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A new comparison principle and its application to nonlinear impulsive functional integro-differential equations

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Abstract

In this article, we first create a new comparison principle for a nonlinear impulsive boundary problem involving different deviating arguments. Then we employ the new result and iterative method to study the existence of the max-minimal solution of a second-order impulsive functional integro-differential equation. The results achieved in this paper are more general and complement many previously known results.

Keywords: Comparison principle; Impulsive functional integro-differential equations; Monotone iterative technique; Upper and lower solutions

1 Introduction

The comparison principle plays an important role since it is one of the basic tools to study ODE and PDE. Thus, how to create a new comparison principle is an interesting and important question. In this paper, we shall create a comparison principle with impulsive effect. By means of the comparison principle and monotone iterative method, the existence of the max-minimal solution of second-order impulsive functional integro-differential Eqs. (1.1) is investigated. Also, the iterative sequences of solutions of the system are given. The importance of this method does not need to be particularly pointed out [1–14]. The theorems achieved in this paper are more general and complement many previously known results.

Impulsive differential equations, arising in the mathematical modeling of complex systems and processes, have drawn more and more attention of the research community due to their numerous applications in various fields of science and engineering such as chemistry, physics, biology, medicine, mechanics, etc. (see [15–24]). Boundary value problems (BVP) of differential equations have been investigated for many years. Now, nonlinear boundary conditions have drawn much attention, there exist many articles dealing with the problem for different kinds of boundary value conditions such as multi-point, integral boundary condition, and other conditions (see [25–35]). On the other hand, deviated arguments also play an important role in nonlinear analysis. It should be noticed that such equations appear often in various fields of science and engineering such as mathematical physics, economics, mechanics, etc. (see [36–38]). However, the relevant theory of this type of problem is still at its developing stage, and a great quantity of aspects remain to be explored. For a detailed description, see [39–43].

Here, we use the new result we achieved in the article to investigate the existence theorems of max–min solutions for impulsive systems of the following:

$$\begin{cases} u''(t) = \Upsilon(t, u(t), u(\phi(t)), Xu(t), Yu(t), u(\psi(t, \mu(t)))), & t \in J', \\ \triangle u(t_k) = I_k(u(t_k)), & k = 1, 2, \dots, m, \\ \triangle u'(t_k) = I_k^*(u(t_k), u'(t_k)), & k = 1, 2, \dots, m, \\ \chi_1(u(0), u(T)) = 0, & \chi_2(u'(0), u'(T)) = 0, \end{cases} \tag{1.1}$$

where $t \in J = [0, T] (T > 0)$, $\Upsilon \in C(J \times R^5, R)$, $I_k \in C(R, R)$, $I_k^*, \chi_i (i = 1, 2) \in C(R \times R, R)$, $\phi \in C(J, J)$, $\mu \in C(J, R)$, $\psi \in C(J \times R, J)$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $J' = J \backslash \{t_1, t_2, \dots, t_m\}$, and

$$Xu(t) = \int_0^t k(t,s)u(s)\, ds, \qquad Yu(t) = \int_0^T h(t,s)u(s)\, ds,$$

$k(t,s) \in C(D, R^+)$, $h(t,s) \in C(J \times J, R^+)$, $D = \{(t,s) \in R^2 | 0 \le s \le t, t \in J\}$, $R^+ = [0, +\infty)$. $\triangle u(t_k) = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ denote the right and the left limits of $u(t)$ at $t = t_k (k = 1, 2, \dots, m)$, respectively. $\triangle u'(t_k)$ has a similar meaning for $u'(t)$. Let $PC(J, R) = \{u : J \to R | u(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$ and $u(t_k^+)$ exists, $k = 1, 2, \dots, m\}$. In addition, $PC^1(J, R) = \{u \in PC(J, R) | u(t)$ is continuously differentiable at $t \neq t_k, u'(t_k^+)$ and $u'(t_k^-)$ exist, $k = 1, 2, \dots, m\}$. Obviously, $PC(J, R)$ and $PC^1(J, R)$ are Banach spaces with respective norms

$$\|u\|_{PC} = \sup_{t \in J} |u(t)|, \qquad \|u\|_{PC^1} = \max\{\|u\|_{PC}, \|u'\|_{PC}\}.$$

2 New comparison principle

Lemma 1 ([16]) *Let $s \in [0, T)$, $c_k \ge 0$, ϕ_k $(k = 1, 2, \dots, m)$ be constants, and let $a, b \in PC(J, R)$, $w \in PC^1(J, R)$. If*

$$\begin{cases} w'(t) \le a(t)w(t) + b(t), & t \in [s, T), t \neq t_k, \\ w(t_k^+) \le c_k w(t_k) + \phi_k, & t_k \in [s, T), \end{cases}$$

then for $t \in [s, T]$

$$w(t) \le w(s^+)\left(\prod_{s<t_k<t} c_k\right) \exp\left(\int_s^t a(r)\, dr\right) + \int_s^t \left(\prod_{r<t_k<t} c_k\right) \exp\left(\int_r^t a(\tau)\, d\tau\right) b(r)\, dr$$
$$+ \sum_{s<t_k<t} \left(\prod_{t_k<t_i<t} c_k\right) \exp\left(\int_{t_k}^t a(\tau)\, d\tau\right) \phi_k.$$

