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Existence and uniqueness results for the coupled systems of implicit fractional differential equations with periodic boundary conditions

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Abstract

In this paper, we study the periodic boundary value problems for the coupled systems of fractional implicit differential equations. Basing on the coincidence degree theory, we establish the existence and uniqueness theorems. Further, we provide several examples to show our main results.

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Keywords: Fractional implicit differential equation; Coupled system; Periodic boundary value problem; Existence and uniqueness; Coincidence degree theory

1 Introduction

In the past two decades, there has been tremendous interest in studying fractional differential equations (FDEs for short) due to their extensive applications in various fields of engineering and scientific disciplines (see [1–8]). For example, in [8], Laskin proposed the following fractional stochastic dynamic model for the considered market:

$$\begin{cases} D_{0+}^{\mu}x(t) = \lambda x(t) + F(t), \quad 0 < \mu \le 1, \\ D_{0+}^{\mu-1}x(0) = x_0, \end{cases}$$

where D_{0+}^{μ} is the standard Riemann–Liouville fractional derivative of order μ, λ and $F(t)$ respectively denote the expected rate and the random force.

As an important issue for the theory of FDEs, the existence, uniqueness, and multiplicity of solutions for the nonlinear fractional initial value problems (FIVPs for short) and fractional boundary value problems (FBVPs for short) have attracted scholars' attention. For some recent work on the topic, see papers [9–19], monographs [1, 2, 20, 21], and the references therein. In particular, many researchers focused on studying the FDEs with periodic boundary conditions (PBCs for short) (see [22–30]).

In [22], Cabada and Kisela discussed the following FDE with PBC:

$$\begin{cases} D_{0+}^{\alpha}u(t) - \lambda u(t) = f(t, t^{1-\alpha}u(t)), & 0 < \alpha \leq 1, \\ \lim_{t\to 0^+} t^{1-\alpha}u(t) = u(1), \end{cases}$$

where $\lambda \neq 0 (\lambda \in \mathbb{R})$, D_{0+}^{α} is the standard Riemann–Liouville fractional derivative of order α. The existence results were based on the fixed point theorems and monotone iterative technique.

In [23], Staněk dealt with the following FDE with PBC:

$$\begin{cases} {}^{c}D^{\alpha}u(t) + q(t, u(t)){}^{c}D^{\beta}u(t) = f(t, u(t)), & 0 < \beta < \alpha \leq 1, \\ u(0) = u(T), \quad T > 0, \end{cases}$$

where ${}^{c}D^{(\cdot)}$ is the Caputo fractional derivative of fractional order. The existence, multiplicity, and uniqueness results were proved by the Schauder fixed point theorem.

Recently, some scholars have considered very interesting aspects of IVPs and BVPs for the implicit FDEs (see [29–39]). For example, Nieto, Ouahab, and Venktesh [32] investigated a class of implicit FIVP:

$$\begin{cases} {}^{c}D^{\alpha}y(t) = f(t, y(t), {}^{c}D^{\alpha}y(t)), & t \in J, 0 < \alpha < 1, \\ y(0) = y_0, \end{cases}$$

where $J = [0, b]$, $b > 0$, ${}^{c}D^{\alpha}$ is the Caputo fractional derivative of order α, and $f : J \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. By using fixed point theory and approximation method, the existence and uniqueness results were obtained.

In [29], Benchohra, Bouriah, and Graef studied the following implicit FDE with PBC:

$$\begin{cases} {}^{c}D^{\alpha}y(t) = f(t, y(t), {}^{c}D^{\alpha}y(t)), & t \in J, 0 < \alpha \leq 1, \\ y(0) = y(T), \quad T > 0, \end{cases}$$

where $J = [0, T]$, ${}^{c}D^{\alpha}$ is the Caputo fractional derivative of order α and $f : J \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. Applying the coincidence degree theory, an existence result was given.

In [33], Ali, Zada, and Shah proved the existence and uniqueness of the solutions for the following implicit FDEs with three-point BCs:

$$\begin{cases} {}^{c}D^{p}y(t) - f(t, z(t), {}^{c}D^{p}y(t)) = 0, & p \in (2, 3], t \in [0, 1], \\ {}^{c}D^{q}z(t) - g(t, y(t), {}^{c}D^{q}z(t)) = 0, & q \in (2, 3], t \in [0, 1], \\ y'(t)|_{t=0} = y''(t)|_{t=0}, \quad y(t)|_{t=1} = \lambda y(\eta), & \lambda, \eta \in (0, 1), \\ z'(t)|_{t=0} = z''(t)|_{t=0}, \quad z(t)|_{t=1} = \lambda z(\eta), & \lambda, \eta \in (0, 1), \end{cases}$$

where ${}^{c}D^{(\cdot)}$ is the Caputo fractional derivative of fractional order and $f, g : J \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions. The results were accomplished by means of the Leray–Schauder fixed point theorem and Banach contraction principle.

Inspired by the above work, in this paper we are mainly concerned with the existence and uniqueness of solutions for the following coupled system of nonlinear implicit FDEs with PBCs:

$$\begin{cases} D_{0+}^{\alpha}x(t)=f(t,t^{1-\beta}y(t),D_{0+}^{\beta}y(t)), & t\in[0,1], 0<\alpha,\beta\leq 1,\\ D_{0+}^{\beta}y(t)=g(t,t^{1-\alpha}x(t),D_{0+}^{\alpha}x(t)), \\ \lim_{t\to 0^+}t^{1-\alpha}x(t)=x(1), \qquad \lim_{t\to 0^+}t^{1-\beta}y(t)=y(1), \end{cases} \tag{1.1}$$

where $D_{0+}^{(\cdot)}$ is the standard Riemann–Liouville fractional derivative of fractional order, $f,g:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ are two continuous functions. To state our main results, we assume that the nonlinear terms f and g satisfy the following general conditions:

(A_1) There exist nonnegative continuous functions $\gamma_i(t),\eta_i(t),\omega_i(t), i=1,2$, such that, for any $t\in[0,1]$, $u_i,v_i\in\mathbb{R}$, $(i=1,2)$,

$$\begin{aligned} \left|f\left(t,t^{1-\beta}u_1,v_1\right)\right| &\leq \gamma_1(t)\left|t^{1-\beta}u_1\right|+\eta_1(t)|v_1|+\omega_1(t),\\ \left|g\left(t,t^{1-\alpha}u_2,v_2\right)\right| &\leq \gamma_2(t)\left|t^{1-\alpha}u_2\right|+\eta_2(t)|v_2|+\omega_2(t). \end{aligned}$$

(A_2) There exist nonnegative continuous functions $p_i(t),q_i(t), i=1,2$, such that, for any $t\in[0,1]$, $u_{ij},v_{ij}\in\mathbb{R}$, $(i,j=1,2)$,

$$\begin{aligned} \left|f\left(t,t^{1-\beta}u_{11},v_{11}\right)-f\left(t,t^{1-\beta}u_{12},v_{12}\right)\right| &\leq p_1(t)t^{1-\beta}|u_{11}-u_{12}|+q_1(t)|v_{11}-v_{12}|,\\ \left|g\left(t,t^{1-\alpha}u_{21},v_{21}\right)-g\left(t,t^{1-\alpha}u_{22},v_{22}\right)\right| &\leq p_2(t)t^{1-\alpha}|u_{21}-u_{22}|+q_2(t)|v_{21}-v_{22}|. \end{aligned}$$

(A_3) There exist constants $a,c>0$, $b,d\geq 0$ such that, for any $t\in[0,1]$, $u_{ij},v_{ij}\in\mathbb{R}$, $(i,j=1,2)$,

$$\begin{aligned} \left|f\left(t,t^{1-\beta}u_{11},v_{11}\right)-f\left(t,t^{1-\beta}u_{12},v_{12}\right)\right| &\geq at^{1-\beta}|u_{11}-u_{12}|-b|v_{11}-v_{12}|,\\ \left|g\left(t,t^{1-\alpha}u_{21},v_{21}\right)-g\left(t,t^{1-\alpha}u_{22},v_{22}\right)\right| &\geq ct^{1-\alpha}|u_{21}-u_{22}|-d|v_{21}-v_{22}|. \end{aligned}$$

Remark 1.1 Condition (A_2) implies condition (A_1).

The objective of this paper is twofold. The first one is to study the existence solutions for BVP (1.1), the other is to consider the uniqueness of solution for (1.1). Our work presented in this paper has the following features. Firstly, this article generalizes the results of papers [29, 30] into coupled systems. Secondly, compared with [29, 30], we not only discuss the existence result but also establish the uniqueness result. In addition, the existence results of papers [29, 30] are based on condition (A_2), in our paper the existence result can also be obtained under condition (A_1). Thirdly, we present two prior estimation ways in using Theorem 2.1 (see Sect. 2) to establish the existence results. It should be pointed out that a number of papers by applying Theorem 2.1 to solve fractional resonance boundary value problems usually used the second way (see Lemma 3.4 in Sect. 3) to estimate the prior bounds. For example [40–43]. Our results show that the first way is better than the second. We finally remark that our paper investigates the FBVP in the frame of Riemann–Liouville fractional derivative which is more complicated than such a problem involving Caputo fractional derivative, and if $\alpha = \beta = 1$, then BVP (1.1) can be reduced to the implicit first order differential systems with PBCs.

The rest of this paper is built up as follows. We devote Sect. 2 to recalling some preliminary definitions and lemmas. We establish the existence and uniqueness theorems for problem (1.1) in Sect. 3. In order to fully explain our main results, we provide three examples in Sect. 4. Finally, we present some conclusions in Sect. 5.

2 Preliminaries

In this section, we recall some basic definitions, lemmas, and theorems which are used throughout this paper. Firstly, we introduce some definitions and results on fractional calculus [1, 2, 44].

Definition 2.1 The Riemann–Liouville fractional integral of order $\alpha > 0$ for a function $x : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\, ds,$$

provided that the right-hand side integral is pointwise defined on $(0, +\infty)$.

Definition 2.2 The Riemann–Liouville fractional derivative of order $\alpha > 0$ for a function $x : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} x(t) = \frac{d^n}{dt^n} I_{0+}^{n-\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} x(s)\, ds,$$

where $n = [\alpha] + 1$, provided that the right-hand side integral is pointwise defined on $(0, +\infty)$.

Lemma 2.1 *Let $\alpha > 0$. If $x, D_{0+}^{\alpha} x \in L^1(0,1)$, then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $n = [\alpha] + 1$, $c_i \in \mathbb{R} (i = 1, 2, \ldots, n)$ are arbitrary constants.

Lemma 2.2 *Let $\alpha > \beta > 0$. If $x \in L^1(0,1)$, then*

$$I_{0+}^{\alpha} I_{0+}^{\beta} x(t) = I_{0+}^{\alpha+\beta} x(t), \qquad D_{0+}^{\beta} I_{0+}^{\alpha} x(t) = I_{0+}^{\alpha-\beta} x(t),$$

in particular $D_{0+}^{\alpha} I_{0+}^{\alpha} x(t) = x(t)$.

Lemma 2.3 (see [44]) *If $\alpha > 0$, $\lambda > -1$, $t > 0$, then*

$$I_{0+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+\alpha)} t^{\alpha+\lambda}, \qquad D_{0+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)} t^{\lambda-\alpha},$$

in particular $D_{0+}^{\alpha} t^{\alpha-m} = 0$, $m = 1, 2, \ldots, n$, where $n = [\alpha] + 1$.

We recall now the basic knowledge on the coincidence degree theory. For more details, we refer the readers to [45–47].

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two real Banach spaces. Suppose $L : \operatorname{dom} L \subset X \to Y$ is a Fredholm operator with index zero, then there exist two continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Im} L = \operatorname{Ker} Q, \qquad X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \qquad Y = \operatorname{Im} L \oplus \operatorname{Im} Q,$$

and $L|_{\operatorname{dom}} L \cap \operatorname{Ker} P : \operatorname{dom} L \to \operatorname{Im} L$ is invertible. We denote by $K_P = (L|_{\operatorname{dom} L \cap \operatorname{Ker} P})^{-1}$. Let Ω be an open bounded subset of X and $\operatorname{dom} L \cap \bar{\Omega} \neq \emptyset$. The map $N : X \to Y$ is called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact.

Theorem 2.1 *Let $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ be L-compact on $\bar{\Omega}$. If the following conditions are satisfied*:

(i) *$Lu \neq \lambda Nu$ for any $u \in (\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega$, $\lambda \in (0, 1)$*;

(ii) *$Nu \notin \operatorname{Im} L$ for any $u \in \operatorname{Ker} L \cap \partial\Omega$*;

(iii) $\deg\{QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$;

then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \bar{\Omega}$.

