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# Nonlocal evolution inclusions under weak conditions

Shamas Bilal[1*], Ovidiu Cârjă[2,3], Tzanko Donchev[3], Nasir Javaid[1] and Alina I. Lazu[4]

*Correspondence:
shams2013sms@gmail.com
[1] Abdus Salam School of Mathematical Sciences, Lahore, Pakistan
Full list of author information is available at the end of the article

**Abstract**
We study evolution inclusions given by multivalued perturbations of m-dissipative operators with nonlocal initial conditions. We prove the existence of solutions. The commonly used Lipschitz hypothesis for the perturbations is weakened to one-sided Lipschitz ones. We prove an existence result for the multipoint problems that cover periodic and antiperiodic cases. We give examples to illustrate the applicability of our results.

**MSC:** 34B15; 34A60; 35R70

**Keywords:** m-dissipative evolution inclusions; Nonlocal initial conditions; One-sided Lipschitz condition

## 1 Introduction

In this paper, we study the nonlinear evolution system with nonlocal initial condition

$$\begin{cases} \dot{y}(s) \in Ay(s) + F(s, y(s)), & s \in I = [t_0, T], \\ y(t_0) = g(y(\cdot)), \end{cases} \tag{1.1}$$

in a Banach space $X$ with uniformly convex dual. Here $A : D(A) \subset X \rightrightarrows X$ is an m-dissipative operator, $F : I \times X \rightrightarrows X$ is a multifunction, and $g : C(I, X) \to \overline{D(A)}$.

Notice that problem (1.1) has a number of applications, since it describes many phenomena better than the local initial value problems. We refer to the recent interesting book [6], where (1.1) is comprehensively studied. See also [2], where the authors assume that $X^*$ is uniformly convex (as in the present paper) and $A$ generates a compact semigroup. Nonautonomous $A$ is also studied. In [21] the author considered the case where $A$ is linear. In [16], it is assumed that $A$ is of complete continuous type and generates a compact semigroup. The conditions on $F(\cdot,\cdot)$ and $g(\cdot)$ are mild, and the problem is studied in arbitrary (separable) Banach spaces. In fact the commonly assumptions used to prove the existence of solutions are either that $A$ is of compact type (in particular, it generates a compact semigroup if $X^*$ is uniformly convex) and $F$ satisfies some upper semicontinuity, or $F$ is Lipschitz continuous. In [22] the authors assumed that $X^*$ is uniformly convex, and two cases are considered: one where $A$ generates an equicontinuous semigroup and $F(\cdot,\cdot)$ is Lipschitz w.r.t. the Hausdorff measure of noncompactness, and the second is where $A$ is m-dissipative and $F(s,\cdot)$ is Lipschitz. In the recent paper [1], the existence of solutions of

(1.1) is proved when $F(s,\cdot)$ is Lipschitz with nonempty closed bounded values in general Banach spaces. The results of [1, 22] are not applicable in the important case of periodic or antiperiodic boundary conditions, that is, $g(y) = y(T)$ or $g(y) = -y(T)$. This problem is especially studied in the present paper. The periodic or antiperiodic boundary conditions were studied in the literature when $A$ generates a compact semigroup (see [6]).

Among others, we recall also [8, 15, 19, 20], where nonlinear delayed evolution inclusions with nonlocal initial conditions are considered. See also [7], where the authors investigated problem (1.2) with state constrains.

In the present paper, we first prove the existence of solutions to problem (1.1), assuming that $F(s,\cdot)$ is one-sided Lipschitz, which is weaker than the commonly used Lipschitz condition. Further, we relax the growth conditions on $g(\cdot)$ used in [22]. Our assumptions are weaker and more flexible. Notice also that even if $F(s,\cdot)$ is Lipschitz, the one-sided Lipschitz constant (or function) is in general smaller (even negative) than the Lipschitz one.

Moreover, for the function $g$ appearing in the nonlocal condition, we consider the particular instance $g(y) = \sum_{i=1}^{k} \alpha_i y(t_i)$, where $t_1, t_2, \ldots, t_k \in I$ are arbitrary but fixed, and $\sum_{i=1}^{k} |\alpha_i| \leq 1$, which covers the remarkable periodic and antiperiodic cases. For this specific case of (1.1), we obtain an existence result under a one-sided Lipschitz condition with negative constant on $F$. Notice that there exist non-Lipschitz multifunctions, which are one-sided Lipschitz with respect to a negative constant.

To prove the existence of solutions to (1.1), we consider the corresponding local Cauchy problem

$$\begin{cases} \dot{y}(s) \in Ay(s) + F(s, y(s)), \\ y(t_0) = x_0 \in \overline{D(A)}, \end{cases} \tag{1.2}$$

for which the existence of the solutions and some properties of the solution set are discussed in [5, 9, 10, 12]. Here we provide estimates on dependence of the solution set of (1.2) on the initial conditions, which will be used to get our existence result for the nonlocal problem.

In the end of the paper, we give two examples to demonstrate the applicability of our results.

## 2 Preliminaries

We start this section by giving the notation and the main definitions used further in this paper. Also, we recall some known results, which will be used in the next sections.

For any nonempty closed bounded subset $C$ of $X$ and $l \in X^*$, we denote by $\sigma(l, C) = \sup_{a\in C}\langle l, a\rangle$ the support function, where $\langle\cdot,\cdot\rangle$ is the duality pairing. Denote by $J(x) = \{z \in X^*; \langle z, x\rangle = |z|^2 = |x|^2\}$ the duality map. Since $X^*$ is uniformly convex, then $J(\cdot)$ is single-valued and uniformly continuous on the bounded sets (see, e.g., [3]). We denote by $\Omega_J(r) = \sup\{|J(x) - J(y)|; |x - y| \leq r, x, y \in X\}$ its modulus of continuity. We define $\operatorname{dist}(x, \mathcal{A}) = \inf_{a\in\mathcal{A}} |x - a|$, the distance from $x \in X$ to $\mathcal{A} \subset X$. The Hausdorff distance between two subsets $\mathcal{A}$ and $\mathcal{B}$ of $X$ is defined by $D_H(\mathcal{A}, \mathcal{B}) = \max\{\operatorname{ex}(\mathcal{A}, \mathcal{B}), \operatorname{ex}(\mathcal{B}, \mathcal{A})\}$, where $\operatorname{ex}(\mathcal{A}, \mathcal{B}) = \sup_{a\in\mathcal{A}} \operatorname{dist}(a, \mathcal{B})$.

