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Existence criteria of solutions for a fractional nonlocal boundary value problem and degeneration to corresponding integer-order case

Chaoqun Gao[1], Zihan Gao[2] and Huihui Pang[1*]

*Correspondence:
phh2000@163.com
[1]College of Science, China Agricultural University, Beijing, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, we mainly discuss the existence and uniqueness results of solutions to fractional differential equations with multi-strip boundary conditions. When the fractional order α becomes integer, the existence theorem of positive solutions can be established by a monotone iterative technique. Also, some examples are presented to illustrate the main results.

Keywords: Fractional differential equations; Multi-strip integral boundary conditions; Green's function; Monotone iterative technique; Leray–Schauder alternative principle

1 Introduction

Differential equations attract many scholars' interest since they can succinctly establish the relationship between variables and their derivatives. And fractional order calculus has been used as an important tool to improve mathematical modeling of many complex problems, such as in fluid mechanics, rheology, fractional model of nerve and fractional regression model; see [1–5], for instance.

In the last decades, fractional order boundary value problems have also received plenty of attention from many researchers. There are many achievements derived from some fractional equations with various boundary conditions, some recent contribution can be found in [6–13].

For example, in [14], authors considered a discrete multi-point boundary value problem such as

$$\begin{cases} {}^cD^\alpha u(t) + f(t,u(t)) = 0, & t \in (0,1), \\ u(0) = 0, \qquad D^\beta_{0+}u(0) = 0, \\ D^\beta_{0+}u(1) = \sum_{i=1}^\infty \xi_i D^\beta_{0+}u(\eta_i), \end{cases}$$

where $2 < \alpha \le 3$, $1 \le \beta \le 2$, $\alpha - \beta \ge 1$ and $0 < \xi_i, \eta_i < 1$ with $\sum_{i=1}^\infty \xi_i \eta_i^{\alpha-\beta-1} < 1$.

In [15], the authors considered the following equation with integral boundary conditions:

$$\begin{cases} {}^cD^q u(t) + f(t,u(t)) = 0, & t \in [0,1], \\ \alpha u(0) - \beta u'(0) = \int_0^1 g(s)u(s)\,ds, \\ \gamma u(1) + \delta u'(1) = \int_0^1 h(s)u(s)\,ds, \end{cases}$$

where $q \in (1,2]$, α, β, γ and δ are nonnegative constants, and ${}^cD^q$ is the standard Caputo fractional derivative of fractional order q.

Different from [14] and [15], some work focused on the solvability of the fractional differential equations with both multi-point and integral boundary conditions. In [16], Ahmad et al. were concerned with the following problem:

$$\begin{cases} ({}^cD^q + k{}^cD^{q-1})x(t) = f(t,x(t),{}^c D^\beta x(t), I^\gamma(t)), & t \in [0,1], \\ x(0) = 0, \qquad x'(0) = 0, \qquad \sum_{i=1}^m x(\zeta_i) = \lambda \int_0^\eta \frac{(\eta-s)^{\delta-1}}{\Gamma(\delta)} x(s)\,ds, \end{cases}$$

where $2 < q \leq 3$, $0 < \beta, \gamma < 1$, $k > 0$, $\delta < 1$, $0 < \eta < \zeta_1 < \zeta_2 < \cdots < \zeta_m < 1$, and λ, a_i, $i = 1,2,\ldots,m$ are real constants.

Motivated by the above works, in this paper, we first deal with the following fractional order differential equation with multi-point and multi-strip boundary conditions:

$$\begin{cases} D_{0+}^\alpha u(t) + q(t)f(t,u(t),D_{0+}^\beta u(t),D_{0+}^\gamma u(t)) = 0, & t \in (0,1), \\ u(0) = D_{0+}^\gamma u(0) = 0, \\ u(1) + \sum_{i=1}^m a_i D_{0+}^\beta u(\xi_i) = \sum_{i=1}^m b_i \int_0^{\xi_i} u(s)\,ds, \end{cases} \tag{1.1}$$

where D_{0+}^α is the Riemann–Liouville fractional derivative of order α, $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function, $2 < \alpha \leq 3$, $0 < \beta \leq 1 < \gamma < \alpha - 1$, $0 < \xi_i \leq 1$, a_i, b_i are nonnegative constants satisfying $a_i \geq \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha+1)} b_i$, for $i = 1,2,\ldots,m$.

It is worth mentioning that the nonlinear term of BVP (1.1) depends on all the lower fractional order derivatives of the unknown function, which implies more complete consideration from the practical application problems' point of view. Although the complexity of the nonlinearity of BVP (1.1) is increased, we still get three Green's functions with concise forms and satisfactory properties. Meanwhile, boundary conditions of (1.1) include both multiple discrete points and multiple band-like integrals, which is a broad generalization of most models in [17–24]. By using the Leray–Schauder alternative theorem and the Banach's contraction mapping principle, existence and uniqueness theorems of solutions to BVP (1.1) are proved.

However, it is known that sometimes only positive solutions are significant in the real world. For this reason, in the second part we degenerate the fractional order model and choose $\alpha = 3$, $\gamma = 2$, $\beta = 1$. Hence, the following integer-order differential equation is discussed:

$$\begin{cases} u'''(t) + q(t)f(t,u(t),u'(t),u''(t)) = 0, & t \in (0,1), \\ u(0) = u''(0) = 0, \qquad u(1) + \sum_{i=1}^m a_i u'(\xi_i) = \sum_{i=1}^m b_i \int_0^{\xi_i} u(s)\,ds, \end{cases} \tag{1.2}$$

where $f : [0,1] \times [0,\infty) \times \mathbb{R} \times (-\infty,0] \to [0,\infty)$ is continuous, a_i, b_i are nonnegative constants satisfying $\frac{1}{6}b_i \le a_i < \frac{\xi_i^2 b_i}{2}$ with $\frac{\sqrt{3}}{3} < \xi_i \le 1$, for $i = 1,2,\dots,m$.

The arguments for BVP (1.2) are based on a monotone iterative technique. It is important that the Green's function associated with BVP (1.2) is nonnegative, which is different from that of BVP (1.1). In this part, not only the existence results of positive solutions are obtained, but also the approximate solutions of BVP (1.2) can be presented.

2 Fractional order differential equation

In this section, we consider the fractional order BVP (1.1) and establish the existence and uniqueness criteria of solutions. We put forward some indispensable definitions and theorems in advance.

Definition 2.1 ([25]) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds,$$

provided the right-hand side is pointwise defined on $(0,\infty)$, where $\Gamma(\alpha)$ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\,dt$, for $\alpha > 0$.

Definition 2.2 ([25]) The Riemann–Liouville fractional derivative of order $\alpha > 0$ for a function $f : (0,\infty) \to \mathbb{R}$ is given on $[0,\infty)$ by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\,ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ stands for the largest integer not greater than α.

From the definitions of Riemann–Liouville's derivative, the following lemmas can be obtained.

Lemma 2.1 ([25]) *For $\alpha > 0$, if we assume that $u \in C[0,\infty) \cap L^1(0,1)$, then*

$$I_{0+}^{\alpha} \big(D_{0+}^{\alpha} u(t)\big) = u(t) + m_1 t^{\alpha-1} + m_2 t^{\alpha-2} + \cdots + m_n t^{\alpha-n},$$

for some $m_i \in \mathbb{R}, i = 1,2,\dots,n$, where n is the smallest integer greater than or equal to α.

Remark 2.1 ([20]) The following properties are useful for our discussion:

(i) As a basic example, we quote, for $\lambda > -1$,

$$D_{0+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha};$$

(ii) $D_{0+}^{\alpha} I_{0+}^{\alpha} u(t) = u(t)$, for $u(t) \in L^1(0,1)$.

Before presenting the main results, we give the following assumptions:

(F1) $2 < \alpha \le 3$, $0 < \beta \le 1 < \gamma < \alpha - 1$, $0 < \xi_i < 1$, for $i = 1,2,\dots,m$;

(F2) a_i, b_i are nonnegative constants and $a_i \ge \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha+1)} b_i$, for $i = 1,2,\dots,m$;

(F3) $q \in L^1[0,1]$ is nonnegative, and $f \in C([0,1] \times \mathbb{R}^3, \mathbb{R})$.

For convenience, denote

$$\begin{aligned} &A = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \sum_{i=1}^{m} a_i \xi_i^{\alpha-\beta-1}, \qquad B = \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} \sum_{i=1}^{m} b_i \xi_i^{\alpha}, \\ &\Delta_1 = 1 + A - B, \qquad \varphi(t) = \frac{t^{\alpha-1}}{\Delta_1}. \end{aligned} \tag{2.1}$$

In view of (F1) and (F2), it is obvious that $A > B \geq 0$, as well as $\Delta_1 > 1$ and $\varphi(t) \geq 0$ for $t \in [0,1]$.

Lemma 2.2 *For $h(t) \in C(0,1) \cap L^1(0,1)$, the boundary value problem*

$$\begin{cases} D_{0+}^{\alpha} u(t) + h(t) = 0, \quad t \in (0,1), \\ u(0) = D_{0+}^{\gamma} u(0) = 0, \\ u(1) + \sum_{i=1}^{m} a_i D_{0+}^{\beta} u(\xi_i) = \sum_{i=1}^{m} b_i \int_0^{\xi_i} u(s)\,ds \end{cases} \tag{2.2}$$

has a unique solution

$$u(t) = \int_0^1 H_0(t,s) h(s)\,ds + P(h)\varphi(t), \tag{2.3}$$

where

$$H_0(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \varphi(t)(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ \varphi(t)(1-s)^{\alpha-1}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{2.4}$$

$$P(h) = \sum_{i=1}^{m} a_i I_{0+}^{\alpha-\beta} h(\xi_i) - \sum_{i=1}^{m} b_i I_{0+}^{\alpha+1} h(\xi_i). \tag{2.5}$$

Proof From Lemma 2.1, we can reduce $D_{0+}^{\alpha} u(t) + h(t) = 0$ to the following equivalent equation:

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,ds + m_1 t^{\alpha-1} + m_2 t^{\alpha-2} + m_3 t^{\alpha-3}, \tag{2.6}$$

where m_1, m_2 and m_3 are arbitrary real constants.