Theorem 2.2 *Let $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero, $\Omega \subset X$ be an open bounded set symmetric with $0 \in \Omega$ and $N : \bar{\Omega} \to Y$ is L-compact. If $Lx - Nx \neq \lambda(-Lx - N(-x))$ for all $(\lambda, x) \in (0, 1] \times \operatorname{dom} L \cap \partial\Omega$, then $Lx = Nx$ has a solution in $\operatorname{dom} L \cap \bar{\Omega}$.*

3 Main results

Take

$$X_1 = \{x : t^{1-\alpha}x, D^{\alpha}_{0+}x \in C[0, 1]\}, \qquad X_2 = \{y : t^{1-\beta}y, D^{\beta}_{0+}y \in C[0, 1]\},$$

endowed with the norms

$$\|x\|_{X_1} = \|t^{1-\alpha}x\|_{\infty} + \|D^{\alpha}_{0+}x\|_{\infty}, \qquad \|y\|_{X_2} = \|t^{1-\beta}y\|_{\infty} + \|D^{\beta}_{0+}y\|_{\infty},$$

respectively, where $\|\cdot\|_{\infty} = \max_{t\in[0,1]} |\cdot|$. We can easily check that $(X_1, \|\cdot\|_{X_1})$ and $(X_2, \|\cdot\|_{X_2})$ are two Banach spaces. Let $Z_1 = C[0, 1]$ with norm $\|z\|_{Z_1} = \max_{t\in[0,1]} |z(t)|$. According to the basic theory of functional analysis, we have $X = X_1 \times X_2$ and $Z = Z_1 \times Z_1$ are also Banach spaces, respectively, with the norms

$$\|(x, y)\|_X = \max\{\|x\|_{X_1}, \|y\|_{X_2}\}, \qquad \|(u, v)\|_Z = \max\{\|u\|_{Z_1}, \|v\|_{Z_1}\}.$$

Define the linear operators $L_i : \operatorname{dom} L_i \subset X_i \to Z_1 (i = 1, 2)$ and the nonlinear operators $N_1 : X_2 \to Z_1$, $N_2 : X_1 \to Z_1$ by

$$L_1x(t) = D^{\alpha}_{0+}x(t), \quad x(t) \in \operatorname{dom} L_1, \qquad N_1y(t) = f(t, t^{1-\beta}y(t), D^{\beta}_{0+}y(t)), \quad y(t) \in X_2,$$

$$L_2y(t) = D^{\beta}_{0+}y(t), \quad y(t) \in \operatorname{dom} L_2, \qquad N_2x(t) = g(t, t^{1-\alpha}x(t), D^{\alpha}_{0+}x(t)), \quad x(t) \in X_1,$$

where

$$\operatorname{dom} L_1 = \Big\{x \in X_1 : \lim_{t\to 0^+} t^{1-\alpha}x(t) = x(1)\Big\}, \qquad \operatorname{dom} L_2 = \Big\{y \in X_2 : \lim_{t\to 0^+} t^{1-\beta}y(t) = y(1)\Big\}.$$

Define the linear operator $L : \operatorname{dom} L \subset X \to Z$ and the nonlinear operator $N : X \to Z$ by

$$L(x,y) = (L_1x, L_2y), \quad (x,y) \in \operatorname{dom} L,$$
$$N(x,y) = (N_1y, N_2x), \quad (x,y) \in X,$$

where

$$\operatorname{dom} L = \big\{(x,y) \in X : x \in \operatorname{dom} L_1, y \in \operatorname{dom} L_2\big\}.$$

Then the coupled system of BVP (1.1) is equivalent to the operator equation $L(x,y) = N(x,y)$, $(x,y) \in \operatorname{dom} L$.

Lemma 3.1 *The mapping $L : \operatorname{dom} L \subset X \to Z$ is a Fredholm operator with index zero.*

Proof First, we claim that the operator L satisfies

$$\operatorname{Ker} L = \big\{(x,y) \in \operatorname{dom} L : x(t) = c_1t^{\alpha-1}, y(t) = c_2t^{\beta-1}, c_1, c_2 \in \mathbb{R}\big\} \cong \mathbb{R}^2, \tag{3.1}$$

$$\operatorname{Im} L = \Big\{(u,v) \in Z : \int_0^1 (1-s)^{\alpha-1}u(s)\,ds = 0, \int_0^1 (1-s)^{\beta-1}v(s)\,ds = 0\Big\}. \tag{3.2}$$

In fact, by Lemma 2.1, it can easily be checked that (3.1) holds. For any $(u,v) \in \operatorname{Im} L$, there exists $(x,y) \in \operatorname{dom} L$ such that $D^{\alpha}_{0+}x(t) = u(t)$, $D^{\beta}_{0+}y(t) = v(t)$. Using Lemma 2.1 and the boundary conditions in (1.1), we find

$$\begin{aligned} &\int_0^1 (1-s)^{\alpha-1}u(s)\,ds = 0, \\ &\int_0^1 (1-s)^{\beta-1}v(s)\,ds = 0. \end{aligned} \tag{3.3}$$

That is,

$$\operatorname{Im} L \subset \Big\{(u,v) \in Z : \int_0^1 (1-s)^{\alpha-1}u(s)\,ds = 0, \int_0^1 (1-s)^{\beta-1}v(s)\,ds = 0\Big\}.$$

Conversely, for any $(u,v) \in Z$ satisfying (3.3), take $x(t) = I^{\alpha}_{0+}u(t)$ and $y(t) = I^{\beta}_{0+}v(t)$, then we have

$$\lim_{t\to 0^+} t^{1-\alpha}x(t) = \lim_{t\to 0^+} t^{1-\alpha}I^{\alpha}_{0+}u(t) = 0 = x(1) = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}u(s)\,ds,$$
$$\lim_{t\to 0^+} t^{1-\beta}y(t) = \lim_{t\to 0^+} t^{1-\beta}I^{\beta}_{0+}v(t) = 0 = y(1) = \frac{1}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}v(s)\,ds,$$

and

$$(x,y) \in X, \quad L(x,y) = (L_1 x, L_2 y) = (u,v),$$

which shows $(u,v) \in \operatorname{Im} L$. Therefore, (3.2) holds.

Second, we prove that $\operatorname{ind} L = \dim \operatorname{Ker} L - co\dim \operatorname{Im} L = 0$. Define the linear operators $Q_i : Z_1 \to Z_1 (i = 1,2)$ and $Q : Z \to Z$ by

$$Q_1 u = \alpha \int_0^1 (1-s)^{\alpha-1} u(s)\, ds,$$
$$Q_2 v = \beta \int_0^1 (1-s)^{\beta-1} v(s)\, ds,$$
$$Q(u,v) = (Q_1 u, Q_2 v).$$

Evidently, Q_1, Q_2, Q are continuous operators and $\operatorname{Im} L = \operatorname{Ker} Q$. For any $(u,v) \in Z$, we have

$$Q_1^2 u = Q_1(Q_1 u) = (Q_1 u)\alpha \int_0^1 (1-s)^{\alpha-1}\, ds = Q_1 u,$$
$$Q_2^2 v = Q_2(Q_2 v) = (Q_2 v)\beta \int_0^1 (1-s)^{\beta-1}\, ds = Q_2 v,$$
$$Q^2(u,v) = Q\bigl(Q(u,v)\bigr) = Q(Q_1 u, Q_2 v) = \bigl(Q_1^2 u, Q_2^2 v\bigr) = Q(u,v).$$

Thus, Q is a continuous linear projector. For $(u,v) \in Z$, set $(u_1, v_1) = (u,v) - Q(u,v)$, then $Q(u_1,v_1) = Q(u,v) - Q^2(u,v) = 0$, i.e., $(u_1,v_1) \in \operatorname{Ker} Q = \operatorname{Im} L$. So, $Z = \operatorname{Im} L + \operatorname{Im} Q$. Besides, for every $(u,v) \in \operatorname{Im} L \cap \operatorname{Im} Q$, we have $(u,v) = Q(u,v) = (0,0)$. Therefore, $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. Furthermore, $\dim \operatorname{Ker} L = \dim \operatorname{Im} Q = co\dim \operatorname{Im} L = 2$, which means L is a Fredholm operator with index zero. □

Lemma 3.2 *Define the linear operator* $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap Y$ *by*

$$K_P(u,v) = \bigl(I_{0+}^{\alpha} u, I_{0+}^{\beta} v\bigr), \quad (u,v) \in \operatorname{Im} L.$$

Then K_P *is the inverse of* $L|_{\operatorname{dom} L \cap Y}$ *and satisfies*

$$\bigl\| K_P(u,v) \bigr\|_X \le \Delta \bigl\| (u,v) \bigr\|_Z \quad \text{for all } (u,v) \in \operatorname{Im} L,$$

where

$$Y = \Bigl\{ (x,y) \in X : \lim_{t \to 0^+} t^{1-\alpha} x(t) = 0, \lim_{t \to 0^+} t^{1-\beta} y(t) = 0 \Bigr\},$$
$$\Delta = \max \biggl\{ 1 + \frac{1}{\Gamma(\alpha+1)}, 1 + \frac{1}{\Gamma(\beta+1)} \biggr\}.$$

Proof Define the linear operators $P_i : X_i \to X_i$ $(i = 1,2)$ and $P : X \to X$ by

$$P_1 x = \Bigl[\lim_{t \to 0^+} t^{1-\alpha} x(t) \Bigr] t^{\alpha-1},$$

$$P_2 y = \Big[\lim_{t\to 0^+} t^{1-\beta} y(t)\Big] t^{\beta-1},$$
$$P(x,y) = (P_1 x, P_2 y), \quad (x,y) \in X.$$

We first claim that P is a continuous linear projector operator. In fact, for any $(x,y)\in X$, we have

$$P_1^2 x = P_1(P_1 x) = \Big[\lim_{t\to 0^+} t^{1-\alpha} P_1 x(t)\Big] t^{\alpha-1} = \Big[\lim_{t\to 0^+} t^{1-\alpha} x(t)\Big] t^{\alpha-1} = P_1 x,$$
$$P_2^2 y = P_2(P_2 y) = \Big[\lim_{t\to 0^+} t^{1-\beta} P_2 y(t)\Big] t^{\beta-1} = \Big[\lim_{t\to 0^+} t^{1-\beta} y(t)\Big] t^{\beta-1} = P_2 y,$$
$$P^2(x,y) = P\big(P(x,y)\big) = P(P_1 x, P_2 y) = \big(P_1^2 x, P_2^2 y\big) = (P_1 x, P_2 y) = P(x,y),$$

and

$$t^{1-\alpha} P_1 x = \lim_{t\to 0^+} t^{1-\alpha} x(t), \qquad t^{1-\beta} P_2 y = \lim_{t\to 0^+} t^{1-\beta} y(t).$$

By Lemma 2.3, we also have

$$D_{0+}^{\alpha} P_1 x(t) = D_{0+}^{\beta} P_2 y(t) = 0.$$

Then

$$\begin{aligned}\big\|P(x,y)\big\|_X &= \big\|(P_1 x, P_2 y)\big\|_X = \max\big\{\|P_1 x\|_{X_1}, \|P_2 y\|_{X_2}\big\} \\ &= \max\Big\{\Big|\lim_{t\to 0^+} t^{1-\alpha} x(t)\Big|, \Big|\lim_{t\to 0^+} t^{1-\beta} y(t)\Big|\Big\}.\end{aligned}$$

Thus, $P: X \to X$ is a bounded linear projector operator, and it is evident that $\operatorname{Im} P = \operatorname{Ker} L$, $Y = \operatorname{Ker} P$.