Lemma 2 (New comparison principle) *Assume that $u \in PC^1(J, R) \cap C^2(J', R)$ satisfies*

$$\begin{cases} u''(t) \le -D_1(t)u(t) - D_2(t)u(\phi(t)) - D_3(t)(Xu)(t) - D_4(t)(Yu)(t), & t \in J', \\ \triangle u(t_k) \le -L_k u(t_k), & k = 1, 2, \dots, m, \\ \triangle u'(t_k) \le -L_k^* u'(t_k), & k = 1, 2, \dots, m, \\ u(0) \le \lambda_1 u(T), & u'(0) \le \lambda_2 u'(T), \end{cases} \tag{2.1}$$

where $D_i \in C(J, R^+)(i = 1, 2, 3, 4)$, $0 \le L_k < 1$, $0 < \lambda_1, \lambda_2, L_k^* < 1$, *and*

$$\lambda_1 \prod_{k=1}^{m}(1-L_k)^2 \prod_{k=1}^{m}\left(1-L_k^*\right)\left(1-\lambda_2 \prod_{k=1}^{m}\left(1-L_k^*\right)\right) \ge \int_0^T q(t)\,dt \int_0^T \prod_{s<t_k<T}(1-L_k)\,ds, \tag{2.2}$$

here $q(t) = D_1(t) + D_2(t) + D_3(t)\int_0^t k(t,s)\,ds + D_4(t)\int_0^T h(t,s)\,ds$. *Then* $u(t) \le 0$, $t \in J$.

Proof Suppose the contrary. Then, for some $t \in J$, $u(t) > 0$, thus there exist two cases:

Case a: For some $\bar{t} \in J$, $u(\bar{t}) > 0$, and $u(t) \ge 0$ for $t \in J$.

Case b: For some $t^*, t_* \in J$ such that $u(t^*) > 0$ and $u(t_*) < 0$.

In *Case a*, it follows from (2.1) that $u''(t) \le 0$ for $t \ne t_k$ and $u'(t_k^+) \le (1-L_k^*)u'(t_k)$. By means of Lemma 1, we have $u'(t) \le u'(0)\prod_{0<t_k<t}(1-L_k^*)$. From this, together with (2.1), we have $u'(0) \le \lambda_2 u'(T) \le \lambda_2 u'(0)\prod_{k=1}^m (1-L_k^*)$, which means $u'(0) \le 0$, thus $u'(t) \le 0$. Meanwhile, $u(t_k^+) \le (1-L_k)u(t_k) \le u(t_k)$. So, $u(t)$ is nonincreasing in J. Then $u(0) \le \lambda_1 u(T) \le \lambda_1 u(0)$, which is a contradiction.

For *Case b*, put $\inf_{t\in J} u(t) = -\gamma$, then $\gamma > 0$, and for some $i \in \{1, 2, \ldots, m\}$, there exists $t_* \in (t_i, t_{i+1}]$ such that $u(t_*) = -\gamma$ or $u(t_i^+) = -\gamma$. Only consider $u(t_*) = -\gamma$, the proof of the case $u(t_i^+) = -\gamma$ is similar.

By (2.1), we have

$$\begin{cases} u''(t) \le \gamma[D_1(t) + D_2(t) + D_3(t)\int_0^t k(t,s)\,ds + D_4(t)\int_0^T h(t,s)\,ds] \equiv \gamma q(t), \\ u'(t_k^+) \le (1-L_k^*)u'(t_k). \end{cases}$$

By Lemma 1, we get

$$u'(t) \le u'(0)\prod_{0<t_k<t}\left(1-L_k^*\right) + \int_0^t \gamma \prod_{s<t_k<t}\left(1-L_k^*\right)q(s)\,ds. \tag{2.3}$$

In (2.3), let $t = T$, then we have

$$\begin{aligned} u'(0) &\le \lambda_2 u'(T) \\ &\le \lambda_2 u'(0)\prod_{k=1}^m\left(1-L_k^*\right) + \lambda_2\int_0^T \gamma \prod_{s<t_k<T}\left(1-L_k^*\right)q(s)\,ds \\ &\le \lambda_2 u'(0)\prod_{k=1}^m\left(1-L_k^*\right) + \lambda_2\int_0^T \gamma q(s)\,ds, \end{aligned}$$

which implies

$$u'(0) \le \lambda_2 \int_0^T \gamma q(s)\,ds\left[1-\lambda_2\prod_{k=1}^m\left(1-L_k^*\right)\right]^{-1}. \tag{2.4}$$

From (2.3) and (2.4), we have that

$$\begin{aligned} u'(t) \le{}& \lambda_2 \int_0^T \gamma q(s)\,ds \left[1-\lambda_2\prod_{k=1}^{m}(1-L_k^*)\right]^{-1} \prod_{0<t_k<t}(1-L_k^*) \\ &+ \int_0^t \gamma \prod_{s<t_k<t}(1-L_k^*) q(s)\,ds \\ \le{}& \int_0^T \gamma q(s)\,ds \left\{\lambda_2 \prod_{0<t_k<t}(1-L_k^*)\left[1-\lambda_2\prod_{k=1}^{m}(1-L_k^*)\right]^{-1} + 1\right\} \\ \le{}& \lambda_2 \prod_{0<t_k<t}(1-L_k^*) \int_0^T \gamma q(s)\,ds \\ &\times \left\{\left[1-\lambda_2\prod_{k=1}^{m}(1-L_k^*)\right]^{-1} + \left[\lambda_2\prod_{k=1}^{m}(1-L_k^*)\right]^{-1}\right\} \\ \le{}& \int_0^T \gamma q(s)\,ds \left[\prod_{k=1}^{m}(1-L_k^*)\left(1-\lambda_2\prod_{k=1}^{m}(1-L_k^*)\right)\right]^{-1}, \end{aligned}$$

