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A fractional order pine wilt disease model with Caputo–Fabrizio derivative

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Abstract

A Caputo–Fabrizio type fractional order mathematical model for the dynamics of pine wilt disease (FPWD) is presented. The basic properties of the model are investigated. The existence and uniqueness of the solution for the proposed FPWD model are given via the fixed point theorem. The numerical simulations for the model are obtained by using particular parameter values. The non-integer order derivative provides more flexible and deeper information about the complexity of the dynamics of the proposed FPWD model than the integer order models established before.

Keywords: Caputo–Fabrizio (CF) fractional derivative; Pint wilt disease; Mathematical model; Fixed point theorem; Numerical simulation

1 Introduction

Mathematical models in epidemiology are used widely in order to understand better the dynamics of an infectious disease [1, 2]. The application of the mathematical models is not limited to only human diseases, but they are also widely applied in other phenomena of biological sciences, such as ecology, forest, etc. In the human life, forest has an important role, therefore, it is necessary to ensure the safety strategies to protect it from being infected with diseases. The forest provides greenery to the environment and pleasant atmosphere for humans. The pine wilt disease (PWD) infects pine trees and is one of the main threats to the ecosystem and forest. The PWD is considered to be the most destructive disease which damages pine trees in a short period of time, that is, a year or sometimes in a few months. The initial symptoms of the PWD include discoloration of needles, which turn from yellow to green then to reddish brown. The main agent of the disease is small worms, known as pinewood nematode (bursaphelenchus xylophilus), causing tree decline [3]. As the trees begin to die, they are attacked by insects, known as sawyers, which are species that transfer the nematode to healthy trees, which is one of the causes of pine wilt disease [4–6]. Native to North America, the PW nematode was introduced while the first epidemic of the PWD was accursed in Japan in 1905 [7], and it has spread in southern China, Korea, Taiwan, and other regions of Europe since the early 1980s [4–6, 8]. The PWD has three main organisms: the gymnosperm host, the pine wood nematode, and the insect vector. At the stage of primary transmission, dauer juveniles (JIV stage) of bursaphelenchus xylophilus are carried vertically in the tracheae of their beetle host to young twigs of susceptible trees, where they enter through resin canals in wounds made during maturation feeding by the insect [8].

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Recently, some mathematical models have been presented to explore the dynamics of PWD consisting of a system of nonlinear differential equations. Lee and Kwang [9] explored the stability analysis of PWD and proposed some suitable controlling strategies for this disease. Khan et al. [10] introduced a model on PWD and its optimal control. A mathematical model with variable population and suggested optimal control was developed in [11]. Most recently, in [12] the dynamics of PWD with saturated incidence rate was explored. All of the above PWD models are restricted to classical integer order differential equations. In the present paper we consider a PWD model with saturated incidence rate in fractional environment using the CF derivative. First, we give an overview of recently published papers on fractional mathematical models using the CF derivative.

Fractional order models are more reliable and helpful in the real phenomena than the classical models due to hereditary properties and the description of memory [13, 14]. Also, in the real world explanation, the integer order derivative does not explore the dynamics between two different points. To deal with such failures of classical local differentiation, different concepts on differentiation with non-local or fractional orders have been developed in the existing literature. For instance, Riemann and Liouville introduced the concept of fractional orders differentiation in [14]. Recently, Caputo and Fabrizio [15] introduced a new derivative with fractional order based on the exponential kernel. The new CF fractional order derivative has been used successfully in modeling of various real phenomena. For example, a fractional Adams–Bashforth technique via the CF derivative was presented in [16]. A study of magnetohydrodynamic electroosmotic flow of Maxwell fluids with CF derivatives was carried out by Abdulh et al. [17]. In [18], the CF fractional derivative was used for numerical approach of the Fokker–Planck equation using Ritz approximation. A mathematical comparative analysis of RL and RC electrical circuits using AB and CF fractional derivatives was recently done in [19]. Mustafa et al.[20] explored the dynamics of the cancer treatment model with the CF fractional derivative. Recently, a new fractional model of hepatitis B virus in the CF derivative sense was presented in [21].

The classical integer order mathematical model is useful for a local dynamic system with no external forces. These models cannot therefore replicate the complexity of the dynamics of the communicable disease like PWD as the model can sometimes have a crossover behavior and this cannot be handled by the classical differential operators. Further, in the literature fractional order models provide a better fit to the real data for different diseases and other experimental work in fluid mechanics. For example, Diethelm [22] provided a good agreement to the real data of the 2009 dengue outbreak in Cape Verde using a non-integer order biological model instead of the ordinary one. A fractional order model for Ebola epidemic was applied to provide a suitable approximation to the real data on Ebola virus [23]. Makris et al. [24] used a fractional order Maxwell model to attain a better fit to the experimental work.

Therefore, motivated by the above work, in this paper, we aim to extend the recently published PWD model [12] to a fractional case by using the newly established derivative known as CF derivative of order $\tau \in (0, 1]$. The details of the remaining sections of this paper are as follows: The basic definition and results of fractional order derivative are stated in Sect. 2. In Sect. 3, we explore the model formulation, model equilibria, and the basic reproduction number. Section 4 deals with the existence of solution in the spread PWD disease model via the fixed point theorem. Also, the uniqueness of a model solution is

obtained. Numerical simulations are presented in Sect. 5. Finally, the concluding remarks are given in Sect. 6.

2 Preliminaries

Here, we give some basic definitions of the fractional calculus that will be used in the onward analysis of the model.

Definition 1 Let $g \in H^1(a,b)$, with b greater than a, $\tau \in [0,1]$, then the CF fractional derivative [15] is given as

$$D_t^\tau \big(g(t)\big) = \frac{M(\tau)}{1-\tau} \int_a^t g'(x) \exp\left[-\tau \frac{t-x}{1-\tau}\right] dx. \tag{1}$$

In Eq. (1) $M(\tau)$ represents a normality with $M(0) = M(1) = 1$ [15]. However, if $g \notin H^1(a,b)$, then the following expression of the derivative is obtained:

$$D_t^\tau \big(g(t)\big) = \frac{\tau M(\tau)}{1-\tau} \int_a^t \big(g(t) - g(x)\big) \exp\left[-\tau \frac{t-x}{1-\tau}\right] dx. \tag{2}$$

Remark 1 If $\sigma = \frac{1-\tau}{\tau} \in [0,\infty)$, $\tau = \frac{1}{1+\sigma} \in [0,1]$, then Eq. (2) gives the following form:

$$D_t^\tau \big(g(t)\big) = \frac{N(\sigma)}{\sigma} \int_a^t g'(x) \exp\left[-\frac{t-x}{\sigma}\right] dx, \qquad N(0) = N(\infty) = 1. \tag{3}$$

Moreover,

$$\lim_{\sigma \to 0} \frac{1}{\sigma} \exp\left[-\frac{t-x}{\sigma}\right] = \delta(x-t). \tag{4}$$

Nieto and Losada [25] give the following definition of the integral.

Definition 2 Let $0 < \tau < 1$, then the fractional integral of the function g having order τ is given below.

$$I_t^\tau \big(g(t)\big) = \frac{2(1-\tau)}{(2-\tau)M(\tau)} g(t) + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t g(s)\, ds, \quad t \geq 0. \tag{5}$$

Remark 2 From Definition 2, we have

$$\frac{2(1-\tau)}{(2-\tau)M(\tau)} + \frac{2\tau}{(2-\tau)M(\tau)} = 1, \tag{6}$$

which implies $M(\tau) = \frac{2}{2-\tau}$, $0 < \tau < 1$. In view of (6), a new Caputo derivative of order $0 < \tau < 1$ is suggested by Nieto and Losada [25], given as follows:

$$D_t^\tau \big(g(t)\big) = \frac{1}{1-\tau} \int_0^t g'(x) \exp\left[\tau \frac{t-x}{1-\tau}\right] dx. \tag{7}$$

The CF derivative [15], given in the above definitions, has been recently used in the mathematical modeling of HBV [21], Maxwell fluid with slip effects [26], and diabetes model [27].