A multimap $G : X \rightrightarrows X$ is called hemicontinuous (upper hemicontinuous) if for every $l \in X^*$, the support function $\sigma(l, G(\cdot))$ is continuous (upper semicontinuous) as a real-valued function. A multifunction $F : I \times X \rightrightarrows X$ is said to be almost upper hemicontinuous if for every $\varepsilon > 0$, there exists a compact $I_\varepsilon \subset I$ with $\operatorname{meas}(I \setminus I_\varepsilon) < \varepsilon$ such that $F|_{I_\varepsilon \times X}$ is upper hemicontinuous.

Let $f \in L^1(I, X)$. The continuous function $x(\cdot) : I \to \overline{D(A)}$ is said to be an (integral) solution of

$$\begin{cases} \dot{x}(t) \in Ax(t) + f(t), \\ x(t_0) = x_0, \end{cases} \tag{2.1}$$

if $x(t_0) = x_0$, and for any $u \in D(A)$, $v \in A(u)$, and $t_0 \le \tau < t \le T$,

$$\bigl|x(t) - u\bigr| \le |x_0 - u| + \int_\tau^t \bigl[x(s) - u, f(s) + v\bigr]_+ \, ds \tag{2.2}$$

(see [4, 14, 17]). Here $[x, u]_+$ denotes the right directional derivative of the norm calculated at $x$ in the direction $u$, i.e.,

$$[x, u]_+ = \lim_{h \to 0^+} \frac{\|x + hu\| - \|x\|}{h}.$$

Concerning the properties of $[\cdot, \cdot]_+$ see, e.g., [14], Section 1.2. Along this paper we will denote, when it is necessary, the solution $x(\cdot)$ of (2.1) as $x(x_0, f)(\cdot)$. Here, of course, $f(\cdot)$ is Bochner integrable and $x_0 \in \overline{D(A)}$.

It is well known that (2.1) has a unique solution.

The following properties of solutions of (2.1) will be essentially used in this paper (see, e.g., [14] for the proof).

**Theorem 2.1** *Let $x(\cdot) = x(x_0, f)(\cdot)$ and $y(\cdot) = y(y_0, g)(\cdot)$, where $x_0, y_0 \in \overline{D(A)}$ and $f, g \in L^1(I, X)$. Then*

$$\bigl|x(t) - y(t)\bigr|^2 \le |x_0 - y_0|^2 + 2 \int_{t_0}^t \bigl\langle J\bigl(x(s) - y(s)\bigr), f(s) - g(s)\bigr\rangle \, ds$$

*and*

$$\bigl|x(t) - y(t)\bigr| \le |x_0 - y_0| + \int_{t_0}^t \bigl|f(s) - g(s)\bigr| \, ds$$

*for any $t \in I$.*

We need the following theorem, which is a reformulation of [3], Theorem 4.1.

**Theorem 2.2** *Let $\omega > 0$, and let $A : D(A) \subset X \rightrightarrows X$ be an m-dissipative operator such that $A + \omega I$ is dissipative. Then*

$$\bigl|x(t) - y(t)\bigr| \le e^{-\omega(t-s)} \bigl|x(s) - y(s)\bigr| + \int_s^t e^{-\omega(t-\tau)} \bigl[x(\tau) - y(\tau), f(\tau) - g(\tau)\bigr]_+ \, d\tau \tag{2.3}$$

*for every $x(\cdot) = x(x_0, f)(\cdot)$, $y(\cdot) = y(y_0, g)(\cdot)$ and each $t_0 \le s < t \le T$.*

**Definition 2.3** A continuous function $y(\cdot)$ is said to be:

(i) a *solution* of (1.2) if $y(\cdot) = y(x_0, f)(\cdot)$ and $f(s) \in F(s, y(s))$ a.e. on $I$,
(ii) an $\varepsilon$-*solution* of (1.2) if $y(\cdot) = y(x_0, f)(\cdot)$ and $f(s) \in F(s, y(s) + \varepsilon\mathbb{B})$ a.e. on $I$, where $\mathbb{B}$ is the open unit ball in $X$.

We state the standard assumptions of our paper.

(F1) $F(\cdot,\cdot)$ has nonempty convex weakly compact values, and there exists a Lebesgue integrable function $\lambda(\cdot)$ such that $\|F(t,x)\| := \max_{v\in F(t,x)} |v| \le \lambda(t)(1 + |x|)$ on $I \times X$.
(F2) $F(\cdot,\cdot)$ is almost upper hemicontinuous, $X$ is not necessarily separable, or $X$ is separable, $F(\cdot, x)$ is measurable, and $F(t,\cdot)$ is hemicontinuous.

Regarding the existence of $\varepsilon$-solutions, we recall the following result proved in [10].

**Proposition 2.4** *Suppose that, for every $x \in \overline{D(A)}$, $F(\cdot, x)$ has a strongly measurable selection. Then, under* (F1), *for every $\varepsilon > 0$, there exists an $\varepsilon$-solution of* (1.2).

**Proposition 2.5** ([10], Proposition 1.5) *Assume* (F1). *Then for every $k > 0$, there exist a constant $M > 0$ and a Lebesgue-integrable function $\mu(\cdot)$ such that, if $y(\cdot) = y(x_0, f)(\cdot)$ with $f(s) \in \overline{co}\, F(s, y(s) + \mathbb{B}) + \mathbb{B}$, then $|y(s)| \le M$ and $\|F(s, y(s))\| \le \mu(s)$ on $I$.*

In the end of this section, we give the following lemma, which is a simplified version of [9], Lemma 1.