which and $u(t_k^+) \le (1-L_k)u(t_k)$ imply for $t \in [t_*, T]$

$$\begin{aligned} u(t) \le{}& u(t_*) \prod_{t_*<t_k<t}(1-L_k) + \int_{t_*}^t \prod_{s<t_k<t}(1-L_k) \int_0^T \gamma q(r)\,dr \\ &\times \left[\prod_{k=1}^{m}(1-L_k^*)\left(1-\lambda_2\prod_{k=1}^{m}(1-L_k^*)\right)\right]^{-1} ds \\ ={}& u(t_*) \prod_{t_*<t_k<t}(1-L_k) + \int_0^T \gamma q(r)\,dr \\ &\times \left[\prod_{k=1}^{m}(1-L_k^*)\left(1-\lambda_2\prod_{k=1}^{m}(1-L_k^*)\right)\right]^{-1} \\ &\times \int_{t_*}^t \prod_{s<t_k<t}(1-L_k)\,ds. \end{aligned} \tag{2.5}$$

If $t^* > t_*$, let $t = t^*$ in (2.5), we have

$$\begin{aligned} 0 <{}& -\gamma \prod_{t_*<t_k<t^*}(1-L_k) \\ &+ \gamma \int_0^T q(r)\,dr \left[\prod_{k=1}^{m}(1-L_k^*)\left(1-\lambda_2\prod_{k=1}^{m}(1-L_k^*)\right)\right]^{-1} \\ &\times \int_{t_*}^{t^*} \prod_{s<t_k<t^*}(1-L_k)\,ds, \end{aligned}$$

so,

$$\frac{\prod_{k=1}^{m}(1-L_k)}{\int_0^T \prod_{s<t_k<T}(1-L_k)\,ds} \le \frac{\prod_{t_*<t_k<t^*}(1-L_k)}{\int_{t_*}^{t^*} \prod_{s<t_k<t^*}(1-L_k)\,ds}$$
$$< \int_0^T q(r)\,dr\left[\prod_{k=1}^{m}(1-L_k^*)\left(1-\lambda_2\prod_{k=1}^{m}(1-L_k^*)\right)\right]^{-1},$$

which contradicts (2.2). Hence, $u(t) \le 0$ on J.

If $t^* < t_*$, without loss of generality, let $t_* \in (t_{p-1}, t_p]$ and $t^* \in (t_q, t_{q+1}]$, $0 \le q \le p-1$, $p, q \in \{1, 2, \dots, m\}$. By Lemma 1, we have

$$\begin{aligned} u(t^*) &\le u(0)\prod_{0<t_k<t^*}(1-L_k) + \int_0^T \gamma q(r)\,dr\left[\prod_{k=1}^{m}(1-L_k^*)\left(1-\lambda_2\prod_{k=1}^{m}(1-L_k^*)\right)\right]^{-1} \\ &\quad \times \int_0^{t^*}\prod_{s<t_k<t^*}(1-L_k)\,ds \\ &= u(0)\prod_{k=1}^{q}(1-L_k) + \int_0^T \gamma q(r)\,dr\left[\prod_{k=1}^{m}(1-L_k^*)\right. \\ &\quad \times \left.\left(1-\lambda_2\prod_{k=1}^{m}(1-L_k^*)\right)\right]^{-1}\int_0^{t^*}\prod_{s<t_k<t^*}(1-L_k)\,ds. \end{aligned} \tag{2.6}$$

On the other hand,

$$\begin{aligned} u(0) &\le \lambda_1 u(T) \le \lambda_1 u(t_*)\prod_{t_*<t_k<T}(1-L_k) + \lambda_1\int_0^T \gamma q(r)\,dr \\ &\quad \times \left[\prod_{k=1}^{m}(1-L_k^*)\left(1-\lambda_2\prod_{k=1}^{m}(1-L_k^*)\right)\right]^{-1}\int_{t_*}^{T}\prod_{s<t_k<T}(1-L_k)\,ds \\ &= -\lambda_1\gamma\prod_{k=p}^{m}(1-L_k) + \lambda_1\int_0^T \gamma q(r)\,dr \\ &\quad \times \left[\prod_{k=1}^{m}(1-L_k^*)\left(1-\lambda_2\prod_{k=1}^{m}(1-L_k^*)\right)\right]^{-1}\int_{t_*}^{T}\prod_{s<t_k<T}(1-L_k)\,ds. \end{aligned} \tag{2.7}$$

From (2.6) and (2.7), we get that

$$\begin{aligned} &\lambda_1\prod_{k=p}^{m}(1-L_k)\prod_{j=1}^{q}(1-L_j) \\ &\quad < \int_0^T q(r)\,dr\left[\prod_{k=1}^{m}(1-L_k^*)\left(1-\lambda_2\prod_{k=1}^{m}(1-L_k^*)\right)\right]^{-1} \\ &\quad\quad \times \left[\lambda_1\prod_{j=1}^{q}(1-L_j)\int_{t_*}^{T}\prod_{s<t_k<T}(1-L_k)\,ds + \int_0^{t^*}\prod_{s<t_k<t^*}(1-L_k)\,ds\right]. \end{aligned}$$