3 Model formulation

Here, in this section, we extend the PWD model [12] to fractional order using a CF derivative of order $\tau \in [0,1]$. The classical integer order PWD model is formulated by the following nonlinear system of differential equations:

$$
\begin{aligned}
\frac{dS_H}{dt} &= \Pi_H - \frac{K_1 S_H I_V}{1 + \theta_1 I_V} - \frac{K_2 \psi S_H I_V}{1 + \theta_1 I_V} - \gamma_H S_H, \\
\frac{dE_H}{dt} &= \frac{K_1 S_H I_V}{1 + \theta_1 I_V} - \gamma_H E_H - \delta_H E_H, \\
\frac{dI_H}{dt} &= \frac{K_2 \psi S_H I_V}{1 + \theta_1 I_V} + \delta_H E_H - \gamma_H I_H, \\
\frac{dS_V}{dt} &= \Pi_V - \frac{\beta_1 S_V I_H}{1 + \theta_2 I_H} - \gamma_V S_V, \\
\frac{dE_V}{dt} &= \frac{\beta_1 S_V I_H}{1 + \theta_2 I_H} - \gamma_V E_V - \delta_V E_V, \\
\frac{dI_V}{dt} &= \delta_V E_V - \gamma_V I_V.
\end{aligned}
\tag{8}
$$

In the above model (8), the total host population (pine trees) is denoted by $N_H(t)$. It is subdivided into three classes: susceptible $S_H(t)$, exposed $E_H(t)$, and infected $I_H(t)$ pine trees. The total vector population (beetles) is further divided into three subclasses: susceptible vector $S_V(t)$, exposed vectors $E_V(t)$, and infected vector $I_V(t)$. The recruitment rates of pine trees and vector population are denoted by Π_H and Π_V, respectively. The rate of contact between suspectable trees and infected vectors is K_1, while K_2 is the contact rate between suspectable trees and infected vectors when the nematode is transmitted by the infected vector at oviposition. The natural death rates of pine trees and vector population are denoted by parameters γ_H and γ_H, respectively. The natural death rate of pine trees which are uninfected through beetles is denoted by parameter ψ. The constants of saturation are θ_1 and θ_2. The exposed pine trees join the infected class at the rate δ_H while the transfer rate of an exposed vector to become an infected vector is denoted by δ_V. The parameter β_1 is the contact rate of a suspectable vector with infected pine trees. We reformulate the classical PWD model (8) by replacing the ordinary integer order derivative by the new CF fractional derivative and it can be written as follows:

$$
\begin{aligned}
{}_0^{CF}D_t^\tau S_H &= \Pi_H - \frac{K_1 S_H I_V}{1 + \theta_1 I_V} - \frac{K_2 \psi S_H I_V}{1 + \theta_1 I_V} - \gamma_H S_H, \\
{}_0^{CF}D_t^\tau E_H &= \frac{K_1 S_H I_V}{1 + \theta_1 I_V} - \gamma_H E_H - \delta_H E_H, \\
{}_0^{CF}D_t^\tau I_H &= \frac{K_2 \psi S_H I_V}{1 + \theta_1 I_V} + \delta_H E_H - \gamma_H I_H, \\
{}_0^{CF}D_t^\tau S_V &= \Pi_V - \frac{\beta_1 S_V I_H}{1 + \theta_2 I_H} - \gamma_V S_V, \\
{}_0^{CF}D_t^\tau E_V &= \frac{\beta_1 S_V I_H}{1 + \theta_2 I_H} - \gamma_V E_V - \delta_V E_V, \\
{}_0^{CF}D_t^\tau I_V &= \delta_V E_V - \gamma_V I_V.
\end{aligned}
\tag{9}
$$

The initial conditions involved in (9) are

$$S_H(0) = c_1, \qquad E_H(0) = c_2, \qquad I_H(0) = c_3, \qquad S_V(0) = c_4,$$

$$E_V(0) = c_5 \quad \text{and} \quad I_V(0) = c_6.$$

3.1 Equilibria and basic reproduction number

Model (9) has a disease free equilibrium $E^0 = (S_H^o, 0, 0, S_V^o, 0, 0)$ and is obtained by solving the system

$$_0^{CF}D_t^\tau S_H = {}_0^{CF}D_t^\tau E_H = {}_0^{CF}D_t^\tau I_H = {}_0^{CF}D_t^\tau C = {}_0^{CF}D_t^\tau S_V = {}_0^{CF}D_t^\tau E_V = {}_0^{CF}D_t^\tau I_V = 0,$$

and is given by

$$E^0 = \left(\frac{\Pi_H}{\gamma_H}, 0, 0, \frac{\Pi_V}{\gamma_V}, 0, 0 \right).$$

The model has a unique endemic equilibrium, denoted by E_1, given by

$$\begin{cases} S_H^* = \frac{\Pi_H(1+\theta_1 I_V^*)}{\gamma_H + I_V^*(K_1 + K_2\psi + \gamma_H\theta_1)}, \\ E_H^* = \frac{\Pi_H K_1 I_V^*}{(\gamma_H+\delta_H)(K_1 I_V^* + K_2\psi I_V^* + \gamma_H + \theta_1\gamma_H I_V^*)}, \\ I_H^* = \frac{\Pi_H I_V^*(K_2\psi\gamma_H + K_1\delta_H + K_2\psi\delta_H)}{\gamma_H(\gamma_H+\delta_H)(K_1 I_V^* + K_2 I_V^* + \gamma_H + \theta_1\gamma_H I_V^*)}, \\ S_V^* = \frac{\Pi_V(1+\theta_2 I_H^*)}{\gamma_V + I_H^*(\beta_1 + \theta_2\gamma_V)}, \\ E_V^* = \frac{\Pi_V\beta_1 I_H^*}{(\gamma_V+\delta_V)(\gamma_V + I_H^*(\beta_1+\theta_2\gamma_V))}, \\ I_V^* = \frac{\Pi_V\beta_1\delta_V I_H^*}{\gamma_V(\gamma_V+\delta_V)(\gamma_V + I_H^*(\beta_1+\theta_2\gamma_V))}. \end{cases}$$

The basic reproduction number $\mathcal{R}_0$ is obtained by using the next generation technique [28] and is given as follows:

$$\mathcal{R}_0 = \sqrt{\frac{\delta_V\beta_1 S_H^0 S_V^0(K_1\delta_H + K_2\psi(\gamma_H + \delta_H))}{\gamma_H\gamma_V(\gamma_H + \delta_H)(\gamma_V + \delta_V)}}.$$

Hence, we state the following theorem.

Theorem 1 *The FPWD model* (9) *has a unique endemic equilibrium if* $\mathcal{R}_0 > 1$.