Since $u(0) = D_{0+}^{\gamma} u(0) = 0$, we have $m_2 = m_3 = 0$. Integrating (2.6) from 0 to ξ_i, for $i = 1, \ldots, m$, we get

$$\int_0^{\xi_i} u(t)\,dt = \int_0^{\xi_i} \left[-I_{0+}^{\alpha} h(s) + m_1 s^{\alpha-1} \right] ds = -I_{0+}^{\alpha+1} h(\xi_i) + \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} m_1 \xi_i^{\alpha}. \tag{2.7}$$

By Remark 2.1, we have

$$D_{0+}^{\beta} u(t) = -\frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} h(s)\,ds + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} m_1 t^{\alpha-\beta-1}. \tag{2.8}$$

Thus, together with (2.6), (2.7) and (2.8), it can be seen that

$$m_1 = \frac{1}{\Delta_1}\bigl[I_{0+}^{\alpha}h(1) + P(h)\bigr], \tag{2.9}$$

where Δ_1 is defined by (2.1), $P(h)$ is given by (2.5). Hence, the solution of problem (2.2) can be expressed as

$$\begin{aligned}
u(t) &= -I_{0+}^{\alpha}h(t) + \frac{t^{\alpha-1}}{\Delta_1}I_{0+}^{\alpha}h(1) + \frac{P(h)}{\Delta_1}t^{\alpha-1} \\
&= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds + \frac{\varphi(t)}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)\,ds + \varphi(t)P(h) \\
&= \int_0^1 H_0(t,s)h(s)\,ds + \varphi(t)P(h),
\end{aligned}$$

where $H_0(t,s)$ is given by (2.4) and $\varphi(t)$ is introduced by (2.1).

This completes the proof of the lemma. □

After replacing m_1 in (2.8), we get

$$D_{0+}^{\beta}u(t) = \int_0^1 H_\beta(t,s)h(s)\,ds + \frac{\Gamma(\alpha)}{\Delta_1\Gamma(\alpha-\beta)}P(h)t^{\alpha-\beta-1}, \tag{2.10}$$

where

$$H_\beta(t,s) = \frac{1}{\Delta_1\Gamma(\alpha-\beta)}\begin{cases} t^{\alpha-\beta-1}(1-s)^{\alpha-1} - \Delta_1(t-s)^{\alpha-\beta-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-\beta-1}(1-s)^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases}$$

Similarly, we have

$$D_{0+}^{\gamma}u(t) = \int_0^1 H_\gamma(t,s)h(s)\,ds + \frac{\Gamma(\alpha)}{\Delta_1\Gamma(\alpha-\gamma)}P(h)t^{\alpha-\gamma-1}, \tag{2.11}$$

where

$$H_\gamma(t,s) = \frac{1}{\Delta_1\Gamma(\alpha-\gamma)}\begin{cases} t^{\alpha-\gamma-1}(1-s)^{\alpha-1} - \Delta_1(t-s)^{\alpha-\gamma-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-\gamma-1}(1-s)^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases}$$

Next, we present some properties of Green's functions and $P(h)$.

Lemma 2.3 *For $t,s \in [0,1]$, the Green functions $H_0(t,s), H_\beta(t,s), H_\gamma(t,s)$ and $P(h)$ satisfy the following properties*:

(a) $|H_0(t,s)| \le H(s)$, *where* $H(s) = \frac{(1-s)^{\alpha-1}}{\Delta_1\Gamma(\alpha)}(1+\Delta_1)$;

(b) $|P(h)| \le \overline{P}(h)$, *for* $h(t) \ge 0$, *where*

$$\overline{P}(h) = \sum_{i=1}^{m}\frac{a_i}{\Gamma(\alpha-\beta)}\int_0^{\xi_i}(\xi_i - s)^{\alpha-\beta-1}\bigl[1 + (\xi_i - s)^{\beta+1}\bigr]h(s)\,ds;$$

(c) $|H_\beta(t,s)| \le \Lambda_\beta(s)$, $|H_\gamma(t,s)| \le \Lambda_\gamma(s)$, *where*

$$\Lambda_\beta(s) = \frac{1}{\Gamma(\alpha-\beta)}(1-s)^{\alpha-\beta-1}\left(\frac{(1-s)^\beta}{\Delta_1}+1\right),$$
$$\Lambda_\gamma(s) = \frac{1}{\Gamma(\alpha-\gamma)}(1-s)^{\alpha-\gamma-1}\left(\frac{(1-s)^\gamma}{\Delta_1}+1\right).$$

Proof (a) For $0 \le s \le t \le 1$, we have

$$\begin{aligned}\left|H_0(t,s)\right| &= \frac{1}{\Gamma(\alpha)}\left|\varphi(t)(1-s)^{\alpha-1}-(t-s)^{\alpha-1}\right| \\ &\le \frac{1}{\Gamma(\alpha)}\left(\frac{t^{\alpha-1}}{\Delta_1}(1-s)^{\alpha-1}+(1-s)^{\alpha-1}\right) \\ &\le \frac{(1-s)^{\alpha-1}}{\Delta_1\Gamma(\alpha)}(1+\Delta_1) = H(s);\end{aligned}$$

while, for $0 \le t \le s \le 1$, we have

$$\left|H_0(t,s)\right| = \frac{1}{\Gamma(\alpha)}\left|\varphi(t)(1-s)^{\alpha-1}\right| \le \frac{(1-s)^{\alpha-1}}{\Delta_1\Gamma(\alpha)}(1+\Delta_1).$$

(b) According to (2.5), (F1) and (F2), for $h(t) \ge 0$, we have

$$\begin{aligned}\left|P(h)\right| &= \left|\sum_{i=1}^m \frac{a_i}{\Gamma(\alpha-\beta)}\int_0^{\xi_i}(\xi_i-s)^{\alpha-\beta-1}h(s)\,ds - \sum_{i=1}^m \frac{b_i}{\Gamma(\alpha+1)}\int_0^{\xi_i}(\xi_i-s)^{\alpha}h(s)\,ds\right| \\ &\le \sum_{i=1}^m\left|\frac{a_i}{\Gamma(\alpha-\beta)}\int_0^{\xi_i}(\xi_i-s)^{\alpha-\beta-1}h(s)\,ds\right| + \sum_{i=1}^m\left|\frac{b_i}{\Gamma(\alpha+1)}\int_0^{\xi_i}(\xi_i-s)^{\alpha}h(s)\,ds\right| \\ &\le \sum_{i=1}^m \frac{a_i}{\Gamma(\alpha-\beta)}\int_0^{\xi_i}(\xi_i-s)^{\alpha-\beta-1}h(s)\,ds + \sum_{i=1}^m \frac{a_i}{\Gamma(\alpha-\beta)}\int_0^{\xi_i}(\xi_i-s)^{\alpha}h(s)\,ds \\ &= \sum_{i=1}^m \frac{a_i}{\Gamma(\alpha-\beta)}\int_0^{\xi_i}(\xi_i-s)^{\alpha-\beta-1}\left[1+(\xi_i-s)^{\beta+1}\right]h(s)\,ds = \overline{P}(h).\end{aligned}$$

(c) For $0 \le s \le t \le 1$, we have

$$\begin{aligned}\left|H_\beta(t,s)\right| &= \frac{1}{\Delta_1\Gamma(\alpha-\beta)}\left|t^{\alpha-\beta-1}(1-s)^{\alpha-1}-\Delta_1(t-s)^{\alpha-\beta-1}\right| \\ &\le \frac{1}{\Delta_1\Gamma(\alpha-\beta)}\left(t^{\alpha-\beta-1}(1-s)^{\alpha-1}+\Delta_1(t-ts)^{\alpha-\beta-1}\right) \\ &= \frac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}(1-s)^{\alpha-\beta-1}\left(\frac{(1-s)^\beta}{\Delta_1}+1\right) \\ &\le \frac{1}{\Gamma(\alpha-\beta)}(1-s)^{\alpha-\beta-1}\left(\frac{(1-s)^\beta}{\Delta_1}+1\right) = \Lambda_\beta(s);\end{aligned}$$

while, for $0 \le t \le s \le 1$, we have

$$\left|H_\beta(t,s)\right| = \frac{1}{\Delta_1\Gamma(\alpha-\beta)}\left|t^{\alpha-\beta-1}(1-s)^{\alpha-1}\right|$$

$$\le \frac{1}{\Gamma(\alpha-\beta)}(1-s)^{\alpha-\beta-1}\cdot\frac{(1-s)^{\beta}}{\Delta_1}$$
$$\le \Lambda_\beta(s).$$

Similarly, we can have

$$\left|H_\gamma(t,s)\right| \le \frac{1}{\Gamma(\alpha-\gamma)}(1-s)^{\alpha-\gamma-1}\left(\frac{(1-s)^{\gamma}}{\Delta_1}+1\right) = \Lambda_\gamma(s).$$

Then the proof is completed. □

Let $E_1 = \{u(t) \mid u(t) \in C[0,1] \text{ and } D^{\beta}_{0+}u(t), D^{\gamma}_{0+}u(t) \in C[0,1]\}$ be endowed with the norm

$$\|u\| = \max\left\{\|u\|_0, \left\|D^{\beta}_{0+}u\right\|_0, \left\|D^{\gamma}_{0+}u\right\|_0\right\},$$

where $\|u\|_0 = \max_{t\in[0,1]}|u(t)|$. In order to ensure the feasibility of the conclusion, we should prove the following lemmas.