By $\prod_{j=q+1}^{m}(1-L_j)$ times the above inequality, then

$$\begin{aligned}
&\lambda_1\prod_{k=p}^{m}(1-L_k)\prod_{j=1}^{m}(1-L_j)\\
&\quad<\int_0^T q(r)\,dr\left[\prod_{k=1}^{m}(1-L_k^*)\left(1-\lambda_2\prod_{k=1}^{m}(1-L_k^*)\right)\right]^{-1}\\
&\quad\quad\times\left[\lambda_1\prod_{j=1}^{m}(1-L_j)\int_{t_*}^{T}\prod_{s<t_k<T}(1-L_k)\,ds+\prod_{j=q+1}^{m}(1-L_j)\int_0^{t^*}\prod_{s<t_k<t^*}(1-L_k)\,ds\right]\\
&\quad\le\int_0^T q(r)\,dr\left[\prod_{k=1}^{m}(1-L_k^*)\left(1-\lambda_2\prod_{k=1}^{m}(1-L_k^*)\right)\right]^{-1}\\
&\quad\quad\times\left[\int_{t_*}^{T}\prod_{s<t_k<T}(1-L_k)\,ds+\int_0^{t^*}\prod_{s<t_k<T}(1-L_k)\,ds\right]\\
&\quad\le\int_0^T q(r)\,dr\left[\prod_{k=1}^{m}(1-L_k^*)\left(1-\lambda_2\prod_{k=1}^{m}(1-L_k^*)\right)\right]^{-1}\\
&\quad\quad\times\int_0^T\prod_{s<t_k<T}(1-L_k)\,ds.
\end{aligned}$$

So, $\lambda_1\prod_{k=1}^{m}(1-L_k)^2<\int_0^T q(r)\,dr[\prod_{k=1}^{m}(1-L_k^*)(1-\lambda_2\prod_{k=1}^{m}(1-L_k^*))]^{-1}\int_0^T\prod_{s<t_k<T}(1-L_k)\,ds$, which contradicts (2.2). Hence, $u(t)\le 0$ on J.

Consider the problem:

$$\begin{cases}
u''(t)=\sigma(t)-D_1(t)u(t)-D_2(t)u(\phi(t))-D_3(t)(Xu)(t)-D_4(t)(Yu)(t), & t\in J',\\
\triangle u(t_k)=\psi_k-L_ku(t_k), \quad k=1,2,\ldots,m,\\
\triangle u'(t_k)=\nu_k-L_k^*u'(t_k), \quad k=1,2,\ldots,m,\\
u(0)=\lambda_1u(T)+m_1, \qquad u'(0)=\lambda_2u'(T)+m_2,
\end{cases}\tag{2.8}$$

where $\sigma\in PC(J,R)$, $\psi_k,\nu_k,m_1,m_2\in R$. □

Lemma 3 *$u(t)\in PC^1(J,R)\cap C^2(J',R)$ is a solution of the impulsive differential system* (2.8) *iff $u(t)\in PC^1(J,R)$ is a solution of the impulsive integral system*

$$\begin{aligned}
u(t)&=G_1+tG_2\\
&\quad+\int_0^t(t-s)\big[\sigma(s)-D_1(s)u(s)-D_2(s)u\big(\phi(s)\big)-D_3(s)(Xu)(s)-D_4(s)(Yu)(s)\big]\,ds\\
&\quad+\sum_{0<t_k<t}\big\{\big[\psi_k-L_ku(t_k)\big]+(t-t_k)\big[\nu_k-L_k^*u'(t_k)\big]\big\},
\end{aligned}\tag{2.9}$$

where

$$G_1 = \frac{\lambda_1}{1-\lambda_1}\left\{\int_0^T (T-s)\big[\sigma(s) - D_1(s)u(s) - D_2(s)u\big(\phi(s)\big) - D_3(s)(Xu)(s) - D_4(s)(Yu)(s)\big]\,ds + \sum_{0<t_k<T}\big\{\big[\psi_k - L_k u(t_k)\big] + (T-t_k)\big[\nu_k - L_k^* u'(t_k)\big]\big\} + TG_2 + m_1\right\} + m_1,$$

$$G_2 = \frac{\lambda_2}{1-\lambda_2}\left\{\int_0^T \big[\sigma(s) - D_1(s)u(s) - D_2(s)u\big(\phi(s)\big) - D_3(s)(Xu)(s) - D_4(s)(Yu)(s)\big]\,ds + \sum_{0<t_k<T}\big[\nu_k - L_k^* u'(t_k)\big] + m_2\right\} + m_2.$$

Lemma 3 is easy, so we omit its proof.

Lemma 4 *For $\sigma \in PC(J,R)$, $\psi_k, \nu_k, m_1, m_2 \in R$, $0 \le L_k < 1$, $0 < \lambda_1, \lambda_2, L_k^* < 1$ and functions $M, K, N, L \in C(J, R^+)$. If*

$$\begin{cases} \frac{1}{1-\lambda_1}\{\int_0^T (T-s)p(s)\,ds + \sum_{k=1}^m [L_k + (T-t_k)L_k^*]\} \\ \qquad + \frac{\lambda_2 T}{(1-\lambda_1)(1-\lambda_2)}[\int_0^T p(s)\,ds + \sum_{k=1}^m L_k^*] < 1, \\ \frac{1}{1-\lambda_2}[\int_0^T p(s)\,ds + \sum_{k=1}^m L_k^*] < 1, \end{cases} \tag{2.10}$$

where $p(t) = D_1(t) + D_2(t) + D_3(t)\int_0^t k(t,s)\,ds + D_4(t)\int_0^T h(t,s)\,ds$. (2.8) has a unique solution $u(t) \in PC^1(J,R) \cap C^2(J',R)$.

A similar proof can be found in [22] (see Lemma 2.3), so we omit it.