Next, we show that $K_P = (L|_{\operatorname{dom} L\cap \operatorname{Ker} P})^{-1}$. In fact, for any $(u,v)\in \operatorname{Im} L$, by the definition of K_P, we can check that $K_P(u,v) \in \operatorname{dom} L \cap \operatorname{Ker} P$, that is, K_P is well defined on $\operatorname{Im} L$. On the one hand, by Lemma 2.2, we have

$$(LK_P)\big(u(t), v(t)\big) = \big(D_{0+}^{\alpha} I_{0+}^{\alpha} u(t), D_{0+}^{\beta} I_{0+}^{\beta} v(t)\big) = \big(u(t), v(t)\big).$$

On the other hand, for every $(x(t), y(t)) \in \operatorname{dom} L \cap \operatorname{Ker} P$, by Lemma 2.1, we get

$$\begin{aligned}(K_P L)\big(x(t), y(t)\big) &= \big(I_{0+}^{\alpha} D_{0+}^{\alpha} x(t), I_{0+}^{\beta} D_{0+}^{\beta} y(t)\big) \\ &= \big(x(t) + c_1 t^{\alpha-1}, y(t) + c_2 t^{\beta-1}\big), \quad (c_1, c_2) \in \mathbb{R}^2.\end{aligned}$$

Because $(K_P L)(x(t), y(t)) \in \operatorname{Ker} P$ and $(c_1 t^{\alpha-1}, c_2 t^{\beta-1}) \in \operatorname{Ker} L = \operatorname{Im} P$, we can obtain

$$\begin{aligned}(0,0) &= P\big[(K_P L)\big(x(t), y(t)\big)\big] = P\big(x(t) + c_1 t^{\alpha-1}, y(t) + c_2 t^{\beta-1}\big) \\ &= \big(P_1 x(t) + c_1 t^{\alpha-1}, P_2 y(t) + c_2 t^{\beta-1}\big) = \big(c_1 t^{\alpha-1}, c_2 t^{\beta-1}\big).\end{aligned}$$

Thus, $(K_PL)(x(t),y(t)) = (x(t),y(t))$. Therefore, $K_P = (L|_{\operatorname{dom} L\cap \operatorname{Ker} P})^{-1}$. Again by Lemma 2.2, for all $(u,v) \in \operatorname{Im} L$, we have

$$\begin{aligned}\|K_P(u,v)\|_X &= \max\{\|I_{0+}^{\alpha}u\|_{X_1}, \|I_{0+}^{\beta}v\|_{X_2}\}\\ &= \max\{\|t^{1-\alpha}I_{0+}^{\alpha}u\|_{\infty} + \|D_{0+}^{\alpha}I_{0+}^{\alpha}u\|_{\infty}, \|t^{1-\beta}I_{0+}^{\beta}v\|_{\infty} + \|D_{0+}^{\beta}I_{0+}^{\beta}v\|_{\infty}\}\\ &\le \max\left\{\left(1+\frac{1}{\Gamma(1+\alpha)}\right)\|u\|_{\infty}, \left(1+\frac{1}{\Gamma(1+\beta)}\right)\|v\|_{\infty}\right\}\\ &\le \max\{\Delta\|u\|_{\infty}, \Delta\|v\|_{\infty}\} = \Delta\|(u,v)\|_Z.\end{aligned}$$

This completes the proof of Lemma 3.2. □

Lemma 3.3 *Assume that* (A_2) *holds,* $\Omega \subset X$ *is an open bounded subset with* $\operatorname{dom} L\cap\bar{\Omega} \neq \emptyset$. *Then* N *is* L*-compact on* $\bar{\Omega}$.

Proof Since $f,g : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous and satisfy (A_2), we claim that $QN(\bar{\Omega})$ and $(I-Q)N(\bar{\Omega})$ are uniformly bounded. In fact, for Ω is bounded in X, there exists a constant $r > 0$ such that $\|(x,y)\|_X \le r$, $\forall (x,y)\in\bar{\Omega}$, by (A_2), we have the following inequalities:

$$\begin{aligned}|N_1y(t)| &\le |f(t,t^{1-\beta}y(t),D_{0+}^{\beta}y(t)) - f(t,0,0)| + |f(t,0,0)|\\ &\le \rho + (p_1(t)t^{1-\beta}|y(t)| + q_1(t)|D_{0+}^{\beta}y(t)|)\\ &\le \rho + (p_1+q_1)r := r_1,\\ |Q_1N_1y(t)| &\le \alpha\int_0^1 (1-s)^{\alpha-1}|N_1y(s)|\,ds\\ &\le r_1\alpha\int_0^1(1-s)^{\alpha-1}\,ds = r_1,\end{aligned}$$

where $\rho = \sup_{t\in[0,1]} f(t,0,0)$, $p_1 = \sup_{t\in[0,1]} p_1(t)$, $q_1 = \sup_{t\in[0,1]} q_1(t)$. In the same way, we have

$$|N_2x(t)| \le \varpi + (p_2+q_2)r := r_2, \qquad |Q_2N_2x(t)| \le r_2,$$

where $\varpi = \sup_{t\in[0,1]} g(t,0,0)$, $p_2 = \sup_{t\in[0,1]} p_2(t)$, $q_2 = \sup_{t\in[0,1]} q_2(t)$. So we get that

$$\begin{aligned}\|QN(x,y)\|_Z &= \max\{\|Q_1N_1y\|_{Z_1}, \|Q_2N_2x\|_{Z_1}\} \le \max\{r_1,r_2\},\\ \|(I-Q)N(x,y)\|_Z &\le \|N(x,y)\|_Z + \|QN(x,y)\|_Z \le 2\max\{r_1,r_2\}.\end{aligned} \tag{3.4}$$

Use of Lemma 3.2 yields

$$\|K_P(I-Q)N(x,y)\|_X \le \Delta\|(I-Q)N(x,y)\|_Z \le 2\Delta\max\{r_1,r_2\}. \tag{3.5}$$

From (3.4), (3.5) it follows that $QN(\bar{\Omega})$, $K_P(I-Q)N(\bar{\Omega})$ are uniformly bounded. Now, we are going to prove that $K_P(I-Q)N(x,y)$ is equicontinuous for all $(x,y)\in\bar{\Omega}$. In fact, take $(x,y)\in\bar{\Omega}$ and $0\le t_1 < t_2 \le 1$. Since t^{α}, t are uniformly continuous on $[t_1,t_2]$ and $f(t,u,v)$,

$g(t,u,v)$ are uniformly continuous on $[t_1,t_2]\times[-r,r]\times[-r,r]$, we have

$$\begin{aligned}
&\left|I_{0+}^{\alpha}(I-Q_1)N_1y(t)|_{t=t_1}-I_{0+}^{\alpha}(I-Q_1)N_1y(t)|_{t=t_2}\right| \\
&\quad=\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}(t_1-s)^{\alpha-1}(I-Q_1)N_1y(s)\,ds-\int_0^{t_2}(t_2-s)^{\alpha-1}(I-Q_1)N_1y(s)\,ds\right| \\
&\quad\le\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\left[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right](I-Q_1)N_1y(s)\,ds\right| \\
&\qquad+\frac{1}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}(I-Q_1)N_1y(s)\,ds\right| \\
&\quad\le\frac{2r_1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right]ds+\frac{2r_1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\,ds \\
&\quad\le\frac{2r_1}{\Gamma(\alpha+1)}\left[\left(t_1^{\alpha}-t_2^{\alpha}\right)+2(t_2-t_1)^{\alpha}\right]\to 0\quad\text{as } t_1\to t_2,
\end{aligned}$$

and

$$\begin{aligned}
&\left|D_{0+}^{\alpha}I_{0+}^{\alpha}(I-Q_1)N_1y(t)|_{t=t_1}-D_{0+}^{\alpha}I_{0+}^{\alpha}(I-Q_1)N_1y(t)|_{t=t_2}\right| \\
&\quad=\left|f\left(t_1,t_1^{1-\beta}y(t_1),D_{0+}^{\beta}y(t_1)\right)-f\left(t_2,t_2^{1-\beta}y(t_2),D_{0+}^{\beta}y(t_2)\right)\right|\to 0\quad\text{as } t_1\to t_2.
\end{aligned}$$

Similarly, it has

$$\begin{aligned}
&\left|I_{0+}^{\beta}(I-Q_2)N_2x(t)|_{t=t_1}-I_{0+}^{\beta}(I-Q_2)N_2x(t)|_{t=t_2}\right| \\
&\quad\le\frac{2r_2}{\Gamma(\beta+1)}\left[\left(t_1^{\beta}-t_2^{\beta}\right)+2(t_2-t_1)^{\beta}\right]\to 0\quad\text{as } t_1\to t_2,
\end{aligned}$$

and

$$\begin{aligned}
&\left|D_{0+}^{\beta}I_{0+}^{\beta}(I-Q_2)N_2x(t)|_{t=t_1}-D_{0+}^{\beta}I_{0+}^{\beta}(I-Q_2)N_2x(t)|_{t=t_2}\right| \\
&\quad=\left|g\left(t_1,t_1^{1-\alpha}x(t_1),D_{0+}^{\alpha}x(t_1)\right)-g\left(t_2,t_2^{1-\alpha}x(t_2),D_{0+}^{\alpha}x(t_2)\right)\right|\to 0\quad\text{as } t_1\to t_2.
\end{aligned}$$

To summarize, we can conclude that $\{K_P(I-Q)N(x,y):(x,y)\in\bar{\Omega}\}$ is equicontinuous. By the Ascoli–Arzelà theorem, it is immediate that $K_P(I-Q)N:\bar{\Omega}\to X$ is compact. Using a similar argument, we can also get N is L-compact if condition (A_1) holds. □

In what follows, we shall give several existence results for BVP (1.1). For simplicity of presentation, we let

$$\begin{aligned}
\rho_1&=\frac{\omega_2+\omega_1\eta_2}{1-\eta_1\eta_2}, &\rho_2&=\frac{\gamma_2}{1-\eta_1\eta_2}, &\rho_3&=\frac{\gamma_1\eta_2}{1-\eta_1\eta_2}, \\
\sigma_1&=\frac{\omega_1+\omega_2\eta_1}{1-\eta_1\eta_2}, &\sigma_2&=\frac{\gamma_2\eta_1}{1-\eta_1\eta_2}, &\sigma_3&=\frac{\gamma_1}{1-\eta_1\eta_2},
\end{aligned}$$

where $\gamma_i=\max_{t\in[0,1]}|\gamma(t)|$, $\eta_i=\max_{t\in[0,1]}|\gamma(t)|$, $\omega_i=\max_{t\in[0,1]}|\gamma(t)|$, $i=1,2$. First, we suppose that the following conditions are satisfied:

(A_4) For $(x,y)\in\operatorname{dom}L$, there exist constants $B_i>0$, $i=1,2$, such that, for all $t\in[0,1]$, if either $|t^{1-\alpha}x(t)|>B_1$ or $|t^{1-\beta}y(t)|>B_2$, then $QN(x,y)\neq(0,0)$.

(A$_5$) For $(c_1 t^{\alpha-1}, c_2 t^{\beta-1}) \in \operatorname{Ker} L$, there exist constants $G_i > 0$, $i = 1, 2$, such that for any $(c_1, c_2) \in \mathbb{R}^2$ satisfying either

$$\begin{cases} c_1 N_2(c_1 t^{\alpha-1}) > 0, & \text{if } |c_1| > G_1, \\ c_2 N_1(c_2 t^{\beta-1}) > 0, & \text{if } |c_2| > G_2, \end{cases} \tag{3.6}$$

or

$$\begin{cases} c_1 N_2(c_1 t^{\alpha-1}) < 0, & \text{if } |c_1| > G_1, \\ c_2 N_1(c_2 t^{\beta-1}) < 0, & \text{if } |c_2| > G_2. \end{cases} \tag{3.7}$$

Lemma 3.4 *Let* (A$_1$) *and* (A$_4$) *hold, set*

$$\bigl\{\Omega_1 = (x, y) \in \operatorname{dom} L \setminus \operatorname{Ker} L : L(x, y) = \lambda N(x, y), \lambda \in (0, 1)\bigr\}.$$

Then Ω_1 *is bounded provided that*

$$\begin{aligned} &\eta_1 \eta_2 < 1, \quad \Gamma(\alpha + 1) > 2\sigma_2, \qquad \Gamma(\beta + 1) > 2\rho_3, \\ &\bigl(\Gamma(\alpha + 1) - 2\sigma_2\bigr)\bigl(\Gamma(\beta + 1) - 2\rho_3\bigr) > 4\rho_2\sigma_3, \end{aligned} \tag{3.8}$$

or

$$\begin{aligned} &\frac{2}{\Gamma(\alpha + 1)}(\gamma_1 + \eta_1) + \Delta(\gamma_2 + \eta_2) < 1, \qquad \left(\frac{2}{\Gamma(\alpha + 1)} + \Delta\right)(\gamma_1 + \eta_1) < 1, \\ &\frac{2}{\Gamma(\beta + 1)}(\gamma_2 + \eta_2) + \Delta(\gamma_1 + \eta_1) < 1, \qquad \left(\frac{2}{\Gamma(\beta + 1)} + \Delta\right)(\gamma_2 + \eta_2) < 1. \end{aligned} \tag{3.9}$$

Proof For $(x, y) \in \Omega_1$, we have $N(x, y) \in \operatorname{Im} L = \operatorname{Ker} Q$. Then $QN(x, y) = (0, 0)$. On the one hand, according to hypothesis (A$_4$), it follows that

$$|t_2^{1-\alpha} x(t_2)| \le B_1, \qquad |t_1^{1-\beta} y(t_1)| \le B_2. \tag{3.10}$$