4 Existence and uniqueness of FPWD model

This section describes the existence of model solutions by using fixed point theory. We use the fractional integral operator in [25] on (9) to obtain

$$S_H(t) - S_H(0) = {}_0^{CF}I_t^\tau \left\{ \Pi_H - \frac{K_1 S_H I_V}{1 + \theta_1 I_V} - \frac{K_2\psi S_H I_V}{1 + \theta_1 I_V} - \gamma_H S_H \right\},$$

$$E_H(t) - E_H(0) = {}_0^{CF}I_t^\tau \left\{ \frac{K_1 S_H I_V}{1 + \theta_1 I_V} - \gamma_H E_H - \delta_H E_H \right\},$$

$$I_H(t) - I_H(0) = {}_0^{CF}I_t^\tau \left\{ \frac{K_2\psi S_H I_V}{1 + \theta_1 I_V} + \delta_H E_H - \gamma_H I_H \right\},$$

$$S_V(t) - S_V(0) = {}_0^{CF}I_t^\tau \left\{ \Pi_V - \frac{\beta_1 S_V I_H}{1 + \theta_2 I_H} - \gamma_V S_V \right\}, \tag{10}$$

$$E_V(t) - E_V(0) = {}_0^{CF}I_t^\tau \left\{ \frac{\beta_1 S_V I_H}{1 + \theta_2 I_H} - \gamma_V E_V - \delta_V E_V \right\},$$

$$I_V(t) - I_V(0) = {}_0^{CF}I_t^\tau \{\delta_V E_V - \gamma_V I_V\}.$$

Applying the idea used in [25], we obtain

$$S_H(t) - S_H(0) = \frac{2(1-\tau)}{(2-\tau)M(\tau)} \left\{ \Pi_H - \frac{K_1 S_H I_V}{1 + \theta_1 I_V} - \frac{K_2 \psi S_H I_V}{1 + \theta_1 I_V} - \gamma_H S_H \right\}$$

$$+ \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t \left\{ \Pi_H - \frac{K_1 S_H I_V}{1 + \theta_1 I_V} - \frac{K_2 \psi S_H I_V}{1 + \theta_1 I_V} - \gamma_H S_H \right\} dy,$$

$$E_H(t) - E_H(0) = \frac{2(1-\tau)}{(2-\tau)M(\tau)} \left\{ \frac{K_1 S_H I_V}{1 + \theta_1 I_V} - \gamma_H E_H - \delta_H E_H \right\}$$

$$+ \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t \left\{ \frac{K_1 S_H I_V}{1 + \theta_1 I_V} - \gamma_H E_H - \delta_H E_H \right\} dy,$$

$$I_H(t) - I_H(0) = \frac{2(1-\tau)}{(2-\tau)M(\tau)} \left\{ \frac{K_2 \psi S_H I_V}{1 + \theta_1 I_V} + \delta_H E_H - \gamma_H I_H \right\}$$

$$+ \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t \left\{ \frac{K_2 \psi S_H I_V}{1 + \theta_1 I_V} + \delta_H E_H - \gamma_H I_H \right\} dy,$$

$$S_V(t) - S_V(0) = \frac{2(1-\tau)}{(2-\tau)M(\tau)} \left\{ \Pi_V - \frac{\beta_1 S_V I_H}{1 + \theta_2 I_H} - \gamma_V S_V \right\} \tag{11}$$

$$+ \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t \left\{ \Pi_V - \frac{\beta_1 S_V I_H}{1 + \theta_2 I_H} - \gamma_V S_V \right\} dy,$$

$$E_V(t) - E_V(0) = \frac{2(1-\tau)}{(2-\tau)M(\tau)} \left\{ \frac{\beta_1 S_V I_H}{1 + \theta_2 I_H} - \gamma_V E_V - \delta_V E_V \right\}$$

$$+ \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t \left\{ \frac{\beta_1 S_V I_H}{1 + \theta_2 I_H} - \gamma_V E_V - \delta_V E_V \right\},$$

$$I_V(t) - I_V(0) = \frac{2(1-\tau)}{(2-\tau)M(\tau)} \{\delta_V E_V - \gamma_V I_V\}$$

$$+ \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t \{\delta_V E_V - \gamma_V I_V\} dy.$$

For simplicity, we replace as follows:

$$F_1(t, S_H) = \Pi_H - \frac{K_1 S_H I_V}{1 + \theta_1 I_V} - \frac{K_2 \psi S_H I_V}{1 + \theta_1 I_V} - \gamma_H S_H,$$

$$F_2(t, E_H) = \frac{K_1 S_H I_V}{1 + \theta_1 I_V} - \gamma_H E_H - \delta_H E_H,$$

$$F_3(t, I_H) = \frac{K_2 \psi S_H I_V}{1 + \theta_1 I_V} + \delta_H E_H - \gamma_H I_H,$$

$$F_4(t, S_V) = \Pi_V - \frac{\beta_1 S_V I_H}{1 + \theta_2 I_H} - \gamma_V S_V, \tag{12}$$

$$F_5(t, E_V) = \frac{\beta_1 S_V I_H}{1 + \theta_2 I_H} - \gamma_V E_V - \delta_V E_V,$$

$$F_6(t, I_V) = \delta_V E_V - \gamma_V I_V.$$

Theorem 2 *The kernels $F_1, F_2, F_3, F_4, F_5,$ and F_6 fulfill the Lipschitz condition and contraction if the following inequality holds:*

$$0 \leq (K_1 + K_2\psi)e + \gamma_H < 1.$$

Proof Here, we start from F_1. Suppose S and S_1 are two functions, then we assess the following:

$$
\begin{aligned}
&\left\| F_1(t, S_H) - F_1(t, S_{1H}) \right\| \\
&= \left\| -\frac{K_1 I_V}{1 + \theta_1 I_V}\left\{ S_H(t) - S_H(t_1) \right\} - \frac{K_2 \psi I_V}{1 + \theta_1 I_V}\left\{ S_H(t) - S_H(t_1) \right\} \right. \\
&\quad \left. - \gamma_H \left\{ S_H(t) - S_H(t_1) \right\} \right\|.
\end{aligned}
\tag{13}
$$

Using the triangular inequality on Eq. (13), we obtain

$$
\begin{aligned}
&\left\| F_1(t, S_H) - F_1(t, S_{1H}) \right\| \\
&\leq \left\| \frac{K_1 I_V}{1 + \theta_1 I_V}\left\{ S_H(t) - S_H(t_1) \right\} \right\| + \left\| \frac{K_2 \psi I_V}{1 + \theta_1 I_V}\left\{ S_H(t) - S_H(t_1) \right\} \right\| \\
&\quad + \gamma_H \left\| \left\{ S_H(t) - S_H(t_1) \right\} \right\| \\
&\leq \left\{ \frac{K_1}{1 + \theta_1 I_V}\left\| I_V(t) \right\| + \frac{K_2 \psi}{1 + \theta_1 I_V}\left\| I_V(t) \right\| + \gamma_H \right\} \left\| \left\{ S_H(t) - S_H(t_1) \right\} \right\| \\
&\leq (K_1 + K_2\psi)\|I_V\| + \gamma_H) \left\| \left\{ S_H(t) - S_H(t_1) \right\} \right\| \\
&\leq \big((K_1 + K_2\psi)e + \gamma_H\big) \left\| \left\{ S_H(t) - S_H(t_1) \right\} \right\| \\
&\leq \mu_1 \left\| \left\{ S(t) - S(t_1) \right\} \right\|.
\end{aligned}
\tag{14}
$$

Taking $\mu_1 = (K_1 + K_2\psi)e + \gamma_H$, where $\|I_H(t)\| \leq e$ is a bounded function, we get

$$\left\| F_1(t, S_H) - F_1(t, S_{1H}) \right\| \leq \mu_1 \left\| S_H(t) - S_H(t_1) \right\|.
\tag{15}$$

Hence, the Lipschitz condition is fulfilled for F_1, and if in addition $0 \leq (K_1 + K_2\psi)e + \gamma_H < 1$, then it is also a contraction. For the remaining cases, in a similar way the Lipschitz conditions are given as follows:

$$
\begin{aligned}
&\left\| F_2(t, E_H) - F_2(t, E_{1H}) \right\| \leq \mu_2 \left\| E_H(t) - E_H(t_1) \right\|, \\
&\left\| F_3(t, I_H) - F_3(t, I_{1H}) \right\| \leq \mu_3 \left\| I_H(t) - I_H(t_1) \right\|, \\
&\left\| F_4(t, S_V) - F_4(t, S_{1V}) \right\| \leq \mu_4 \left\| S_V(t) - S_V(t_1) \right\|, \\
&\left\| F_5(t, E_V) - F_5(t, E_{1V}) \right\| \leq \mu_5 \left\| E_V(t) - E_V(t_1) \right\|, \\
&\left\| F_6(t, I_V) - F_6(t, I_{1V}) \right\| \leq \mu_6 \left\| I_V(t) - I_V(t_1) \right\|.
\end{aligned}
\tag{16}
$$