3 Main results

Theorem 1 *Assume that condition (2.10) holds. In addition, assume that*

(H_1) There exist $u_0(t) \le v_0(t) \in PC^1(J,R) \cap C^2(J',R)$ such that

$$\begin{cases} u_0''(t) \le \Upsilon(t, u_0(t), u_0(\phi(t)), Xu_0(t), Yu_0(t), u_0(\psi(t,\mu(t)))), \\ \triangle u_0(t_k) \le I_k(u_0(t_k)), \quad k = 1,2,\ldots,m, \\ \triangle u_0'(t_k) \le I_k^*(u_0(t_k), u_0'(t_k)), \quad k = 1,2,\ldots,m, \\ \chi_1(u_0(0), u_0(T)) \le 0, \qquad \chi_2(u_0'(0), u_0'(T)) \le 0, \end{cases}$$

and

$$\begin{cases} v_0''(t) \ge \Upsilon(t, v_0(t), v_0(\phi(t)), Xv_0(t), Yv_0(t), v_0(\psi(t,\mu(t)))), \\ \triangle v_0(t_k) \ge I_k(v_0(t_k)), \quad k = 1,2,\ldots,m, \\ \triangle v_0'(t_k) \ge I_k^*(v_0(t_k), v_0'(t_k)), \quad k = 1,2,\ldots,m, \\ \chi_1(v_0(0), v_0(T)) \ge 0, \qquad \chi_2(v_0'(0), v_0'(T)) \ge 0. \end{cases}$$

(H_2) Functions $D_i \in C(J, R^+)(i = 1,2,3,4)$, which satisfy (2.2) such that

$$\begin{aligned}\Upsilon(t,u,v,w,z,\xi) - \Upsilon(t,\overline{u},\overline{v},\overline{w},\overline{z},\overline{\xi}) \geq{}& -D_1(t)(u-\overline{u}) - D_2(t)(v-\overline{v}) \\ & - D_3(t)(w-\overline{w}) - D_4(t)(z-\overline{z}),\end{aligned}$$

where $u_0(t) \leq \overline{u} \leq u \leq v_0(t)$, $u_0(\phi(t)) \leq \overline{v} \leq v \leq v_0(\phi(t))$, $Xu_0(t) \leq \overline{w} \leq w \leq Xv_0(t)$, $Yu_0(t) \leq \overline{z} \leq z \leq Yv_0(t)$, $u_0(\psi(t,\mu(t))) \leq \overline{\xi} \leq \xi \leq v_0(\psi(t,\mu(t)))$, $\forall t \in J$.

(H_3) There exist constants $0 \leq L_k < 1$, $0 < L_k^* < 1$ $(k = 1,2,\dots,m)$, and $0 < b_1 < a_1$, $0 < b_2 < a_2$ such that

$$\begin{aligned}&I_k(u) - I_k(\overline{u}) \geq -L_k(u-\overline{u}),\\ &I_k^*(u,u') - I_k^*(\overline{u},\overline{u}') \geq -L_k^*(u'-\overline{u}'),\\ &\chi_1(u,v) - \chi_1(\overline{u},\overline{v}) \leq a_1(u-\overline{u}) - b_1(v-\overline{v}),\\ &\chi_2(u',v') - \chi_2(\overline{u}',\overline{v}') \leq a_2(u'-\overline{u}') - b_2(v'-\overline{v}'),\end{aligned}$$

where $u_0(t_k) \leq \overline{u} \leq u \leq v_0(t_k)$ $(k = 1,2,\dots,m)$, $u_0(0) \leq \overline{u} \leq u \leq v_0(0)$, $u_0(T) \leq \overline{v} \leq v \leq v_0(T)$.

Then the impulsive system (1.1) has the min-maximal solutions u^*, v^* in $[u_0, v_0]$, respectively. Moreover, there exist monotone iterative sequences $\{u_n(t)\}, \{v_n(t)\} \subset [u_0, v_0]$ such that $u_n \to u^*, v_n \to v^*(n \to \infty)$ uniformly on $t \in J$, where $\{u_n(t)\}, \{v_n(t)\}$ satisfy

$$\begin{cases} u_n''(t) = \Upsilon(t, u_{n-1}(t), u_{n-1}(\phi(t)), Xu_{n-1}(t), Yu_{n-1}(t), u_{n-1}(\psi(t,\mu(t))))\\ \qquad - D_1(t)(u_n - u_{n-1})(t) - D_2(t)(u_n - u_{n-1})(\phi(t)) - D_3(t)X(u_n - u_{n-1})(t)\\ \qquad - D_4(t)Y(u_n - u_{n-1})(t), \quad t \in J',\\ \triangle u_n(t_k) = I_k(u_{n-1}(t_k)) - L_k(u_n - u_{n-1})(t_k), \quad k = 1,2,\dots,m,\\ \triangle u_n'(t_k) = I_k^*(u_{n-1}(t_k), u_{n-1}'(t_k)) - L_k^*(u_n' - u_{n-1}')(t_k), \quad k = 1,2,\dots,m,\\ u_n(0) = u_{n-1}(0) + \lambda_1[u_n(T) - u_{n-1}(T)] - \frac{1}{a_1}\chi_1(u_{n-1}(0), u_{n-1}(T)), \quad n = 1,2,\dots,\\ u_n'(0) = u_{n-1}'(0) + \lambda_2[u_n'(T) - u_{n-1}'(T)] - \frac{1}{a_2}\chi_2(u_{n-1}'(0), u_{n-1}'(T)), \quad n = 1,2,\dots, \end{cases} \tag{3.1}$$