Since

$$I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) = x(t) + c_1 t^{\alpha-1}, \quad c_1 \in \mathbb{R}, \qquad I_{0+}^{\beta} D_{0+}^{\beta} y(t) = y(t) + c_2 t^{\beta-1}, \quad c_2 \in \mathbb{R}. \tag{3.11}$$

By substituting (3.10) into (3.11), we obtain

$$c_1 = t^{1-\alpha} I_{0+}^{\alpha} D_{0+}^{\alpha} x(t)|_{t=t_2} - t_2^{1-\alpha} x(t_2), \qquad c_2 = t^{1-\beta} I_{0+}^{\beta} D_{0+}^{\beta} y(t)|_{t=t_1} - t_1^{1-\beta} y(t_1). \tag{3.12}$$

From (3.10), (3.11), and (3.12) we have

$$\bigl|t^{1-\alpha} x(t)\bigr| \le B_1 + \frac{2}{\Gamma(\alpha + 1)} \bigl\|D_{0+}^{\alpha} x\bigr\|_\infty, \qquad \bigl|t^{1-\beta} y(t)\bigr| \le B_2 + \frac{2}{\Gamma(\beta + 1)} \bigl\|D_{0+}^{\beta} y\bigr\|_\infty. \tag{3.13}$$

On the other hand, by (A$_1$), we have

$$\begin{aligned} \bigl|D_{0+}^{\alpha} x\bigr| = \bigl|f\bigl(t, t^{1-\beta} y, D_{0+}^{\beta} y\bigr)\bigr| &\le \gamma_1(t)\bigl|t^{1-\beta} y\bigr| + \eta_1(t)\bigl|D_{0+}^{\beta} y\bigr| + w_1(t) \\ &\le \gamma_1\bigl|t^{1-\beta} y\bigr| + \eta_1\bigl|D_{0+}^{\beta} y\bigr| + \omega_1, \end{aligned} \tag{3.14}$$

$$
\begin{aligned}
\left|D_{0+}^{\beta}y\right| &= \left|g\left(t,t^{1-\alpha}x,D_{0+}^{\alpha}x\right)\right| \le \gamma_2(t)\left|t^{1-\alpha}x\right| + \eta_2(t)\left|D_{0+}^{\alpha}x\right| + w_2(t) \\
&\le \gamma_2\left|t^{1-\alpha}x\right| + \eta_2\left|D_{0+}^{\alpha}x\right| + \omega_2.
\end{aligned} \tag{3.15}
$$

We now estimate Ω_1 is bounded under conditions (3.8) and (3.9), respectively.

First. We show that Ω_1 is bounded if condition (3.8) holds. In fact, (3.14) and (3.15) imply that

$$
\left|D_{0+}^{\alpha}x\right| \le \sigma_1 + \sigma_2\left|t^{1-\alpha}x\right| + \sigma_3\left|t^{1-\beta}y\right|, \qquad \left|D_{0+}^{\beta}y\right| \le \rho_1 + \rho_2\left|t^{1-\alpha}x\right| + \rho_3\left|t^{1-\beta}y\right|. \tag{3.16}
$$

If we plug (3.16) back into (3.13), we get

$$
\begin{aligned}
\left|t^{1-\alpha}x\right| &\le B_1 + \frac{2}{\Gamma(\alpha+1)}\left(\sigma_1 + \sigma_2\left|t^{1-\alpha}x\right| + \sigma_3\left|t^{1-\beta}y\right|\right), \\
\left|t^{1-\beta}y\right| &\le B_2 + \frac{2}{\Gamma(\beta+1)}\left(\rho_1 + \rho_2\left|t^{1-\alpha}x\right| + \rho_3\left|t^{1-\beta}y\right|\right).
\end{aligned}
$$

So that

$$
\left|t^{1-\alpha}x\right| \le \frac{B_1\Gamma(\alpha+1)+2\sigma_1}{\Gamma(\alpha+1)-2\sigma_2} + \frac{2\sigma_3}{\Gamma(\alpha+1)-2\sigma_2}\left|t^{1-\beta}y\right| := \ell_1 + \ell_2\left|t^{1-\beta}y\right|, \tag{3.17}
$$

$$
\left|t^{1-\beta}y\right| \le \frac{B_2\Gamma(\beta+1)+2\rho_1}{\Gamma(\beta+1)-2\rho_3} + \frac{2\rho_2}{\Gamma(\beta+1)-2\rho_3}\left|t^{1-\alpha}x\right| := \delta_1 + \delta_2\left|t^{1-\alpha}x\right|. \tag{3.18}
$$

It follows from (3.17) and (3.18) that

$$
\left|t^{1-\beta}y\right| \le \frac{\delta_1+\delta_2\ell_1}{1-\delta_2\ell_2} := \tau, \quad \left|t^{1-\alpha}x\right| \le \ell_1 + \ell_2\tau. \tag{3.19}
$$

Substituting (3.19) into (3.16), we obtain

$$
\left|D_{0+}^{\alpha}x\right| \le \sigma_1 + \sigma_2(\ell_1+\ell_2\tau) + \sigma_3\tau, \qquad \left|D_{0+}^{\beta}y\right| \le \rho_1 + \rho_2(\ell_1+\ell_2\tau) + \rho_3\tau.
$$

Thus, Ω_1 is bounded.

Second. We prove that Ω_1 is bounded under condition (3.9). In such a case, by Lemma 3.2, one has

$$
\begin{aligned}
\left\|(I-P)(x,y)\right\|_X &= \left\|K_P L(I-P)(x,y)\right\|_X \le \Delta\left\|L(x,y)\right\|_Z \\
&= \Delta\max\left\{\left\|D_{0+}^{\alpha}x\right\|_\infty, \left\|D_{0+}^{\beta}y\right\|_\infty\right\}.
\end{aligned}
$$

Therefore, from (3.13)–(3.15), we can derive that

$$
\begin{aligned}
\left\|(x,y)\right\|_X &\le \left\|P(x,y)\right\|_X + \left\|(I-P)(x,y)\right\|_X \\
&\le \max\left\{\left\|t^{1-\beta}y\right\|_\infty, \left\|t^{1-\alpha}x\right\|_\infty\right\} + \Delta\max\left\{\left\|D_{0+}^{\alpha}x\right\|_\infty, \left\|D_{0+}^{\beta}y\right\|_\infty\right\} \\
&\le \max\left\{B_2 + \frac{2}{\Gamma(\beta+1)}\left\|D_{0+}^{\beta}y\right\|_\infty, B_1 + \frac{2}{\Gamma(\alpha+1)}\left\|D_{0+}^{\alpha}x\right\|_\infty\right\} \\
&\quad + \Delta\max\left\{\left\|D_{0+}^{\alpha}x\right\|_\infty, \left\|D_{0+}^{\beta}y\right\|_\infty\right\}
\end{aligned}
$$

$$\le \max\left\{B_1 + \frac{2}{\Gamma(\alpha+1)}\left\|D_{0+}^{\alpha}x\right\|_\infty + \Delta\left\|D_{0+}^{\beta}y\right\|_\infty, \right.$$
$$B_1 + \left(\frac{2}{\Gamma(\alpha+1)} + \Delta\right)\left\|D_{0+}^{\alpha}x\right\|_\infty,$$
$$B_2 + \frac{2}{\Gamma(\beta+1)}\left\|D_{0+}^{\beta}y\right\|_\infty + \Delta\left\|D_{0+}^{\alpha}x\right\|_\infty,$$
$$\left. B_2 + \left(\frac{2}{\Gamma(\beta+1)} + \Delta\right)\left\|D_{0+}^{\beta}y\right\|_\infty\right\}.$$

Next, we separate the proof into four cases.

Case 1. $\|(x,y)\|_X \le B_1 + \frac{2}{\Gamma(\alpha+1)}\|D_{0+}^{\alpha}x\|_\infty + \Delta\|D_{0+}^{\beta}y\|_\infty$ because

$$\max\left\{\left\|t^{1-\alpha}x\right\|_\infty, \left\|D_{0+}^{\alpha}x\right\|_\infty, \left\|t^{1-\beta}y\right\|_\infty, \left\|D_{0+}^{\beta}y\right\|_\infty\right\} \le \left\|(x,y)\right\|_X. \tag{3.20}$$

By (3.14) and (3.15), one gets

$$\left\|(x,y)\right\|_X \le B_1 + \frac{2}{\Gamma(\alpha+1)}\left\|D_{0+}^{\alpha}x\right\|_\infty + \Delta\left\|D_{0+}^{\beta}y\right\|_\infty$$
$$\le B_1 + \frac{2}{\Gamma(\alpha+1)}\left[(\gamma_1+\eta_1)\left\|(x,y)\right\|_X + \omega_1\right] + \Delta\left[(\gamma_2+\eta_2)\left\|(x,y)\right\|_X + \omega_2\right],$$

that is,

$$\left\|(x,y)\right\|_X \le \frac{B_1 + (2\omega_1/\Gamma(\alpha+1)) + \Delta\omega_2}{1 - [(2(\gamma_1+\eta_1)/\Gamma(\alpha+1)) + \Delta(\gamma_2+\eta_2)]}.$$

Case 2. $\|(x,y)\|_X \le B_1 + (\frac{2}{\Gamma(\alpha+1)} + \Delta)\|D_{0+}^{\alpha}x\|_\infty$. By (3.14) and (3.20), we have

$$\left\|(x,y)\right\|_X \le B_1 + \left(\frac{2}{\Gamma(\alpha+1)} + \Delta\right)\left\|D_{0+}^{\alpha}x\right\|_\infty$$
$$\le B_1 + \left(\frac{2}{\Gamma(\alpha+1)} + \Delta\right)\left[(\gamma_1+\eta_1)\left\|(x,y)\right\|_X + \omega_1\right],$$

which implies that

$$\left\|(x,y)\right\|_X \le \frac{B_1 + ((2/\Gamma(\alpha+1)) + \Delta)\omega_1}{1 - ((2/\Gamma(\alpha+1)) + \Delta)(\gamma_1+\eta_1)}.$$

Case 3. $\|(x,y)\|_X \le B_2 + \frac{2}{\Gamma(\beta+1)}\|D_{0+}^{\beta}y\|_\infty + \Delta\|D_{0+}^{\alpha}x\|_\infty$. Using a similar proof as that in Case 1, we can get

$$\left\|(x,y)\right\|_X \le \frac{B_2 + (2\omega_2/\Gamma(\beta+1)) + \Delta\omega_1}{1 - [(2(\gamma_2+\eta_2)/\Gamma(\beta+1)) + \Delta(\gamma_1+\eta_1)]}.$$

Case 4. $\|(x,y)\|_X \le B_2 + (\frac{2}{\Gamma(\beta+1)} + \Delta)\|D_{0+}^{\beta}y\|_\infty$. By applying a method similar to Case 2, we can obtain

$$\left\|(x,y)\right\|_X \le \frac{B_2 + ((2/\Gamma(\beta+1)) + \Delta)\omega_2}{1 - ((2/\Gamma(\beta+1)) + \Delta)(\gamma_2+\eta_2)}.$$

To summarize, Ω_1 is bounded and the proof is completed. □

Remark 3.1 If $\alpha = \beta$, then condition (3.8) can be derived by (3.9).

