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Nonlinear sequential Riemann–Liouville and Caputo fractional differential equations with generalized fractional integral conditions

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Abstract

In this paper, we discuss the existence and uniqueness of solutions for two new classes of sequential fractional differential equations of Riemann–Liouville and Caputo types with generalized fractional integral boundary conditions, by using standard fixed point theorems. In addition, we also demonstrate the application of the obtained results with the aid of examples.

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1 Introduction

In this paper, we investigate the following two nonlinear sequential fractional differential equations of Riemann–Liouville and Caputo fractional derivatives subject to the generalized fractional integral boundary conditions of the forms

$$^{\mathrm{RL}}D^{q}\big(^{C}D^{r}x\big)(t)=f\big(t,x(t)\big),\quad t\in(0,T),\tag{1}$$

$$x(0)=\sum_{i=1}^{m}\gamma_{i}{}^{\bar\rho_i}I^{\bar\alpha_i,\bar\beta_i}_{\bar\eta_i,\bar\kappa_i}x(\xi_i),\qquad x(T)=\sum_{j=1}^{n}\sigma_{j}{}^{\rho_j}I^{\alpha_j,\beta_j}_{\eta_j,\kappa_j}x(\delta_j),\tag{2}$$

and

$$^{C}D^{q}\big(^{\mathrm{RL}}D^{r}x\big)(t)=f\big(t,x(t)\big),\quad t\in(0,T),\tag{3}$$

$$x(0)=0,\qquad x(T)=\sum_{j=1}^{n}\sigma_{j}{}^{\rho_j}I^{\alpha_j,\beta_j}_{\eta_j,\kappa_j}x(\delta_j),\tag{4}$$

where $^{\mathrm{RL}}D^{q}$ and $^{C}D^{r}$ denote the Riemann–Liouville and Caputo fractional derivatives of order $0<q,r\leq 1$, respectively, with $1<q+r\leq 2$, $^{\tilde\rho}I^{\tilde\alpha,\tilde\beta}_{\tilde\eta,\tilde\kappa}$ denote the generalized fractional integral of order $\tilde\alpha>0$, $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, $\xi_i,\delta_j\in(0,T)$,

$\tilde{\alpha} \in \{\bar{\alpha}_i, \alpha_j\} > 0$, $\tilde{\rho} \in \{\bar{\rho}_i, \rho_j\}$, $\tilde{\beta} \in \{\bar{\beta}_i, \beta_j\}$, $\tilde{\eta} \in \{\bar{\eta}_i, \eta_j\}$, $\tilde{\kappa} \in \{\bar{\kappa}_i, \kappa_j\} \in \mathbb{R}$, $\gamma_i, \sigma_j \in \mathbb{R}$, for all $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. Observe that interchanging the sequence of fractional derivatives in (1) and (3) has an effect on the boundary conditions which are seen in (2) and (4), namely the Caputo fractional derivative of a constant is zero while the Riemann–Liouville derivative is not.

The subject of fractional differential equations has emerged as an interesting and popular field of research in view of its extensive applications in applied and technical sciences. One can easily observe the role and importance of fractional calculus in several diverse disciplines such as physics, chemical processes, population dynamics, biotechnology, economics, etc. For examples and recent development on the topic, see [1–21] and the references cited therein. The significance of fractional derivatives owes to the fact that they serve as an excellent tool for the description of memory and hereditary properties of various materials and processes. One can notice that fractional derivatives are defined via fractional integrals. Among several types of fractional integral found in the literature, Riemann–Liouville and Hadamard fractional integrals are the most extensively studied. A new fractional integral, called *generalized Riemann–Liouville fractional integral*, which generalizes the Riemann–Liouville and Hadamard integrals into a single form, was introduced in [22] (see Definition 5). For more details of this integral and similar ones, we refer the reader to [23] and [24–27].

Several new existence and uniqueness results for problems (1)–(2) and (3)–(4) are proved by using a variety of fixed point theorems (such as Banach contraction principle, Krasnoselskii's fixed point theorem, Leray–Schauder nonlinear alternative). The rest of the paper is organized as follows: in Sect. 2 we recall some preliminary facts that we need in the sequel. In Sect. 3 we present our existence and uniqueness results. Examples illustrating the obtained results are presented in Sect. 4.

2 Preliminaries

In this section, we recall some basic concepts of fractional calculus [1, 2] and present known results needed in our forthcoming analysis.

Definition 1 The Riemann–Liouville fractional derivative of order q for a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$^{\mathrm{RL}}D^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_{0+}^t (t-s)^{n-q-1} f(s)\, ds, \quad q > 0, n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q, provided the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2 The Riemann–Liouville fractional integral of order q for a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$^{\mathrm{RL}}I^q f(t) = \frac{1}{\Gamma(q)} \int_{0+}^t (t-s)^{q-1} f(s)\, ds, \quad q > 0,$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

Definition 3 The Caputo derivative of fractional order q for an n-times differentiable function $f:(0,\infty)\to\mathbb{R}$ is defined as

$$^{C}D^{q}f(t)=\frac{1}{\Gamma(n-q)}\int_{0+}^{t}(t-s)^{n-q-1}\left(\frac{d}{ds}\right)^{n}f(s)\,ds,\quad q>0, n=[q]+1.$$

Definition 4 The Hadamard fractional integral of order q for a function $f:(0,\infty)\to\mathbb{R}$ is defined by

$$^{H}J^{q}f(t)=\frac{1}{\Gamma(q)}\int_{0+}^{t}\left(\log\frac{t}{s}\right)^{q-1}\frac{f(s)}{s}\,ds,\quad q>0,$$

provided the integral exists, where $\log(\cdot)=\log_{e}(\cdot)$.

Definition 5 The Katugampola fractional integral of order $q>0$ and $\rho>0$ of a function $f(t)$ for all $0<t<\infty$ is defined by

$$^{\rho}\tilde{J}^{q}f(t)=\frac{\rho^{1-q}}{\Gamma(q)}\int_{0+}^{t}\frac{s^{\rho-1}f(s)}{(t^{\rho}-s^{\rho})^{1-q}}\,ds,$$

provided the integral exists.

Remark 1 For $\rho=1$ in the above definition, we arrive at the standard Riemann–Liouville fractional integral, which is used to define both the Riemann–Liouville and Caputo fractional derivatives, while in the limit $\rho\to0^{+}$ we have

$$\lim_{\rho\to0^{+}}{}^{\rho}\tilde{J}^{q}f(t)=\frac{1}{\Gamma(q)}\int_{0+}^{t}\left(\log\frac{t}{s}\right)^{q-1}\frac{f(s)}{s}\,ds,$$

which is the famous Hadamard fractional integral; see [22].

Definition 6 The Erdélyi–Kober fractional integral of order $\delta>0$ with $\eta>0$ and $\gamma\in\mathbb{R}$ of a function $f:(0,\infty)\to\mathbb{R}$ is defined by

$$\hat{J}_{\eta}^{\gamma,\delta}f(t)=\frac{\eta t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)}\int_{0+}^{t}\frac{s^{\eta\gamma+\eta-1}f(s)}{(t^{\eta}-s^{\eta})^{1-\delta}}\,ds,$$

provided the integral exists.

Let $X_{c}^{p}(a,b), c\in\mathbb{R}, 1\le p\le\infty$ be the space of all complex-valued Lebesgue measurable functions ϕ on (a,b) for which $\|\phi\|_{X_{c}^{p}}<\infty$, with

$$\|\phi\|_{X_{c}^{p}}=\left(\int_{a}^{b}\left|x^{c}\phi(x)\right|^{p}\frac{dx}{x}\right)^{1/p},\quad 1\le p<\infty.$$

Definition 7 ([28]) Let $f\in X_{c}^{p}(a,b)$ with $a=0^{+}$. The generalized fractional integral of order $\alpha>0$ and constants $\beta,\rho,\eta,\kappa\in\mathbb{R}$ for a function $f:(0,\infty)\to\mathbb{R}$ is defined by

$$\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f\right)(t)=\frac{\rho^{1-\beta}t^{\kappa}}{\Gamma(\alpha)}\int_{0+}^{t}\frac{\tau^{\rho(\eta+1)-1}}{(t^{\rho}-\tau^{\rho})^{1-\alpha}}f(\tau)\,d\tau,\tag{5}$$

provided the integral exists.