Using notations for kernels, Eq. (11) becomes

$$S_H(t) = S_H(0) + \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_1(t, S_H) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_1(y, S_H)\right)dy,$$

$$E_H(t) = E_H(0) + \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_2(t, E_H) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_2(y, E_H)\right)dy,$$

$$I_H(t) = I_H(0) + \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_3(t, I_H) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_3(y, I_H)\right)dy,$$

$$S_V(t) = S_V(0) + \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_4(t, S_V) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_4(y, S_V)\right)dy, \qquad (17)$$

$$E_V(t) = E_V(0) + \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_5(t, E_V) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_5(y, E_V)\right)dy,$$

$$I_V(t) = I_V(0) + \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_6(t, I_V) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_6(y, I_V)\right)dy.$$

The following recursive formula is presented:

$$S_{Hn}(t) = \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_1(t, S_{H(n-1)}) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_1(y, S_{H(n-1)})\right)dy,$$

$$E_{Hn}(t) = \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_2(t, E_{H(n-1)}) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_2(y, E_{H(n-1)})\right)dy,$$

$$I_{Hn}(t) = \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_3(t, I_{H(n-1)}) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_3(y, I_{H(n-1)})\right)dy,$$

$$S_{Vn}(t) = \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_4(t, S_{V(n-1)}) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_4(y, S_{V(n-1)})\right)dy, \qquad (18)$$

$$E_{Vn}(t) = \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_5(t, E_{V(n-1)}) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_5(y, E_{V(n-1)})\right)dy,$$

$$I_{Vn}(t) = \frac{2(1-\tau)}{(2-\tau)M(\tau)}F_6(t, I_{V(n-1)}) + \frac{2\tau}{(2-\tau)M(\tau)}\int_0^t \left(F_6(y, I_{V(n-1)})\right)dy$$

with the initial conditions given below

$$\begin{cases} S_H^0(t) = S_H(0), \\ E_H^0(t) = E_H(0), \\ I_H^0(t) = I_H(0), \\ S_V^0(t) = S_V(0), \\ E_V^0(t) = E_V(0), \\ I_V^0(t) = I_V(0). \end{cases} \qquad (19)$$

The difference of the successive terms is calculated as follows:

$$\omega_{1n}(t) = S_{Hn}(t) - S_{H(n-1)}(t)$$

$$= \frac{2(1-\tau)}{(2-\tau)M(\tau)}\left(F_1(t, S_{H(n-1)}) - F_1(t, S_{H(n-2)})\right)$$

$$+ \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t \left(F_1(y, S_{H(n-1)}) - F_1(y, S_{H(n-2)}) \right) dy,$$

$$\omega_{2n}(t) = E_{Hn}(t) - E_{H(n-1)}(t)$$

$$= \frac{2(1-\tau)}{(2-\tau)M(\tau)} \left(F_2(t, S_{H(n-1)}) - F_2(t, S_{H(n-2)}) \right)$$

$$+ \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t \left(F_2(y, E_{H(n-1)}) - F_2(y, E_{H(n-2)}) \right) dy,$$

$$\omega_{3n}(t) = I_{Hn}(t) - I_{H(n-1)}(t)$$

$$= \frac{2(1-\tau)}{(2-\tau)M(\tau)} \left(F_3(t, I_{H(n-1)}) - F_3(t, I_{H(n-2)}) \right)$$

$$+ \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t \left(F_3(y, I_{H(n-1)}) - F_3(y, I_{H(n-2)}) \right) dy,$$

$$\omega_{4n}(t) = S_{Vn}(t) - S_{V(n-1)}(t)$$

$$= \frac{2(1-\tau)}{(2-\tau)M(\tau)} \left(F_4(t, S_{V(n-1)}) - F_4(t, S_{V(n-2)}) \right)$$

$$+ \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t \left(F_4(y, S_{V(n-1)}) - F_4(y, S_{V(n-2)}) \right) dy,$$

$$\omega_{5n}(t) = E_{Vn}(t) - E_{V(n-1)}(t)$$

$$= \frac{2(1-\tau)}{(2-\tau)M(\tau)} \left(F_5(t, E_{V(n-1)}) - F_5(t, E_{V(n-2)}) \right)$$

$$+ \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t \left(F_5(y, E_{V(n-1)}) - F_5(y, E_{V(n-2)}) \right) dy,$$

$$\omega_{6n}(t) = I_{Vn}(t) - I_{V(n-1)}(t)$$

$$= \frac{2(1-\tau)}{(2-\tau)M(\tau)} \left(F_6(t, I_{V(n-1)}) - F_6(t, I_{V(n-2)}) \right)$$

$$+ \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t \left(F_6(y, I_{V(n-1)}) - F_6(y, I_{V(n-2)}) \right) dy.$$

(20)

Notice that

$$\begin{cases} S_{Hn}(t) = \sum_{i=1}^n \omega_{1i}(t), \\ E_{Hn}(t) = \sum_{i=1}^n \omega_{2i}(t), \\ I_{Hn}(t) = \sum_{i=1}^n \omega_{3i}(t), \\ S_{Vn}(t) = \sum_{i=1}^n \omega_{4i}(t), \\ E_{Vn}(t) = \sum_{i=1}^n \omega_{5i}(t), \\ I_{Vn}(t) = \sum_{i=1}^n \omega_{6i}(t). \end{cases}$$

(21)

On continuing the same process, we assess

$$\left\| \omega_{1n}(t) \right\| = \left\| S_{Hn}(t) - S_{H(n-1)}(t) \right\|$$

$$= \left\| \frac{2(1-\tau)}{(2-\tau)M(\tau)} \left(F_1(t, S_{H(n-1)}) - F_1(t, S_{H(n-2)}) \right) \right.$$

$$+ \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t \left(F_1(y, S_{H(n-1)}) - F_1(y, S_{H(n-2)}) \right) dy \Bigg\|. \tag{22}$$

Using the triangular inequality, Eq. (22) is simplified to

$$\left\| S_{Hn}(t) - S_{H(n-1)}(t) \right\| \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)} \left\| \left(F_1(t, S_{H(n-1)}) - F_1(t, S_{H(n-2)}) \right) \right\|$$
$$+ \frac{2\tau}{(2-\tau)M(\tau)} \left\| \int_0^t \left(F_1(y, S_{H(n-1)}) - F_1(y, S_{H(n-2)}) \right) dy \right\|. \tag{23}$$

As the kernel fulfills the Lipschitz condition, then we have

$$\left\| S_n(t) - S_{n-1}(t) \right\| \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)} \mu_1 \| S_{H(n-1)} - S_{H(n-2)} \| + \frac{2\tau}{(2-\tau)M(\tau)} \mu_1$$
$$\times \int_0^t \| S_{H(n-1)} - S_{H(n-2)} \| dy. \tag{24}$$

Then we have

$$\left\| \omega_{1n}(t) \right\| \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)} \mu_1 \left\| \omega_{1(n-1)}(t) \right\| + \frac{2\tau}{(2-\tau)M(\tau)} \mu_1 \int_0^t \left\| \omega_{1(n-1)}(y) \right\| dy. \tag{25}$$