Lemma 2.4 $(E_1, \|\cdot\|)$ *is a Banach space.*

Proof Let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the space $(E_1, \|\cdot\|)$. It is clear that $\{u_n\}_{n=1}^{\infty}$, $\{D^{\beta}_{0+}u_n\}_{n=1}^{\infty}$ and $\{D^{\gamma}_{0+}u_n\}_{n=1}^{\infty}$ are Cauchy sequences in the space $C[0,1]$. Accordingly, $\{u_n\}_{n=1}^{\infty}$, $\{D^{\beta}_{0+}u_n\}_{n=1}^{\infty}$ and $\{D^{\gamma}_{0+}u_n\}_{n=1}^{\infty}$ uniformly converge to some u, v and w on $[0,1]$ while $u, v, w \in C[0,1]$. Now we should prove that $v = D^{\beta}_{0+}u$ and $w = D^{\gamma}_{0+}u$.

For $t \in [0,1]$, we notice that

$$\begin{aligned}\left|I^{\beta}_{0+}D^{\beta}_{0+}u_n(t) - I^{\beta}_{0+}v(t)\right| &\le \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\left|D^{\beta}_{0+}u_n(s)-v(s)\right|ds \\ &\le \frac{1}{\Gamma(\beta+1)}\max_{s\in[0,1]}\left|D^{\beta}_{0+}u_n(s)-v(s)\right|.\end{aligned}$$

Due to the convergence of $\{D^{\beta}_{0+}u_n\}_{n=1}^{\infty}$, we obtain that $\lim_{n\to\infty} I^{\beta}_{0+}D^{\beta}_{0+}u_n(t) = I^{\beta}_{0+}v(t)$ uniformly for $t \in [0,1]$. Meanwhile, by Lemma 2.1, we get $I^{\beta}_{0+}D^{\beta}_{0+}u_n(t) = u_n(t) + m_1t^{\beta-1}$, for $t \in [0,1]$ and some $m_1 \in \mathbb{R}$. These two facts yield

$$\lim_{n\to\infty} I^{\beta}_{0+}D^{\beta}_{0+}u_n(t) = \lim_{n\to\infty} u_n(t) + m_1t^{\beta-1} = I^{\beta}_{0+}v(t), \quad \text{for } t\in[0,1].$$

Together with $\lim_{n\to\infty} u_n(t) = u(t)$ for $t \in [0,1]$, we have

$$u(t) + m_1t^{\beta-1} = I^{\beta}_{0+}v(t), \quad \text{for } t\in[0,1]. \tag{2.12}$$

Taking the derivative of order β of both sides of Eq. (2.12), as a result we have

$$D^{\beta}_{0+}I^{\beta}_{0+}v(t) = D^{\beta}_{0+}\left(u(t)+m_1t^{\beta-1}\right) = D^{\beta}_{0+}u(t), \quad \text{for } t\in[0,1].$$

From Remark 2.1, it is easy to see that

$$v(t) = D^{\beta}_{0+}u(t), \quad \text{for } t\in[0,1].$$

Also, for $t \in [0,1]$, $\omega(t) = D_{0+}^{\gamma} u(t)$ can be proved using similar steps.
The proof of this lemma is completed. □

For $u \in E_1$, we define an operator T_1 as follows:

$$(T_1 u)(t) = \int_0^1 H_0(t,s)q(s)f_u(s)\,ds + \varphi(t)P\big[q(s)f_u(s)\big],$$

where

$$P\big[q(s)f_u(s)\big] = \sum_{i=1}^{m} a_i I_{0+}^{\alpha-\beta} q(\xi_i)f_u(\xi_i) - \sum_{i=1}^{m} b_i I_{0+}^{\alpha+1} q(\xi_i)f_u(\xi_i),$$

and $f_u(s) = f(s,u(s),D_{0+}^{\beta}u(s),D_{0+}^{\gamma}u(s))$. Also, from (2.10) and (2.11), we have

$$D_{0+}^{\beta}(T_1 u)(t) = \int_0^1 H_\beta(t,s)q(s)f_u(s)\,ds + \frac{\Gamma(\alpha)}{\Delta_1\Gamma(\alpha-\beta)}P\big[q(s)f_u(s)\big]t^{\alpha-\beta-1},$$
$$D_{0+}^{\gamma}(T_1 u)(t) = \int_0^1 H_\gamma(t,s)q(s)f_u(s)\,ds + \frac{\Gamma(\alpha)}{\Delta_1\Gamma(\alpha-\gamma)}P\big[q(s)f_u(s)\big]t^{\alpha-\gamma-1}.$$

Lemma 2.5 *$T_1 : E_1 \to E_1$ is completely continuous.*

Proof By the continuity of Green's function $H_0(t,s)$ and $f(t,u(t),D_{0+}^{\beta}u(t),D_{0+}^{\gamma}u(t))$, T_1 is continuous.

Then, we show T_1 is uniformly bounded. Let $\Omega \subset E_1$ be bounded. We set $\|u\| \le K$, with $K > 0$, for all $u \in \Omega$. Let $F = \max\{|f_u(t)| \mid 0 \le t \le 1, -K \le u(t) \le K, -K \le D_{0+}^{\beta}u(t) \le K, -K \le D_{0+}^{\gamma}u(t) \le K\}$. Then from Lemma 2.3, we have

$$\begin{aligned}\big|(T_1 u)(t)\big| &\le \int_0^1 H_0(t,s)q(s)\big|f_u(s)\big|\,ds + \frac{1}{\Delta_1}P\big[q(s)\big|f_u(s)\big|\big]\\ &\le F\left(\int_0^1 H(s)q(s)\,ds + \frac{1}{\Delta_1}\overline{P}\big[q(s)\big]\right)\end{aligned}$$

and

$$\begin{aligned}\big|D_{0+}^{\beta}(T_1 u)(t)\big| &\le \int_0^1 \Lambda_\beta(s)q(s)\big|f_u(s)\big|\,ds + \frac{\Gamma(\alpha)}{\Delta_1\Gamma(\alpha-\beta)}t^{\alpha-\beta-1}P\big[q(s)\big|f_u(s)\big|\big]\\ &\le F\left(\int_0^1 \Lambda_\beta(s)q(s)\,ds + \frac{\Gamma(\alpha)}{\Delta_1\Gamma(\alpha-\beta)}\overline{P}\big[q(s)\big]\right),\\ \big|D_{0+}^{\gamma}(T_1 u)(t)\big| &\le \int_0^1 \Lambda_\gamma(s)q(s)\big|f_u(s)\big|\,ds + \frac{\Gamma(\alpha)}{\Delta_1\Gamma(\alpha-\gamma)}t^{\alpha-\gamma-1}P\big[q(s)\big|f_u(s)\big|\big]\\ &\le F\left(\int_0^1 \Lambda_\gamma(s)q(s)\,ds + \frac{\Gamma(\alpha)}{\Delta_1\Gamma(\alpha-\gamma)}\overline{P}\big[q(s)\big]\right).\end{aligned}$$

Hence, we can find upper bounds of $|(T_1 u)(t)|$, $|D_{0+}^{\beta}(T_1 u)(t)|$ and $|D_{0+}^{\gamma}(T_1 u)(t)|$, for $t \in [0,1]$. Thus, $\|T_1 u\|$ is bounded, which implies that the operator T_1 is uniformly bounded.

Finally, we show T_1 is equicontinuous. Indeed, for any $u \in \Omega$, $t_1, t_2 \in [0,1]$, with $t_1 < t_2$, we can infer that

$$\begin{aligned}
&\bigl|(T_1u)(t_2) - (T_1u)(t_1)\bigr| \\
&\quad \le \left|\int_0^1 H_0(t_2,s)q(s)f_u(s)\,ds - \int_0^1 H_0(t_1,s)q(s)f_u(s)\,ds\right| + \bigl|\varphi(t_2) - \varphi(t_1)\bigr| P\bigl[q(s)\bigl|f_u(s)\bigr|\bigr] \\
&\quad \le \frac{F}{\Delta_1\Gamma(\alpha)}\int_0^1 \bigl(\bigl|\bigl(t_2^{\alpha-1} - t_1^{\alpha-1}\bigr)(1-s)^{\alpha-1}\bigr| + \Delta_1\bigl|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\bigr|\bigr)q(s)\,ds \\
&\qquad + \frac{F}{\Delta_1}\bigl|t_2^{\alpha-1} - t_1^{\alpha-1}\bigr|\overline{P}\bigl[q(s)\bigr].
\end{aligned}$$

Applying the mean value theorem, the following inequalities hold:

$$\begin{aligned}
&t_2^{\alpha-1} - t_1^{\alpha-1} \le (\alpha-1)(t_2-t_1), \\
&(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \le (\alpha-1)(t_2-t_1),
\end{aligned}$$

from which we can deduce that

$$\begin{aligned}
&\bigl|(T_1u)(t_2) - (T_1u)(t_1)\bigr| \\
&\quad \le \frac{(\alpha-1)F}{\Delta_1\Gamma(\alpha)}\left\{\int_0^1 \bigl[(1-s)^{\alpha-1} + \Delta_1\bigr]q(s)\,ds + \Gamma(\alpha)\overline{P}\bigl[q(s)\bigr]\right\}(t_2-t_1) \\
&\quad \to 0 \quad \text{as } t_2 \to t_1.
\end{aligned}$$