Remark 2 The fractional integral (5) contains six well-known fractional integrals as its particular cases (see also [28]).

From (5) we have the following special cases:

(i) If $\beta = \alpha$, $\kappa = 0$, $\eta = 0$, then (2) can be reduced to

$$x(0) = \sum_{i=1}^{m} \gamma_i {}^{\bar{\rho}_i}\tilde{J}^{\bar{\alpha}_i} x(\xi_i), \qquad x(T) = \sum_{j=1}^{n} \sigma_j {}^{\rho_j}\tilde{J}^{\alpha_j} x(\delta_j), \tag{6}$$

which are the Katugampola fractional integral boundary conditions;

(ii) If $\rho = 1$, $\beta = \alpha$, $\kappa = 0$, $\eta = 0$, then (2) can be reduced to

$$x(0) = \sum_{i=1}^{m} \gamma_i {}^{\mathrm{RL}}I^{\bar{\alpha}_i} x(\xi_i), \qquad x(T) = \sum_{j=1}^{n} \sigma_j {}^{\mathrm{RL}}I^{\alpha_j} x(\delta_j), \tag{7}$$

which are the Riemann–Liouville fractional integral boundary conditions;

(iii) If $\rho \to 0$, $\beta = \alpha$, $\kappa = 0$, $\eta = 0$, then (2) can be reduced

$$x(0) = \sum_{i=1}^{m} \gamma_i {}^{H}J^{\bar{\alpha}_i} x(\xi_i), \qquad x(T) = \sum_{j=1}^{n} \sigma_j {}^{H}J^{\alpha_j} x(\delta_j), \tag{8}$$

which are the Hadamard fractional integral boundary conditions;

(iv) If $\beta = 0$, $\kappa = -\rho(\alpha + \eta)$, then (2) can be reduced to

$$x(0) = \sum_{i=1}^{m} \gamma_i \hat{J}_{\bar{\eta}_i}^{\bar{\alpha}_i,\bar{\rho}_i} x(\xi_i), \qquad x(T) = \sum_{j=1}^{n} \sigma_j \hat{J}_{\eta_j}^{\alpha_j,\rho_j} x(\delta_j), \tag{9}$$

which are the Erdélyi–Kober fractional integral boundary conditions.

Lemma 1 ([2]) *Let $q > 0$. Then for $y \in C(0,T) \cap L(0,T)$ it holds*

$${}^{\mathrm{RL}}I^q\big({}^{\mathrm{RL}}D^q y\big)(t) = y(t) + c_1 t^{q-1} + c_2 t^{q-2} + \cdots + c_n t^{q-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$ and $n - 1 < q < n$.

Lemma 2 ([2]) *Let $q > 0$. Then for $y \in C(0,T) \cap L(0,T)$ it holds*

$${}^{\mathrm{RL}}I^q\big({}^{C}D^q y\big)(t) = y(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$ and $n = [q] + 1$.

Lemma 3 *Let α, $\rho > 0$ and $\beta, \rho, \eta, \kappa \in \mathbb{R}$, $m > 0$ and $\rho(\alpha + \eta) + m + \kappa \geq 0$. Then we have*

$${}^{\rho}I_{\eta,\kappa}^{\alpha,\beta} t^m = \rho^{-\beta} \frac{\Gamma(\frac{\rho\eta+\rho+m}{\rho})}{\Gamma(\frac{\rho\eta+\rho\alpha+\rho+m}{\rho})} t^{\rho(\alpha+\eta)+m+\kappa}. \tag{10}$$

Proof Now we state the definition of the beta function and its property, which for $x, y > 0$ read

$$B(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}\,du \quad \text{and} \quad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

From Definition 7 and by changing the variable of integration, we can compute the following formula:

$$\begin{aligned}
{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta} t^m &= \frac{\rho^{1-\beta}t^{\kappa}}{\Gamma(\alpha)}\int_0^t \frac{\tau^{\rho(\eta+1)-1}}{(t^{\rho}-\tau^{\rho})^{1-\alpha}}\tau^m\,d\tau \\
&= \frac{\rho^{-\beta}t^{\rho(\alpha+\eta)+m+\kappa}}{\Gamma(\alpha)}\int_0^1 u^{(\frac{\rho\eta+\rho+m}{\rho})-1}(1-u)^{\alpha-1}\,du \\
&= \frac{\rho^{-\beta}t^{\rho(\alpha+\eta)+m+\kappa}}{\Gamma(\alpha)}B\left(\frac{\rho\eta+\rho+m}{\rho},\alpha\right) \\
&= \rho^{-\beta}\frac{\Gamma(\frac{\rho\eta+\rho+m}{\rho})}{\Gamma(\frac{\rho\eta+\rho\alpha+\rho+m}{\rho})}t^{\rho(\alpha+\eta)+m+\kappa}.
\end{aligned}$$

The proof is completed. □

Before going to prove the next lemma, for convenience, we set constants

$$\begin{aligned}
\Omega_1 &= \frac{\Gamma(q)}{\Gamma(q+r)}\sum_{i=1}^{m}\gamma_i\pi_{q+r-1}^{\bar{\rho}_i,\bar{\alpha}_i,\bar{\beta}_i,\bar{\eta}_i,\bar{\kappa}_i}(\xi_i), \\
\Omega_2 &= \sum_{i=1}^{m}\gamma_i\pi_0^{\bar{\rho}_i,\bar{\alpha}_i,\bar{\beta}_i,\bar{\eta}_i,\bar{\kappa}_i}(\xi_i) - 1, \\
\Omega_3 &= \frac{\Gamma(q)}{\Gamma(q+r)}\left(\sum_{j=1}^{n}\sigma_j\pi_{q+r-1}^{\rho_j,\alpha_j,\beta_j,\eta_j,\kappa_j}(\delta_j) - T^{q+r-1}\right), \\
\Omega_4 &= \sum_{j=1}^{n}\sigma_j\pi_0^{\rho_j,\alpha_j,\beta_j,\eta_j,\kappa_j}(\delta_j) - 1
\end{aligned}$$

and

$$\Omega = \Omega_1\Omega_4 - \Omega_2\Omega_3 \neq 0, \tag{11}$$

where

$$\pi_m^{\rho,\alpha,\beta,\eta,\kappa}(t) = \rho^{-\beta}\frac{\Gamma(\frac{\rho\eta+\rho+m}{\rho})}{\Gamma(\frac{\rho\eta+\rho\alpha+\rho+m}{\rho})}t^{\rho(\alpha+\eta)+\kappa+m}. \tag{12}$$

Lemma 4 *Let $0 < q, r \le 1$ with $1 < q + r \le 2$, $\bar{\rho}_i$, ρ_j, q, r, $\bar{\alpha}_i$, $\alpha_j > 0$, ξ_i, $\delta_j \in (0,T)$, $\bar{\beta}_i$, $\bar{\eta}_i$, $\bar{\kappa}_i$, β_j, η_j, $\kappa_j \in \mathbb{R}$ for $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, $\Omega \neq 0$ and $y \in C([0,T],\mathbb{R})$. The unique solution of the following linear sequential Riemann–Liouville and Caputo fractional differential equation*

$${}^{\mathrm{RL}}D^q\left({}^{C}D^r x\right)(t) = y(t), \quad t \in (0,T), \tag{13}$$

subject to the generalized fractional integral boundary conditions

$$x(0) = \sum_{i=1}^{m} \gamma_i {}^{\bar{\rho}_i} I_{\bar{\eta}_i,\bar{\kappa}_i}^{\bar{\alpha}_i,\bar{\beta}_i} x(\xi_i), \qquad x(T) = \sum_{j=1}^{n} \sigma_j {}^{\rho_j} I_{\eta_j,\kappa_j}^{\alpha_j,\beta_j} x(\delta_j), \tag{14}$$

is given by the integral equation

$$\begin{aligned} x(t) = \frac{1}{\Omega}\Bigg(& \left(\Omega_1 - \Omega_2 \frac{\Gamma(q)}{\Gamma(q+r)} t^{q+r-1}\right)\left({}^{\mathrm{RL}}I^{q+r}y\right)(T) \\ & + \left(\Omega_2 \frac{\Gamma(q)}{\Gamma(q+r)} t^{q+r-1} - \Omega_1\right) \sum_{j=1}^{n} \sigma_j \left[{}^{\rho_j} I_{\eta_j,\kappa_j}^{\alpha_j,\beta_j} \left({}^{\mathrm{RL}}I^{q+r}y\right)\right](\delta_j) \\ & + \left(\Omega_3 - \Omega_4 \frac{\Gamma(q)}{\Gamma(q+r)} t^{q+r-1}\right) \sum_{i=1}^{m} \gamma_i \left[{}^{\bar{\rho}_i} I_{\bar{\eta}_i,\bar{\kappa}_i}^{\bar{\alpha}_i,\bar{\beta}_i} \left({}^{\mathrm{RL}}I^{q+r}y\right)\right](\xi_i)\Bigg) + \left({}^{\mathrm{RL}}I^{q+r}y\right)(t). \end{aligned} \tag{15}$$