Similarly, we get the following results:

$$\left\| \omega_{2n}(t) \right\| \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)} \mu_2 \left\| \omega_{2(n-1)}(t) \right\| + \frac{2\tau}{(2-\tau)M(\tau)} \mu_2 \int_0^t \left\| \omega_{2(n-1)}(y) \right\| dy,$$

$$\left\| \omega_{3n}(t) \right\| \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)} \mu_3 \left\| \omega_{3(n-1)}(t) \right\| + \frac{2\tau}{(2-\tau)M(\tau)} \mu_1 \int_0^t \left\| \omega_{3(n-1)}(y) \right\| dy,$$

$$\left\| \omega_{4n}(t) \right\| \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)} \mu_4 \left\| \omega_{4(n-1)}(t) \right\| + \frac{2\tau}{(2-\tau)M(\tau)} \mu_4 \int_0^t \left\| \omega_{4(n-1)}(y) \right\| dy, \tag{26}$$

$$\left\| \omega_{5n}(t) \right\| \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)} \mu_5 \left\| \omega_{5(n-1)}(t) \right\| + \frac{2\tau}{(2-\tau)M(\tau)} \mu_5 \int_0^t \left\| \omega_{5(n-1)}(y) \right\| dy,$$

$$\left\| \omega_{6n}(t) \right\| \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)} \mu_6 \left\| \omega_{6(n-1)}(t) \right\| + \frac{2\tau}{(2-\tau)M(\tau)} \mu_6 \int_0^t \left\| \omega_{6(n-1)}(y) \right\| dy. \qquad \Box$$

Now we state the theorem below.

Theorem 3 *The FPWD model* (9) *has exact coupled solutions if the conditions below hold. That is, we can find t_0 such that*

$$\frac{2(1-\tau)}{(2-\tau)M(\tau)} \mu_1 + \frac{2\tau}{(2-\tau)M(\tau)} \mu_1 t_0 < 1.$$

Proof Since all the functions $S_H(t)$, $E_H(t)$, $I_H(t)$ and $S_V(t)$, $E_V(t)$, $I_V(t)$ are bounded, we have shown that the kernels fulfill the Lipschitz condition, thus on using of Eqs. (25) and

([26](26)) and by using the recursive method, we obtain the succeeding relation as follows:

$$
\begin{aligned}
\|\omega_{1n}(t)\| &\leq \|S_{Hn}(0)\| \left[\left(\frac{2(1-\tau)}{(2-\tau)M(\tau)} \mu_1 \right) + \left(\frac{2\tau}{(2-\tau)M(\tau)} \mu_1 t \right) \right]^n, \\
\|\omega_{2n}(t)\| &\leq \|E_{Hn}(0)\| \left[\left(\frac{2(1-\tau)}{(2-\tau)M(\tau)} \mu_2 \right) + \left(\frac{2\tau}{(2-\tau)M(\tau)} \mu_2 t \right) \right]^n, \\
\|\omega_{3n}(t)\| &\leq \|I_{Hn}(0)\| \left[\left(\frac{2(1-\tau)}{(2-\tau)M(\tau)} \mu_3 \right) + \left(\frac{2\tau}{(2-\tau)M(\tau)} \mu_3 t \right) \right]^n, \\
\|\omega_{4n}(t)\| &\leq \|S_{Vn}(0)\| \left[\left(\frac{2(1-\tau)}{(2-\tau)M(\tau)} \mu_4 \right) + \left(\frac{2\tau}{(2-\tau)M(\tau)} \mu_4 t \right) \right]^n, \\
\|\omega_{5n}(t)\| &\leq \|E_{Vn}(0)\| \left[\left(\frac{2(1-\tau)}{(2-\tau)M(\tau)} \mu_5 \right) + \left(\frac{2\tau}{(2-\tau)M(\tau)} \mu_5 t \right) \right]^n, \\
\|\omega_{6n}(t)\| &\leq \|I_{Vn}(0)\| \left[\left(\frac{2(1-\tau)}{(2-\tau)M(\tau)} \mu_6 \right) + \left(\frac{2\tau}{(2-\tau)M(\tau)} \mu_6 t \right) \right]^n.
\end{aligned}
\tag{27}
$$

Hence, the existence and continuity of the said solutions is proved. Furthermore, to ensure that the above function is a solution of Eq. ([9](9)), we proceed as follows:

$$
\begin{aligned}
S_H(t) - S_H(0) &= S_{Hn}(t) - B_n(t), \\
E_H(t) - E_H(0) &= E_{Hn}(t) - C_n(t), \\
I_H(t) - I_H(0) &= I_{Hn}(t) - D_n(t), \\
S_V(t) - S_V(0) &= S_{Vn}(t) - F_n(t), \\
E_V(t) - E_V(0) &= E_{Vn}(t) - G_n(t), \\
I_V(t) - I_V(0) &= I_{Vn}(t) - H_n(t).
\end{aligned}
\tag{28}
$$

Therefore, we have

$$
\begin{aligned}
\|B_n(t)\| &= \left\| \frac{2(1-\tau)}{(2-\tau)M(\tau)} \left(F_1(t, S_{Hn}) - F_1(t, S_{H(n-1)}) \right) + \frac{2\tau}{(2-\tau)M(\tau)} \right. \\
&\quad \left. \times \int_0^t \left(F_1(y, S_{Hn}) - F_1(y, S_{H(n-1)}) \right) dy \right\| \\
&\leq \frac{2(1-\tau)}{(2-\tau)M(\tau)} \left\| \left(F_1(t, S_{Hn}) - \left(F_1(t, S_{H(n-1)}) \right) \right) \right\| + \frac{2\tau}{(2-\tau)M(\tau)} \\
&\quad \times \int_0^t \left\| \left(F_1(y, S_H) - F_1(y, S_{H(n-1)}) \right) \right\| dy \\
&\leq \frac{2(1-\tau)}{(2-\tau)M(\tau)} \mu_1 \|S_H - S_{H(n-1)}\| + \frac{2\tau}{(2-\tau)M(\tau)} \mu_1 \|S_H - S_{H(n-1)}\| t.
\end{aligned}
\tag{29}
$$

Using the process in a recursive manner gives

$$
\|B_n(t)\| \leq \left(\frac{2(1-\tau)}{(2-\tau)M(\tau)} + \frac{2\tau}{(2-\tau)M(\tau)} t \right)^{n+1} \mu_1^{n+1} a.
\tag{30}
$$

Then at t_0 we have

$$\|B_n(t)\| \le \left(\frac{2(1-\tau)}{(2-\tau)M(\tau)} + \frac{2\tau}{(2-\tau)M(\tau)} t_0 \right)^{n+1} \mu_1^{n+1} a. \tag{31}$$

By applying the limit on Eq. (31) as n tends to infinity, we get

$$\|B_n(t)\| \longrightarrow 0.$$

Similarly, we get

$$\|C_n(t)\| \longrightarrow 0, \qquad \|D_n(t)\| \longrightarrow 0, \qquad \|F_n(t)\| \longrightarrow 0,$$
$$\|G_n(t)\| \longrightarrow 0, \qquad \|H_n(t)\| \longrightarrow 0. \qquad \qquad \square$$

For the uniqueness system (9) solution, we take on contrary that there exists another solution of (9) given by $S_{1H}(t)$, $E_{1H}(t)$, $I_{H1}(t)$, $S_{1V}(t)$, $E_{1V}(t)$, and $I_{1V}(t)$. Then

$$S_H(t) - S_{1H}(t) = \frac{2(1-\tau)}{(2-\tau)M(\tau)} \big(F_1(t, S_H) - F_1(t, S_{1H}) \big) + \frac{2\tau}{(2-\tau)M(\tau)}$$
$$\times \int_0^t \big(F_1(y, S_H) - F_1(y, S_{1H}) \big) \, dy. \tag{32}$$

Taking norm on Eq. (32), we get

$$\|S_H(t) - S_{1H}(t)\| \le \frac{2(1-\tau)}{(2-\tau)M(\tau)} \|F_1(t, S_H) - F_1(t, S_{1H})\| + \frac{2\tau}{(2-\tau)M(\tau)}$$
$$\times \int_0^t \|F_1(y, S_H) - U_1(y, S_{1H})\| \, dy. \tag{33}$$