$$\begin{cases} v_n''(t) = \Upsilon(t, v_{n-1}(t), v_{n-1}(\phi(t)), Xv_{n-1}(t), Yv_{n-1}(t), v_{n-1}(\psi(t,\mu(t))))\\ \qquad - D_1(t)(v_n - v_{n-1})(t) - D_2(t)(v_n - v_{n-1})(\phi(t)) - D_3(t)X(v_n - v_{n-1})(t)\\ \qquad - D_4(t)Y(v_n - v_{n-1})(t), \quad t \in J',\\ \triangle v_n(t_k) = I_k(v_{n-1}(t_k)) - L_k(v_n - v_{n-1})(t_k), \quad k = 1,2,\dots,m,\\ \triangle v_n'(t_k) = I_k^*(v_{n-1}(t_k), v_{n-1}'(t_k)) - L_k^*(v_n' - v_{n-1}')(t_k), \quad k = 1,2,\dots,m,\\ v_n(0) = v_{n-1}(0) + \lambda_1[v_n(T) - v_{n-1}(T)] - \frac{1}{a_1}\chi_1(v_{n-1}(0), v_{n-1}(T)), \quad n = 1,2,\dots,\\ v_n'(0) = v_{n-1}'(0) + \lambda_2[v_n'(T) - v_{n-1}'(T)] - \frac{1}{a_2}\chi_2(v_{n-1}'(0), v_{n-1}'(T)), \quad n = 1,2,\dots, \end{cases} \tag{3.2}$$

and

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq u^* \leq v^* \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0, \tag{3.3}$$

here $\lambda_i = b_i/a_i(i = 1,2)$.

Proof For any $u_{n-1}, v_{n-1} \in PC^1(J,R) \cap C^2(J',R)$, it follows from Lemma 4 that (3.1) and (3.2) have unique solutions u_n and v_n in $PC^1(J,R) \cap C^2(J',R)$, respectively.

Now, we verify that

$$u_{n-1} \le u_n \le v_n \le v_{n-1}, \quad n = 1, 2, \ldots. \tag{3.4}$$

Let $p(t) = u_0(t) - u_1(t)$, $q(t) = v_1(t) - v_0(t)$, $w(t) = u_1(t) - v_1(t)$, by (3.1), (3.2) and (H_1) – (H_4), we have that

$$\begin{cases} p''(t) \le -D_1(t)p(t) - D_2(t)p(\phi(t)) - D_3(t)(Xp)(t) - D_4(t)(Yp)(t), & t \in J', \\ \triangle p(t_k) \le -L_k p(t_k), \quad k = 1, 2, \ldots, m, \\ \triangle p'(t_k) \le -L_k^* p'(t_k), \quad k = 1, 2, \ldots, m, \\ p(0) \le \lambda_1 p(T), \qquad p'(0) \le \lambda_2 p'(T), \end{cases}$$

$$\begin{cases} q''(t) \le -D_1(t)q(t) - D_2(t)q(\phi(t)) - D_3(t)(Xq)(t) - D_4(t)(Yq)(t), & t \in J', \\ \triangle q(t_k) \le -L_k q(t_k), \quad k = 1, 2, \ldots, m, \\ \triangle q'(t_k) \le -L_k^* q'(t_k), \quad k = 1, 2, \ldots, m, \\ q(0) \le \lambda_1 q(T), \qquad q'(0) \le \lambda_2 q'(T), \end{cases}$$

$$\begin{cases} w''(t) = \Upsilon(t, u_0(t), u_0(\phi(t)), Xu_0(t), Yu_0(t), u_0(\psi(t,\mu(t)))) - D_1(t)(u_1 - u_0)(t) \\ \qquad - D_2(t)(u_1 - u_0)(\phi(t)) - D_3(t)X(u_1 - u_0)(t) - D_4(t)Y(u_1 - u_0)(t) \\ \qquad - \Upsilon(t, v_0(t), v_0(\phi(t)), Xv_0(t), Yv_0(t), v_0(\psi(t,\mu(t)))) + D_1(t)(v_1 - v_0)(t) \\ \qquad + D_2(t)(v_1 - v_0)(\phi(t)) + D_3(t)X(v_1 - v_0)(t) + D_4(t)Y(v_1 - v_0)(t) \\ \quad \le -D_1(t)w(t) - D_2(t)w(\phi(t)) - D_3(t)(Xw)(t) - D_4(t)(Yw)(t), \quad t \in J', \\ \triangle w(t_k) = I_k(u_0(t_k)) - I_k(v_0(t_k)) - L_k(u_1 - u_0)(t_k) + L_k(v_1 - v_0)(t_k) \\ \quad \le -L_k w(t_k), \quad k = 1, 2, \ldots, m, \\ \triangle w'(t_k) = I_k^*(u_0(t_k), u_0'(t_k)) - I_k^*(v_0(t_k), v_0'(t_k)) - L_k^*(u_1' - u_0')(t_k) + L_k^*(v_1' - v_0')(t_k) \\ \quad \le -L_k^* w'(t_k), \quad k = 1, 2, \ldots, m, \\ w(0) = u_0(0) - v_0(0) + \lambda_1[u_1(T) - u_0(T)] - \lambda_1[v_1(T) - v_0(T)] \\ \qquad - \frac{1}{a_1}\chi_1(u_0(0), u_0(T)) + \frac{1}{a_1}\chi_1(v_0(0), v_0(T)) \\ \quad \le u_0(0) - v_0(0) + \lambda_1 w(T) - \lambda_1[u_0(T) - v_0(T)] \\ \qquad + [v_0(0) - u_0(0)] - \lambda_1[v_0(T) - u_0(T)] \\ \quad \le \lambda_1 w(T), \\ w'(0) \le \lambda_2 w'(T). \end{cases}$$

Thus, by means of Lemma 2, we have $p(t) \le 0$, $q(t) \le 0$, $w(t) \le 0$, $\forall t \in J$, *i.e.*, $u_0 \le u_1 \le v_1 \le v_0$.