**Lemma 2.6** *Let $G : I \rightrightarrows X$ be a measurable multifunction, integrally bounded with convex weakly compact values. Then for all strongly measurable $u : I \to X^*$, there exists a strongly measurable $h(\cdot)$ such that $h(s) \in G(s)$ and $\langle u(s), h(s)\rangle = \sigma(u(s), G(s))$ for a.e. $s \in I$.*

## 3 An existence result

In this section, we prove the first main result of this paper, namely the existence of solutions for the nonlocal problem (1.1). We add another assumption on $F$, much weaker than the Lipschitz continuity, called one-sided Lipschitz condition.

(F3) There exists a Lebesgue-integrable function $L : I \to \mathbb{R}_+$ such that, for all $x, y \in X$ and $t \in I$,

$$\sigma\big(J(x-y), F(t,x)\big) - \sigma\big(J(x-y), F(t,y)\big) \le L(t)|x-y|^2.$$

**Theorem 3.1** *Assume* (F1)–(F3). *Moreover, assume that $g$ is $K$-Lipschitz and*

$$K \exp\left(\int_{t_0}^{T} L(s)\,ds\right) < 1. \tag{3.1}$$

*Then the nonlocal problem* (1.1) *has at least a solution.*

To prove the theorem, we need some auxiliary results.

**Lemma 3.2** *Assume* (F1)–(F3) *and let $x_0, y_0 \in k\overline{\mathbb{B}} \cap \overline{D(A)}$, $k > 0$. There exists a constant $C > 0$ such that, for any $0 < \delta < \varepsilon$ and any $\varepsilon$-solution $x(\cdot)$ of* (1.2) *with $x(t_0) = x_0$, there exists a $\delta$-solution $y(\cdot)$ of* (1.2) *with $y(t_0) = y_0$ such that*

$$\big|x(t) - y(t)\big| \le |x_0 - y_0| \exp\left(\int_{t_0}^{t} L(s)\,ds\right) + C\big(\Omega_J(\varepsilon+\delta) + \varepsilon + \delta\big)^{1/2}$$

*for all $t \in I$.*

A version of this lemma was proved in [9], Lemma 2. Our assumptions are however stronger and allow us to obtain a more relevant result to the problems considered here.

*Proof* Let $0 < \delta < \varepsilon$, and let $x(\cdot)$ be an $\varepsilon$-solution of (1.2). Then $x(\cdot) = x(x_0, f_x)(\cdot)$ with $f_x(t) \in F(t, x(t) + \varepsilon\mathbb{B})$ a.e. on $I$. Let $h(\cdot)$ be such that $|h(t)| \leq \varepsilon$ and $f_x(t) \in F(t, x(t) + h(t))$ for a.e. $t \in [t_0, T]$. Suppose that $y_\tau(\cdot)$ is a $\delta$-solution of (1.2) defined on some interval $[t_0, \tau]$, $\tau < T$, with $y_\tau(t_0) = y_0$ satisfying the condition of the lemma. Using (F3), we have that, for a.e. $t \in [t_0, T]$,

$$\begin{aligned}
&\sigma\bigl(J\bigl(x(t) + h(t) - y_\tau(\tau)\bigr), F\bigl(t, x(t) + h(t)\bigr)\bigr) - \sigma\bigl(J\bigl(x(t) + h(t) - y_\tau(\tau)\bigr), F\bigl(t, y_\tau(\tau)\bigr)\bigr) \\
&\quad \leq L(t)\bigl|x(t) + h(t) - y_\tau(\tau)\bigr|^2.
\end{aligned}$$

By Lemma 2.6 there exists a (strongly) measurable $f_y(\cdot)$ such that $f_y(t) \in F(t, y_\tau(\tau))$ and

$$\bigl\langle J\bigl(x(t) + h(t) - y_\tau(\tau)\bigr), f_y(t)\bigr\rangle = \sigma\bigl(J\bigl(x(t) + h(t) - y_\tau(\tau)\bigr), F\bigl(t, y_\tau(\tau)\bigr)\bigr)$$

for a.e. $t \in [t_0, T]$. Therefore, we get the following inequality

$$\bigl\langle J\bigl(x(t) + h(t) - y_\tau(\tau)\bigr), f_x(t) - f_y(t)\bigr\rangle \leq L(t)\bigl|x(t) + h(t) - y_\tau(\tau)\bigr|^2 \tag{3.2}$$

for a.e. $t \in [t_0, T]$. There exists $\theta > 0$ such that if $y(\cdot)$ is an extension of $y_\tau(\cdot)$ on $[\tau, \tau + \theta]$ such that $\dot{y}(t) \in Ay(t) + f_y(t)$, then $|y(t) - y(\tau)| \leq \delta$ for $t \in [\tau, \tau + \theta]$, and (3.2) holds with $y(\tau)$ instead of $y_\tau(\tau)$. We have that

$$\begin{aligned}
\bigl\langle J\bigl(x(t) - y(t)\bigr), f_x(t) - f_y(t)\bigr\rangle \leq{}& L(t)\bigl|x(t) - y(t)\bigr|^2 \\
&+ \bigl|\bigl\langle J\bigl(x(t) - y(t)\bigr) - J\bigl(x(t) + h(t) - y(\tau)\bigr), f_x(t) - f_y(t)\bigr\rangle\bigr| \\
&+ L(t)\bigl(\bigl|x(t) + h(t) - y(\tau)\bigr|^2 - \bigl|x(t) - y(t)\bigr|^2\bigr).
\end{aligned}$$

Since $|x(t) - y(t) - x(t) - h(t) + y(\tau)| \leq \varepsilon + \delta$, we get that

$$\bigl|J\bigl(x(t) - y(t)\bigr) - J\bigl(x(t) + h(t) - y(\tau)\bigr)\bigr| \leq \Omega_J(\varepsilon + \delta).$$

On another hand,

$$\begin{aligned}
\bigl|\bigl|x(t) + h(t) - y(\tau)\bigr|^2 - \bigl|x(t) - y(t)\bigr|^2\bigr| &\leq \bigl|y(t) + h(t) - y(\tau)\bigr|\bigl|2x(t) + h(t) - y(t) - y(\tau)\bigr| \\
&\leq (\varepsilon + \delta)(4M + \varepsilon).
\end{aligned}$$