In addition,

$$\begin{aligned}
&\bigl|D_{0+}^{\beta}(T_1u)(t_2) - D_{0+}^{\beta}(T_1u)(t_1)\bigr| \\
&\quad \le F\int_0^1 \left|\frac{(t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1})}{\Delta_1\Gamma(\alpha-\beta)}(1-s)^{\alpha-1} - \frac{(t_2-s)^{\alpha-\beta-1} - (t_1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right| q(s)\,ds \\
&\qquad + F\frac{\Gamma(\alpha)}{\Delta_1\Gamma(\alpha-\beta)}\bigl|t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1}\bigr|\overline{P}\bigl[q(s)\bigr] \\
&\quad \le \frac{F}{\Delta_1\Gamma(\alpha-\beta)}\left[\int_0^1 \left|\bigl(t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1}\bigr)(1-s)^{\alpha-1}\right| q(s)\,ds \right. \\
&\qquad \left. + \Delta_1\int_0^1 \left|(t_2-s)^{\alpha-\beta-1} - (t_1-s)^{\alpha-\beta-1}\right| q(s)\,ds\right] \\
&\qquad + \frac{\Gamma(\alpha)F}{\Delta_1\Gamma(\alpha-\beta)}\bigl|t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1}\bigr|\overline{P}\bigl[q(s)\bigr] \\
&\quad \le \frac{(\alpha-\beta-1)F}{\Delta_1\Gamma(\alpha-\beta)}\left\{\int_0^1 \bigl[(1-s)^{\alpha-1} + \Delta_1\bigr]q(s)\,ds + \Gamma(\alpha)\overline{P}\bigl[q(s)\bigr]\right\}(t_2-t_1) \\
&\quad \to 0 \quad \text{as } t_2 \to t_1.
\end{aligned}$$

This also leads to $|D_{0+}^{\gamma}(T_1u)(t_2) - D_{0+}^{\gamma}(T_1u)(t_1)| \to 0$, as $t_2 \to t_1$.

Therefore, T_1 is equicontinuous for all $u \in \Omega$. Thus, by means of the Arzelà–Ascoli Theorem, we obtain that $T_1 : E_1 \to E_1$ is completely continuous. □

Lemma 2.6 (Leray–Schauder alternative theorem) *Let $T : E \to E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let*

$$\varepsilon(T) = \{u \in E : u = \lambda T(u), 0 < \lambda < 1\}.$$

Then, either the set is unbounded, or T has at least one fixed point.

From the above facts, if operator T_1 has fixed points, we can observe that BVP (1.1) has solutions.

The next stage is devoted to obtaining the existence result.

For convenience, set

$$\begin{aligned}
L_P &= \sum_{i=1}^{m} \frac{a_i}{\Gamma(\alpha-\beta)} \int_0^{\xi_i} (\xi_i - s)^{\alpha-\beta-1}\left[1 + (\xi_i - s)^{\beta+1}\right] q(s)\,ds, \\
L_\beta &= \int_0^1 (1-s)^{\alpha-\beta-1}\left(\frac{(1-s)^\beta}{\Delta_1} + 1\right) q(s)\,ds, \\
L_\gamma &= \int_0^1 (1-s)^{\alpha-\gamma-1}\left(\frac{(1-s)^\gamma}{\Delta_1} + 1\right) q(s)\,ds, \\
L_H &= \int_0^1 \left(\frac{1}{\Delta_1} + 1\right)(1-s)^{\alpha-1} q(s)\,ds, \\
M_1 &= \max\left\{\frac{L_H}{\Gamma(\alpha)} + \frac{L_P}{\Delta_1}, \frac{L_\beta}{\Gamma(\alpha-\beta)} + \frac{\Gamma(\alpha)L_P}{\Delta_1\Gamma(\alpha-\beta)}, \frac{L_\gamma}{\Gamma(\alpha-\gamma)} + \frac{\Gamma(\alpha)L_P}{\Delta_1\Gamma(\alpha-\gamma)}\right\}.
\end{aligned} \tag{2.13}$$

Theorem 2.7 *Assume there exist real constants $\mu_i \geq 0$ ($i = 1, 2, 3$) and $\mu_0 > 0$ such that for $t \in [0, 1]$, $u, v, w \in \mathbb{R}$, we have*

$$\left|f(t, u, v, w)\right| \leq \mu_0 + \mu_1|u| + \mu_2|v| + \mu_3|w|.$$

If $(\mu_1 + \mu_2 + \mu_3)M_1 < 1$, then BVP (1.1) has at least one solution.

Proof In order to verify that problem (1.1) has at least one solution by Lemma 2.6, we should prove that the set $\varepsilon = \{u \in E_1 \mid u = \lambda T_1(u), 0 \leq \lambda \leq 1\}$ is bounded.

Let $u \in \varepsilon$, for any $t \in [0, 1]$, we have

$$\begin{aligned}
\left|u(t)\right| &= \left|\lambda(T_1u)(t)\right| \leq \left|(T_1u)(t)\right| \\
&\leq \left|\int_0^1 H_0(t,s)q(s)f_u(s)\,ds\right| + \left|\varphi(t)P\left[q(s)f_u(s)\right]\right| \\
&\leq \left(\mu_0 + \mu_1|u| + \mu_2|v| + \mu_3|w|\right)\int_0^1 H(s)q(s)\,ds \\
&\quad + \left(\mu_0 + \mu_1|u| + \mu_2|v| + \mu_3|w|\right)\frac{\overline{P}[q(s)]}{\Delta_1} \\
&\leq \left(\mu_0 + \mu_1\|u\|_0 + \mu_2\|v\|_0 + \mu_3\|w\|_0\right)\left[\frac{L_H}{\Gamma(\alpha)} + \frac{L_P}{\Delta_1}\right].
\end{aligned}$$

Also, we have

$$\begin{aligned}\left|D_{0+}^{\beta}u(t)\right| &= \left|D_{0+}^{\beta}\lambda(T_1u)(t)\right| \le \left|D_{0+}^{\beta}(T_1u)(t)\right| \\ &\le \left|\int_0^1 H_\beta(t,s)q(s)f_u(s)\,ds\right| + \left|\frac{\Gamma(\alpha)}{\Delta_1\Gamma(\alpha-\beta)}P\big[q(s)f_u(s)\big]t^{\alpha-\beta-1}\right| \\ &\le \big(\mu_0+\mu_1|u|+\mu_2|v|+\mu_3|w|\big)\left[\int_0^1 \Lambda_\beta(s)q(s)\,ds + \frac{\Gamma(\alpha)}{\Delta_1\Gamma(\alpha-\beta)}\overline{P}\big[q(s)\big]\right] \\ &\le \big(\mu_0+\mu_1\|u\|_0+\mu_2\|v\|_0+\mu_3\|w\|_0\big)\left[\frac{L_\beta}{\Gamma(\alpha-\beta)}+\frac{\Gamma(\alpha)L_P}{\Delta_1\Gamma(\alpha-\beta)}\right]\end{aligned}$$

and

$$\left|D_{0+}^{\gamma}u(t)\right| \le \big(\mu_0+\mu_1\|u\|_0+\mu_2\|v\|_0+\mu_3\|w\|_0\big)\left[\frac{L_\gamma}{\Gamma(\alpha-\gamma)}+\frac{\Gamma(\alpha)L_P}{\Delta_1\Gamma(\alpha-\gamma)}\right].$$

Thus, we obtain

$$\begin{aligned}\|u\|_0 &\le \big(\mu_0+(\mu_1+\mu_2+\mu_3)\|u\|\big)\left[\frac{L_H}{\Gamma(\alpha)}+\frac{L_P}{\Delta_1}\right], \\ \left\|D_{0+}^{\beta}u\right\|_0 &\le \big(\mu_0+(\mu_1+\mu_2+\mu_3)\|u\|\big)\left[\frac{L_\beta}{\Gamma(\alpha-\beta)}+\frac{\Gamma(\alpha)L_P}{\Delta_1\Gamma(\alpha-\beta)}\right], \\ \left\|D_{0+}^{\gamma}u\right\|_0 &\le \big(\mu_0+(\mu_1+\mu_2+\mu_3)\|u\|\big)\left[\frac{L_\gamma}{\Gamma(\alpha-\gamma)}+\frac{\Gamma(\alpha)L_P}{\Delta_1\Gamma(\alpha-\gamma)}\right].\end{aligned}$$

Hence,

$$\|u\| \le \frac{\mu_0 M_1}{1-(\mu_1+\mu_2+\mu_3)M_1},$$

where M_1 has been given in (2.13), which proves that $\|u\|$ is bounded. Thus, operator T_1 has at least one fixed point and, consequently, we can derive that BVP (1.1) has at least one solution. □

In the following, we should verify the uniqueness of the solution to BVP (1.1) by Banach's contraction mapping principle.

Theorem 2.8 *Let $f:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ be a continuous function. For all $t\in[0,1]$, $u_i,v_i,w_i\in\mathbb{R}$ $(i=1,2)$, we have*

$$\big|f(t,u_2,v_2,w_2)-f(t,u_1,v_1,w_1)\big| \le M\big(|u_2-u_1|+|v_2-v_1|+|w_2-w_1|\big),$$

where M is the Lipschitz constant. If $3MM_1\le 1$, BVP (1.1) has a unique solution, where M_1 is given by (2.13).