By applying the Lipschitz condition of kernel, we have

$$\|S_H(t) - S_{1H}(t)\| \le \frac{2(1-\tau)}{(2-\tau)M(\tau)} \mu_1 \|S_H(t) - S_{1H}(t)\| + \frac{2\tau}{(2-\tau)M(\tau)}$$
$$\times \int_0^t \mu_1 t \|S_H(t) - S_{1H}(t)\| \, dy. \tag{34}$$
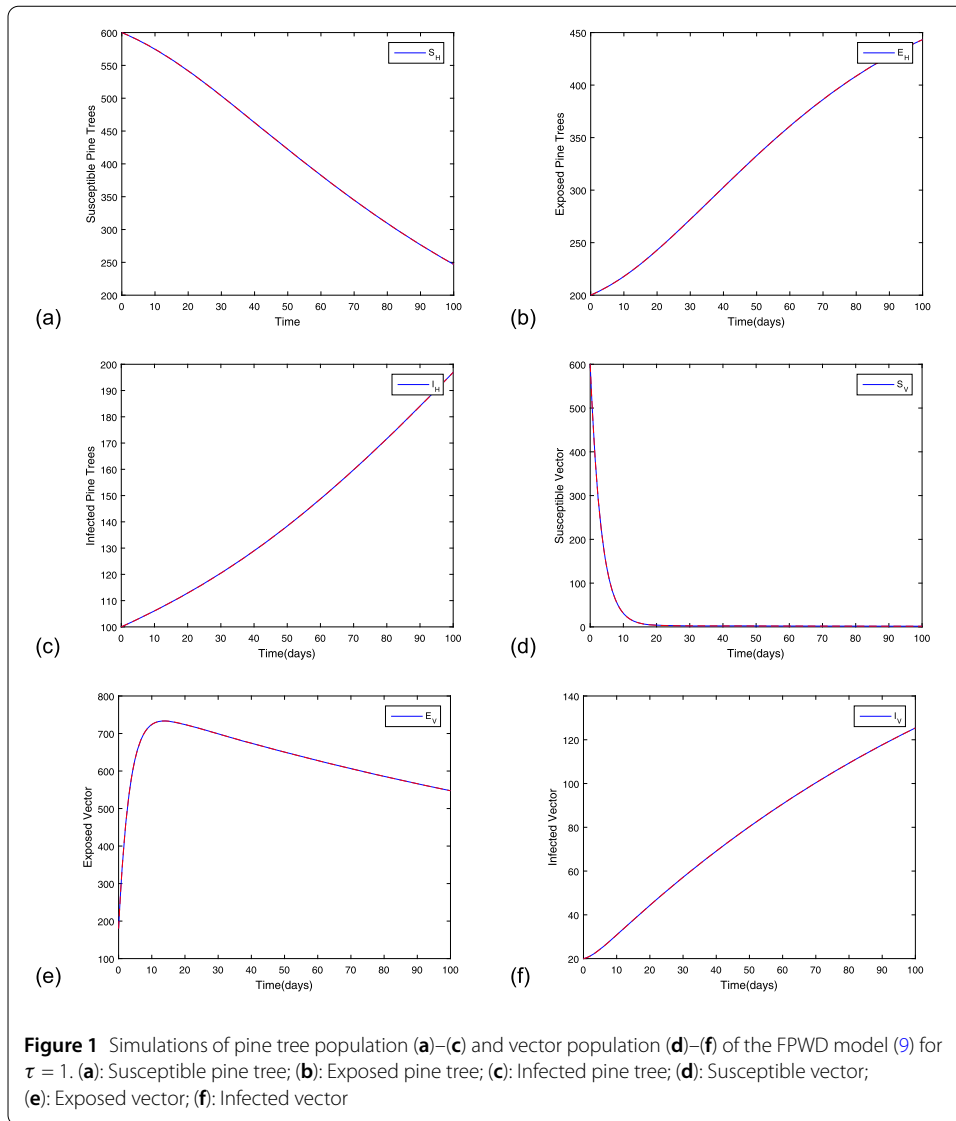
It gives

$$\|S_H(t) - S_{1H}(t)\| \left(1 - \frac{2(1-\tau)}{(2-\tau)M(\tau)} \mu_1 - \frac{2\tau}{(2-\tau)M(\tau)} \mu_1 t \right) \le 0. \tag{35}$$

Theorem 4 *The model* (9) *solution will be unique if*

$$\left(1 - \frac{2(1-\tau)}{(2-\tau)M(\tau)} \mu_1 - \frac{2\tau}{(2-\tau)M(\tau)} \mu_1 t \right) > 0. \tag{36}$$

Proof If condition (36) holds, then (35) implies that

$$\|S_H(t) - S_{1H}(t)\| = 0. \tag{37}$$

Figure 1 Simulations of pine tree population (**a**)–(**c**) and vector population (**d**)–(**f**) of the FPWD model (9) for $\tau = 1$. (**a**): Susceptible pine tree; (**b**): Exposed pine tree; (**c**): Infected pine tree; (**d**): Susceptible vector; (**e**): Exposed vector; (**f**): Infected vector
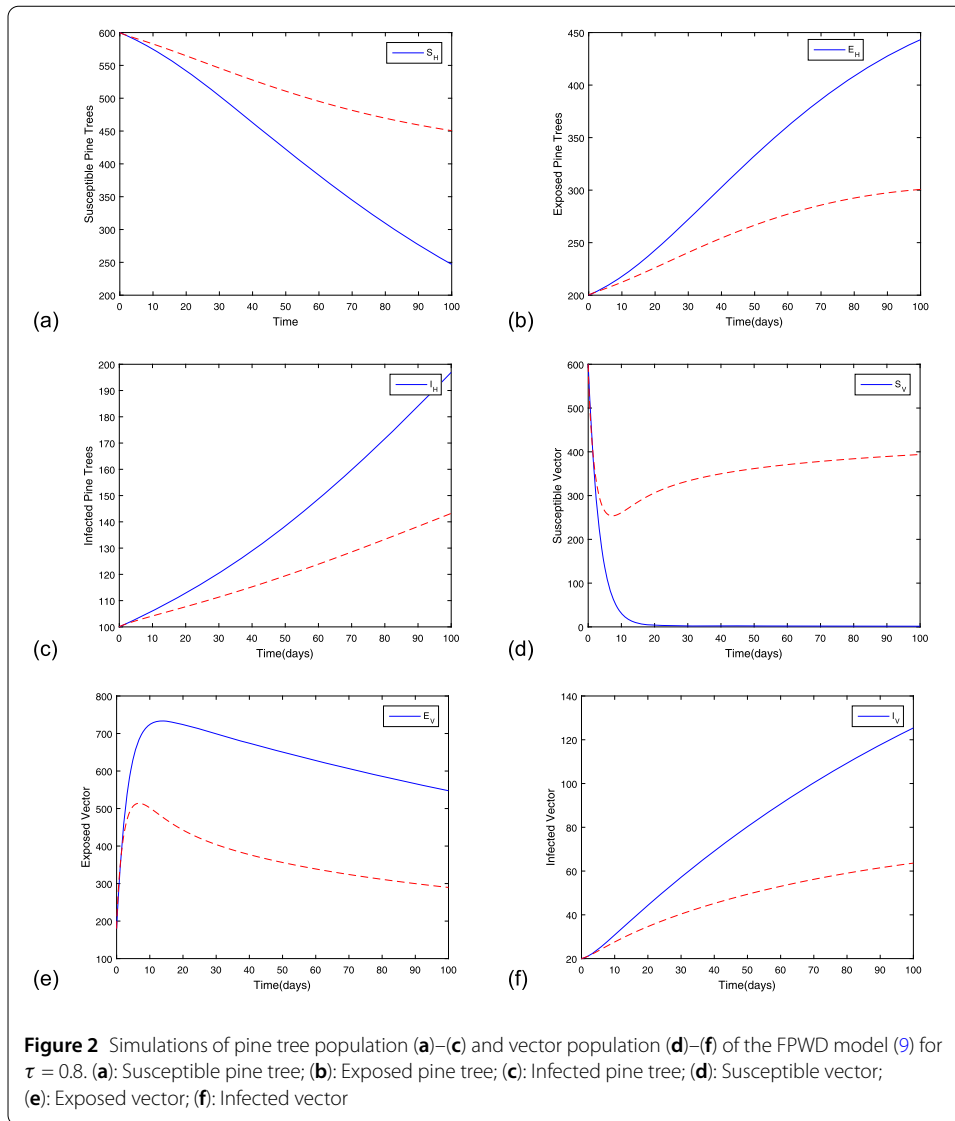
Hence, we get

