differential equations of infinite order and with constant coefficients. Int. J. Math. **1**(1), 63–82 (1990)
14. Kawai, T., Struppa, D.: Overconvergence phenomena and grouping in exponential representation of solutions of linear differential equations of infinite order. Adv. Math. **161**(2), 131–140 (2001)
15. Lastra, A., Malek, S.: Parametric Gevrey asymptotics for some nonlinear initial value Cauchy problems. J. Differ. Equ. **259**(10), 5220–5270 (2015)
16. Lastra, A., Malek, S.: On parametric multisummable formal solutions to some nonlinear initial value Cauchy problems. Adv. Differ. Equ. **2015**, Article ID 200 (2015)
17. Lastra, A., Malek, S.: Parametric Borel summability for linear singularly perturbed Cauchy problems with linear fractional transforms (2018). ArXiv preprint
18. Malek, S.: On Gevrey asymptotics for some nonlinear integro-differential equations. J. Dyn. Control Syst. **16**(3), 377–406 (2010)
19. Malek, S.: Singularly perturbed small step size difference-differential nonlinear PDEs. J. Differ. Equ. Appl. **20**(1), 118–168 (2014)
20. Mandai, T.: Existence and nonexistence of null-solutions for some non-Fuchsian partial differential operators with T-dependent coefficients. Nagoya Math. J. **122**, 115–137 (1991)
21. Ōuchi, S.: On some functional equations with Borel summable solutions. Funkc. Ekvacioj **58**(2), 223–251 (2015)
22. Tahara, H., Yamazawa, H.: Multisummability of formal solutions to the Cauchy problem for some linear partial differential equations. J. Differ. Equ. **255**(10), 3592–3637 (2013)