Proof Denote $M_0=\sup_{t\in[0,1]}|f(t,0,0,0)|$. Set $r\ge\frac{M_0M_1}{1-3MM_1}$ subject to the above mentioned M, M_1 and M_0. The set $B_r\subset E_1$ is defined by $B_r=\{u\in E_1 \mid \|u\|\le r\}$, and we will show that $T_1B_r\subset B_r$. For $u\in B_r$, we obtain

$$\big|f\big(t,u(t),D_{0+}^{\beta}u(t),D_{0+}^{\gamma}u(t)\big)\big|$$

$$
\begin{aligned}
&\leq \left|f\left(t,u(t),D_{0+}^{\beta}u(t),D_{0+}^{\gamma}u(t)\right)-f(t,0,0,0)\right|+\left|f(t,0,0,0)\right| \\
&\leq M\left(\|u\|_0+\left\|D_{0+}^{\beta}u\right\|_0+\left\|D_{0+}^{\gamma}u\right\|_0\right)+M_0 \\
&\leq 3Mr+M_0.
\end{aligned}
\tag{2.14}
$$

For $u \in B_r$, from (2.14) and similar to the proof of Theorem 2.7, we have

$$
\begin{aligned}
\left|(T_1u)(t)\right| &= \left|\int_0^1 H_0(t,s)q(s)f_u(s)\,ds+\varphi(t)P\left[q(s)f_u(s)\right]\right| \\
&\leq (3Mr+M_0)\left[\frac{L_H}{\Gamma(\alpha)}+\frac{L_P}{\Delta_1}\right], \\
\left|D_{0+}^{\beta}(T_1u)(t)\right| &= \left|\int_0^1 H_\beta(t,s)q(s)f_u(s)\,ds+\frac{\Gamma(\alpha)}{\Delta_1\Gamma(\alpha-\beta)}P\left[q(s)f_u(s)\right]t^{\alpha-\beta-1}\right| \\
&\leq (3Mr+M_0)\left[\frac{L_\beta}{\Gamma(\alpha-\beta)}+\frac{\Gamma(\alpha)L_P}{\Delta_1\Gamma(\alpha-\beta)}\right], \\
\left|D_{0+}^{\gamma}(T_1u)(t)\right| &\leq (3Mr+M_0)\left[\frac{L_\gamma}{\Gamma(\alpha-\gamma)}+\frac{\Gamma(\alpha)L_P}{\Delta_1\Gamma(\alpha-\gamma)}\right],
\end{aligned}
$$

which yields $\|T_1u\| \leq (3Mr+M_0)M_1 < r$. This shows that T_1 maps B_r into itself. Now, setting $u_1,u_2 \in E_1$, for each $t \in [0,1]$, we get

$$
\begin{aligned}
&\left|(T_1u_2)(t)-(T_1u_1)(t)\right| \\
&= \left|\int_0^1 H_0(t,s)q(s)\left(f_{u_2}(s)-f_{u_1}(s)\right)ds+\varphi(t)\left(P\left[q(s)f_{u_2}(s)\right]-P\left[q(s)f_{u_1}(s)\right]\right)\right| \\
&\leq M\left(\|u_2-u_1\|_0+\left\|D_{0+}^{\beta}u_2-D_{0+}^{\beta}u_1\right\|_0+\left\|D_{0+}^{\gamma}u_2-D_{0+}^{\gamma}u_1\right\|_0\right)\left(\frac{L_H}{\Gamma(\alpha)}+\frac{L_P}{\Delta_1}\right) \\
&\leq 3M\cdot\left(\frac{L_H}{\Gamma(\alpha)}+\frac{L_P}{\Delta_1}\right)\|u_2-u_1\| \\
&\leq 3MM_1\|u_2-u_1\|.
\end{aligned}
$$

Additionally, we obtain

$$
\begin{aligned}
&\left|D_{0+}^{\beta}(T_1u_2)(t)-D_{0+}^{\beta}(T_1u_1)(t)\right| \\
&= \left|\int_0^1 H_\beta(t,s)q(s)\left(f_{u_2}(s)-f_{u_1}(s)\right)ds+\frac{t^{\alpha-\beta-1}\Gamma(\alpha)}{\Delta_1\Gamma(\alpha-\beta)}\left(P\left[q(s)f_{u_2}(s)\right]-P\left[q(s)f_{u_1}(s)\right]\right)\right| \\
&\leq M\left(\|u_2-u_1\|_0+\left\|D_{0+}^{\beta}u_2-D_{0+}^{\beta}u_1\right\|_0\right. \\
&\quad\left.+\left\|D_{0+}^{\gamma}u_2-D_{0+}^{\gamma}u_1\right\|_0\right)\left[\frac{L_\beta}{\Gamma(\alpha-\beta)}+\frac{\Gamma(\alpha)L_P}{\Delta_1\Gamma(\alpha-\beta)}\right] \\
&\leq 3M\cdot\left[\frac{L_\beta}{\Gamma(\alpha-\beta)}+\frac{\Gamma(\alpha)L_P}{\Delta_1\Gamma(\alpha-\beta)}\right]\|u_2-u_1\| \\
&\leq 3MM_1\|u_2-u_1\|
\end{aligned}
$$

and

$$\left|D_{0+}^{\gamma}(T_1u_2)(t)-D_{0+}^{\gamma}(T_1u_1)(t)\right| \le 3M\cdot\left[\frac{L_\gamma}{\Gamma(\alpha-\gamma)}+\frac{\Gamma(\alpha)L_P}{\Delta_1\Gamma(\alpha-\gamma)}\right]\|u_2-u_1\|$$
$$\le 3MM_1\|u_2-u_1\|.$$

From the above, we have $\|T_1u_2-T_1u_1\|\le 3MM_1\|u_2-u_1\|$. In view of $3MM_1<1$, the operator T_1 is a contraction. Thus, the uniqueness of solution to BVP (1.1) follows from Banach's contraction mapping principle. □

Example 2.1 Consider the nonlinear fractional differential equation

$$\begin{cases} D_{0+}^{2.2}u(t)+[\frac{2}{5}t+\frac{1}{48}(2u(t)+\frac{8}{5}D_{0+}^{0.7}u(t)+\frac{4}{3}D_{0+}^{1.1}u(t))]=0, & t\in(0,1),\\ u(0)=D_{0+}^{1.1}u(0)=0,\\ u(1)+\sum_{i=1}^{m}a_iD_{0+}^{0.7}u(\xi_i)=\sum_{i=1}^{m}b_i\int_0^{\xi_i}u(s)\,ds. \end{cases} \tag{2.15}$$

In this model, we set

$$m=2,\qquad q(t)=1,\qquad \xi_1=\frac{2}{3},\qquad \xi_2=\frac{5}{6},$$
$$a_1=\frac{3}{5},\qquad a_2=\frac{2}{5},\qquad b_1=\frac{3}{2},\qquad b_2=1.$$

It is easy to verify that (F1)–(F3) hold. By calculation, we have

$$A=1.063,\qquad B=0.5838,\qquad \Delta_1=1.4793,$$
$$L_P=0.6111,\qquad L_\beta=0.9739,\qquad L_\gamma=1.2164,\qquad L_H=0.7618,$$
$$M_1=\max\left\{\frac{L_H}{\Gamma(\alpha)}+\frac{L_P}{\Delta_1},\frac{L_\beta}{\Gamma(\alpha-\beta)}+\frac{\Gamma(\alpha)L_P}{\Delta_1\Gamma(\alpha-\beta)},\frac{L_\gamma}{\Gamma(\alpha-\gamma)}+\frac{\Gamma(\alpha)L_P}{\Delta_1\Gamma(\alpha-\gamma)}\right\}=1.757.$$

Meanwhile, we see

$$\left|f\big(t,u,D_{0+}^{0.7}u(t),D_{0+}^{1.1}u(t)\big)\right|$$
$$\le\frac{2}{5}+\frac{1}{24}|u|+\frac{1}{30}\left|D_{0+}^{0.7}u(t)\right|+\frac{1}{36}\left|D_{0+}^{1.1}u(t)\right|,$$
$$\left|f\big(t,u_2,D_{0+}^{0.7}u_2(t),D_{0+}^{1.1}u_2(t)\big)-f\big(t,u_1,D_{0+}^{0.7}u_1(t),D_{0+}^{1.1}u_1(t)\big)\right|$$
$$\le\frac{1}{9}\big(\|u_2-u_1\|_0+\left\|D_{0+}^{0.7}u_2-D_{0+}^{0.7}u_1\right\|_0+\left\|D_{0+}^{1.1}u_2-D_{0+}^{1.1}u_1\right\|_0\big).$$

Therefore, $(\mu_1+\mu_2+\mu_3)M_1<1$ and $3MM_1<1$.

Thus, all the conditions of the above theorems are satisfied. Hence, by Theorem 2.7 problem (2.15) has at least one solution, and by Theorem 2.8 it has a unique solution.

3 Integer-order differential equation

In this section, in order to establish the existence results of positive solutions, we try to degenerate the fractional order problem into a corresponding integer-order differential model.

Necessarily, we give the following assumptions:

(H1) a_i, b_i are nonnegative constants satisfying $\frac{1}{6}b_i \le a_i < \frac{\xi_i^2 b_i}{2}$, with $\frac{\sqrt{3}}{3} < \xi_i \le 1$ for $i = 1, 2, \ldots, m$;

(H2) $q \in L^1[0,1]$ is nonnegative and $f \in C([0,1] \times [0,\infty) \times \mathbb{R} \times (-\infty, 0], [0,\infty))$.