Proof Applying the Riemann–Liouville fractional integral of orders q and r, respectively, to both sides of (13) and using Lemmas 1 and 2, we have

$$x(t) = \left({}^{\mathrm{RL}}I^{q+r}y\right)(t) + c_1 \frac{\Gamma(q)}{\Gamma(q+r)} t^{q+r-1} + c_2, \tag{16}$$

where constants $c_1, c_2 \in \mathbb{R}$.

Using the nonlocal boundary condition (14) to the above equation with Lemma 3 and the above-set constants, we obtain the following linear system of constants c_1 and c_2:

$$\Omega_1 c_1 + \Omega_2 c_2 = -\sum_{i=1}^{m} \gamma_i \left[{}^{\bar{\rho}_i} I_{\bar{\eta}_i,\bar{\kappa}_i}^{\bar{\alpha}_i,\bar{\beta}_i} \left({}^{\mathrm{RL}}I^{q+r}y\right)\right](\xi_i),$$

$$\Omega_3 c_1 + \Omega_4 c_2 = {}^{\mathrm{RL}}I^{q+r}y(T) - \sum_{j=1}^{n} \sigma_j \left[{}^{\rho_j} I_{\eta_j,\kappa_j}^{\alpha_j,\beta_j} \left({}^{\mathrm{RL}}I^{q+r}y\right)\right](\delta_j).$$

Solving the above system of linear equations for the constants c_1, c_2, we have

$$\begin{aligned} c_1 &= \frac{1}{\Omega}\Bigg[\Omega_2 \sum_{j=1}^{n} \sigma_j \left[{}^{\rho_j} I_{\eta_j,\kappa_j}^{\alpha_j,\beta_j} \left({}^{\mathrm{RL}}I^{q+r}y\right)\right](\delta_j) - \Omega_2 \left({}^{\mathrm{RL}}I^{q+r}y\right)(T) \\ &\quad - \Omega_4 \sum_{i=1}^{m} \gamma_i \left[{}^{\bar{\rho}_i} I_{\bar{\eta}_i,\bar{\kappa}_i}^{\bar{\alpha}_i,\bar{\beta}_i} \left({}^{\mathrm{RL}}I^{q+r}y\right)\right](\xi_i)\Bigg], \\ c_2 &= \frac{1}{\Omega}\Bigg[\Omega_1 \left({}^{\mathrm{RL}}I^{q+r}y\right)(T) - \Omega_1 \sum_{j=1}^{n} \sigma_j \left[{}^{\rho_j} I_{\eta_j,\kappa_j}^{\alpha_j,\beta_j} \left({}^{\mathrm{RL}}I^{q+r}y\right)\right](\delta_j) \\ &\quad + \Omega_3 \sum_{i=1}^{m} \gamma_i \left[{}^{\bar{\rho}_i} I_{\bar{\eta}_i,\bar{\kappa}_i}^{\bar{\alpha}_i,\bar{\beta}_i} \left({}^{\mathrm{RL}}I^{q+r}y\right)\right](\xi_i)\Bigg]. \end{aligned}$$

Substituting constants c_1 and c_2 into (16), we obtain integral equation (15). The converse follows by direct computation. The proof is completed. □

Remark 3 Since $q + r > 1$, equation (16) is well defined when $t = 0$.

Lemma 5 *The linear sequential Caputo and Riemann–Liouville fractional differential equation*

$$ {}^{C}D^{q}\big({}^{\mathrm{RL}}D^{r}x\big)(t) = y(t), \quad t \in (0,T), \tag{17}$$

assuming (4), *can be written as an integral equation*

$$\begin{aligned} x(t) ={}& \frac{t^{q}}{\Gamma(q+1)\Omega^{*}}\left(\sum_{j=1}^{n}\sigma_{j}\big[{}^{\rho_{j}}I_{\eta_{j},\kappa_{j}}^{\alpha_{j},\beta_{j}}\big({}^{\mathrm{RL}}I^{q+r}y\big)\big](\delta_{j}) - \big({}^{\mathrm{RL}}I^{q+r}y\big)(T)\right) \\ &+ {}^{\mathrm{RL}}I^{q+r}y(t), \end{aligned} \tag{18}$$

where the constant $\Omega^{*} \neq 0$ *is defined by*

$$\Omega^{*} = T^{q} - \sum_{j=1}^{n}\sigma_{j}\pi_{q}^{\rho_{j},\alpha_{j},\beta_{j},\eta_{j},\kappa_{j}}(\delta_{j}).$$

Proof By taking the Riemann–Liouville fractional derivative of orders q and r, respectively, of (17), we obtain

$$x(t) = \big({}^{\mathrm{RL}}I^{q+r}y\big)(t) + c_{1}\frac{t^{q}}{\Gamma(q+1)} + c_{2}t^{q-1}. \tag{19}$$

Condition $x(0) = 0$ implies $c_2 = 0$. Applying the boundary condition (4) and using the same method as in Lemma 4 for finding a constant c_1, we obtain (18) as desired. This completes the proof. □

Remark 4 If $c_2 \neq 0$, then (19) is singular in the case $t = 0$ and $q \in (0,1)$.

The following fixed point theorems are fundamental in the proofs of our main results.

Lemma 6 (Krasnoselskii's fixed point theorem, [29]) *Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that* (a) $Ax + By \in M$ *whenever* $x, y \in M$; (b) *A is compact and continuous*; (c) *B is a contraction mapping. Then there exists $z \in M$ such that $z=Az+Bz$.*

Lemma 7 (Nonlinear alternative for single-valued maps, [30]) *Let E be a Banach space, C be a closed, convex subset of E, U be an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

3 Main results

In this section, we will use fixed point theorems to prove the existence and uniqueness of solutions for problems (1)–(2) and (3)–(4). Throughout this paper, for convenience, we use the abbreviate notations

$$\big({}^{\mathrm{RL}}I^{q+r}f_{x}\big)(z) = \frac{1}{\Gamma(q+r)}\int_{0+}^{z}(z-s)^{q+r-1}f\big(s,x(s)\big)\,ds \quad \text{for } z \in [0,T]$$

and

$$\left[{}^{\tilde{\rho}}I_{\tilde{\eta},\tilde{\kappa}}^{\tilde{\alpha},\tilde{\beta}}\left({}^{\mathrm{RL}}I^{q+r}f_x\right)\right](\nu) = \frac{\tilde{\rho}\nu^{\tilde{\kappa}}}{\Gamma(\tilde{\alpha})\Gamma(q+r)}\int_{0+}^{\nu}\int_{0+}^{t}\frac{\nu^{\tilde{\rho}(\tilde{\eta}+1)-1}}{(t^{\tilde{\rho}}-\nu^{\tilde{\rho}})^{1-\tilde{\alpha}}}(t-s)^{q+r-1}f\bigl(s,x(s)\bigr)\,ds\,dt,$$

for $\nu \in [0,T]$, where $z \in \{t,T\}$, $\nu \in \{\xi_i,\delta_j\}$, $\tilde{\rho} \in \{\bar{\rho}_i,\rho_j\}$, $\tilde{\alpha} \in \{\bar{\alpha}_i,\alpha_j\}$, $\tilde{\beta} \in \{\bar{\beta}_i,\beta_j\}$, $\tilde{\eta} \in \{\bar{\eta}_i,\eta_j\}$, $\tilde{\kappa} \in \{\bar{\kappa}_i,\kappa_j\}$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$.