In fact, from

$$\frac{2}{\Gamma(\alpha+1)}(\gamma_1+\eta_1)+\Delta(\gamma_2+\eta_2)<1, \qquad \frac{2}{\Gamma(\beta+1)}(\gamma_2+\eta_2)+\Delta(\gamma_1+\eta_1)<1,$$

we can obtain

$$\gamma_1+\eta_1<1, \qquad \gamma_2+\eta_2<1, \qquad 2\eta_1<\Gamma(\alpha+1), \qquad 2\eta_2<\Gamma(\beta+1). \tag{3.21}$$

On the other hand, by

$$\left(\frac{2}{\Gamma(\alpha+1)}+\Delta\right)(\gamma_1+\eta_1)<1, \qquad \left(\frac{2}{\Gamma(\beta+1)}+\Delta\right)(\gamma_2+\eta_2)<1,$$

we have

$$\left(\frac{2}{\Gamma(\alpha+1)}+1\right)(\gamma_1+\eta_1)<1, \qquad \left(\frac{2}{\Gamma(\beta+1)}+1\right)(\gamma_2+\eta_2)<1.$$

Using the fact $\alpha = \beta$, we also have

$$\left(\frac{2}{\Gamma(\beta+1)}+1\right)(\gamma_1+\eta_1)<1, \qquad \left(\frac{2}{\Gamma(\alpha+1)}+1\right)(\gamma_2+\eta_2)<1.$$

Then it follows that

$$\bigl(2+\Gamma(\alpha+1)\bigr)\zeta<\Gamma(\alpha+1), \qquad \bigl(2+\Gamma(\beta+1)\bigr)\zeta<\Gamma(\beta+1), \tag{3.22}$$

where $\zeta = \max\{\gamma_1,\eta_1,\gamma_2,\eta_2\}$. From (3.21) and (3.22), we have

$$\begin{aligned}
&\Gamma(\alpha+1)\eta_1\eta_2+2\gamma_2\eta_1=\eta_1\bigl(\Gamma(\alpha+1)\eta_2+2\gamma_2\bigr)<2\eta_1(\eta_2+\gamma_2)<2\eta_1<\Gamma(\alpha+1),\\
&\Gamma(\beta+1)\eta_1\eta_2+2\gamma_1\eta_2=\eta_2\bigl(\Gamma(\beta+1)\eta_1+2\gamma_1\bigr)<2\eta_2(\eta_1+\gamma_1)<2\eta_2<\Gamma(\beta+1),\\
&2\sigma_2+2\rho_2=\frac{2\gamma_2\eta_1}{1-\eta_1\eta_2}+\frac{2\gamma_2}{1-\eta_1\eta_2}=\frac{2\gamma_2(1+\eta_1)}{1-\eta_1\eta_2}<\frac{2\zeta(1+\zeta)}{1-\zeta^2}=\frac{2\zeta}{1-\zeta}<\Gamma(\alpha+1),\\
&2\sigma_3+2\rho_3=\frac{2\gamma_1}{1-\eta_1\eta_2}+\frac{2\gamma_1\eta_2}{1-\eta_1\eta_2}=\frac{2\gamma_1(1+\eta_2)}{1-\eta_1\eta_2}<\frac{2\zeta(1+\zeta)}{1-\zeta^2}=\frac{2\zeta}{1-\zeta}<\Gamma(\beta+1).
\end{aligned}$$

According to the above inequalities, it follows (3.8) holds.

Lemma 3.5 *Let* (A_4) *hold, set*

$$\Omega_2=\bigl\{(x,y)\in \operatorname{Ker}L : N(x,y)\in \operatorname{Im}L\bigr\}.$$

Then Ω_2 *is bounded.*

Proof For $(x,y)\in \operatorname{Ker}L$, then we can write $x=c_1t^{\alpha-1}$, $y=c_2t^{\beta-1}$, $(c_1,c_2)\in\mathbb{R}^2$, and $N(c_1t^{\alpha-1},c_2t^{\beta-1})\in \operatorname{Im}L=\operatorname{Ker}Q$, that is, $QN(c_1t^{\alpha-1},c_2t^{\beta-1})=(0,0)$. By (A_4), there exist

$t_3, t_4 \in [0,1]$ such that $|t_3^{1-\alpha} x(t_3)| = |c_1| \le B_1$, $|t_4^{1-\beta} y(t_4)| = |c_2| \le B_2$. Therefore,

$$\|(x,y)\|_X = \max\{\|x\|_{X_1}, \|y\|_{X_2}\} = \max\{|c_1|, |c_2|\} \le \max\{B_1, B_2\}.$$

The proof is completed. □

Lemma 3.6 *Let* (A$_5$) *hold, set*

$$\Omega_3 = \{(x,y) \in \operatorname{Ker} L : \vartheta \lambda J(x,y) + (1-\lambda) QN(x,y) = (0,0), \lambda \in [0,1]\}.$$

Then Ω_3 *is bounded, where*

$$\vartheta = \begin{cases} 1, & \text{if (3.6) holds,} \\ -1, & \text{if (3.7) holds,} \end{cases}$$

and $J : \operatorname{Ker} L \to \operatorname{Im} Q$ *is the linear isomorphism given by*

$$J(c_1 t^{\alpha-1}, c_2 t^{\beta-1}) = (c_2, c_1), \quad \forall c_1, c_2 \in \mathbb{R}^2, t \in [0,1].$$

Proof Without loss of generality, we suppose that (3.7) holds, then for $(x,y) \in \Omega_3$, we have

$$\lambda c_2 = (1-\lambda)\alpha \int_0^1 (1-s)^{\alpha-1} f(s, c_2, 0)\, ds,$$
$$\lambda c_1 = (1-\lambda)\beta \int_0^1 (1-s)^{\beta-1} g(s, c_1, 0)\, ds.$$

By the preceding lemma, it suffices to show that $|c_1|$, $|c_2|$ are bounded. In fact, if $\lambda = 1$, then $c_1 = c_2 = 0$. Otherwise, for $\lambda \in [0,1)$, we get

$$0 \le \lambda c_2^2 = (1-\lambda)\alpha \int_0^1 (1-s)^{\alpha-1} c_2 f(s, c_2, 0)\, ds,$$
$$0 \le \lambda c_1^2 = (1-\lambda)\beta \int_0^1 (1-s)^{\beta-1} c_1 g(s, c_1, 0)\, ds.$$

If $|c_1| > G_1$ or $|c_2| > G_2$, by (3.7), it is easy to verify that at least one of the above equations is not true. Therefore, $|c_1|$, $|c_2|$ are bounded, which completes the proof of Lemma 3.6. □

Lemma 3.7 *Let* (A$_2$) *hold, set*

$$\Omega_4 = \{(x,y) \in \operatorname{dom} L \setminus \operatorname{Ker} L : L(x,y) - N(x,y) = -\lambda(L(x,y) + N(-x,-y)), \lambda \in (0,1]\}.$$

Then Ω_4 *is bounded provided that*

$$\Gamma(\alpha+1)\Gamma(\beta+1) > (1 + \Gamma(\alpha+1))(1 + \Gamma(\beta+1))(p_1 + q_1)(p_2 + q_2), \tag{3.23}$$

where $p_i = \sup_{t \in [0,1]} p_i(t)$, $q_i = \sup_{t \in [0,1]} q_i(t)$, $i = 1, 2$.

Proof For $(x,y)\in\Omega_4$, we have

$$L(x,y)=\frac{1}{1+\lambda}N(x,y)-\frac{\lambda}{1+\lambda}N(-x,-y),$$

that is,

$$L_1x=\frac{1}{1+\lambda}N_1y-\frac{\lambda}{1+\lambda}N_1(-y), \tag{3.24}$$

$$L_2y=\frac{1}{1+\lambda}N_2x-\frac{\lambda}{1+\lambda}N_2(-x). \tag{3.25}$$

From (3.24) it follows that, for any $t\in[0,1]$,

$$\begin{aligned}|L_1x|=\left|D^{\alpha}_{0+}x\right|&\le\frac{1}{1+\lambda}|N_1y|+\frac{\lambda}{1+\lambda}\left|N_1(-y)\right|\\&=\frac{1}{1+\lambda}\left|f\left(t,t^{1-\beta}y(t),D^{\beta}_{0+}y(t)\right)\right|+\frac{\lambda}{1+\lambda}\left|f\left(t,-t^{1-\beta}y(t),-D^{\beta}_{0+}y(t)\right)\right|\\&\le\frac{1}{1+\lambda}\left[\left|f\left(t,t^{1-\beta}y(t),D^{\beta}_{0+}y(t)\right)-f(t,0,0)\right|+\rho\right]\\&\quad+\frac{\lambda}{1+\lambda}\left[\left|f\left(t,-t^{1-\beta}y(t),-D^{\beta}_{0+}y(t)\right)-f(t,0,0)\right|+\rho\right]\\&\le\rho+p_1\left|t^{1-\beta}y(t)\right|+q_1\left|D^{\beta}_{0+}y(t)\right|\\&\le\rho+p_1\left\|t^{1-\beta}y\right\|_\infty+q_1\left\|D^{\beta}_{0+}y\right\|_\infty,\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}\left|t^{1-\alpha}x(t)\right|&=\frac{1}{1+\lambda}\left|t^{1-\alpha}I^{\alpha}_{0+}N_1y-\lambda t^{1-\alpha}I^{\alpha}_{0+}N_1(-y)\right|\\&\le\frac{\rho}{\Gamma(\alpha+1)}+\frac{t^{1-\alpha}}{(1+\lambda)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left|f\left(s,s^{1-\beta}y(s),D^{\beta}_{0+}y(s)\right)-f(t,0,0)\right|ds\\&\quad+\frac{\lambda t^{1-\alpha}}{(1+\lambda)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left|f\left(s,-s^{1-\beta}y(s),-D^{\beta}_{0+}y(s)\right)-f(t,0,0)\right|ds\\&\le\frac{1}{\Gamma(\alpha+1)}\left[p_1\left\|t^{1-\beta}y\right\|_\infty+q_1\left\|D^{\beta}_{0+}y\right\|_\infty\right]+\frac{\rho}{\Gamma(\alpha+1)}.\end{aligned} \tag{3.27}$$

Taking account of (3.26) and (3.27), we derive

$$\left\|D^{\alpha}_{0+}x\right\|_\infty\le\rho+p_1\left\|t^{1-\beta}y\right\|_\infty+q_1\left\|D^{\beta}_{0+}y\right\|_\infty, \tag{3.28}$$

$$\left\|t^{1-\alpha}x\right\|_\infty\le\frac{1}{\Gamma(\alpha+1)}\left[p_1\left\|t^{1-\beta}y\right\|_\infty+q_1\left\|D^{\beta}_{0+}y\right\|_\infty+\rho\right]. \tag{3.29}$$

Similarly, by (3.25), it can be shown that, for any $t\in[0,1]$,

$$\left\|D^{\beta}_{0+}y\right\|_\infty\le\varpi+p_2\left\|t^{1-\alpha}x\right\|_\infty+q_2\left\|D^{\alpha}_{0+}x\right\|_\infty, \tag{3.30}$$

$$\left\|t^{1-\beta}y\right\|_\infty\le\frac{1}{\Gamma(\beta+1)}\left[p_2\left\|t^{1-\alpha}x\right\|_\infty+q_2\left\|D^{\alpha}_{0+}x\right\|_\infty+\varpi\right]. \tag{3.31}$$

According to (3.28)–(3.31), we get

$$\|x\|_{X_1} \le \frac{1+\Gamma(\alpha+1)}{\Gamma(\alpha+1)}\left[p_1\left\|t^{1-\beta}y\right\|_\infty + q_1\left\|D_{0+}^{\beta}y\right\|_\infty + \rho\right]$$
$$\le \frac{1+\Gamma(\alpha+1)}{\Gamma(\alpha+1)}\left[(p_1+q_1)\|y\|_{X_2} + \rho\right], \tag{3.32}$$
$$\|y\|_{X_2} \le \frac{1+\Gamma(\beta+1)}{\Gamma(\beta+1)}\left[p_2\left\|t^{1-\alpha}x\right\|_\infty + q_2\left\|D_{0+}^{\alpha}x\right\|_\infty + \varpi\right]$$
$$\le \frac{1+\Gamma(\beta+1)}{\Gamma(\beta+1)}\left[(p_2+q_2)\|x\|_{X_1} + \varpi\right]. \tag{3.33}$$

Now, by using (3.32) and (3.33), we obtain

$$\|y\|_{X_2} \le \frac{(1+\Gamma(\alpha+1))(1+\Gamma(\beta+1))(p_2+q_2)\rho + \Gamma(\alpha+1)(1+\Gamma(\beta+1))\varpi}{\Gamma(\alpha+1)\Gamma(\beta+1) - (1+\Gamma(\alpha+1))(1+\Gamma(\beta+1))(p_1+q_1)(p_2+q_2)} := m_1,$$
$$\|x\|_{X_1} \le \frac{1+\Gamma(\alpha+1)}{\Gamma(\alpha+1)}\left[(p_1+q_1)m_1 + \rho\right] := m_2.$$

So we get that

$$\left\|(x,y)\right\|_X = \max\left\{\|x\|_{X_1}, \|y\|_{X_2}\right\} \le \max\{m_1, m_2\}$$
$$= \frac{1}{2}\left(m_1 + m_2 + |m_1 - m_2|\right) := m.$$

This completes the proof of the lemma. □

Theorem 3.1 *Assume that* (A_1), (A_4), (A_5), *and* (3.8) *hold or* (A_1), (A_4), (A_5), *and* (3.9) *hold. Then BVP* (1.1) *has at least one solution in* X.