For convenience, we denote

$$\begin{aligned} &E = \sum_{i=1}^{m} a_i, \qquad F = \sum_{i=1}^{m} \frac{b_i \xi_i^2}{2}, \\ &\Delta_2 = 1 + E - F, \qquad \psi(t) = \frac{t}{\Delta_2}. \end{aligned} \tag{3.1}$$

From (H1), it is easy to see that $F > E \ge 0$. In the following, we always assume that $0 < F - E < 1$. Hence, we have $0 < \Delta_2 < 1$, and $\psi(t) \ge 0$ for $t \in [0,1]$.

Lemma 3.1 *Let $h(t) \in C[0,1] \cap L^1(0,1)$. Then the boundary value problem*

$$\begin{cases} u'''(t) + h(t) = 0, \quad t \in (0,1), \\ u(0) = u''(0) = 0, \\ u(1) + \sum_{i=1}^{m} a_i u'(\xi_i) = \sum_{i=1}^{m} b_i \int_0^{\xi_i} u(s)\,ds \end{cases} \tag{3.2}$$

has an integral representation

$$u(t) = \int_0^1 G_0(t,s)h(s)\,ds + Q(h)\psi(t), \tag{3.3}$$

where

$$G_0(t,s) = \frac{1}{2} \begin{cases} \psi(t)(1-s)^2 - (t-s)^2, & 0 \le s \le t \le 1, \\ \psi(t)(1-s)^2, & 0 \le t \le s \le 1, \end{cases} \tag{3.4}$$

$$Q(h) = \sum_{i=1}^{m} a_i \int_0^{\xi_i} (\xi_i - s)h(s)\,ds - \sum_{i=1}^{m} \frac{b_i}{6} \int_0^{\xi_i} (\xi_i - s)^3 h(s)\,ds. \tag{3.5}$$

The proof is similar to that of Lemma 2.2, so we omit it. Moreover, one has

$$u'(t) = \int_0^1 g(t,s)h(s)\,ds + \frac{Q(h)}{\Delta_2}, \tag{3.6}$$

where

$$g(t,s) = \begin{cases} \frac{1}{2\Delta_2}(1-s)^2 - (t-s), & 0 \le s \le t \le 1, \\ \frac{1}{2\Delta_2}(1-s)^2, & 0 \le t \le s \le 1. \end{cases} \tag{3.7}$$

Now, we will provide some properties of $G_0(t,s)$, $Q(h)$ and $g(t,s)$.

Lemma 3.2 *For $(t,s) \in [0,1] \times [0,1]$, the functions $G_0(t,s)$, $Q(h)$ and $g(t,s)$ satisfy the following properties*:

(a) $0 \le \frac{t}{2\Delta_2}(1-s)^2(1-\Delta_2 t) \le G_0(t,s) \le \frac{t}{2\Delta_2}(1-s)^2 + t^2(1-s)$;
(b) $0 \le \underline{Q}(h) \le Q(h) \le \overline{Q}(h)$, *for* $h(t) \ge 0$, *where*

$$\underline{Q}(h) = \sum_{i=1}^{m} \frac{b_i}{6} \int_0^{\xi_i} (\xi_i - s)\big[1 - (\xi_i - s)^2\big] h(s)\, ds,$$

$$\overline{Q}(h) = \sum_{i=1}^{m} a_i \int_0^{\xi_i} (\xi_i - s)\big[1 + (\xi_i - s)^2\big] h(s)\, ds;$$

(c) $|g(t,s)| \le \frac{(1-s)^2}{2\Delta_2} + (1-s)$.

Proof (a) For $0 \le s \le t \le 1$, we have

$$\begin{aligned} G_0(t,s) &= \frac{1}{2}\big[\psi(t)(1-s)^2 - (t-s)^2\big] \\ &\ge \frac{1}{2}\left[\frac{t}{\Delta_2}(1-s)^2 - (t-ts)^2\right] \\ &= \frac{t}{2\Delta_2}(1-s)^2(1-\Delta_2 t) \ge 0, \end{aligned}$$

while, for $0 \le t \le s \le 1$, we have

$$\begin{aligned} G_0(t,s) &= \frac{1}{2}\psi(t)(1-s)^2 \\ &\ge \frac{1}{2}\left[\frac{t}{\Delta_2}(1-s)^2 - (t-ts)^2\right] \\ &= \frac{t}{2\Delta_2}(1-s)^2(1-\Delta_2 t). \end{aligned}$$

On the other hand, for $0 \le s \le t \le 1$, we have

$$\begin{aligned} G_0(t,s) &= \frac{1}{2}\big[\psi(t)(1-s)^2 - (t-s)^2\big] \\ &\le \frac{1}{2}\left[\frac{t}{\Delta_2}(1-s)^2 + t^2(1-s)\right] \\ &\le \frac{t}{2\Delta_2}(1-s)^2 + t^2(1-s), \end{aligned}$$

while, for $0 \le t \le s \le 1$, we have

$$G_0(t,s) = \frac{1}{2}\psi(t)(1-s)^2 \le \frac{t}{2\Delta_2}(1-s)^2 + t^2(1-s).$$

(b) From (3.5) and (H1), for $h(t) \ge 0$, we have

$$\begin{aligned} Q(h) &= \sum_{i=1}^{m} a_i \int_0^{\xi_i} (\xi_i - s) h(s)\, ds - \sum_{i=1}^{m} \frac{b_i}{6} \int_0^{\xi_i} (\xi_i - s)^3 h(s)\, ds \\ &\ge \sum_{i=1}^{m} \frac{b_i}{6} \int_0^{\xi_i} (\xi_i - s) h(s)\, ds - \sum_{i=1}^{m} \frac{b_i}{6} \int_0^{\xi_i} (\xi_i - s)^3 h(s)\, ds \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^{m} \frac{b_i}{6} \int_0^{\xi_i} (\xi_i - s)\big[1 - (\xi_i - s)^2\big] h(s)\,ds \\
&= \underline{Q}(h) \geq 0.
\end{aligned}$$

For $h(t) \geq 0$, we also have

$$\begin{aligned}
Q(h) &= \sum_{i=1}^{m} a_i \int_0^{\xi_i} (\xi_i - s) h(s)\,ds - \sum_{i=1}^{m} \frac{b_i}{6} \int_0^{\xi_i} (\xi_i - s)^3 h(s)\,ds \\
&\leq \sum_{i=1}^{m} a_i \int_0^{\xi_i} (\xi_i - s) h(s)\,ds + \sum_{i=1}^{m} a_i \int_0^{\xi_i} (\xi_i - s)^3 h(s)\,ds \\
&= \sum_{i=1}^{m} a_i \int_0^{\xi_i} (\xi_i - s)\big[1 + (\xi_i - s)^2\big] h(s)\,ds \\
&= \overline{Q}(h).
\end{aligned}$$

(c) From (3.7), for $0 \leq s \leq t \leq 1$, we have

$$\big|g(t,s)\big| = \left| \frac{1}{2\Delta_2}(1-s)^2 - (t-s) \right| \leq \frac{(1-s)^2}{2\Delta_2} + (1-s),$$

and, for $0 \leq t \leq s \leq 1$, we have

$$\big|g(t,s)\big| = \left| \frac{1}{2\Delta_2}(1-s)^2 \right| \leq \frac{(1-s)^2}{2\Delta_2} + (1-s).$$

This completes the proof of the lemma. □

In this section, we introduce the Banach space $E_2 = C^2[0,1]$ equipped with the norm

$$\|u\| := \max\Big\{ \max_{0\leq t\leq 1} \big|u(t)\big|, \max_{0\leq t\leq 1} \big|u'(t)\big|, \max_{0\leq t\leq 1} \big|u''(t)\big| \Big\}$$

and define a cone $P \subset E_2$ by $P = \{u \in E_2 : u(t) \geq 0, u''(t) \leq 0\}$. Then, for all $u \in E_2$, we define an integral operator $T_2 : P \to E_2$ by

$$(T_2 u)(t) = \int_0^1 G_0(t,s) q(s) f_u(s)\,ds + Q\big[q(s) f_u(s)\big] \psi(t),$$

where $f_u(s) = f(s, u(s), u'(s), u''(s))$ and

$$Q\big[q(s) f_u(s)\big] = \sum_{i=1}^{m} a_i \int_0^{\xi_i} (\xi_i - s) q(s) f_u(s)\,ds - \sum_{i=1}^{m} \frac{b_i}{6} \int_0^{\xi_i} (\xi_i - s)^3 q(s) f_u(s)\,ds.$$

Lemma 3.3 *If* (H1) *and* (H2) *are satisfied*, $T_2 : P \to P$ *is completely continuous*.

Proof After introducing the operator T_2, for $u \in P$, we can get that

$$\begin{aligned} &(T_2u)'(t) = \int_0^1 g(t,s)q(s)f_u(s)\,ds + \frac{Q[q(s)f_u(s)]}{\Delta_2}, \qquad (T_2u)''(t) \le 0, \\ &(T_2u)(0) = 0, \qquad (T_2u)(1) = \int_0^1 G_0(1,s)q(s)f_u(s)\,ds + \frac{Q[q(s)f_u(s)]}{\Delta_2} \ge 0. \end{aligned} \tag{3.8}$$

Thus, $(T_2u)(t)$ is concave and $(T_2u)(t) \ge 0$, for $0 \le t \le 1$, which implies that operator T_2 maps P into P.