Let $\mathcal{C} = C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup_{t\in[0,T]}|x(t)|$. By Lemma 4, we define an operator $\mathcal{F}:\mathcal{C}\to\mathcal{C}$ by

$$\begin{aligned}
(\mathcal{F}x)(t) &= \frac{1}{\Omega}\Biggl(\biggl(\Omega_1 - \Omega_2\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1}\biggr)\left({}^{\mathrm{RL}}I^{q+r}f_x\right)(T) \\
&\quad + \biggl(\Omega_2\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} - \Omega_1\biggr)\sum_{j=1}^{n}\sigma_j\left[{}^{\rho_j}I_{\eta_j,\kappa_j}^{\alpha_j,\beta_j}\left({}^{\mathrm{RL}}I^{q+r}f_x\right)\right](\delta_j) \\
&\quad + \biggl(\Omega_3 - \Omega_4\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1}\biggr)\sum_{i=1}^{m}\gamma_i\left[{}^{\bar{\rho}_i}I_{\bar{\eta}_j,\bar{\kappa}_i}^{\bar{\alpha}_i,\bar{\beta}_i}\left({}^{\mathrm{RL}}I^{q+r}f_x\right)\right](\xi_i)\Biggr) \\
&\quad + \left({}^{\mathrm{RL}}I^{q+r}f_x\right)(t),
\end{aligned} \tag{20}$$

with $\Omega \neq 0$. It should be noticed that problem (1)–(2) has solutions if and only if the operator $\mathcal{F}$ has fixed points. For the sake of convenience, we put a constant

$$\begin{aligned}
\Phi &= \frac{1}{\Gamma(q+r+1)|\Omega|}\Biggl(|\Omega_1|T^{q+r} + |\Omega_2|\frac{\Gamma(q)}{\Gamma(q+r)}T^{2q+2r-1} \\
&\quad + \biggl[|\Omega_2|\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1} + |\Omega_1|\biggr]\sum_{j=1}^{n}|\sigma_j|\pi_{q+r}^{\rho_j,\alpha_j,\beta_j,\eta_j,\kappa_j}(\delta_j) \\
&\quad + \biggl[|\Omega_3| + |\Omega_4|\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1}\biggr]\sum_{i=1}^{m}|\gamma_i|\pi_{q+r}^{\bar{\rho}_i,\bar{\alpha}_i,\bar{\beta}_i,\bar{\eta}_i,\bar{\kappa}_i}(\xi_i)\Biggr) \\
&\quad + \frac{T^{q+r}}{\Gamma(q+r+1)}.
\end{aligned} \tag{21}$$

To prove the existence theorems for problem (3)–(4), by Lemma 5, we define an operator $\mathcal{H}:\mathcal{C}\to\mathcal{C}$ by

$$\begin{aligned}
(\mathcal{H}x)(t) &= \frac{t^q}{\Gamma(q+1)\Omega^*}\Biggl(\sum_{j=1}^{n}\sigma_j\left[{}^{\rho_j}I_{\eta_j,\kappa_j}^{\alpha_j,\beta_j}\left({}^{\mathrm{RL}}I^{q+r}f_x\right)\right](\delta_j) - \left({}^{\mathrm{RL}}I^{q+r}f_x\right)(T)\Biggr) \\
&\quad + \left({}^{\mathrm{RL}}I^{q+r}f_x\right)(t), \quad \Omega^* \neq 0.
\end{aligned} \tag{22}$$

The first existence and uniqueness result is based on the Banach contraction mapping principle.

Theorem 1 *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the following assumption*:

(H_1) *There exists a constant $L > 0$ such that $|f(t,x) - f(t,y)| \leq L|x-y|$, for each $t \in [0,T]$ and $x, y \in \mathbb{R}$.*

If

$$L\Phi < 1, \tag{23}$$

where a constant Φ is given by (21), then the boundary value problem (1)–(2) has a unique solution on $[0,T]$.

Proof Problem (1)–(2) can be transformed into a fixed point problem, $x = \mathcal{F}x$, where the operator $\mathcal{F}$ is defined by (20). By using the Banach's contraction mapping principle, we shall show that $\mathcal{F}$ has a fixed point which is the unique solution of problem (1)–(2).

Let us set $\sup_{t\in[0,T]} |f(t,0)| = M < \infty$ and choose

$$r \geq \frac{M\Phi}{1 - L\Phi},$$

as a radius of the ball B_r, where $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$. From inequality (23), a constant r is well defined. Now, we show that $\mathcal{F}B_r \subset B_r$. For any $x \in B_r$, and taking into account Lemma 3, we obtain

$$\begin{aligned}
\|\mathcal{F}x\| = {} & \sup_{t\in[0,T]} \Bigg| \frac{1}{\Omega}\Bigg(\bigg(\Omega_1 - \Omega_2 \frac{\Gamma(q)}{\Gamma(q+r)} t^{q+r-1}\bigg)\big({}^{\mathrm{RL}}I^{q+r} f_x\big)(T) \\
& + \bigg(\Omega_2 \frac{\Gamma(q)}{\Gamma(q+r)} t^{q+r-1} - \Omega_1\bigg) \sum_{j=1}^{n} \sigma_j \big[{}^{\rho_j}I_{\eta_j,\kappa_j}^{\alpha_j,\beta_j}\big({}^{\mathrm{RL}}I^{q+r} f_x\big)\big](\delta_j) \\
& + \bigg(\Omega_3 - \Omega_4 \frac{\Gamma(q)}{\Gamma(q+r)} t^{q+r-1}\bigg) \sum_{i=1}^{m} \gamma_i \big[{}^{\bar{\rho}_i}I_{\bar{\eta}_i,\bar{\kappa}_i}^{\bar{\alpha}_i,\bar{\beta}_i}\big({}^{\mathrm{RL}}I^{q+r} f_x\big)\big](\xi_i)\Bigg) + \big({}^{\mathrm{RL}}I^{q+r} f_x\big)(t)\Bigg| \\
\leq {} & \frac{1}{|\Omega|}\Bigg(\bigg[|\Omega_1| + |\Omega_2| \frac{\Gamma(q)}{\Gamma(q+r)} T^{q+r-1}\bigg]\big[\big({}^{\mathrm{RL}}I^{q+r}\big(|f_x - f_0| + |f_0|\big)\big)(T)\big] \\
& + \bigg[|\Omega_2| \frac{\Gamma(q)}{\Gamma(q+r)} T^{q+r-1} + |\Omega_1|\bigg] \\
& \times \sum_{j=1}^{n} |\sigma_j| \big[{}^{\rho_j}I_{\eta_j,\kappa_j}^{\alpha_j,\beta_j}\big({}^{\mathrm{RL}}I^{q+r}\big(|f_x - f_0| + |f_0|\big)\big)\big](\delta_j) \\
& + \bigg[|\Omega_3| + |\Omega_4| \frac{\Gamma(q)}{\Gamma(q+r)} T^{q+r-1}\bigg] \\
& \times \sum_{i=1}^{m} |\gamma_i| \big[{}^{\bar{\rho}_i}I_{\bar{\eta}_i,\bar{\kappa}_i}^{\bar{\alpha}_i,\bar{\beta}_i}\big({}^{\mathrm{RL}}I^{q+r}\big(|f_x - f_0| + |f_0|\big)\big)\big](\xi_i)\Bigg) + \big({}^{\mathrm{RL}}I^{q+r}\big(|f_x - f_0| + |f_0|\big)\big)(t) \\
\leq {} & (Lr + M)\Bigg\{\frac{1}{\Gamma(q+r+1)|\Omega|}\Bigg(|\Omega_1| T^{q+r} + |\Omega_2| \frac{\Gamma(q)}{\Gamma(q+r)} T^{2q+2r-1} \\
& + \bigg[|\Omega_2| \frac{\Gamma(q)}{\Gamma(q+r)} T^{q+r-1} + |\Omega_1|\bigg] \sum_{j=1}^{n} |\sigma_j| \pi_{q+r}^{\rho_j,\alpha_j,\beta_j,\eta_j,\kappa_j}(\delta_j)
\end{aligned}$$

$$
\begin{aligned}
&+\left[|\Omega_3|+|\Omega_4|\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1}\right]\sum_{i=1}^{m}|\gamma_i|\pi_{q+r}^{\bar{\rho}_i,\bar{\alpha}_i,\bar{\beta}_i,\bar{\eta}_i,\bar{\kappa}_i}(\xi_i)\Bigg)+\frac{T^{q+r}}{\Gamma(q+r+1)}\Bigg\}\\
&=(Lr+M)\Phi\leq r,
\end{aligned}
$$

which gives $\mathcal{F}B_r\subset B_r$.