Hence

$$\bigl\langle J\bigl(x(t) - y(t)\bigr), f_x(t) - f_y(t)\bigr\rangle \leq L(t)\bigl|x(t) - y(t)\bigr|^2 + 2\mu(t)\Omega_J(\varepsilon + \delta) + L(t)c_1(\varepsilon + \delta)$$

for a.e. $t \in [\tau, \tau + \theta]$, where $c_1 = 4M + \varepsilon$. Using now Theorem 2.1, we get that, for $t \in [\tau, \tau + \theta]$,

$$\begin{aligned} \bigl|x(t) - y(t)\bigr|^2 \le |x_0 - y_0|^2 + 2\int_{t_0}^{t} \bigl(L(s)\bigl|x(s) - y(s)\bigr|^2 \\ + 2\mu(s)\Omega_J(\varepsilon + \delta) + L(s)c_1(\varepsilon + \delta)\bigr)\,ds. \end{aligned}$$

Therefore $|x(t) - y(t)|^2 \le r(t)$ on $[t_0, \tau + \theta]$, where $r(\cdot)$ is the maximal solution of

$$\begin{cases} \dot{r}(t) = 2L(t)r(t) + 4\mu(t)\Omega_J(\varepsilon + \delta) + 2c_1(\varepsilon + \delta)L(t), \\ r(t_0) = |x_0 - y_0|^2. \end{cases}$$

It follows that

$$\begin{aligned} r(t) &\le \exp\biggl(2\int_{t_0}^{t} L(s)\,ds\biggr)\biggl(|x_0 - y_0|^2 + 4\Omega_J(\varepsilon + \delta)\int_{t_0}^{t} \mu(s)\,ds + 2c_1(\varepsilon + \delta)\int_{t_0}^{t} L(s)\,ds\biggr) \\ &\le \exp\biggl(2\int_{t_0}^{t} L(s)\,ds\biggr)\bigl(|x_0 - y_0|^2 + c_2\bigl(\Omega_J(\varepsilon + \delta) + \varepsilon + \delta\bigr)\bigr) \end{aligned}$$

for some positive constant $c_2$. Clearly,

$$\bigl|x(t) - y(t)\bigr| \le \sqrt{r(t)} \le |x_0 - y_0| \exp\biggl(\int_{t_0}^{t} L(s)\,ds\biggr) + C\sqrt{\Omega_J(\varepsilon + \delta) + \varepsilon + \delta}$$

on $[t_0, \tau + \theta]$. We have used the fact that $\exp(2\int_{t_0}^{t} L(s)\,ds)$ is bounded. Applying now Zorn's lemma we get the existence of the $\delta$-solution $y(\cdot)$ on the whole interval $I$. □

Denote by $\mathrm{Sol}(x_0)$ the solution set of (1.2), which, under (F1)–(F3), is nonempty (see [9]).

The next result is a variant of the well-known lemma of Filippov and Plis.

**Lemma 3.3** *Assume* (F1)–(F3). *Let* $x_0, y_0 \in k\overline{\mathbb{B}} \cap \overline{D(A)}$, $k > 0$, *and let* $y(\cdot)$ *be an* $\varepsilon$*-solution of* (1.2) *with* $y(t_0) = y_0$. *Then there exists* $c > 0$ *such that*

$$\operatorname{dist}\bigl(y(\cdot), \mathrm{Sol}(x_0)\bigr) \le |x_0 - y_0| \exp\biggl(\int_{t_0}^{t} L(s)\,ds\biggr) + c\bigl(\Omega_J(\varepsilon) + \varepsilon\bigr)^{1/2}.$$

*Proof* Let $\eta > 0$ and choose $\varepsilon_n \downarrow 0^+$ with $\varepsilon_0 = \varepsilon$ such that $C\sum_{n=1}^{\infty}(\Omega_J(2\varepsilon_n) + 2\varepsilon_n)^{1/2} < \eta/2$, where $C$ is the constant given in Lemma 3.2. Applying Lemma 3.2 for $\varepsilon = \varepsilon_n$ and $\delta = \varepsilon_{n+1}$, for every natural $n \ge 0$, we obtain a sequence $(x_n(\cdot))$ of $\varepsilon_n$-solutions such that

$$\bigl|y(t) - x_1(t)\bigr| \le |x_0 - y_0| \exp\biggl(\int_{t_0}^{t} L(s)\,ds\biggr) + C\bigl(\Omega_J(\varepsilon + \varepsilon_1) + \varepsilon + \varepsilon_1\bigr)^{1/2}$$

and, for any natural number $n$,

$$\bigl|x_n(t) - x_{n+1}(t)\bigr| \le C\bigl(\Omega_J(\varepsilon_n + \varepsilon_{n+1}) + \varepsilon_n + \varepsilon_{n+1}\bigr)^{1/2}$$

for any $t \in I$. The latter implies that $(x_n(\cdot))$ converges uniformly on $I$ to some function $z(\cdot)$.

For any positive number $n$, let $f_n(\cdot) \in L^1(I,X)$ with $f_n(t) \in F(t, x_n(t) + \varepsilon_n \mathbb{B})$ for a.e. $t \in I$ be such that $x_n(\cdot)$ is a solution of $\dot{x}(t) \in Ax(t) + f_n(t)$. Due to the growth condition (F1), the sequence $(f_n(\cdot))$ is integrally bounded and hence $L^1$-weakly precompact. Passing to subsequences if necessary, we get that $(f_n(\cdot))$ converges $L^1$-weakly to some function $f(\cdot) \in L^1(I,X)$. By [5], Prop. 1, we get that $z(\cdot)$ is a solution of (1.2) with $z(t_0) = x_0$.