$$S_H(t) = S_{1H}(t). \tag{38}$$
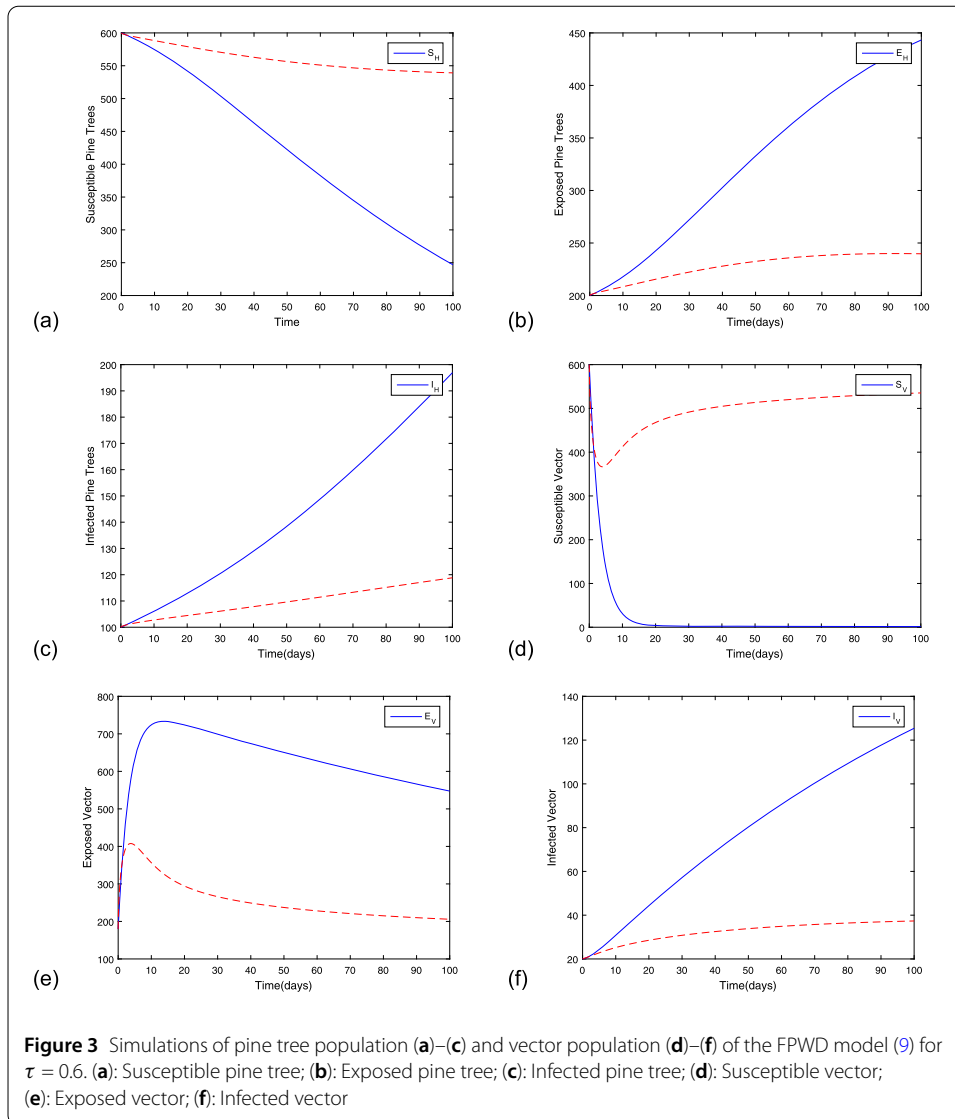
On employing the same procedure, we get

$$
\begin{aligned}
E_H(t) &= E_{1H}(t), \\
I_H(t) &= I_{1H}(t), \\
S_V(t) &= S_{1V}(t), \\
E_V(t) &= E_{1V}(t), \\
I_V(t) &= I_{1V}(t).
\end{aligned}
\tag{39}
$$
□

Figure 2 Simulations of pine tree population (**a**)–(**c**) and vector population (**d**)–(**f**) of the FPWD model (9) for $\tau = 0.8$. (**a**): Susceptible pine tree; (**b**): Exposed pine tree; (**c**): Infected pine tree; (**d**): Susceptible vector; (**e**): Exposed vector; (**f**): Infected vector
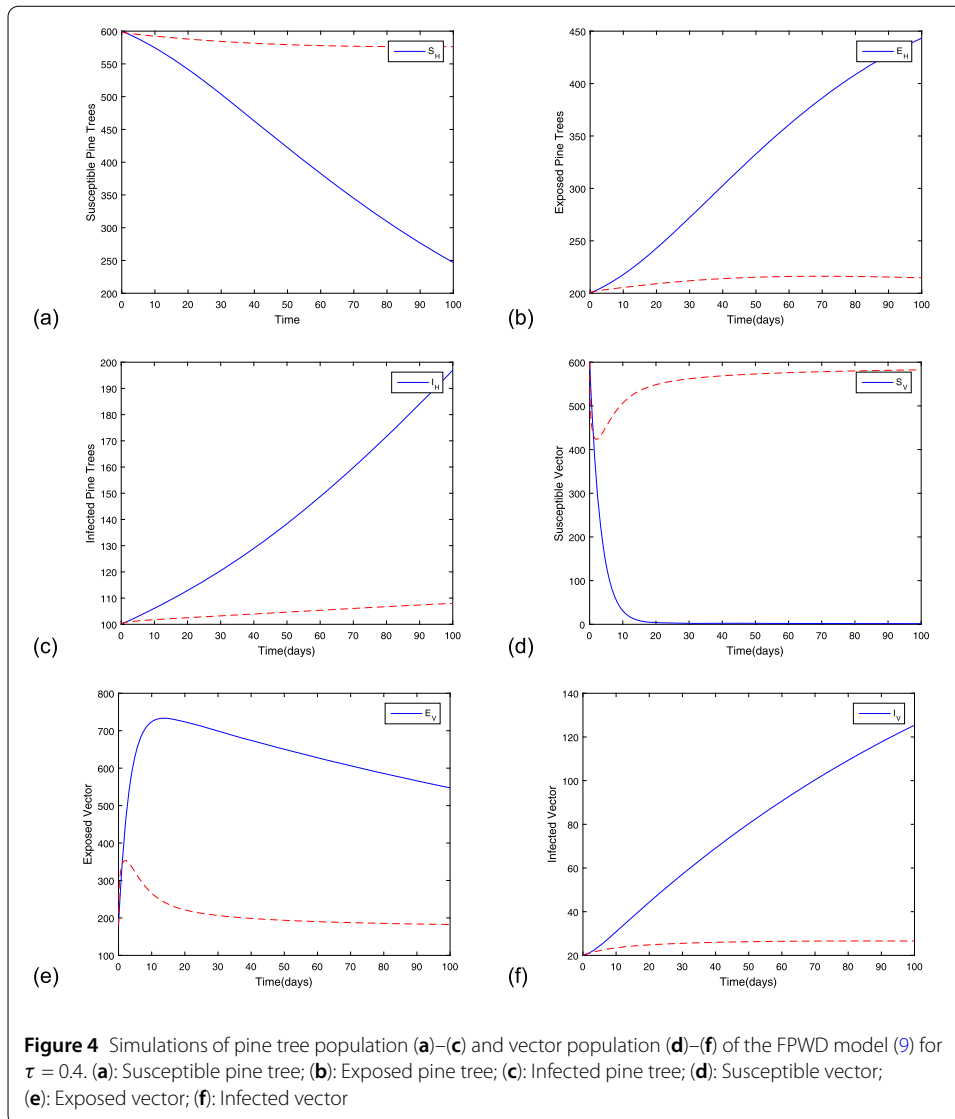
5 Numerical results