For $x,y\in\mathcal{C}$ and for each $t\in[0,T]$, we obtain

$$
\begin{aligned}
&\big|\mathcal{F}x(t)-\mathcal{F}y(t)\big|\\
&\leq\frac{1}{|\Omega|}\Bigg(\left[|\Omega_1|+|\Omega_2|\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1}\right]\big({}^{\mathrm{RL}}I^{q+r}\big(|f_x-f_y|\big)\big)(T)\\
&\quad+\left[|\Omega_2|\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1}+|\Omega_1|\right]\\
&\quad\times\sum_{j=1}^{n}|\sigma_j|\big[{}^{\rho_j}I_{\eta_j,\kappa_j}^{\alpha_j,\beta_j}\big({}^{\mathrm{RL}}I^{q+r}\big(|f_x-f_y|\big)\big)\big](\delta_j)\\
&\quad+\left[|\Omega_3|+|\Omega_4|\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1}\right]\\
&\quad\times\sum_{i=1}^{m}|\gamma_i|\big[{}^{\bar{\rho}_i}I_{\bar{\eta}_i,\bar{\kappa}_i}^{\bar{\alpha}_i,\bar{\beta}_i}\big({}^{\mathrm{RL}}I^{q+r}\big(|f_x-f_y|\big)\big)\big](\xi_i)\Bigg)\\
&\quad+\big({}^{\mathrm{RL}}I^{q+r}\big(|f_x-f_y|\big)\big)(t)\\
&\leq L\|x-y\|\Bigg\{\frac{1}{\Gamma(q+r+1)|\Omega|}\Bigg(|\Omega_1|T^{q+r}+|\Omega_2|\frac{\Gamma(q)}{\Gamma(q+r)}T^{2q+2r-1}\\
&\quad+\left[|\Omega_2|\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1}+|\Omega_1|\right]\sum_{j=1}^{n}|\sigma_j|\pi_{q+r}^{\rho_j,\alpha_j,\beta_j,\eta_j,\kappa_j}(\delta_j)\\
&\quad+\left[|\Omega_3|+|\Omega_4|\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1}\right]\sum_{i=1}^{m}|\gamma_i|\pi_{q+r}^{\bar{\rho}_i,\bar{\alpha}_i,\bar{\beta}_i,\bar{\eta}_i,\bar{\kappa}_i}(\xi_i)\Bigg)+\frac{T^{q+r}}{\Gamma(q+r+1)}\Bigg\}\\
&=L\Phi\|x-y\|.
\end{aligned}
$$

The above result leads to $\|\mathcal{F}x-\mathcal{F}y\|\leq L\Phi\|x-y\|$. As $L\Phi<1$, therefore the operator $\mathcal{F}$ is a contraction. Hence, by the Banach contraction mapping principle, we deduce that $\mathcal{F}$ has a fixed point which is the unique solution of the problem (1)–(2). The proof is completed. □

Corollary 1 *Let condition* (H_1) *in Theorem* (1) *hold. If* $L\Phi^*<1$, *where* Φ^* *is defined by*

$$
\Phi^*=\frac{T^q}{|\Omega^*|\Gamma(q+1)\Gamma(q+r+1)}\left(\sum_{j=1}^{n}|\sigma_j|\pi_{q+r}^{\rho_j,\alpha_j,\beta_j,\eta_j,\kappa_j}(\delta_j)+T^{q+r}\right)+\frac{T^{q+r}}{\Gamma(q+r+1)},
$$

then the boundary value problem (3)–(4) *has a unique solution on* $[0,T]$.

Next, we give the second existence theorem by using Krasnoselskii's fixed point theorem.

Setting a constant

$$\Phi_1 = \frac{T^{q+r}}{\Gamma(q+r+1)} + \frac{1}{|\Omega|\Gamma(q+r+1)}\left(|\Omega_1|T^{q+r} + |\Omega_2|\frac{\Gamma(q)}{\Gamma(q+r)}T^{2q+2r+1}\right).$$

Note that $\Phi_1 \leq \Phi$. Now, we state and prove the second result.

Theorem 2 *Assume that $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies assumption (H_1) of Theorem 1. In addition we suppose that:*

(H_2) $|f(t,x)| \leq \phi(t)$, $\forall (t,x) \in [0,T] \times \mathbb{R}$ *and* $\phi \in C([0,T],\mathbb{R}^+)$.

If the inequality

$$\Phi_1 L < 1 \tag{24}$$

holds, then the boundary value problem (1)–(2) has at least one solution on $[0,T]$.

Proof Let us define a suitable ball $B_{\bar{r}} = \{x \in \mathcal{C} : \|x\| \leq \bar{r}\}$, where the radius $\bar{r}$ is defined by

$$\bar{r} \geq \|\phi\|\Phi,$$

with $\sup_{t\in[0,T]} |\phi(t)| = \|\phi\|$ and Φ defined by (21). Furthermore, we define two operators $\mathcal{P}$ and $\mathcal{Q}$ on $B_{\bar{r}}$ as

$$\begin{aligned}(\mathcal{P}x)(t) &= \frac{1}{\Omega}\left(\left(\Omega_2\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} - \Omega_1\right)\sum_{j=1}^{n}\sigma_j\left[{}^{\rho_j}I_{\eta_j,\phi_j}^{\alpha_j,\beta_j}\left({}^{\mathrm{RL}}I^{q+r}f_x\right)\right](\delta_j)\right. \\ &\quad \left. + \left(\Omega_3 - \Omega_4\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1}\right)\sum_{i=1}^{m}\gamma_i\left[{}^{\bar{\rho}_i}I_{\bar{\eta}_i,\bar{\kappa}_i}^{\bar{\alpha}_i,\bar{\beta}_i}\left({}^{\mathrm{RL}}I^{q+r}f_x\right)\right](\xi_j)\right), \\ (\mathcal{Q}x)(t) &= \frac{1}{\Omega}\left(\left(\Omega_1 - \Omega_2\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1}\right)\left({}^{\mathrm{RL}}I^{q+r}f_x\right)(T)\right) \\ &\quad + \left({}^{\mathrm{RL}}I^{q+r}f_x\right)(t), \quad t \in [0,T].\end{aligned}$$

Observe that $\mathcal{F}x = \mathcal{P}x + \mathcal{Q}x$. For $x, y \in B_{\bar{r}}$, we have

$$\begin{aligned}&\|\mathcal{P}x + \mathcal{Q}y\| \\ &\quad \leq \|\phi\|\left\{\frac{1}{\Gamma(q+r+1)|\Omega|}\left(|\Omega_1|T^{q+r} + |\Omega_2|\frac{\Gamma(q)}{\Gamma(q+r)}T^{2q+2r-1}\right.\right. \\ &\qquad + \left[|\Omega_2|\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1} + |\Omega_1|\right]\sum_{j=1}^{n}|\sigma_j|\pi_{q+r}^{\rho_j,\alpha_j,\beta_j,\eta_j,\kappa_j}(\delta_j) \\ &\qquad \left.\left. + \left[|\Omega_3| + |\Omega_4|\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1}\right]\sum_{i=1}^{m}|\gamma_i|\pi_{q+r}^{\bar{\rho}_i,\bar{\alpha}_i,\bar{\beta}_i,\bar{\eta}_i,\bar{\kappa}_i}(\xi_i)\right) + \frac{T^{q+r}}{\Gamma(q+r+1)}\right\} \\ &\quad = \|\phi\|\Phi \\ &\quad \leq \bar{r}.\end{aligned}$$

This shows that $\mathcal{P}x + \mathcal{Q}y \in B_{\bar{r}}$. Therefore, condition (a) of Lemma 6 holds. Using assumption (H_1) with the inequality in (24), we deduce that operator $\mathcal{Q}$ is a contraction mapping which satisfies condition (c) of Lemma 6.