Let $n \in \mathbb{N}$ be such that $|x_n(t) - z(t)| < \eta/2$ for any $t \in I$. We have that

$$\begin{aligned}\bigl|y(t) - z(t)\bigr| &\leq |x_0 - y_0| \exp\biggl(\int_{t_0}^t L(s)\,ds\biggr) + C\bigl(\Omega_J(2\varepsilon) + 2\varepsilon\bigr)^{1/2} + C\sum_{n=1}^{\infty}\bigl(\Omega_J(2\varepsilon_n) + 2\varepsilon_n\bigr)^{1/2} \\ &\quad + \bigl|x_n(t) - z(t)\bigr| < |x_0 - y_0| \exp\biggl(\int_{t_0}^t L(s)\,ds\biggr) + C\bigl(\Omega_J(2\varepsilon) + 2\varepsilon\bigr)^{1/2} + \eta\end{aligned}$$

for any $t \in I$. Since $\eta$ is arbitrary, we get the conclusion. □

The following theorem is crucial in the proof of the main result.

**Theorem 3.4** *Assume* (F1)–(F3). *For any* $x_0, y_0 \in \overline{D(A)}$,

$$D_H\bigl(\mathrm{Sol}(x_0), \mathrm{Sol}(y_0)\bigr) \leq |x_0 - y_0| \exp\biggl(\int_{t_0}^t L(s)\,ds\biggr).$$

The proof is very similar to the proof of Corollary 1 in [9] and is omitted.

*Proof of Theorem* 3.1 Consider the map $\mathcal{S} : C(I,X) \rightrightarrows C(I,X)$, where $\mathcal{S}(z(\cdot))$ is the solution set of the Cauchy problem

$$\begin{cases} \dot{x}(t) \in Ax(t) + F(t, x(t)), \\ x(t_0) = g(z(\cdot)), \end{cases} \tag{3.3}$$

for any continuous function $z(\cdot)$. It follows from Theorem 3.4 that, for any $y(\cdot), z(\cdot) \in C(I,X)$,

$$\begin{aligned} D_H\bigl(\mathcal{S}(z(\cdot)), \mathcal{S}(y(\cdot))\bigr) &\leq \bigl|g(z(\cdot)) - g(y(\cdot))\bigr| \exp\biggl(\int_{t_0}^t L(s)\,ds\biggr) \\ &\leq K\bigl\|z(\cdot) - y(\cdot)\bigr\|_{C(I,X)} \exp\biggl(\int_{t_0}^t L(s)\,ds\biggr) < \alpha\bigl\|z(\cdot) - y(\cdot)\bigr\|_{C(I,X)}, \end{aligned}$$

where $\alpha = K \exp(\int_{t_0}^t L(s)\,ds) < 1$. Consequently, the map $\mathcal{S}$ is a set-valued contraction with closed values. Thus, there exists a fixed point $x(\cdot) \in \mathcal{S}(x(\cdot))$. Clearly, this fixed point $x(\cdot)$ is a solution of (1.1). □

## 4 A multipoint problem

Our target now is to investigate inclusion (1.1) with a particular choice of the function $g$. More precisely, we consider the nonlocal problem

$$\begin{cases} \dot{y}(s) \in Ay(s) + F(s, y(s)), \quad s \in I, \\ y(t_0) = \sum_{i=1}^k \alpha_i y(t_i), \end{cases} \tag{4.1}$$

where $t_0 < t_1 < \cdots < t_k \leq T$ are arbitrary but fixed, and $\alpha_i \in \mathbb{R}$ with $\sum_{i=1}^k |\alpha_i| = \kappa \leq 1$.

Clearly, we can apply Theorem 3.1 to this problem in the case where $\kappa < 1$. However, this theorem is not applicable for $\kappa = 1$, since, in this case, (3.1) does not hold. We mention that the case $\kappa = 1$ includes periodic and antiperiodic boundary conditions, that is, $y(t_0) = \pm y(T)$.

We further provide an existence result the problem (4.1) that covers also the case $\kappa = 1$. To this aim, we assume the following stronger form of condition (F3).

(F3′) There exists a positive constant $m$ such that, for all $x, y \in X$ and $t \in I$,

$$\sigma\big(J(x-y), F(t,x)\big) - \sigma\big(J(x-y), F(t,y)\big) \leq -m|x-y|^2.$$

**Theorem 4.1** *Under* (F1), (F2), *and* (F3′), *system* (4.1) *has at least a solution.*

To prove this theorem, we need the following lemma, which is used implicitly in [9] when the right-hand side is autonomous (see the proof of Theorem 4 therein).

**Lemma 4.2** *Assume* (F1), (F2), *and* (F3′). *Then for every* $\tau \in (t_0, T)$ *there exists a constant* $\bar{\alpha} = \bar{\alpha}(\tau) \in (0,1)$ *such that, for every solution* $x(\cdot)$ *of* (1.2) *with initial condition* $x_0 \in \overline{D(A)}$ *and every* $z_0 \in \overline{D(A)}$, *there exists a solution* $z(\cdot)$ *of* (1.2) *with* $z(t_0) = z_0$ *such that*

$$\big|x(t) - z(t)\big| \leq \bar{\alpha}|x_0 - z_0| \tag{4.2}$$

*for every* $\tau \leq t \leq T$.

*Proof* To use Theorem 2.2, we define $\bar{A} = A - mI$ and $H(t,x) = F(t,x) + mx$. Clearly, $\bar{A} + mI$ is m-dissipative, and $H(t,\cdot)$ is one-sided Lipschitz with the constant 0. System (1.2) can be rewritten as

$$\begin{cases} \dot{x}(t) \in \bar{A}x(t) + H(t,x(t)), \\ x(t_0) = x_0. \end{cases}$$

However, we do not want to change the notation and assume that $A + mI$ is m-dissipative and $F(t,\cdot)$ is one-sided Lipschitz with the constant 0.

Let $x_0, z_0 \in \overline{D(A)}$, $x_0 \neq z_0$, and let $x(\cdot) = x(x_0, f_x)(\cdot)$ with $f_x(s) \in F(s, x(s))$ a.e. on $I$.