Proof Set Ω be a bounded open set of X such that $\bigcup_{i=1}^{3}\bar{\Omega}_i \subset \Omega$. By Lemma 3.3, N is L-compact on $\bar{\Omega}$. Lemmas 3.4 and 3.5 imply that (i) and (ii) of Theorem 2.1 are satisfied. In order to achieve the thesis, we have to prove that condition (iii) of Theorem 2.1 holds. Define the homotopy mapping as follows:

$$H\big((x,y),\lambda\big) = \vartheta\lambda J(x,y) + (1-\lambda)QN(x,y).$$

By Lemma 3.6, we get $H((x,y),\lambda) \neq (0,0)$ for all $(x,y) \in \operatorname{Ker} L \cap \partial\Omega$. Using the homotopy invariance of the topological degree,

$$\deg\left\{QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, (0,0)\right\} = \deg\left\{H(\cdot,0), \Omega \cap \operatorname{Ker} L, (0,0)\right\}$$
$$= \deg\left\{H(\cdot,1), \Omega \cap \operatorname{Ker} L, (0,0)\right\}$$
$$= \deg\left\{\vartheta J, \Omega \cap \operatorname{Ker} L, (0,0)\right\} \neq 0.$$

Then, by Theorem 2.1, BVP (1.1) has at least one solution in X. Thus the theorem is proved. □

Theorem 3.2 *If* (A_2) *and* (3.23) *hold, then BVP* (1.1) *has at least one solution in* X.

Proof Set $\Omega = \{(x,y) \in X : \|(x,y)\|_X < m+1\}$. Obviously, Ω is symmetric with $(0,0) \in \Omega$ and $X \cap \bar{\Omega} \neq \emptyset$. By Lemma 3.7, we get, for every $(x,y) \in \partial\Omega$ and $\lambda \in (0,1]$,

$$L(x,y) - N(x,y) \neq -\lambda\big(L(x,y) + N(-x,-y)\big),$$

which together with Theorem 2.2 yields that problem (1.1) has at least one solution in X. □

Theorem 3.3 *If* (A_2), (A_3), *and* (3.23) *hold, then BVP* (1.1) *has exactly one solution in* X *provided that*

$$\begin{aligned}&\max\big\{(\kappa+\Delta)(p_1+q_1), \kappa(p_1+q_1)+\Delta(p_2+q_2),\\&\quad \mu(p_2+q_2)+\Delta(p_1+q_1), (\mu+\Delta)(p_2+q_2)\big\} < 1,\end{aligned} \tag{3.34}$$

where

$$\kappa = \left(\frac{2}{\Gamma(\alpha+1)} + \frac{d}{c}\right), \qquad \mu = \left(\frac{2}{\Gamma(\beta+1)} + \frac{b}{a}\right).$$

Proof By Theorem 3.2, we obtain that BVP (1.1) has at least one solution in X. Now, we prove the uniqueness result. Suppose that BVP (1.1) has two solutions $(x_1,y_1),(x_2,y_2) \in \operatorname{dom} L$. Then, for $i = 1,2$, we have

$$\begin{aligned}&D_{0+}^{\alpha}x_i(t) = f\big(t, t^{1-\beta}y_i(t), D_{0+}^{\beta}y_i(t)\big),\\&D_{0+}^{\beta}y_i(t) = g\big(t, t^{1-\alpha}x_i(t), D_{0+}^{\alpha}x_i(t)\big),\end{aligned}$$

and

$$\lim_{t\to 0^+} t^{1-\alpha}x_i(t) = x_i(1), \qquad \lim_{t\to 0^+} t^{1-\beta}y_i(t) = y_i(1).$$

Let $x = x_1 - x_2$, $y = y_1 - y_2$. Then x, y satisfy the equations

$$D_{0+}^{\alpha}x(t) = f\big(t, t^{1-\beta}y_1(t), D_{0+}^{\beta}y_1(t)\big) - f\big(t, t^{1-\beta}y_2(t), D_{0+}^{\beta}y_2(t)\big), \tag{3.35}$$

$$D_{0+}^{\beta}y(t) = g\big(t, t^{1-\alpha}x_1(t), D_{0+}^{\alpha}x_1(t)\big) - g\big(t, t^{1-\alpha}x_2(t), D_{0+}^{\alpha}x_2(t)\big). \tag{3.36}$$

Noting that $\operatorname{Im} L = \operatorname{Ker} Q$, we have

$$\begin{aligned}&\int_0^1 (1-s)^{\alpha-1}\big[f\big(s, s^{1-\beta}y_1(s), D_{0+}^{\beta}y_1(s)\big) - f\big(s, s^{1-\beta}y_2(s), D_{0+}^{\beta}y_2(s)\big)\big]\,ds = 0,\\&\int_0^1 (1-s)^{\beta-1}\big[g\big(s, s^{1-\alpha}x_1(s), D_{0+}^{\alpha}x_1(s)\big) - g\big(s, s^{1-\alpha}x_2(s), D_{0+}^{\alpha}x_2(s)\big)\big]\,ds = 0,\end{aligned}$$

which imply there exist $t_5, t_6 \in [0,1]$ such that

$$\begin{aligned}&f\big(t_5, t_5^{1-\beta}y_1(t_5), D_{0+}^{\beta}y_1(t_5)\big) - f\big(t_5, t_5^{1-\beta}y_2(t_5), D_{0+}^{\beta}y_2(t_5)\big) = 0,\\&g\big(t_6, t_6^{1-\alpha}x_1(t_6), D_{0+}^{\alpha}x_1(t_6)\big) - g\big(t_6, t_6^{1-\alpha}x_2(t_6), D_{0+}^{\alpha}x_2(t_6)\big) = 0.\end{aligned}$$

Basing on condition (A_3), we conclude that

$$\left|t_5^{1-\beta}y(t_5)\right| \le \frac{b}{a}\left\|D_{0+}^{\beta}y\right\|_\infty, \quad \left|t_6^{1-\alpha}x(t_6)\right| \le \frac{d}{c}\left\|D_{0+}^{\alpha}x\right\|_\infty. \tag{3.37}$$

Considering that

$$I_{0+}^{\alpha}D_{0+}^{\alpha}x(t) = x(t) + c_1 t^{\alpha-1}, \quad c_1 \in \mathbb{R}, \qquad I_{0+}^{\beta}D_{0+}^{\beta}y(t) = y(t) + c_2 t^{\beta-1}, \quad c_2 \in \mathbb{R},$$

thus,

$$c_1 = t^{1-\alpha}I_{0+}^{\alpha}D_{0+}^{\alpha}x(t)|_{t=t_6} - t_6^{1-\alpha}x(t_6), \qquad c_2 = t^{1-\beta}I_{0+}^{\beta}D_{0+}^{\beta}y(t)|_{t=t_5} - t_5^{1-\beta}y(t_5).$$

Therefore, we can draw a fact

$$\left|t^{1-\alpha}x(t)\right| \le \kappa\left\|D_{0+}^{\alpha}x\right\|_\infty, \qquad \left|t^{1-\beta}y(t)\right| \le \mu\left\|D_{0+}^{\beta}y\right\|_\infty. \tag{3.38}$$

On the other hand, using hypothesis (A_2) and (3.35)–(3.36), we find that

$$\left|D_{0+}^{\alpha}x(t)\right| \le p_1\left|t^{1-\beta}y(t)\right| + q_1\left|D_{0+}^{\beta}y(t)\right|, \qquad \left|D_{0+}^{\beta}y(t)\right| \le p_2\left|t^{1-\alpha}x(t)\right| + q_2\left|D_{0+}^{\alpha}x(t)\right|.$$

Consequently, we infer that

$$\begin{aligned}
&\left|D_{0+}^{\alpha}x(t)\right| \le p_1\left\|t^{1-\beta}y\right\|_\infty + q_1\left\|D_{0+}^{\beta}y\right\|_\infty, \\
&\left|D_{0+}^{\beta}y(t)\right| \le p_2\left\|t^{1-\alpha}x\right\|_\infty + q_2\left\|D_{0+}^{\alpha}x\right\|_\infty.
\end{aligned} \tag{3.39}$$

By Lemma 3.2, we obtain

$$\begin{aligned}
\left\|(I-P)(x,y)\right\|_X &= \left\|K_P L(I-P)(x,y)\right\|_X \le \Delta\left\|L(x,y)\right\|_Z \\
&= \Delta\max\left\{\left\|D_{0+}^{\alpha}x\right\|_\infty, \left\|D_{0+}^{\beta}y\right\|_\infty\right\}.
\end{aligned} \tag{3.40}$$

From (3.38)–(3.40), one has

$$\begin{aligned}
\left\|(x,y)\right\|_X &\le \left\|P(x,y)\right\|_X + \left\|(I-P)(x,y)\right\|_X \\
&\le \max\left\{\left\|t^{1-\beta}y\right\|_\infty, \left\|t^{1-\alpha}x\right\|_\infty\right\} + \Delta\max\left\{\left\|D_{0+}^{\alpha}x\right\|_\infty, \left\|D_{0+}^{\beta}y\right\|_\infty\right\} \\
&\le \max\left\{\mu\left\|D_{0+}^{\beta}y\right\|_\infty, \kappa\left\|D_{0+}^{\alpha}x\right\|_\infty\right\} + \Delta\max\left\{\left\|D_{0+}^{\alpha}x\right\|_\infty, \left\|D_{0+}^{\beta}y\right\|_\infty\right\} \\
&= \max\left\{\kappa\left\|D_{0+}^{\alpha}x\right\|_\infty + \Delta\left\|D_{0+}^{\beta}y\right\|_\infty, (\kappa+\Delta)\left\|D_{0+}^{\alpha}x\right\|_\infty,\right. \\
&\quad \left.\mu\left\|D_{0+}^{\beta}y\right\|_\infty + \Delta\left\|D_{0+}^{\alpha}x\right\|_\infty, (\mu+\Delta)\left\|D_{0+}^{\beta}y\right\|_\infty\right\}.
\end{aligned} \tag{3.41}$$

Proceeding as in the proof of Lemma 3.4, we divide the proof in four cases.

Case 1. $\|(x,y)\|_X \le \kappa\|D_{0+}^{\alpha}x\|_\infty + \Delta\|D_{0+}^{\beta}y\|_\infty$. By (3.20) and (3.39), we get

$$\begin{aligned}
\left\|(x,y)\right\|_X &\le \kappa\left\|D_{0+}^{\alpha}x\right\|_\infty + \Delta\left\|D_{0+}^{\beta}y\right\|_\infty \\
&\le \kappa\left(p_1\left\|t^{1-\beta}y\right\|_\infty + q_1\left\|D_{0+}^{\beta}y\right\|_\infty\right) + \Delta\left(p_2\left\|t^{1-\alpha}x\right\|_\infty + q_2\left\|D_{0+}^{\alpha}x\right\|_\infty\right)
\end{aligned}$$

$$\le \kappa\left(p_1\left\|t^{1-\beta}y\right\|_\infty + q_1\left\|(x,y)\right\|_X\right) + \Delta(p_2+q_2)\left\|(x,y)\right\|_X,$$

and

$$\begin{aligned}\left\|(x,y)\right\|_X &\le \kappa\left\|D_{0+}^{\alpha}x\right\|_\infty + \Delta\left\|D_{0+}^{\beta}y\right\|_\infty \\ &\le \kappa\left(p_1\left\|t^{1-\beta}y\right\|_\infty + q_1\left\|D_{0+}^{\beta}y\right\|_\infty\right) + \Delta\left(p_2\left\|t^{1-\alpha}x\right\|_\infty + q_2\left\|D_{0+}^{\alpha}x\right\|_\infty\right) \\ &\le \kappa(p_1+q_1)\left\|(x,y)\right\|_X + \Delta p_2\left\|t^{1-\alpha}x\right\|_\infty + \Delta q_2\left\|(x,y)\right\|_X.\end{aligned}$$

Again, by (3.20), we obtain

$$\begin{aligned}\left\|t^{1-\beta}y\right\|_\infty &\le \frac{\kappa p_1}{1-[\kappa q_1 + \Delta(p_2+q_2)]}\left\|t^{1-\beta}y\right\|_\infty, \\ \left\|t^{1-\alpha}x\right\|_\infty &\le \frac{\Delta p_2}{1-[\kappa(p_1+q_1)+\Delta q_2]}\left\|t^{1-\alpha}x\right\|_\infty.\end{aligned}$$

In view of condition (3.34), we have

$$\left\|t^{1-\alpha}x\right\|_\infty = 0, \qquad \left\|t^{1-\beta}y\right\|_\infty = 0.$$

As a result, we get $x_1 = x_2$, $y_1 = y_2$.