Assume that $u_{k-1} \le u_k \le v_k \le v_{k-1}$ for some $k \ge 1$. Thus, employing the same technique once again, by Lemma 2, one can get $u_k \le u_{k+1} \le v_{k+1} \le v_k$. Thus, one can easily show that

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_1 \le v_0, \quad n = 1, 2, \ldots. \tag{3.5}$$

Employing the standard arguments, we have

$$\lim_{n\to\infty} u_n(t) = u^*(t),$$
$$\lim_{n\to\infty} v_n(t) = v^*(t)$$

uniformly on $t \in J$, and the limit functions u^*, v^* satisfy (1.1). Moreover, $u^*, v^* \in [u_0, v_0]$.

Next, we prove that u^*, v^* are the min-maximal solutions of impulsive differential system (1.1) in $[u_0, v_0]$. If $w \in [u_0, v_0]$ is any solution of (1.1). Let $u_{n-1}(t) \le w(t) \le v_{n-1}(t), \forall t \in J$, for some positive integer n. Put $p = u_n - w$. Then

$$\begin{cases} p''(t) = \Upsilon(t, u_{n-1}(t), u_{n-1}(\phi(t)), Xu_{n-1}(t), Yu_{n-1}(t), u_{n-1}(\psi(t,\mu(t)))) \\ \qquad - D_1(t)(u_n - u_{n-1})(t) - D_2(t)(u_n - u_{n-1})(\phi(t)) \\ \qquad - D_3(t)X(u_n - u_{n-1})(t) - D_4(t)Y(u_n - u_{n-1})(t) \\ \qquad - \Upsilon(t, w(t), w(\phi(t)), Xw(t), Yw(t), w(\psi(t,\mu(t)))) \\ \quad \le D_1(t)(w - u_{n-1})(t) + D_2(t)(w - u_{n-1})(\phi(t)) + D_3(t)X(w - u_{n-1})(t) \\ \qquad + D_4(t)Y(w - u_{n-1})(t) - D_1(t)(u_n - u_{n-1})(t) - D_2(t)(u_n - u_{n-1})(\phi(t)) \\ \qquad - D_3(t)X(u_n - u_{n-1})(t) - D_4(t)Y(u_n - u_{n-1})(t) \\ \quad = -D_1(t)p(t) - D_2(t)p(\phi(t)) - D_3(t)(Xp)(t) - D_4(t)(Yp)(t), \quad t \in J', \\ \triangle p(t_k) = I_k(u_{n-1}(t_k)) - I_k(w(t_k)) - L_k(u_n - u_{n-1})(t_k) \\ \qquad \le -L_k p(t_k), \quad k = 1, 2, \ldots, m, \\ \triangle p'(t_k) = I_k^*(u_{n-1}(t_k), u'_{n-1}(t_k)) - I_k^*(w(t_k), w'(t_k)) - L_k^*(u'_n - u'_{n-1})(t_k) \\ \qquad \le -L_k^* p'(t_k), \quad k = 1, 2, \ldots, m, \\ p(0) = u_{n-1}(0) + \lambda_1[u_n(T) - u_{n-1}(T)] - \frac{1}{a_1}\chi_1(u_{n-1}(0), u_{n-1}(T)) \\ \qquad + \frac{1}{a_1}\chi_1(w(0), w(T)) - w(0) \\ \quad \le u_{n-1}(0) + \lambda_1[u_n(T) - u_{n-1}(T)] + [w(0) - u_{n-1}(0)] \\ \qquad - \lambda_1[w(T) - u_{n-1}(T)] - w(0) \\ \quad = \lambda_1 p(T), \\ p'(0) = u'_{n-1}(0) + \lambda_2[u'_n(T) - u'_{n-1}(T)] - \frac{1}{a_2}\chi_2(u'_{n-1}(0), u'_{n-1}(T)) \\ \qquad + \frac{1}{a_2}\chi_2(w(0), w(T)) - w(0) \\ \quad \le \lambda_2 p'(T). \end{cases}$$

By Lemma 2, we have $u_n(t) \le w(t), \forall t \in J$. By the same way as above, we can show $w(t) \le v_n(t), \forall t \in J$. That is, $u_n(t) \le w(t) \le v_n(t)$, $\forall t \in J$.

Now, if $n \to \infty$, then

$$u_0(t) \le u^*(t) \le w(t) \le v^*(t) \le v_0(t), \quad \forall t \in J.$$

That is, u^*, v^* are the min-maximal solutions of (1.1) in $[u_0, v_0]$. □

4 Example

Consider

$$\begin{cases} u''(t) = \frac{1}{100}t^3[t-u(t)] + \frac{1}{300}t^3[t-u(t^2)]^3 + \frac{1}{500}t[t^3 - \int_0^t tsu(s)\,ds]^5 \\ \qquad + \frac{1}{700}t^2[t^2 - \int_0^1 t^2 su(s)\,ds]^7 - \frac{1}{100}t^4 e^{-u(\frac{1}{2}t^3+\frac{1}{2}t^2e^{-t})}, \quad t \in [0,1], t \neq \frac{1}{2}, \\ \triangle u(\frac{1}{2}) = -\frac{1}{4}u(\frac{1}{2}), \\ \triangle u'(\frac{1}{2}) = bu(\frac{1}{2}) - \frac{3}{8}u'(\frac{1}{2}), \\ \chi_1(u(0),u(1)) = u^2(0) - \frac{1}{2}u(1) = 0, \\ \chi_2(u'(0),u'(1)) = u'(0) - \frac{1}{3}u'(1) = 0, \end{cases} \tag{4.1}$$

where $0 \le b \le \frac{1}{26}$, $m = 1$, $t_1 = \frac{1}{2}$, $\phi(t) = t^2$, $\psi(t,\mu(t)) = \frac{1}{2}t^3 + \frac{1}{2}t^2e^{-t}$, $\forall t \in J$.