We define the multifunction

$$G(s,u) = \big\{v \in F(s,u); \big\langle J\big(x(s) - u\big), f_x(s) - v\big\rangle \leq 0\big\}.$$

It is easy to show that $G(\cdot,u)$ has a strongly measurable selection. Then, by Proposition 2.4, for every $\delta > 0$, there exists a $\delta$-solution $z(\cdot)$ of

$$\begin{cases} \dot{z}(t) \in Az(t) + G(t,z(t)), \\ z(t_0) = z_0. \end{cases}$$

Then $z(\cdot)$ is a solution of $\dot{z}(t) \in Az(t) + f_z(t)$ for some function $f_z(\cdot) \in L^1(I,X)$ with $f_z(t) \in G(t, z(t) + h(t))$ a.e. on $I$, where $|h(t)| \leq \delta$. It follows that

$$\big\langle J\big(x(t) - \big(z(t) + h(t)\big)\big), f_x(t) - f_z(t)\big\rangle \leq 0.$$

Then we have that

$$\begin{aligned}\langle J(x(s)-z(s)),f_x(s)-f_z(s)\rangle &\le \left|\langle J(x(s)-z(s)-h(s))-J(x(s)-z(s)),f_x(s)-f_z(s)\rangle\right| \\ &\le \Omega_J(\delta)\left(\left|f_x(s)\right|+\left|f_z(s)\right|\right)\le 2\Omega_J(\delta)\mu(s).\end{aligned}$$

It is well known that $\langle J(u),v\rangle = |u|[u,v]_+$. Hence, for $|x(s)-z(s)|>0$, we have

$$\left[x(s)-z(s),f_x(s)-f_z(s)\right]_+ \le 2\frac{\Omega_J(\delta)\mu(s)}{|x(s)-z(s)|}.$$

Let $t\in I$ be fixed. Then, either $|x(t)-z(t)|\le \frac{1}{2}|x_0-z_0|$, or

$$\left[x(s)-z(s),f_x(s)-f_z(s)\right]_+ \le \frac{4\Omega_J(\delta)\mu(s)}{|x_0-z_0|} = \nu(\delta)\mu(s),$$

where $\nu(\delta)=\frac{4\Omega_J(\delta)}{|x_0-z_0|}$. Denote $N(t)=\int_{t_0}^t \mu(s)\,ds$.

The set $B=\{t\in(t_0,T); |x(t)-z(t)|>\frac{1}{2}|x_0-z_0|\}$ is open, and hence it is a countable union of pairwise disjoin open intervals, that is, $B=\bigcup_{i=1}^{\infty}(a_i,b_i)$, where $a_1=t_0$. Applying now Theorem 2.2, we obtain that

$$\left|x(t)-z(t)\right|\le e^{-m(t-t_0)}|x_0-z_0|+\nu(\delta)N(t) \tag{4.3}$$

for $t\in(a_1,b_1)$ and

$$\left|x(t)-z(t)\right|\le \frac{1}{2}e^{-m(t-a_i)}|x_0-z_0|+\nu(\delta)\left(N(t)-N(a_i)\right)\le\frac{1}{2}|x_0-z_0|+\nu(\delta)N(T)$$

for $t\in(a_i,b_i)$, $i>1$.

Since $\lim_{\delta\to 0}\Omega_J(\delta)=0$, for $\delta$ small enough, $\nu(\delta)N(T)<\frac{1}{4}|x_0-z_0|$. Therefore, we get that $|x(t)-z(t)|\le\frac{3}{4}|x_0-z_0|$ for every $t\ge b_1$. Let $\tau\in(t_0,T)$. If $\tau\ge b_1$, then we are done. If $\tau<b_1$, then we will use estimate (4.3). We can choose $\delta$ so small that $\nu(\delta)N(T)\le\frac{1-e^{-m(\tau-t_0)}}{2}|x_0-z_0|$. Consequently, $|x(t)-z(t)|\le\frac{1+e^{-m(\tau-t_0)}}{2}|x_0-z_0|$ for every $t\in[\tau,b_1)$. Hence, we have proved that we can choose $\delta$ so small that $|x(t)-z(t)|\le\gamma|x_0-z_0|$ for all $t\ge\tau$, where $\gamma<1$ does not depend on $|x_0-z_0|$.

The latter, together with the lemma of Filippov–Plis [9], Lemma 2, finishes the proof. □

Note that in [9], Lemma 2, we assumed that $A$ generates an equicontinuous semigroup. This fact, however, is not used in the proof there.

Now we are ready to prove the second main result of this paper.

*Proof of Theorem* 4.1 We will use the successive approximations method. We start with a point $x_0\in\overline{D(A)}$ and let $y^0(\cdot)$ be a solution of the local problem

$$\begin{cases}\dot{y}(t)\in Ay(t)+F(t,y(t)),\\ y(t_0)=x_0.\end{cases} \tag{4.4}$$

For the existence of such a solution, see, for example, [9]. Consider now the problem

$$\begin{cases} \dot{y}(t) \in Ay(t) + F(t, y(t)), \\ y(t_0) = \sum_{i=1}^{k} \alpha_i y^0(t_i). \end{cases} \tag{4.5}$$

Let $\bar{\alpha} = \bar{\alpha}(t_1)$ given by Lemma 4.2. Furthermore, there exists a solution $y^1(\cdot)$ of (4.5) such that

$$\left|y^1(t) - y^0(t)\right| \leq \bar{\alpha} \left| x_0 - \sum_{i=1}^{k} \alpha_i y^0(t_i) \right|$$

for all $t \geq t_1$.

We continue by applying Lemma 4.2 and define a sequence $(y^n(\cdot))$ such that, for every positive number $n$, $y^{n+1}(\cdot)$ is a solution of

$$\begin{cases} \dot{y}(t) \in Ay(t) + F(t, y(t)), \\ y(t_0) = \sum_{i=1}^{k} \alpha_i y^n(t_i), \end{cases} \tag{4.6}$$

and

$$\left|y^{n+1}(t) - y^n(t)\right| \leq \bar{\alpha} \sum_{i=1}^{k} \left|\alpha_i\left[y^n(t_i) - y^{n-1}(t_i)\right]\right|$$

for every $t \geq t_1$. It follows that $|y^{n+1}(t) - y^n(t)| \leq \bar{\alpha}\|y^n(\cdot) - y^{n-1}(\cdot)\|_{C(I,X)}$ for every $t \geq t_1$.