It is obvious that T_2 is continuous, but we need to prove that T_2 is also compact. Let $\Omega \subset P$ be a bounded set. Similar to Lemma 2.5, we can easily prove that $T_2(\Omega)$ is bounded and equicontinuous. Thus, by the Arzelà–Ascoli Theorem, $T_2(\Omega)$ is relatively compact, which implies T_2 is compact. Consequently, we get that $T_2 : P \to P$ is completely continuous. □

For convenience, we denote

$$\begin{aligned} &L_1 = \int_0^1 (1-s)^2 q(s)\,ds, \qquad L_2 = \int_0^1 (1-s)q(s)\,ds, \qquad L_3 = \int_0^1 q(s)\,ds, \\ &L_Q = \sum_{i=1}^m a_i \int_0^{\xi_i} (\xi_i - s)\bigl[1 + (\xi_i - s)^2\bigr]q(s)\,ds. \end{aligned} \tag{3.9}$$

According to (H1) and (H2), it is obviously that L_1, L_2, L_3 and L_Q are nonnegative.

Now, based on Lemmas 3.2 and 3.3, in what follows, we show that there exist positive extremal solutions for BVP (1.2) by a monotone iterative method.

Theorem 3.4 *Assume that* (H1) *and* (H2) *hold, let l_1 and l be two positive numbers, satisfying $l = \max\{\frac{1}{2\Delta_2}L_1 + L_2 + \frac{1}{\Delta_2}L_Q + L_3, \frac{L_3}{\Delta_2}\}l_1$, and*

(S1) $f(t,u_1,v_1,w_1) \le f(t,u_2,v_2,w_2)$, *for* $0 \le t \le 1, 0 \le u_1 \le u_2 \le l, 0 \le |v_1| \le |v_2| \le l, -l \le w_2 \le w_1 \le 0$;

(S2) $\max_{0\le t\le 1} f(t,l,l,-l) \le l_1$;

(S3) $f(t,0,0,0) \not\equiv 0$, *for* $0 \le t \le 1$.

Then BVP (1.2) *has concave positive solutions v^* and ω^*, which satisfy*

$$\begin{aligned} &0 \le \|v^*\| \le l, \qquad 0 \le \|\omega^*\| \le l, \\ &v_0(t) = 0, \qquad \omega_0(t) = \left[\left(\frac{1}{2\Delta_2}L_1 + L_2 + \frac{L_Q}{\Delta_2}\right)t + \left(t - \frac{t^2}{2}\right)L_3\right]l_1, \\ &v_n = T_2v_{n-1}, \qquad \lim_{n\to\infty} v_n = v^*, \qquad \omega_n = T_2\omega_{n-1}, \qquad \lim_{n\to\infty} \omega_n = \omega^*, \quad n = 1,2,\ldots, \\ &v_n' = (T_2v_{n-1})', \qquad \lim_{n\to\infty} (v_n)' = (v^*)', \qquad \omega_n' = (T_2\omega_{n-1})', \qquad \lim_{n\to\infty} (\omega_n)' = (\omega^*)', \\ &v_n'' = (T_2v_{n-1})'', \qquad \lim_{n\to\infty} (v_n)'' = (v^*)'', \qquad \omega_n'' = (T_2\omega_{n-1})'', \qquad \lim_{n\to\infty} (\omega_n)' = (\omega^*)''. \end{aligned}$$

Proof Denote $P_l = \{u \in P \mid \|u\| \le l\}$. In the following, we first prove that $T_2 : P_l \to P_l$. Let $u \in P_l$. Then for $t \in [0,1]$, we have

$$0 \le u(t) \le \|u\| \le l, \qquad 0 \le |u'(t)| \le \|u\| \le l, \qquad -l \le -\|u\| \le u''(t) \le 0.$$

So, for $0 \le t \le 1$, by (S1) and (S2), we get

$$0 \le f\big(t, u(t), u'(t), u''(t)\big) \le \max_{0\le t\le 1} f(t, l, l, -l) \le l_1.$$

Consequently, for $t \in [0,1]$, we have

$$\begin{aligned}
\big|(T_2u)(t)\big| &= \left|\int_0^1 G_0(t,s)q(s)f_u(s)\,ds + \frac{t}{\Delta_2}Q\big[q(s)f_u(s)\big]\right| \\
&\le \int_0^1 \left[\frac{t}{2\Delta_2}(1-s)^2 + t^2(1-s)\right]q(s)f_u(s)\,ds + \frac{\overline{Q}[q(s)f_u(s)]}{\Delta_2} \\
&\le \left(\frac{1}{2\Delta_2}L_1 + L_2 + \frac{L_Q}{\Delta_2}\right)l_1 \le l, \\
\big|(T_2u)'(t)\big| &= \left|\int_0^1 g(t,s)q(s)f_u(s)\,ds + \frac{Q[q(s)f_u(s)]}{\Delta_2}\right| \\
&\le \int_0^1 \frac{1}{2\Delta_2}(1-s)^2 q(s)f_u(s)\,ds + \int_0^1 (1-s)q(s)f_u(s)\,ds + \frac{\overline{Q}[q(s)f_u(s)]}{\Delta_2} \\
&\le \left(\frac{1}{2\Delta_2}L_1 + L_2 + \frac{L_Q}{\Delta_2}\right)l_1 \le l, \\
\big|(T_2u)''(t)\big| &\le \left|\int_0^1 q(s)f_u(s)\,ds\right| \le L_3 l_1 \le l.
\end{aligned}$$

To sum up, we obtain

$$\|T_2u\| = \left\{\max_{0\le t\le 1}\big|(T_2u)(t)\big|, \max_{0\le t\le 1}\big|(T_2u)'(t)\big|, \max_{0\le t\le 1}\big|(T_2u)''(t)\big|\right\} \le l$$

and $T_2 : P_l \to P_l$.

Set $\omega_0 = [\frac{t}{\Delta_2}(\frac{1}{2}L_1 + L_Q + \Delta_2 L_2) + (t - \frac{t^2}{2})L_3]l_1$ and $\nu_0 = 0$. Obviously, $\omega_0, \nu_0 \in P_l$. By using the completely continuous operator T_2, we define the sequences $\{\omega_n\}$ and $\{\nu_n\}$ as $\omega_n = T_2\omega_{n-1}$, $\nu_n = T_2\nu_{n-1}$, for $n = 1, 2, \ldots$. Since $T_2 : P_l \to P_l$, we get that $\omega_n, \nu_n \in P_l$, $n = 1, 2, \ldots$. Also we assert that $\{\omega_n\}$ and $\{\nu_n\}$ have relatively compact subsequences, for $n = 0, 1, 2, \ldots$. Hence, we prove that there exist ω^*, ν^*, satisfying $\lim_{n\to\infty}\omega_n = \omega^*$ and $\lim_{n\to\infty}\nu_n = \nu^*$, which are monotone positive solutions of problem (1.2).

For $t \in [0,1]$, according to the definition of the iterative scheme, we have

$$\begin{aligned}
\omega_1(t) &= T_2\omega_0(t) \\
&= \int_0^1 G_0(t,s)q(s)f_{\omega_0}(s)\,ds + \frac{Q[q(s)f_{\omega_0}(s)]t}{\Delta_2} \\
&\le \int_0^1 \left[\frac{t}{2\Delta_2}(1-s)^2 + t^2(1-s)\right]q(s)f_{\omega_0}(s)\,ds + \frac{\overline{Q}[q(s)f_{\omega_0}(s)]t}{\Delta_2} \\
&\le \left[\left(\frac{1}{2\Delta_2}L_1 + tL_2\right)t + \frac{L_Q t}{\Delta_2}\right]l_1 \\
&\le \left[\left(\frac{1}{2\Delta_2}L_1 + L_2 + \frac{L_Q}{\Delta_2}\right)t + \left(t - \frac{t^2}{2}\right)L_3\right]l_1 = \omega_0(t);
\end{aligned}$$

$$\begin{aligned}
\left|\omega_1'(t)\right| &= \left|(T_2\omega_0)'(t)\right| = \left|\int_0^1 g(t,s)q(s)f_{\omega_0}(s)\,ds + \frac{Q[q(s)f_{\omega_0}(s)]}{\Delta_2}\right| \\
&\le \int_0^1 \left[\frac{1}{2\Delta_2}(1-s)^2 + (1-s)\right] q(s)f_{\omega_0}(s)\,ds + \frac{\overline{Q}[q(s)f_{\omega_0}(s)]}{\Delta_2} \\
&\le \left(\frac{1}{2\Delta_2}L_1 + L_2 + \frac{L_Q}{\Delta_2}\right) l_1 \\
&\le \left[\left(\frac{1}{2\Delta_2}L_1 + L_2 + \frac{L_Q}{\Delta_2}\right) + (1-t)L_3\right] l_1 = \left|\omega_0'(t)\right|; \\
\omega_1''(t) &= (T_2\omega_0)''(t) = -\int_0^t q(s)f_{\omega_0}(s)\,ds \ge -L_3 l_1 = \omega_0''(t).
\end{aligned}$$

Thus, for $0 \le t \le 1$, we have

$$\begin{aligned}
&\omega_2(t) = T_2\omega_1(t) \le T_2\omega_0(t) = \omega_1(t), \\
&\left|\omega_2'(t)\right| = \left|(T_2\omega_1)'(t)\right| \le \left|(T_2\omega_0)'(t)\right| = \left|\omega_1'(t)\right|, \\
&\omega_2''(t) = (T_2\omega_1)''(t) \ge (T_2\omega_0)''(t) = \omega_1''(t).
\end{aligned}$$

By induction, one has

$$\begin{aligned}
&\omega_{n+1}(t) \le \omega_n(t), \qquad \left|\omega_{n+1}'(t)\right| \le \left|\omega_n'(t)\right|, \\
&\omega_{n+1}''(t) \ge \omega_n''(t), \quad 0 \le t \le 1, n = 0,1,2,\ldots.
\end{aligned}$$

So, we assert that $\omega_n \to \omega^*$ and $T_2\omega^* = \omega^*$, since T_2 is completely continuous and $\omega_{n+1} = T_2\omega_n$.