Now, we will show that operator $\mathcal{P}$ satisfies condition (b) of Lemma 6. Since f is a continuous function, we have that operator $\mathcal{P}$ is continuous. Next, we prove compactness of operator $\mathcal{P}$. It is easy to verify that

$$\begin{aligned}\|\mathcal{P}x\| \le \|\phi\| \Bigg\{ \frac{1}{\Gamma(q+r+1)|\Omega|} \Bigg(\bigg[|\Omega_2| \frac{\Gamma(q)}{\Gamma(q+r)} T^{q+r-1} + |\Omega_1| \bigg] \sum_{j=1}^{n} |\sigma_j| \pi_{q+r}^{\rho_j,\alpha_j,\beta_j,\eta_j,\kappa_j}(\delta_j) \\ + \bigg[|\Omega_3| + |\Omega_4| \frac{\Gamma(q)}{\Gamma(q+r)} T^{q+r-1} \bigg] \sum_{i=1}^{m} |\gamma_i| \pi_{q+r}^{\bar{\rho}_i,\bar{\alpha}_i,\bar{\beta}_i,\bar{\eta}_i,\bar{\kappa}_i}(\xi_i) \Bigg) \Bigg\}.\end{aligned}$$

Hence, $\mathcal{P}(B_{\bar{r}})$ is a uniformly bounded set. Let us put $\sup_{(t,x)\in[0,T]\times B_{\bar{r}}} |f(t,x)| = \bar{f} < \infty$. Consequently, we get

$$\begin{aligned}&\big|(\mathcal{P}x)(t_1) - (\mathcal{P}x)(t_2)\big| \\ &= \Bigg| \frac{1}{\Omega} \Bigg\{ \bigg(\Omega_2 \frac{\Gamma(q)}{\Gamma(q+r)} t_1^{q+r-1} - \Omega_1 \bigg) \sum_{j=1}^{n} \sigma_j \big[{}^{\rho_j} I_{\eta_j,\kappa_j}^{\alpha_j,\beta_j} \big({}^{\mathrm{RL}} I^{q+r} f_x \big) \big](\delta_j) \\ &\quad + \bigg(\Omega_3 - \Omega_4 \frac{\Gamma(q)}{\Gamma(q+r)} t_1^{q+r-1} \bigg) \sum_{i=1}^{m} \gamma_i \big[{}^{\bar{\rho}_i} I_{\bar{\eta}_i,\bar{\kappa}_i}^{\bar{\alpha}_i,\bar{\beta}_i} \big({}^{\mathrm{RL}} I^{q+r} f_x \big) \big](\xi_i) \Bigg\} \\ &\quad - \frac{1}{\Omega} \Bigg\{ \bigg(\Omega_2 \frac{\Gamma(q)}{\Gamma(q+r)} t_2^{q+r-1} - \Omega_1 \bigg) \sum_{j=1}^{n} \sigma_j \big[{}^{\rho_j} I_{\eta_j,\kappa_j}^{\alpha_j,\beta_j} \big({}^{\mathrm{RL}} I^{q+r} f_x \big) \big](\delta_j) \\ &\quad + \bigg(\Omega_3 - \Omega_4 \frac{\Gamma(q)}{\Gamma(q+r)} t_2^{q+r-1} \bigg) \sum_{i=1}^{m} \gamma_i \big[{}^{\bar{\rho}_i} I_{\bar{\eta}_i,\bar{\kappa}_i}^{\bar{\alpha}_i,\bar{\beta}_i} \big({}^{\mathrm{RL}} I^{q+r} f_x \big) \big](\xi_i) \Bigg\} \Bigg| \\ &\le \frac{1}{|\Omega|} \Bigg[|\Omega_2| \frac{\Gamma(q)}{\Gamma(q+r)} \big| t_1^{q+r-1} - t_2^{q+r-1} \big| \bar{f} \sum_{j=1}^{n} |\sigma_j| \pi_{q+r}^{\rho_j,\alpha_j,\beta_j,\eta_j,\kappa_j}(\delta_j) \\ &\quad + |\Omega_4| \frac{\Gamma(q)}{\Gamma(q+r)} \big| t_2^{q+r-1} - t_1^{q+r-1} \big| \bar{f} \sum_{i=1}^{m} |\gamma_i| \pi_{q+r}^{\bar{\rho}_i,\bar{\alpha}_i,\bar{\beta}_i,\bar{\eta}_i,\bar{\kappa}_i}(\xi_i) \Bigg],\end{aligned}$$

which is independent of x and tends to zero as $t_2 \to t_1$. Hence, the set $\mathcal{P}(B_{\bar{r}})$ is equicontinuous. Hence, by the Arzelá–Ascoli theorem, the set $\mathcal{P}(B_{\bar{r}})$ is relatively compact. Therefore, the operator $\mathcal{P}$ is compact on $B_{\bar{r}}$. Thus all the assumptions of Lemma 6 are satisfied. Then the boundary value problem (1)–(2) has at least one solution on $[0, T]$. The proof is completed. □

Remark 5 In the above theorem, we can interchange the roles of operators $\mathcal{P}$ and $\mathcal{Q}$ to obtain a second result, replacing (24) by the following condition:

$$\begin{aligned}&\frac{L}{|\Omega|\Gamma(q+r+1)} \Bigg(\bigg[|\Omega_2| \frac{\Gamma(q)}{\Gamma(q+r)} T^{q+r-1} + |\Omega_1| \bigg] \sum_{j=1}^{n} |\sigma_j| \pi_{q+r}^{\rho_j,\alpha_j,\beta_j,\eta_j,\kappa_j}(\delta_j) \\ &\quad + \bigg[|\Omega_3| + |\Omega_4| \frac{\Gamma(q)}{\Gamma(q+r)} T^{q+r-1} \bigg] \sum_{i=1}^{m} |\gamma_i| \pi_{q+r}^{\bar{\rho}_i,\bar{\alpha}_i,\bar{\beta}_i,\bar{\eta}_i,\bar{\kappa}_i}(\xi_i) \Bigg) < 1.\end{aligned}$$

Corollary 2 *Assume that* (H_1) *and* (H_2) *are fulfilled. If either*

$$\frac{LT^{q+r}}{\Gamma(q+r+1)} < 1,$$

or

$$\frac{LT^{q}}{|\Omega^*|\Gamma(q+1)\Gamma(q+r+1)}\left(\sum_{j=1}^{n}|\sigma_j|\pi_{q+r}^{\rho_j,\alpha_j,\beta_j,\eta_j,\kappa_j}(\delta_j)+T^{q+r}\right) < 1,$$

holds, then the boundary value problem (3)–(4) *has at least one solution on* $[0,T]$.

Now, our third existence theorem will be proved by using the Leray–Schauder's Nonlinear Alternative.

Theorem 3 *Assume that* $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ *is a continuous function. In addition, we suppose that*:

(H_3) *There exist a continuous nondecreasing function* $\psi:[0,\infty)\to(0,\infty)$ *and a function* $p\in C([0,T],\mathbb{R}^+)$ *such that*

$$\left|f(t,x)\right|\le p(t)\psi\left(\|x\|\right)\quad \textit{for each } (t,x)\in[0,T]\times\mathbb{R};$$

(H_4) *There exists a constant* $N>0$ *such that*

$$\frac{N}{\|p\|\psi(N)\Phi} > 1,$$

where Φ *is defined by* (21).

Then the boundary value problem (1)–(2) *has at least one solution on* $[0,T]$.