Take

$$u_0(t) = 0, \quad \forall t \in J, \qquad v_0(t) = \begin{cases} 1 + t + \frac{1}{2}t^2, & t \in [0, \frac{1}{2}], \\ 1 + t^2, & t \in (\frac{1}{2}, 1]. \end{cases} \tag{4.2}$$

Then

$$u_0(t) \le v_0(t), \qquad u_0'(t) = 0, \qquad v_0'(t) = \begin{cases} 1 + t, & t \in [0, \frac{1}{2}], \\ 2t, & t \in (\frac{1}{2}, 1], \end{cases} \tag{4.3}$$

and

$$\begin{cases} u_0''(t) = 0 \le \frac{1}{100}t^4 + \frac{1}{300}t^6 + \frac{1}{500}t^{16} + \frac{1}{700}t^{16} - \frac{1}{100}t^4 \\ \qquad = \Upsilon(t, u_0(t), u_0(\phi(t)), Xu_0(t), Yu_0(t), u_0(\psi(t,\mu(t)))), \\ \triangle u_0(\frac{1}{2}) = 0 = I_1(u_0(\frac{1}{2})), \\ \triangle u_0'(\frac{1}{2}) = 0 = I_1^*(u_0(\frac{1}{2}), u_0'(\frac{1}{2})), \\ \chi_1(u_0(0),u_0(1)) = u_0^2(0) - \frac{1}{2}u_0(1) = 0, \\ \chi_2(u_0'(0),u_0'(1)) = u_0'(0) - \frac{1}{3}u_0'(1) = 0, \end{cases}$$

$$\begin{cases} v_0''(t) \ge 1 > \frac{1}{100} + \frac{1}{300} + \frac{1}{500} + \frac{1}{700} \\ \qquad \ge \Upsilon(t, v_0(t), v_0(\phi(t)), Xv_0(t), Yv_0(t), v_0(\psi(t,\mu(t)))), \\ \triangle v_0(\frac{1}{2}) = -\frac{3}{8} \ge -\frac{13}{32} = I_1(v_0(\frac{1}{2})), \\ \triangle v_0'(\frac{1}{2}) = -\frac{1}{2} \ge \frac{13b}{8} - \frac{9}{16} = I_1^*(v_0(\frac{1}{2}), v_0'(\frac{1}{2})), \\ \chi_1(v_0(0),v_0(1)) = v_0^2(0) - \frac{1}{2}v_0(1) = 0, \\ \chi_2(v_0'(0),v_0'(1)) = v_0'(0) - \frac{1}{3}v_0'(1) = \frac{1}{3} > 0. \end{cases}$$

Consequently, u_0, v_0 satisfy (H_1). Let

$$\Upsilon(t,u,v,w,z,\xi) = \frac{1}{100}t^3(t-u) + \frac{1}{300}t^3(t-v)^3 + \frac{1}{500}t\big(t^3 - w\big)^5 + \frac{1}{700}t^2\big(t^2 - z\big)^7 - \frac{1}{100}t^4e^{-\xi},$$

we have

$$\begin{aligned}
&\Upsilon(t,u,v,w,z,\xi)-\Upsilon(t,\overline{u},\overline{v},\overline{w},\overline{z},\overline{\xi})\\
&\quad=\frac{1}{100}t^3\big[(t-u)-(t-\overline{u})\big]+\frac{1}{300}t^3\big[(t-v)^3-(t-\overline{v})^3\big]+\frac{1}{500}t\big[\big(t^3-w\big)^5-\big(t^3-\overline{w}\big)^5\big]\\
&\qquad+\frac{1}{700}t^2\big[\big(t^2-z\big)^7-\big(t^2-\overline{z}\big)^7\big]-\frac{1}{100}t^4\big(e^{-\xi}-e^{-\overline{\xi}}\big)\\
&\quad\geq-\frac{1}{100}t^3(u-\overline{u})-\frac{1}{25}t^3(v-\overline{v})-\frac{1}{100}t^{13}(w-\overline{w})-\frac{1}{100}t^{14}(z-\overline{z}),
\end{aligned}$$

where $u_0(t)\leq\overline{u}\leq u\leq v_0(t)$, $u_0(\phi(t))\leq\overline{v}\leq v\leq v_0(\phi(t))$, $Xu_0(t)\leq\overline{w}\leq w\leq Xv_0(t)$, $Yu_0(t)\leq\overline{z}\leq z\leq Yv_0(t)$, $u_0(\psi(t,\mu(t)))\leq\overline{\xi}\leq\xi\leq v_0(\psi(t,\mu(t)))$, $\forall t\in J$. For $L_1=\frac{1}{4}$, $L_1^*=\frac{3}{8}$, $a_1=2$, $b_1=\frac{1}{2}$, $a_2=1$, $b_2=\frac{1}{3}$, obviously, (H_3) and (H_4) hold. On the other hand, put $D_1(t)=\frac{1}{100}t^3$, $D_2(t)=\frac{1}{25}t^3$, $D_3(t)=\frac{1}{100}t^{13}$, $D_4(t)=\frac{1}{100}t^{14}$, $\lambda_1=\frac{1}{4}$, $\lambda_2=\frac{1}{3}$, $L_1=\frac{1}{4}$, $L_1^*=\frac{3}{8}$, it is easy to see that conditions (2.2) and (2.10) hold. So, (H_2) also holds.

Thus, Theorem 1 is satisfied. Therefore, our conclusions come from Theorem 1 that (4.1) has the min-maximal solution $u^*,v^*\in[u_0,v_0]$.

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