Thus, the sequence $(y^n(\cdot))$ converges uniformly to some continuous function $y(\cdot)$. The latter, together with $y^{n+1}(t_0) = \sum_{i=1}^{k} \alpha_i y^n(t_i)$, shows that $y(\cdot)$ is the required solution. □

**Remark 4.3** Using more carefully the estimations, we can prove the conclusion of Theorem 4.1 when $\kappa > 1$ and

$$\sum_{i=1}^{k} e^{-m(t_i - t_0)}|\alpha_i| < 1.$$

## 5 Examples

In this section we give two examples to apply the abstract results to partial differential inclusions.

The first one, inspired by [13], Section 5, illustrates the applicability of Theorem 3.1.

*Example* 5.1 Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary $\partial\Omega$. We consider the following boundary value problem:

$$\begin{cases} u_t(t,x) \in \Delta_x u(t,x) - \partial\varphi(u(t,x)) + G(t, u(t,x)), & x \in \Omega, t \in (0,T], \\ \frac{\partial u}{\partial n}(t,x) \in \partial\psi(u(t,x)), & x \in \partial\Omega, t \in (0,T], \\ u(0,x) = \int_\Omega \int_0^T h(s,x,\lambda,u(s,\lambda))\,ds\,d\lambda, & x \in \Omega. \end{cases} \tag{5.1}$$

Here $\varphi : \mathbb{R} \to \mathbb{R}$ is a proper lower semicontinuous convex function with $\varphi(0) = 0$, $\psi : \mathbb{R} \to \mathbb{R}$ is a convex continuous function with $0 \leq \psi(t) \leq C(1 + t^2)$, $t \in \mathbb{R}$, for some constant $C > 0$,

$G(t,u) = [f_1(t,u), f_2(t,u)]$ with $f_i : [0,T] \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$, and $h : [0,T] \times \Omega \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a given function.

Let $X = L^2(\Omega)$. Following [13], we define

$$\Phi(v) = \int_\Omega \varphi\bigl(v(x)\bigr)\,dx,$$

$$\Psi(v) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla(v(x))|^2\,dx + \int_{\partial\Omega} \psi(v(x))\,ds, & v \in H^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then $\Phi$ and $\Psi$ are proper lower semicontinuous convex functions with the domains $D(\Phi) = \{v \in L^2(\Omega); \varphi \circ v \in L^1(\Omega)\}$ and $D(\Psi) = H^1(\Omega)$. Moreover, $f \in \partial\Phi(v)$ if and only if $v, f \in L^2(\Omega), f(x) \in \partial\varphi(v(x))$ for a.e. $x \in \Omega$, and $g \in \partial\Psi(v)$ if and only if $-\Delta v = g$ in $L^2(\Omega)$ and $\frac{\partial v}{\partial n} + \partial\psi(v) \ni 0$ in $L^2(\partial\Omega)$ (see [18], Examples 2.B and 2.E, p. 163–164). Further, $\partial\Phi + \partial\Psi$ is m-dissipative and equal to $\partial(\Phi + \Psi)$ (see [18], Example 2.F, p. 167).

Then problem (5.1) can be rewritten in the abstract form (1.1) with the operator $A = \partial(\Phi + \Psi)$, the multifunction $F : [0,T] \times L^2(\Omega) \rightrightarrows L^2(\Omega)$ given by

$$F(t,u) = \bigl\{v \in X; f_1\bigl(t,u(x)\bigr) \le v(x) \le f_2\bigl(t,u(x)\bigr), x \in \Omega\bigr\},$$

and

$$g(u)(x) = \int_\Omega \int_0^T h\bigl(s,x,\lambda,u(s)(\lambda)\bigr)\,ds\,d\lambda,$$

for $u \in L^2(\Omega)$ and $x \in \Omega$. It is known that $A$ generates an equicontinuous (but not compact) semigroup on $L^2(\Omega)$ (see, e.g., [13]).

We suppose that the following hypotheses are satisfied.

(F) The functions $f_i$, $i = 1,2$, satisfy the following conditions:

(1) $f_1(t,u) \le f_2(t,u)$ for every $(t,u) \in [0,T] \times \mathbb{R}$;

(2) $f_1(\cdot,\cdot)$ is almost lower semicontinuous, and $f_2(\cdot,\cdot)$ is almost upper semicontinuous;

(3) there exists two positive Lebesgue-integrable functions $a(\cdot)$ and $b(\cdot)$ such that $|f_i(t,u)| \le a(t)|u| + b(t)$ on $[0,T] \times \mathbb{R}$ for $i = 1,2$;

(4) $f_i(t,u) = f_i'(t,u) + f_i''(t,u)$, where $f_i'(t,\cdot)$ is Lipschitz continuous with respect to a Lebesgue-integrable function $L(\cdot)$, and $f_i''(t,\cdot)$ are decreasing for $i = 1,2$.

(H) The function $h$ satisfies the following conditions:

(1) there exist a function $H(\cdot) \in L^2(\Omega, \mathbb{R}_+)$ and a positive Lebesgue-integrable function $\nu(\cdot)$ such that $|h(t,x,\lambda,r)| \le \nu(t)H(\lambda)$ for all $(t,x,\lambda,r) \in [0,T] \times \Omega \times \Omega \times \mathbb{R}$;

(2) $h(t,x,\lambda,r)$ is measurable in $(t,x,\lambda)$ for all $r \in \mathbb{R}$;

(3) $|h(t,x,\lambda,u) - h(t,x,\lambda,v)| \le \frac{K}{T\mu(\Omega)}|u - v|$ for all $(t,x,\lambda,u), (t,x,\lambda,v) \in [0,T] \times \Omega \times \Omega \times \mathbb{R}$.

If hypothesis (F) holds, then it is easy to prove that the multifunction $F(\cdot,\cdot)$ satisfies (F1)–(F3). Due to hypothesis (H), the function $g(\cdot)$ is well defined, and

$$\bigl|g(u_1) - g(u_2)\bigr|_{L^2(\Omega)} \le K\bigl\|u_1(\cdot) - u_2(\cdot)\bigr\|_{C([0,T];L^2(\Omega))}$$

for all $u_1, u_2 \in C([0,T]; L^2(\Omega))$. Applying Theorem 3.1, we get the following result.