Proof To apply Lemma 7, we define a bounded ball in $\mathcal{C}$ by $B_R=\{x\in\mathcal{C}:\|x\|\le R\}$, $R>0$. Now, we shall show that the operator $\mathcal{F}$ defined by (20), maps bounded sets B_R into bounded sets in $\mathcal{C}$. For $t\in[0,T]$ we have

$$\begin{aligned}
&\left|\mathcal{F}x(t)\right|\\
&\quad\le \frac{1}{|\Omega|}\Bigg(\left[|\Omega_1|+|\Omega_2|\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1}\right]\left({}^{\mathrm{RL}}I^{q+r}\left(|f_x|\right)\right)(T)\\
&\qquad+\left[|\Omega_2|\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1}+|\Omega_1|\right]\sum_{j=1}^{n}|\sigma_j|\left[{}^{\rho_j}I_{\eta_j,\kappa_j}^{\alpha_j,\beta_j}\left({}^{\mathrm{RL}}I^{q+r}\left(|f_x|\right)\right)\right](\delta_j)\\
&\qquad+\left[|\Omega_3|+|\Omega_4|\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1}\right]\sum_{i=1}^{m}|\gamma_i|\left[{}^{\bar{\rho}_i}I_{\bar{\eta}_i,\bar{\kappa}_i}^{\bar{\alpha}_i,\bar{\beta}_i}\left({}^{\mathrm{RL}}I^{q+r}\left(|f_x|\right)\right)\right](\xi_i)\Bigg)\\
&\qquad+\left({}^{\mathrm{RL}}I^{q+r}\left(|f_x|\right)\right)(t)\\
&\quad\le\|p\|\psi\left(\|x\|\right)\Bigg\{\frac{1}{\Gamma(q+r+1)|\Omega|}\Bigg(|\Omega_1|T^{q+r}+|\Omega_2|\frac{\Gamma(q)}{\Gamma(q+r)}T^{2q+2r-1}
\end{aligned}$$

$$
\begin{aligned}
&+\left[|\Omega_2|\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1}+|\Omega_1|\right]\sum_{j=1}^{n}|\sigma_j|\pi_{q+r}^{\rho_j,\alpha_j,\beta_j,\eta_j,\kappa_j}(\delta_j)\\
&+\left[|\Omega_3|+|\Omega_4|\frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1}\right]\sum_{i=1}^{m}|\gamma_i|\pi_{q+r}^{\bar{\rho}_i,\bar{\alpha}_i,\bar{\beta}_i,\bar{\eta}_i,\bar{\kappa}_i}(\xi_i)\Bigg)+\frac{T^{q+r}}{\Gamma(q+r+1)}\Bigg\}\\
&\leq \|p\|\psi(R)\Phi.
\end{aligned}
$$

Therefore, we conclude that $\|\mathcal{F}x\| \leq \|p\|\psi(R)\Phi$, which implies that the set $\mathcal{F}(B_R)$ is uniformly bounded.

Next, we will show that $\mathcal{F}$ maps a bounded set B_R into an equicontinuous set in $\mathcal{C}$. Let $\nu_1, \nu_2 \in [0, T]$ with $\nu_1 < \nu_2$ and for any $x \in B_R$. Then we have

$$
\begin{aligned}
&\big|(\mathcal{F}x)(\nu_2)-(\mathcal{F}x)(\nu_1)\big|\\
&\leq \frac{1}{|\Omega|}\Bigg(\bigg(|\Omega_2|\frac{\Gamma(q)}{\Gamma(q+r)}\big|{\nu_1}^{q+r-1}-{\nu_2}^{q+r-1}\big|\bigg)\big({}^{\mathrm{RL}}I^{q+r}|f_x|\big)(T)\\
&+\bigg(|\Omega_2|\frac{\Gamma(q)}{\Gamma(q+r)}\big|{\nu_2}^{q+r-1}-{\nu_1}^{q+r-1}\big|\bigg)\sum_{j=1}^{n}\sigma_j\big[{}^{\rho_j}I_{\eta_j,\kappa j}^{\alpha_j,\beta_j}\big({}^{\mathrm{RL}}I^{q+r}|f_x|\big)\big](\delta_j)\\
&+\bigg(|\Omega_4|\frac{\Gamma(q)}{\Gamma(q+r)}\big|{\nu_1}^{q+r-1}-{\nu_2}^{q+r-1}\big|\bigg)\sum_{i=1}^{m}\gamma_i\big[{}^{\bar{\rho}_i}I_{\bar{\eta}_i,\bar{\kappa}_i}^{\bar{\alpha}_i,\bar{\beta}_i}\big({}^{\mathrm{RL}}I^{q+r}|f_x|\big)\big](\xi_i)\Bigg)\\
&+\frac{\|p\|\psi(R)}{\Gamma(q+r)}\bigg|\int_0^{\nu_1}\big[(\nu_2-s)^{q+r-1}-(\nu_1-s)^{q+r-1}\big]\,ds\bigg|\\
&+\frac{\|p\|\psi(R)}{\Gamma(q+r)}\bigg|\int_{\nu_1}^{\nu_2}(\nu_2-s)^{q+r-1}\,ds\bigg|\\
&\leq \|p\|\psi(R)\Bigg\{\frac{1}{|\Omega|}\Bigg(\bigg(|\Omega_2|\frac{\Gamma(q)}{\Gamma(q+r)}\big|{\nu_1}^{q+r-1}-{\nu_2}^{q+r-1}\big|\bigg)\frac{T^{q+r}}{\Gamma(q+r+1)}\\
&+\bigg(|\Omega_2|\frac{\Gamma(q)}{\Gamma(q+r)}\big|{\nu_2}^{q+r-1}-{\nu_1}^{q+r-1}\big|\bigg)\sum_{j=1}^{n}|\sigma_j|\pi_{q+r}^{\rho_j,\alpha_j,\beta_j,\eta_j,\kappa_j}(\delta_j)\\
&+\bigg(|\Omega_4|\frac{\Gamma(q)}{\Gamma(q+r)}\big|{\nu_1}^{q+r-1}-{\nu_2}^{q+r-1}\big|\bigg)\sum_{i=1}^{m}|\gamma_i|\pi_{q+r}^{\bar{\rho}_i,\bar{\alpha}_i,\bar{\beta}_i,\bar{\eta}_i,\bar{\kappa}_i}(\xi_i)\Bigg)\\
&+\frac{1}{\Gamma(q+r+1)}\big[\big|\nu_2^{q+r}-\nu_1^{q+r}\big|+2(\nu_2-\nu_1)^{q+r}\big]\Bigg\}.
\end{aligned}
$$

Obviously, the right-hand side of the above inequality tends to zero independently of $x \in B_R$ as $\nu_2 \to \nu_1$. Thus $\mathcal{F}(B_R)$ is an equicontinuous set. Therefore, it follows by the Arzelá–Ascoli theorem that $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ is completely continuous.

Let x be a solution of boundary value problem (1)–(2). Hence, for $t \in [0, T]$, and using the above method, we have

$$
\|x\| \leq \|p\|\psi\big(\|x\|\big)\Phi,
$$

which can be written as

$$\frac{\|x\|}{\|p\|\psi(\|x\|)\Phi} \leq 1.$$

In view of (H_4), there exists an N such that $\|x\| \neq N$. Now we define a set

$$U = \{x \in B_R : \|x\| < N\}. \tag{25}$$

Note that the operator $\mathcal{F} : \overline{U} \to \mathcal{C}$ is continuous and compact. From the choice of U, there is no $x \in \partial U$ such that $x = \theta \mathcal{F} x$ for some $\theta \in (0,1)$. Consequently, by the nonlinear alternative of Leray–Schauder type (Lemma 7) we get that $\mathcal{F}$ has a fixed point in $\overline{U}$, which is a solution of the boundary value problem (1)–(2). This completes the proof. □

Corollary 3 *Let condition* (H_3) *in Theorem* 3 *hold. If there exists a constant* $N > 0$ *such that*

$$\frac{N}{\|p\|\psi(N)\Phi^*} > 1,$$

then the boundary value problem (3)–(4) *has at least one solution on* $[0,T]$.

The following corollary is obtained by substituting $p(t) \equiv 1$ and $\psi(|x|) = M|x| + K$. Then we can use the following assumption.

(H_5) There exist constants $M > 0$ and $K \geq 0$ such that

$$|f(t,x)| \leq M|x| + K \quad \textit{for each } (t,x) \in [0,T] \times \mathbb{R}.$$

Corollary 4 *Assume that a continuous function* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *satisfies condition* (H_5).

(i) *If* $M\Phi < 1$, *then boundary value problem* (1)–(2) *has at least one solution on* $[0,T]$.