The present section aims to find the numerical simulations of the extended fractional order PWD model (9) by variation of the non-integer derivative of order $\tau \in [0, 1]$. The values of the parameters used in the numerical simulation are $\Pi_H = 0.3$, $K_1 = 0.00022$, $K_2 = 0.00043$, $\theta_1 = 0.01$, $\gamma_H = 0.00048$, $\delta_H = 0.00317$, $\psi = 0.0000083$, $\theta_2 = 0.001$, $\gamma_V = 0.0028$, $\delta_V = 0.002$, $\Pi_V = 0.8$, $\beta_1 = 0.003183$. The behavior of the individuals of model (9) for $\tau = 1, 0.80, 0.60, 0.40, 0.20$ is simulated in Figs. 1–5, respectively. In each Figure, the behavior of host population (pine trees) is shown by sub-plots (a)–(c), while the graphical behavior of vector population (beetles) is given by sub-plots (d)–(f). Also in each case the solid blue line represents the model simulations plot for $\tau = 1$, and the doted red line is the behavior of the model for distinct values of τ other than 1. From the numerical results in Figs. 1–5, it is clear that when we decrease the order of fractional derivative τ, the number of both susceptible pine trees and beetles increases, while the exposed and infected pine trees and beetles decrease significantly. From the graphical results we can say

Figure 3 Simulations of pine tree population (**a**)–(**c**) and vector population (**d**)–(**f**) of the FPWD model (9) for $\tau = 0.6$. (**a**): Susceptible pine tree; (**b**): Exposed pine tree; (**c**): Infected pine tree; (**d**): Susceptible vector; (**e**): Exposed vector; (**f**): Infected vector

that the model depends notably on the fractional order derivative, which provides more biologically reasonable results.

6 Conclusion

In the present work, we extended the PWD model [12] to fractional order using the Caputo–Fabrizio fractional derivative. The model equilibria and basic reproduction number are explored. The existence and uniqueness of the solution for the FPWD model with CF derivative are proved in detail. Some numerical simulations are carried out to explore the effect of fractional order. From numerical simulations one can see that when fractional order of derivative τ decreases, the CF derivative provides more biologically feasible behavior about the dynamic of pine wilt disease. Therefore, we concluded that the newly fractional derivative is very useful for modeling such phenomena. Also, from the graphical behavioral we conclude that the proposed fractional order model provides richer and more flexible results when compared with the corresponding integer order PWD model.

Figure 4 Simulations of pine tree population (**a**)–(**c**) and vector population (**d**)–(**f**) of the FPWD model (9) for $\tau = 0.4$. (**a**): Susceptible pine tree; (**b**): Exposed pine tree; (**c**): Infected pine tree; (**d**): Susceptible vector; (**e**): Exposed vector; (**f**): Infected vector

Thus, we suggest that this new fractional model gives more effective results which could be implemented for the elimination of the pine tree infection.

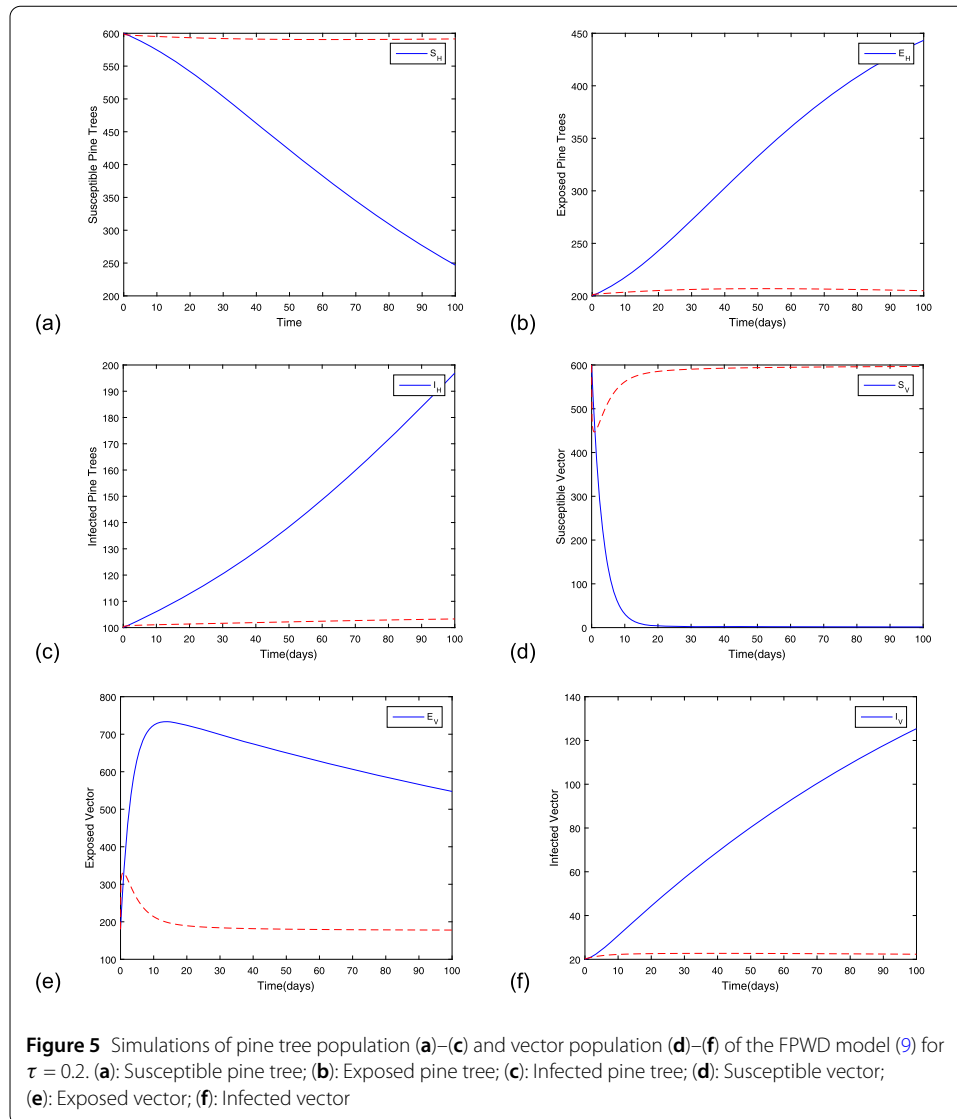**Availability of data and materials**
The data is available without any restrictions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
MAK and KS formulated the fractional order model and obtained the numerical results. SU obtained the existence results for the model. MAK and SU wrote the introduction, abstract, and conclusion. MAK, KOO revised the manuscript. All authors read and approved the final manuscript.

Figure 5 Simulations of pine tree population (**a**)–(**c**) and vector population (**d**)–(**f**) of the FPWD model (9) for $\tau = 0.2$. (**a**): Susceptible pine tree; (**b**): Exposed pine tree; (**c**): Infected pine tree; (**d**): Susceptible vector; (**e**): Exposed vector; (**f**): Infected vector

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