Case 2. $\|(x,y)\|_X \le (\kappa+\Delta)\|D_{0+}^{\alpha}x\|_\infty$. Then (3.39) and (3.20) imply

$$\begin{aligned}\left\|(x,y)\right\|_X &\le (\kappa+\Delta)\left\|D_{0+}^{\alpha}x\right\|_\infty \\ &\le (\kappa+\Delta)\left[p_1\left\|t^{1-\beta}y\right\|_\infty + q_1\left\|D_{0+}^{\beta}y\right\|_\infty\right] \\ &\le (\kappa+\Delta)p_1\left\|t^{1-\beta}y\right\|_\infty + q_1(\kappa+\Delta)\left\|(x,y)\right\|_X,\end{aligned}$$

and

$$\begin{aligned}\left\|(x,y)\right\|_X &\le (\kappa+\Delta)\left\|D_{0+}^{\alpha}x\right\|_\infty \\ &\le (\kappa+\Delta)\left[p_1\left\|t^{1-\beta}y\right\|_\infty + q_1p_2\left\|t^{1-\alpha}x\right\|_\infty + q_1q_2\left\|D_{0+}^{\alpha}x\right\|_\infty\right] \\ &\le (\kappa+\Delta)q_1p_2\left\|t^{1-\alpha}x\right\|_\infty + (\kappa+\Delta)(p_1+q_1q_2)\left\|(x,y)\right\|_X.\end{aligned}$$

Using (3.20), we derive

$$\begin{aligned}\left\|t^{1-\beta}y\right\|_\infty &\le \frac{(\kappa+\Delta)p_1}{1-(\kappa+\Delta)q_1}\left\|t^{1-\beta}y\right\|_\infty, \\ \left\|t^{1-\alpha}x\right\|_\infty &\le \frac{(\kappa+\Delta)q_1p_2}{1-(\kappa+\Delta)(p_1+q_1q_2)}\left\|t^{1-\alpha}x\right\|_\infty.\end{aligned}$$

According to assumption (3.34), we obtain

$$\left\|t^{1-\alpha}x\right\|_\infty = 0, \qquad \left\|t^{1-\beta}y\right\|_\infty = 0.$$

Consequently, $x_1 = x_2$, $y_1 = y_2$.

Case 3. $\|(x,y)\|_X \le \mu\|D_{0+}^{\beta}y\|_\infty + \Delta\|D_{0+}^{\alpha}x\|_\infty$. By a method similar to that used in Case 1, we can conclude that

$$\begin{aligned} \|t^{1-\beta}y\|_\infty &\le \frac{\Delta p_1}{1-[\Delta q_1+\mu(p_2+q_2)]}\|t^{1-\beta}y\|_\infty, \\ \|t^{1-\alpha}x\|_\infty &\le \frac{\mu p_2}{1-[\Delta(p_1+q_1)+\mu q_2]}\|t^{1-\alpha}x\|_\infty. \end{aligned} \tag{3.42}$$

Case 4. $\|(x,y)\|_X \le (\mu+\Delta)\|D_{0+}^{\beta}y\|_\infty$. Similar to the analysis in Case 2, we can deduce that

$$\begin{aligned} \|t^{1-\beta}y\|_\infty &\le \frac{(\mu+\Delta)p_1q_2}{1-(\mu+\Delta)(p_2+q_1q_2)}\|t^{1-\beta}y\|_\infty, \\ \|t^{1-\alpha}x\|_\infty &\le \frac{(\mu+\Delta)p_2}{1-(\mu+\Delta)q_2}\|t^{1-\alpha}x\|_\infty. \end{aligned} \tag{3.43}$$

From (3.34), (3.42), and (3.43), we also obtain that

$$\|t^{1-\alpha}x\|_\infty = 0, \qquad \|t^{1-\beta}y\|_\infty = 0,$$

that is, $x_1 = x_2, y_1 = y_2$. In summary, BVP (1.1) has a unique continuous solution in X. □

4 Example

Example 4.1 Consider the boundary value problem

$$\begin{cases} D_{0+}^{1/2}x(t) = f(t, t^{1/2}y(t), D_{0+}^{1/2}y(t)), \quad t \in [0,1], \\ D_{0+}^{1/2}y(t) = g(t, t^{1/2}x(t), D_{0+}^{1/2}x(t)), \\ \lim_{t\to 0^+} t^{1/2}x(t) = x(1), \qquad \lim_{t\to 0^+} t^{1/2}y(t) = y(1). \end{cases} \tag{4.1}$$

Corresponding to problem (1.1), here

$$\begin{aligned} &\alpha = \beta = \frac{1}{2}, \\ &f\bigl(t, t^{1/2}y(t), D_{0+}^{1/2}y(t)\bigr) = \begin{cases} \frac{11}{50}t^2 \sin|t^{1/2}y(t)| + \frac{t}{2}\sin D_{0+}^{1/2}y(t) + \frac{4}{5}, & |t^{1/2}y(t)| \le 3, \\ \frac{11}{50}t^2 \sin 3 + \frac{t}{2}\sin D_{0+}^{1/2}y(t) + \frac{4}{5}, & |t^{1/2}y(t)| \ge 3, \end{cases} \\ &g\bigl(t, t^{1/2}x(t), D_{0+}^{1/2}x(t)\bigr) = \begin{cases} \frac{1}{5}t^2 |t^{1/2}x(t)| + \frac{t}{2}\sin D_{0+}^{1/2}x(t) + \frac{2}{3}, & |t^{1/2}y(t)| \le 1, \\ \frac{1}{5}t^2 + \frac{t}{2}\sin D_{0+}^{1/2}x(t) + \frac{2}{3}, & |t^{1/2}y(t)| \ge 1. \end{cases} \end{aligned}$$

Let

$$\begin{aligned} &\gamma_1(t) = \frac{11}{50}t^2, \qquad \eta_1(t) = \frac{t}{2}, \qquad \omega_1(t) = \frac{4}{5}, \\ &\gamma_2(t) = \frac{1}{5}t^2, \qquad \eta_2(t) = \frac{t}{2}, \qquad \omega_2(t) = \frac{2}{3}. \end{aligned}$$

Then (A_1) holds and

$$\gamma_1 = \frac{11}{50}, \qquad \eta_1 = \eta_2 = \frac{1}{2}, \qquad \omega_1 = \frac{4}{5}, \qquad \gamma_2 = \frac{1}{5}, \qquad \omega_2 = \frac{2}{3},$$

$$\sigma_2 = \frac{\gamma_2\eta_1}{1-\eta_1\eta_2} = \frac{2}{15}, \qquad \rho_3 = \frac{\gamma_1\eta_2}{1-\eta_1\eta_2} = \frac{11}{75}, \qquad \rho_2 = \frac{\gamma_2}{1-\eta_1\eta_2} = \frac{4}{15},$$
$$\sigma_3 = \frac{\gamma_1}{1-\eta_1\eta_2} = \frac{22}{75}.$$

Accordingly,

$$\eta_1\eta_2 = \frac{1}{4} < 1, \qquad \Gamma(\alpha+1) = \frac{\sqrt{\pi}}{2} > 2\sigma_2 = \frac{4}{15}, \qquad \Gamma(\beta+1) = \frac{\sqrt{\pi}}{2} > 2\rho_3 = \frac{22}{75},$$
$$\bigl(\Gamma(\alpha+1) - 2\sigma_2\bigr)\bigl(\Gamma(\beta+1) - 2\rho_3\bigr) = \left(\frac{\sqrt{\pi}}{2} - \frac{4}{15}\right)\left(\frac{\sqrt{\pi}}{2} - \frac{22}{75}\right) > \frac{352}{1125} = 4\rho_2\sigma_3.$$

Consequently, (3.8) holds. Since

$$N_1 y = \begin{cases} \frac{11}{50}t^2 \sin|t^{1/2}y(t)| + \frac{t}{2}\sin D_{0+}^{1/2}y(t) + \frac{4}{5}, & |t^{1/2}y(t)| \le 3, \\ \frac{11}{50}t^2 \sin 3 + \frac{t}{2}\sin D_{0+}^{1/2}y(t) + \frac{4}{5}, & |t^{1/2}y(t)| \ge 3, \end{cases}$$
$$N_2 x = \begin{cases} \frac{1}{5}t^2 |t^{1/2}x(t)| + \frac{t}{2}\sin D_{0+}^{1/2}x(t) + \frac{2}{3}, & |t^{1/2}y(t)| \le 1, \\ \frac{1}{5}t^2 + \frac{t}{2}\sin D_{0+}^{1/2}x(t) + \frac{2}{3}, & |t^{1/2}y(t)| \ge 1, \end{cases}$$

and

$$N_1\bigl(c_2 t^{\beta-1}\bigr) = N_1\bigl(c_2 t^{-1/2}\bigr) = \begin{cases} \frac{11}{50}t^2 \sin|c_2| + \frac{4}{5}, & |c_2| \le 3, \\ \frac{11}{50}t^2 \sin 3 + \frac{4}{5}, & |c_2| \ge 3, \end{cases}$$
$$N_2\bigl(c_1 t^{\alpha-1}\bigr) = N_1\bigl(c_1 t^{-1/2}\bigr) = \begin{cases} \frac{1}{5}t^2 |c_1| + \frac{2}{3}, & |c_1| \le 1, \\ \frac{1}{5}t^2 + \frac{2}{3}, & |c_1| \ge 1. \end{cases}$$

So if we put $B_1 = G_1 = 1$, $B_2 = G_2 = 3$, then we have

$$N_1 y = \frac{11}{50}t^2 \sin 3 + \frac{t}{2}\sin D_{0+}^{1/2}y(t) + \frac{4}{5} \ge \frac{2}{25} > 0, \quad \bigl|t^{1/2}y(t)\bigr| \ge 3,$$
$$N_1\bigl(c_2 t^{\beta-1}\bigr) = N_1\bigl(c_2 t^{-1/2}\bigr) = \frac{11}{50}t^2 \sin 3 + \frac{4}{5} \ge \frac{29}{50} > 0, \quad |c_2| \ge 3,$$
$$N_2 x = \frac{1}{5}t^2 + \frac{t}{2}\sin D_{0+}^{1/2}x(t) + \frac{2}{3} \ge \frac{1}{6} > 0, \quad \bigl|t^{1/2}y(t)\bigr| \ge 1,$$
$$N_2\bigl(c_1 t^{\alpha-1}\bigr) = N_1\bigl(c_1 t^{-1/2}\bigr) = \frac{1}{5}t^2 + \frac{2}{3} \ge \frac{2}{3} > 0, \quad |c_1| \ge 1.$$

Therefore, (A_4) and (A_5) hold. By Theorem 3.1, we can conclude that BVP (4.1) has at least one solution.

Remark 4.1 Obviously, for BVP (4.1), condition (A_2) is not valid and (3.9) does not hold.

In fact, we can obtain that

$$\frac{2}{\Gamma(\alpha+1)}(\gamma_1 + \eta_1) + \Delta(\gamma_2 + \eta_2) = \frac{4}{\sqrt{\pi}} \cdot \frac{18}{25} + \frac{\pi + 2\sqrt{\pi}}{\pi} \cdot \frac{7}{10} > 1.$$

So that (3.9) is not true.

Example 4.2 Consider the boundary value problem

$$\begin{cases} D_{0+}^{1/2}x(t) = \frac{t^2}{5}\sin t^{1/2}y(t) + \frac{4t}{15}\sin D_{0+}^{1/2}y(t) + \frac{1}{2}, \quad t \in [0,1], \\ D_{0+}^{1/2}y(t) = \frac{t^2}{7}\sin t^{1/2}x(t) + \frac{2t}{7}\sin D_{0+}^{1/2}x(t) + \frac{1}{2}, \\ \lim_{t\to 0^+} t^{1/2}x(t) = x(1), \qquad \lim_{t\to 0^+} t^{1/2}y(t) = y(1). \end{cases} \tag{4.2}$$

Corresponding to problem (1.1), here

$$\alpha = \beta = \frac{1}{2},$$
$$f\big(t, t^{1-\beta}y(t), D_{0+}^{\beta}y(t)\big) = \frac{t^2}{5}\sin t^{1/2}y(t) + \frac{4t}{15}\sin D_{0+}^{1/2}y(t) + \frac{1}{2},$$
$$g\big(t, t^{1-\alpha}x(t), D_{0+}^{\alpha}x(t)\big) = \frac{t^2}{7}\sin t^{1/2}x(t) + \frac{2t}{7}\sin D_{0+}^{1/2}x(t) + \frac{1}{2}.$$

Let

$$p_1(t) = \frac{t^2}{5}, \qquad q_1(t) = \frac{4t}{15}, \qquad p_2(t) = \frac{t^2}{7}, \qquad q_2(t) = \frac{2t}{7},$$

then $p_1 = \frac{1}{5}$, $q_1 = \frac{4}{15}$, $p_2 = \frac{1}{7}$, $q_2 = \frac{2}{7}$. We can easily check that (A_2) holds and

$$\Gamma(\alpha+1)\Gamma(\beta+1) = \frac{\pi}{4} > \frac{(2+\sqrt{\pi})^2}{20} = \big(1+\Gamma(\alpha+1)\big)\big(1+\Gamma(\beta+1)\big)(p_1+q_1)(p_2+q_2).$$