**Theorem 5.2** *Assume that* (F) *and* (H) *hold*. *If*

$$K\exp\biggl(\int_0^T L(s)\,ds\biggr) < 1,$$

*then the nonlocal problem* (5.1) *has at least one solution*.

In the second example, we will use Theorem 4.1.

*Example* 5.3 Let $\Omega$, $\varphi$, and $\psi$ be as in the previous example, and let $G : [0,T] \times \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be a given multifunction. We consider the following system:

$$\begin{pmatrix} u_t(t,x) \\ \dot{v}(t) \end{pmatrix} \in \begin{pmatrix} B \\ 0 \end{pmatrix} + G\bigl(t, u(t,x), v(t)\bigr), \quad x \in \Omega, t \in (0,T), \tag{5.2}$$

$$\begin{cases} \frac{\partial u}{\partial n}(t,x) \in \partial\psi(u(t,x)), & x \in \partial\Omega, t \in (0,T), \\ u(0,x) = \frac{1}{2}(u(\frac{T}{2},x) + u(T,x)), & x \in \Omega, \\ v(0) = \frac{1}{2}(v(\frac{T}{2}) + v(T)). \end{cases} \tag{5.3}$$

Here $B = \partial(\Phi + \Psi)$ with $\Phi$ and $\Psi$ as in the first example.

Let $X = L^2(\Omega) \times H^1(0,T)$ with the norm $|(u,v)|_X = \sqrt{|u|^2_{L^2(\Omega)} + |v|^2_{H^1(0,T)}}$. Problem (5.2)–(5.3) can be rewritten in the abstract form (4.1) with the m-dissipative operator $A = \binom{B}{0}$ and the multifunction $F : [0,T] \times X \rightrightarrows X$ given by

$$F(t,u,v) = \bigl\{(y_1,y_2) \in X; \bigl(y_1(x), y_2(t)\bigr) \in G\bigl(t,u(x),v(t)\bigr) \text{ for a.e. } x \in \Omega \text{ and } t \in [0,T]\bigr\}.$$

We suppose that the multifunction $G$ satisfies the following conditions:

(G1) there exist $a(\cdot), b(\cdot) \in L^1(0,T)$ such that $\|G(t,z)\| \le a(t) + b(t)|z|$ for all $(t,z) \in [0,T] \times X$;

(G2) $G$ is almost upper semicontinuous with nonempty closed convex values;

(G3) $G(t,\cdot)$ is one-sided Lipschitz with negative constant.

Under these hypotheses, it is easy to prove that (F1), (F2), and (F3′) hold. Then, due to Theorem 4.1, we obtain the following result.

**Theorem 5.4** *Under assumptions* (G1)–(G3), *the nonlocal problem* (5.2)–(5.3) *has at least one solution*.

## 6 Concluding remarks

In this paper, we investigate the nonlocal problem (1.1) and prove two existence results.

The first one extends Theorem 4.1 of [22] in several directions. We recall that, in [22], the authors established an existence result for the nonlocal differential inclusion (1.1) assuming that $X$ is separable with uniformly convex dual, $F(\cdot,x)$ is measurable, $F(t,\cdot)$ is Lipschitz with the Lipschitz function $p(\cdot) \in L^1(I,\mathbb{R}_+)$, and

$$K + \int_{t_0}^T p(s)\,ds < 1, \tag{6.1}$$

where $K$ is the Lipschitz constant of $g(\cdot)$. Our condition (3.1) is weaker than (6.1), as can be seen from [1], Lemma 2.7. In fact, this condition (3.1) was used in [1] to prove an existence result for the nonlocal problem (1.1) in general Banach spaces under the assumption that $F(t,\cdot)$ is Lipschitz continuous. Clearly, the one-sided Lipschitz condition assumed in this paper is much weaker than the Lipschitz one, and, moreover, if $F(t,\cdot)$ is $p(t)$ Lipschitz, then it is $L(t)$ one-sided Lipschitz with $L(t) \le p(t)$. We further give two simple examples of maps that are one-sided Lipschitz but not Lipschitz. We also refer the reader to [11], where the advantages of the one-sided Lipschitz condition are shown.

*Example* 6.1 Let $X$ be a Hilbert space. We define the map

$$f(x) = \begin{cases} -\frac{x}{\sqrt{|x|}}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Clearly, $f(\cdot)$ is continuous and one-sided Lipschitz with the constant 0, but it is not Lipschitz.

Another example is in $X = L^3(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain. The dual space is $X^* = L^{3/2}(\Omega)$, and the duality map is

$$J(x)(\omega) = \begin{cases} \frac{x(\omega)|x(\omega)|}{\|x\|_{L^3(\Omega)}}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

for a.e. $\omega \in \Omega$ and for all $x \in L^3(\Omega)$. The map $F : L^3(\Omega) \rightrightarrows L^3(\Omega)$ given by

$$F(x) = -x + \begin{cases} -\frac{x}{|x|}, & x \neq 0, \\ \bar{\mathbb{B}}, & x = 0, \end{cases}$$

is upper semicontinuous at 0 and one-sided Lipschitz with the constant $-1$, but it is not Lipschitz, and even discontinuous at 0.

The second main result of this paper is devoted to the so-called multipoint problem. Note that this problem cannot be studied under the assumptions of Theorem 3.1. Such a kind of problems is studied in the literature under compactness-type assumptions or under stronger assumptions on the right-hand side. We refer the reader to [6]. Another approach is assuming that $F(t,\cdot)$ is $m$-Lipschitz, $A$ is m-dissipative, and, moreover, $A + \lambda I$ is dissipative with $m < \lambda$. We can see that our assumptions are more clear than the assumptions in [6] (although their results are applicable in more general Banach spaces).

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**Author details**

[1]Abdus Salam School of Mathematical Sciences, Lahore, Pakistan. [2]"Octav Mayer" Mathematics Institute, Romanian Academy, Iaşi, Romania. [3]Department of Mathematics, "Al. I. Cuza" University, Iaşi, Romania. [4]Department of Mathematics, "Gh. Asachi" Technical University, Iaşi, Romania.

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