For the sequence $\{v_n\}_{n=1}^{\infty}$, we apply a similar argument. For $t \in [0,1]$, we have

$$\begin{aligned}
v_1(t) &= T_2v_0(t) = \int_0^1 G_0(t,s)q(s)f_{v_0}(s)\,ds + \frac{Q[q(s)f_{v_0}(s)]t}{\Delta_2} \\
&\ge \int_0^1 \left[\frac{t}{2}(1-s)^2\left(\frac{1}{\Delta_2} - t\right)\right] q(s)f_{v_0}(s)\,ds + \frac{\underline{Q}[q(s)f_{v_0}(s)]t}{\Delta_2} \\
&\ge 0 = v_0(t); \\
\left|v_1'(t)\right| &= \left|(T_2v_0)'(t)\right| = \left|\int_0^1 g(t,s)q(s)f_{v_0}(s)\,ds + \frac{Q[q(s)f_{v_0}(s)]}{\Delta_2}\right| \ge 0 = \left|v_0'(t)\right|; \\
v_1''(t) &= (T_2v_0)''(t) = -\int_0^1 q(s)f_{v_0}(s)\,ds \le 0 = v_0''(t).
\end{aligned}$$

So, for $0 \le t \le 1$, we have

$$\begin{aligned}
&v_2(t) = T_2v_1(t) \ge T_2v_0(t) = v_1(t), \quad \text{while } v_1(t) = T_2v_0(t) \ge 0, \\
&\left|v_2'(t)\right| = \left|(T_2v_1)'(t)\right| \ge \left|(T_2v_0)'(t)\right| = \left|v_1'(t)\right|, \quad \text{while } \left|v_1'(t)\right| = \left|(T_2v_0)'(t)\right| \ge 0, \\
&v_2''(t) = (T_2v_1)''(t) \le (T_2v_0)''(t) = v_1''(t), \quad \text{while } v_1''(t) = (T_2v_0)''(t) \le 0.
\end{aligned}$$

Similarly, one has

$$v_{n+1}(t) \ge v_n(t), \qquad \left|v_{n+1}'(t)\right| \ge \left|v_n'(t)\right|, \qquad v_{n+1}''(t) \le v_n''(t), \quad 0 \le t \le 1, n = 0,1,2,\ldots$$

And, we assert that $v_n \to v^*$ and $T_2 v^* = v^*$, since T_2 is completely continuous and $v_{n+1} = T_2 v_n$.

Consequently, there exist ω^* and v^* in P_l, which are nonnegative solutions of BVP (1.2). From (S3), it is obvious that $\omega^*(t) > 0, v^*(t) > 0$, for $t \in [0,1]$, since zero is not a solution of Eq. (1.2). The proof is completed. □

Example 3.1 Consider the following third-order boundary value problem:

$$\begin{cases} u'''(t) + (\frac{1}{2}t^2 + \frac{1}{10}u(t) + \frac{1}{50}u'^2(t) + \sin\frac{2}{u''(t)-2} + \frac{11}{10}) = 0, & t \in (0,1), \\ u(0) = u''(0) = 0, \qquad u(1) + \sum_{i=1}^m a_i u'(\xi_i) = \sum_{i=1}^m b_i \int_0^{\xi_i} u(s)\,ds. \end{cases} \tag{3.10}$$

In this model, we set

$$\begin{aligned} &m = 2, \qquad q(s) = 1, \qquad \xi_1 = \frac{3}{5}, \qquad \xi_2 = \frac{4}{5}, \\ &a_1 = \frac{3}{7}, \qquad a_2 = \frac{2}{7}, \quad b_1 = \frac{5}{2}, \qquad b_2 = \frac{5}{3}. \end{aligned}$$

It is obvious that (H1) and (H2) hold. By calculation, we get

$$\begin{aligned} &E = \frac{5}{7}, \qquad F = \frac{295}{300}, \qquad \Delta_2 = 0.731, \\ &L_1 = \frac{1}{3}, \qquad L_2 = \frac{1}{2}, \qquad L_3 = 1, \qquad L_Q = 0.2117, \end{aligned}$$

and $l = \max\{\frac{1}{2\Delta_2}L_1 + L_2 + \frac{1}{\Delta_2}L_Q + L_3, \frac{L_3}{\Delta_2}\}l_1 \approx 2.0176 l_1$. Set $l_1 = 5$. Then all the hypotheses of Theorem 3.4 are satisfied with $l = 10$. Hence, BVP (3.10) has monotone positive solutions v^* and ω^*, which satisfy, for $t \in [0,1]$,

$$v_0(t) = 0 \quad \text{and} \quad \omega_0(t) = \left[\left(\frac{1}{2\Delta_2}L_1 + L_2 + \frac{L_Q}{\Delta_2}\right)t + \left(t - \frac{t^2}{2}\right)L_3\right]l_1 = -\frac{5}{2}t^2 + 10t.$$

For $n = 1, 2, \ldots$, the two iterative schemes are

$$\begin{aligned} &\omega_0(t) = -\frac{5}{2}t^2 + 10t, \qquad \omega_1(t) = -\frac{1}{80}t^5 + \frac{1}{24}t^4 - \frac{4697}{10{,}000}t^3 + \frac{383}{500}t, \qquad \ldots, \\ &\omega_{n+1}(t) = \int_0^t G_0(t,s) f_{\omega_n}(s)\,ds + \frac{Q[f_{\omega_n(s)}]t}{\Delta_2} \\ &\quad = -\frac{1}{2}\int_0^t (t-s)^2 \left(\frac{1}{2}s^2 + \frac{1}{10}\omega_n(s) + \frac{1}{50}\omega_n'^2(s) + \sin\frac{2}{\omega_n''(s) - 2} + \frac{11}{10}\right) ds \\ &\qquad + \frac{t}{2\Delta_2}\int_0^1 (1-s)^2 \left(\frac{1}{2}s^2 + \frac{1}{10}\omega_n(s) + \frac{1}{50}\omega_n'^2(s) + \sin\frac{2}{\omega_n''(s) - 2} + \frac{11}{10}\right) ds \\ &\qquad + \frac{t}{\Delta_2}\left\{\int_0^{\frac{3}{5}} \left(\frac{5}{12}s^3 - \frac{3}{4}s^2 + \frac{3}{140}s + \frac{117}{700}\right)\left(\frac{1}{2}s^2 + \frac{1}{10}\omega_n(s) + \frac{1}{50}\omega_n'^2(s)\right.\right. \\ &\qquad + \left.\sin\frac{2}{\omega_n''(s) - 2} + \frac{11}{10}\right) ds \\ &\qquad + \int_0^{\frac{4}{5}} \left(\frac{5}{18}s^3 - \frac{2}{3}s^2 + \frac{26}{105}s + \frac{136}{1575}\right)\left(\frac{1}{2}s^2 + \frac{1}{10}\omega_n(s) + \frac{1}{50}\omega_1'^2(s)\right. \end{aligned}$$

$$+\sin\frac{2}{\omega_n''(s)-2}+\frac{11}{10}\biggr)\,ds\biggr\}$$

and

$$\begin{aligned}
\nu_0(t)&=0,\qquad \nu_1(t)=-\frac{1}{120}t^5-\frac{53}{1000}t^3+\frac{9}{100}t,\qquad \ldots,\\
\nu_{n+1}(t)&=\int_0^t G_0(t,s)f_{\nu_n}(s)\,ds+\frac{Q[f_{\nu_n(s)}]t}{\Delta_2}\\
&=-\frac{1}{2}\int_0^t (t-s)^2\biggl(\frac{1}{2}s^2+\frac{1}{10}\nu_n(s)+\frac{1}{50}\nu_n'^2(s)+\sin\frac{2}{\nu_n''(s)-2}+\frac{11}{10}\biggr)\,ds\\
&\quad+\frac{t}{2\Delta_2}\int_0^1 (1-s)^2\biggl(\frac{1}{2}s^2+\frac{1}{10}\nu_n(s)+\frac{1}{50}\nu_n'^2(s)+\sin\frac{2}{\nu_n''(s)-2}+\frac{11}{10}\biggr)\,ds\\
&\quad+\frac{t}{\Delta_2}\biggl\{\int_0^{\frac{3}{5}}\biggl(\frac{5}{12}s^3-\frac{3}{4}s^2+\frac{3}{140}s+\frac{117}{700}\biggr)\biggl(\frac{1}{2}s^2+\frac{1}{10}\nu_n(s)\\
&\quad+\frac{1}{50}\nu_n'^2(s)+\sin\frac{2}{\nu_n''(s)-2}+\frac{11}{10}\biggr)\,ds\\
&\quad+\int_0^{\frac{4}{5}}\biggl(\frac{5}{18}s^3-\frac{2}{3}s^2+\frac{26}{105}s+\frac{136}{1575}\biggr)\biggl(\frac{1}{2}s^2+\frac{1}{10}\nu_n(s)\\
&\quad+\frac{1}{50}\nu_n'^2(s)+\sin\frac{2}{\nu_n''(s)-2}+\frac{11}{10}\biggr)\,ds\biggr\}.
\end{aligned}$$

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Author details
[1]College of Science, China Agricultural University, Beijing, P.R. China. [2]College of Economics and Management, China Agricultural University, Beijing, P.R. China.

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