By Theorem 3.2, BVP (4.2) has at least one solution. If we let

$$\gamma_1(t) = \frac{t^2}{5}, \qquad \eta_1(t) = \frac{4t}{15}, \qquad \omega_1(t) = \frac{1}{2},$$
$$\gamma_2(t) = \frac{t^2}{7}, \qquad \eta_2(t) = \frac{2t}{7}, \qquad \omega_2(t) = \frac{1}{2},$$

then (A_1) holds and

$$\gamma_1 = \frac{1}{5}, \qquad \eta_1 = \frac{4}{15}, \qquad \omega_1 = \omega_2 = \frac{1}{2}, \qquad \gamma_2 = \frac{1}{7}, \qquad \eta_2 = \frac{2}{7},$$
$$\sigma_2 = \frac{\gamma_2\eta_1}{1-\eta_1\eta_2} = \frac{4}{97}, \qquad \rho_3 = \frac{\gamma_1\eta_2}{1-\eta_1\eta_2} = \frac{6}{97}, \qquad \rho_2 = \frac{\gamma_2}{1-\eta_1\eta_2} = \frac{15}{97},$$
$$\sigma_3 = \frac{\gamma_1}{1-\eta_1\eta_2} = \frac{21}{97}.$$

Therefore,

$$\eta_1\eta_2 = \frac{8}{105} < 1, \qquad \Gamma(\alpha+1) = \frac{\sqrt{\pi}}{2} > 2\sigma_2 = \frac{8}{97}, \qquad \Gamma(\beta+1) = \frac{\sqrt{\pi}}{2} > 2\rho_3 = \frac{12}{97},$$
$$\big(\Gamma(\alpha+1) - 2\sigma_2\big)\big(\Gamma(\beta+1) - 2\rho_3\big) = \left(\frac{\sqrt{\pi}}{2} - \frac{8}{97}\right)\left(\frac{\sqrt{\pi}}{2} - \frac{12}{97}\right) > \frac{1260}{9409} = 4\rho_2\sigma_3,$$

that is, (3.8) holds. Since

$$N_1 y = \frac{t^2}{5}\sin t^{1/2}y(t) + \frac{4t}{15}\sin D_{0+}^{1/2}y(t) + \frac{1}{2} \geq \frac{1}{30} > 0,$$

$$N_1\bigl(c_2 t^{\beta-1}\bigr) = N_1\bigl(c_2 t^{-1/2}\bigr) = \frac{t^2}{5}\sin c_2 + \frac{1}{2} \geq \frac{3}{10} > 0,$$

$$N_2 x = \frac{t^2}{7}\sin t^{1/2}x(t) + \frac{2t}{7}\sin D_{0+}^{1/2}x(t) + \frac{1}{2} \geq \frac{1}{14} > 0,$$

$$N_2\bigl(c_1 t^{\alpha-1}\bigr) = N_1\bigl(c_1 t^{-1/2}\bigr) = \frac{t^2}{7}\sin c_1 + \frac{1}{2} \geq \frac{4}{15} > 0.$$

Then, for any $B_i > 0$ and $G_i > 0$, $(i = 1, 2)$, we have (A_4) and (A_5) hold. By Theorem 3.1, we can also obtain that BVP (4.2) has at least one solution.

Remark 4.2 The existence result of BVP (4.2) cannot be obtained by verifying conditions (A_1), (A_4), (A_5), and (3.9) of Theorem 3.1.

In fact, we can check that

$$\frac{2}{\Gamma(\alpha+1)}(\gamma_1 + \eta_1) + \Delta(\gamma_2 + \eta_2) = \frac{4}{\sqrt{\pi}} \cdot \frac{7}{15} + \frac{\sqrt{\pi}+2}{\sqrt{\pi}} \cdot \frac{3}{7} = \frac{286 + 45\sqrt{\pi}}{105\sqrt{\pi}} > 1.$$

This implies that (3.9) does not hold.

Example 4.3 Consider the following fractional boundary value problem:

$$\begin{cases} D_{0+}^{1/2}x(t) = \frac{t^2+3}{30}\sin t^{1/2}y(t) + \frac{t}{20}\sin D_{0+}^{1/2}y(t) + \frac{1}{4}, & t \in [0,1], \\ D_{0+}^{1/2}y(t) = \frac{t^2+1}{14}\sin t^{1/2}x(t) + \frac{t}{28}\sin D_{0+}^{1/2}x(t) + \frac{1}{5}, \\ \lim_{t\to 0^+} t^{1/2}x(t) = x(1), \qquad \lim_{t\to 0^+} t^{1/2}y(t) = y(1). \end{cases} \tag{4.3}$$

Corresponding to problem (1.1), here

$$\alpha = \beta = \frac{1}{2},$$

$$f\bigl(t, t^{1-\beta}y(t), D_{0+}^{\beta}y(t)\bigr) = \frac{t^2+3}{30}\sin t^{1/2}y(t) + \frac{t}{20}\sin D_{0+}^{1/2}y(t) + \frac{1}{4},$$

$$g\bigl(t, t^{1-\alpha}x(t), D_{0+}^{\alpha}x(t)\bigr) = \frac{t^2+1}{14}\sin t^{1/2}x(t) + \frac{t}{28}\sin D_{0+}^{1/2}x(t) + \frac{1}{5}.$$

Let

$$p_1(t) = \frac{1}{30}t^2 + \frac{1}{10}, \qquad q_1(t) = \frac{1}{20}t, \qquad p_2(t) = \frac{1}{14}t^2 + \frac{1}{14}, \qquad q_2(t) = \frac{1}{28}t.$$

Then

$$p_1 = \frac{2}{15}, \qquad q_1 = \frac{1}{20}, \qquad p_2 = \frac{1}{7}, \qquad q_2 = \frac{1}{28}, \qquad \Delta = \frac{\pi + 2\sqrt{\pi}}{\pi},$$

$$p_1 + q_1 = \frac{11}{60} > \frac{5}{28} = p_2 + q_2.$$

Choose $a = \frac{1}{10}$, $b = \frac{1}{20}$, $c = \frac{1}{14}$, $d = \frac{1}{28}$. It is easy to show that (A_2) and (A_3) hold. Since

$$\begin{aligned}\Gamma(\alpha+1)\Gamma(\beta+1) = \frac{\pi}{4} &> \frac{11}{1344}(2+\sqrt{\pi})^2 \\ &= \bigl(1+\Gamma(\alpha+1)\bigr)\bigl(1+\Gamma(\beta+1)\bigr)(p_1+q_1)(p_2+q_2).\end{aligned}$$

By Theorem 3.2, BVP (4.3) has at least one solution. We also can check that

$$\begin{aligned}(\kappa+\Delta)(p_1+q_1) &= \left(\frac{2}{\Gamma(3/2)} + \frac{1}{2} + \frac{\pi+2\sqrt{\pi}}{\pi}\right)\left(\frac{2}{15}+\frac{1}{20}\right) \\ &= \frac{11\sqrt{\pi}+44}{40\sqrt{\pi}} := \Xi < 1, \\ \kappa(p_1+q_1) &+ \Delta(p_2+q_2) < \Xi < 1, \\ \mu(p_2+q_2) &+ \Delta(p_1+q_1) = \kappa(p_2+q_2) + \Delta(p_1+q_1) < \Xi < 1, \\ (\mu+\Delta)&(p_2+q_2) = (\kappa+\Delta)(p_2+q_2) < \Xi < 1.\end{aligned}$$

By Theorem 3.3, BVP (4.3) has a unique solution. If we let

$$\begin{aligned}\gamma_1(t) &= \frac{t^2+3}{30}, \qquad \eta_1(t) = \frac{t}{20}, \qquad \omega_1(t) = \frac{1}{4}, \\ \gamma_2(t) &= \frac{t^2+1}{14}, \qquad \eta_2(t) = \frac{t}{28}, \qquad \omega_2(t) = \frac{1}{5},\end{aligned}$$

then (A_1) holds and

$$\begin{aligned}&\gamma_1 = \frac{2}{15}, \qquad \eta_1 = \frac{1}{20}, \qquad \omega_1 = \frac{1}{4}, \qquad \gamma_2 = \frac{1}{7}, \qquad \eta_2 = \frac{1}{28}, \qquad \omega_2 = \frac{1}{5}, \\ &\gamma_2+\eta_2 = \frac{5}{28} < \frac{11}{60} = \gamma_1+\eta_1, \\ &\sigma_2 = \frac{\gamma_2\eta_1}{1-\eta_1\eta_2} = \frac{4}{559}, \qquad \rho_3 = \frac{\gamma_1\eta_2}{1-\eta_1\eta_2} = \frac{8}{1677}, \qquad \rho_2 = \frac{\gamma_2}{1-\eta_1\eta_2} = \frac{80}{559}, \\ &\sigma_3 = \frac{\gamma_1}{1-\eta_1\eta_2} = \frac{224}{1677}.\end{aligned}$$

Thus,

$$\begin{aligned}&\eta_1\eta_2 = \frac{1}{560} < 1, \qquad \Gamma(\alpha+1) = \frac{\sqrt{\pi}}{2} > 2\sigma_2 = \frac{8}{559}, \qquad \Gamma(\beta+1) = \frac{\sqrt{\pi}}{2} > 2\rho_3 = \frac{16}{1677}, \\ &\bigl(\Gamma(\alpha+1)-2\sigma_2\bigr)\bigl(\Gamma(\beta+1)-2\rho_3\bigr) \\ &\quad = \left(\frac{\sqrt{\pi}}{2}-\frac{8}{559}\right)\left(\frac{\sqrt{\pi}}{2}-\frac{16}{1677}\right) > \frac{71{,}680}{937{,}443} = 4\rho_2\sigma_3,\end{aligned}$$

and

$$\begin{aligned}\frac{2}{\Gamma(\alpha+1)}(\gamma_1+\eta_1) + \Delta(\gamma_2+\eta_2) &\le \left(\frac{2}{\Gamma(\alpha+1)}+\Delta\right)(\gamma_1+\eta_1) \\ &= \left(\frac{2}{\Gamma(3/2)} + \frac{\pi+2\sqrt{\pi}}{\pi}\right)\left(\frac{2}{15}+\frac{1}{20}\right) < \Xi < 1,\end{aligned}$$

$$\frac{2}{\Gamma(\beta+1)}(\gamma_2+\eta_2)+\Delta(\gamma_1+\eta_1)\leq\frac{2}{\Gamma(\alpha+1)}(\gamma_1+\eta_1)+\Delta(\gamma_2+\eta_2)<1,$$
$$\left(\frac{2}{\Gamma(\beta+1)}+\Delta\right)(\gamma_2+\eta_2)\leq\left(\frac{2}{\Gamma(\alpha+1)}+\Delta\right)(\gamma_1+\eta_1)<1.$$

That is, both (3.8) and (3.9) hold because

$$N_1y=\frac{t^2+3}{30}\sin t^{1/2}y(t)+\frac{t}{20}\sin D_{0+}^{1/2}y(t)+\frac{1}{4}\geq\frac{1}{15}>0,$$
$$N_1\left(c_2t^{\beta-1}\right)=N_1\left(c_2t^{-1/2}\right)=\frac{t^2+3}{30}\sin c_2+\frac{1}{4}\geq\frac{7}{60}>0,$$
$$N_2x=\frac{t^2+1}{14}\sin t^{1/2}x(t)+\frac{t}{28}\sin D_{0+}^{1/2}x(t)+\frac{1}{5}\geq\frac{3}{140}>0,$$
$$N_2\left(c_1t^{\alpha-1}\right)=N_1\left(c_1t^{-1/2}\right)=\frac{t^2+1}{14}\sin c_1+\frac{1}{5}\geq\frac{2}{35}>0.$$

Therefore, for any $B_i>0$ and $G_i>0$, $(i=1,2)$ (A_4) and (A_5) hold, which means the existence result of BVP (4.3) can be obtained by Theorem 3.1.

5 Conclusion

In the present paper, we investigate the existence and uniqueness of solutions for the coupled systems of nonlinear implicit fractional periodic boundary value problems in the frame of Riemann–Liouville fractional derivative. By using Theorems 2.1 and 2.2, the new existence and uniqueness results are established. The results in papers [29, 30] are improved and extended in this paper. First, we extend the results of [29, 30] to coupled systems; second, in [29, 30], the authors only studied the existence results based on Lemma 2.1 and established existence theorems under condition (A_2). Our results show that the existence results can also be obtained under condition (A_1). Besides, compared with [40–43], we used a different technique to prove that Ω_1 is bounded (see Lemma 3.4, the first way). By Remark 3.1, we show that the first way is superior to the second way used by [40–43]. Finally, our main results are well illustrated with the aid of several interesting examples.

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