(ii) *If* $M\Phi^* < 1$, *then the boundary value problem* (3)–(4) *has at least one solution on* $[0,T]$.

4 Examples

Example 1 Consider the following nonlinear sequential Riemann–Liouville and Caputo fractional differential equation with generalized fractional integral conditions:

$$\begin{cases} {}^{\mathrm{RL}}D^{\frac{1}{2}}({}^{C}D^{\frac{3}{4}}x)(t) = \frac{\cos^2(2\pi t)}{(t^2+5)^2+35} \cdot (\frac{x^2(t)+2|x(t)|}{|x(t)|+1}) + e^t, \quad 0 < t < 5, \\ x(0) = \frac{1}{3}\,{}^{\frac{1}{2}}I^{\frac{1}{2},\frac{3}{4}}_{\frac{1}{2},\frac{3}{4}} x(\frac{3}{2}) + \frac{1}{2}\,{}^{\frac{5}{2}}I^{\frac{3}{2},\frac{1}{2}}_{\frac{1}{2},\frac{1}{3}} x(\frac{5}{2}), \\ x(5) = \frac{2}{3}\,{}^{\frac{1}{3}}I^{\frac{1}{4},\frac{1}{4}}_{\frac{1}{2},\frac{1}{3}} x(\frac{1}{2}) + \frac{5}{7}\,{}^{\frac{3}{2}}I^{\frac{1}{3},\frac{1}{4}}_{\frac{1}{2},\frac{3}{2}} x(\frac{7}{2}) + \frac{11}{16}\,{}^{\frac{5}{2}}I^{\frac{5}{2},\frac{1}{2}}_{\frac{3}{2},\frac{1}{2}} x(\frac{9}{2}). \end{cases} \tag{26}$$

Here $q = 1/2$, $r = 3/4$, $m = 2$, $n = 3$, $T = 5$, $\gamma_1 = 1/3$, $\gamma_2 = 1/2$, $\bar{\rho_1} = 1/2$, $\bar{\rho_2} = 3/2$, $\bar{\alpha_1} = 1/2$, $\bar{\alpha_2} = 3/2$, $\bar{\beta}_1 = 3/4$, $\bar{\beta}_2 = 1/2$, $\bar{\eta_1} = 3/2$, $\bar{\eta_2} = 1/2$, $\bar{\kappa_1} = 1/2$, $\bar{\kappa_2} = 1/3$, $\xi_1 = 3/2$, $\xi_2 = 5/2$, $\sigma_1 = 2/3$, $\sigma_2 = 5/7$, $\sigma_3 = 11/16$, $\rho_1 = 1/3$, $\rho_2 = 3/2$, $\rho_3 = 5/2$, $\alpha_1 = 1/4$, $\alpha_2 = 1/3$, $\alpha_3 = 5/2$, $\beta_1 = 1/4$, $\beta_2 = 1/4$, $\beta_3 = 1/2$, $\eta_1 = 1/2$, $\eta_2 = 1/2$, $\eta_3 = 3/2$, $\kappa_1 = 1/3$, $\kappa_2 = 3/2$, $\kappa_3 = 1/2$, $\delta_1 = 1/2$, $\delta_2 = 7/2$,

$\delta_3 = 9/2$ and $f(t,x) = ((\cos^2(2\pi t))/((t^2+5)^2+35)) \cdot ((x^2+2|x|)/(|x|+1)) + e^t$. From the given information, we find that $\Phi = 26.98694773$. Since

$$\bigl|f(t,x) - f(t,y)\bigr| \le \frac{1}{30}|x-y|,$$

condition (H_1) is satisfied with $L = 1/30$. Thus

$$L\Phi = 0.8995649243 < 1.$$

Hence, by Theorem 1, the boundary value problem (26) has a unique solution on $[0,5]$.

Example 2 Consider the following nonlinear sequential Riemann–Liouville and Caputo fractional differential equation with generalized fractional integral conditions:

$$\begin{cases} {}^{\mathrm{RL}}D^{\frac{4}{5}}({}^{C}D^{\frac{1}{2}}x)(t) = \frac{e^{-t^2}}{(t+2)^2+2} \cdot \frac{|x(t)|}{|x(t)|+1} + \frac{t}{t+1}, \quad 0 < t < 3, \\ x(0) = \frac{1}{2}\,{}^{\frac{1}{2}}I^{\frac{1}{2},\frac{1}{4}}_{\frac{1}{2},\frac{1}{4}}x(\frac{1}{2}) + \frac{1}{3}\,{}^{\frac{1}{3}}I^{\frac{1}{3},\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}x(\frac{2}{3}) + \frac{5}{6}\,{}^{\frac{1}{4}}I^{\frac{1}{2},\frac{1}{2}}_{\frac{1}{4},\frac{1}{4}}x(\frac{3}{2}), \\ x(3) = \frac{1}{4}\,{}^{\frac{1}{4}}I^{\frac{1}{3},\frac{1}{2}}_{\frac{1}{4},\frac{1}{4}}x(\frac{4}{3}) + \frac{1}{3}\,{}^{\frac{1}{3}}I^{\frac{1}{3},\frac{1}{3}}_{\frac{1}{2},\frac{1}{2}}x(\frac{7}{3}) + \frac{5}{6}\,{}^{\frac{1}{4}}I^{\frac{1}{4},\frac{1}{2}}_{\frac{1}{2},\frac{1}{4}}x(\frac{5}{2}). \end{cases} \tag{27}$$

Here $q = 4/5$, $r = 1/2$, $m = 3$, $n = 3$, $T = 3$, $\gamma_1 = 1/2$, $\gamma_2 = 1/3$, $\gamma_3 = 5/6$, $\bar{\rho_1} = 1/2$, $\bar{\rho_2} = 1/3$, $\bar{\rho_3} = 1/4$, $\bar{\alpha_1} = 1/2$, $\bar{\alpha_2} = 1/3$, $\bar{\alpha_3} = 1/2$ $\bar{\beta_1} = 1/4$, $\bar{\beta_2} = 1/2$, $\bar{\beta_3} = 1/2$ $\bar{\eta_1} = 1/2$, $\bar{\eta_2} = 1/2$, $\bar{\eta_3} = 1/4$, $\bar{\kappa_1} = 1/4$, $\bar{\kappa_2} = 1/2$, $\bar{\kappa_3} = 1/4$, $\xi_1 = 1/2$, $\xi_2 = 2/3$, $\xi_3 = 3/2$, $\sigma_1 = 1/4$, $\sigma_2 = 1/3$, $\sigma_3 = 5/6$, $\rho_1 = 1/4$, $\rho_2 = 1/3$, $\rho_3 = 1/4$, $\alpha_1 = 1/3$, $\alpha_2 = 1/3$, $\alpha_3 = 1/4$, $\beta_1 = 1/2$, $\beta_2 = 1/3$, $\beta_3 = 1/2$, $\eta_1 = 1/4$, $\eta_2 = 1/2$, $\eta_3 = 1/2$, $\kappa_1 = 1/4$, $\kappa_2 = 1/2$, $\kappa_3 = 1/4$, $\delta_1 = 4/3$, $\delta_2 = 7/3$, $\delta_3 = 5/2$ and $f(t,x) = ((e^{-t^2})/((t+2)^2+2)) \cdot ((|x|)/(|x|+1)) + (t/(t+1))$. From the above information, we can find that $\Phi = 11.03750380$ and $\Phi_1 = 5.898666195$. From $|f(t,x) - f(t,y)| \le (1/6)|x-y|$, we set $L = 1/6$, which is a constant satisfying (H_1). Since $L\Phi = 1.839583967 > 1$, Theorem 1 cannot be used in this example. However, we can check that

$$L\Phi_1 = 0.9831110325 < 1$$

and

$$\bigl|f(t,x)\bigr| = \left|\frac{e^{-t^2}}{(t+2)^2} \cdot \frac{|x(t)|}{|x(t)+1|} + \frac{t}{t+1}\right| \le \frac{e^{-t^2}}{6} + \frac{t}{t+1},$$

which is needed in condition (H_2) in Theorem 2. Hence, by Theorem 2, the boundary value problem (27) has at least one solution on $[0,3]$.

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Authors' contributions
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