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# Dynamical analysis and optimal harvesting of a stochastic three-species cooperative system with delays and Lévy jumps

Yuanfu Shao<sup>1,2\*</sup>, Yuming Chen<sup>2</sup> and Binxiang Dai<sup>3</sup>

\*Correspondence: [shaoyuanfu@163.com](mailto:shaoyuanfu@163.com)

<sup>1</sup>College of Science, Guilin University of Technology, Guilin, China

<sup>2</sup>Department of Mathematics, Wilfrid Laurier University, Waterloo, Canada

Full list of author information is available at the end of the article

## Abstract

A three-species cooperative system with time delays and Lévy jumps is proposed in this paper. Firstly, by comparison method and inequality techniques, we discuss the stability in mean and extinction of species, and the stochastic permanence of this system. Secondly, by applying asymptotic method, we investigate the stability in distribution of solutions. Thirdly, utilizing ergodic method, we obtain the optimal harvesting policy of this system. Finally, some numerical examples are given to illustrate our main results.

**Keywords:** Delay; Lévy; Stability; Optimal harvesting

## 1 Introduction

The Lotka–Volterra model, proposed by Lotka [1] and Volterra [2], is used to describe the evolutionary process in population dynamics, physics, and economics. In the last decades, many modifications of it have been investigated (see, for example, [3–7]).

In [3], Goh proposed the following typical cooperative Lotka–Volterra system:

$$\begin{cases} \frac{dx_1(t)}{dt} = x_1(t)(r_1 - a_{11}x_1(t) + a_{12}x_2(t)), \\ \frac{dx_2(t)}{dt} = x_2(t)(r_2 + a_{21}x_1(t) - a_{22}x_2(t)), \end{cases} \quad (1.1)$$

where  $x_i(t)$  denotes the density of the  $i$ th species at time  $t$ ,  $r_i > 0$  is its intrinsic growth rate,  $a_{ii}$  is the intra-specific competition rate ( $i = 1, 2$ ),  $a_{12}$  and  $a_{21}$  are the inter-specific cooperation rates. The author obtained the globally asymptotic attractivity of the positive equilibrium point of this system.

However, on the one hand, in the real world, the growth of a population is often subject to environmental perturbations, and hence it is necessary to consider stochastic perturbations in the progress of mathematical modeling [8–15]. There are many kinds of stochastic perturbations. When the stochastic influence on the intrinsic growth rates in (1.1) is considered, one way is

$$r_i \rightarrow r_i + \sigma_i \dot{\omega}_i(t), \quad i = 1, 2,$$

where  $\omega_1(t)$  and  $\omega_2(t)$  are independent Brownian motions, and the parameters  $\sigma_1^2$  and  $\sigma_2^2$  represent the intensities of the white noises. Then the following stochastic Lotka–Volterra cooperative system:

$$\begin{cases} dx_1(t) = x_1(t)(r_1 - a_{11}x_1(t) + a_{12} \int_{-\infty}^t f_2(t-s)x_2(s) ds) dt + \sigma_1 d\omega_1(t), \\ dx_2(t) = x_2(t)(r_2 + a_{21} \int_{-\infty}^t f_1(t-s)x_1(s) ds - a_{22}x_2(s)) dt + \sigma_2 d\omega_2(t), \end{cases}$$

was studied by Zuo *et al.* [16].

On the other hand, a population’s growth is also affected by some sudden random perturbations such as earthquakes, epidemics, hurricanes, harvesting, and so on. To deal with such phenomena, Lévy jump processes are introduced [17–20]. Generally, a Lévy process can be decomposed into the sum of a linear drift, a Brownian motion, and a superposition of independent Poisson processes with different jump sizes. For example, Liu and Wang [20] proposed and investigated a stochastic and cooperative model with Lévy jumps as follows:

$$\begin{cases} dx_1(t) = x_1(t^-)(r_1 - a_{11}x_1(t^-) + a_{12}x_2(t^-)) dt + \sigma_1 x_1(t^-) d\omega_1(t) \\ \quad + x_1(t^-) \int_{\mathbb{Z}} \gamma_1(u) \tilde{N}(du, dt), \\ dx_2(t) = x_2(t^-)(r_2 + a_{21}x_1(t^-) - a_{22}x_2(t^-)) dt + \sigma_2 x_2(t^-) d\omega_2(t) \\ \quad + x_2(t^-) \int_{\mathbb{Z}} \gamma_2(u) \tilde{N}(du, dt), \end{cases}$$

where  $x_i(t^-)$  is the left limit of  $x_i(t)$  ( $i = 1, 2$ ),  $\tilde{N}(du, dt) = N(du, dt) - \lambda(du) dt$  is a compensated Poisson process,  $N$  represents a Poisson counting measure with characteristic measure  $\lambda$  on a measurable subset  $\mathbb{Z}$  of  $R_+ = (0, \infty)$  with  $\lambda(\mathbb{Z}) < \infty$ . For more details of the Lévy jump process, see Gihman and Skorohod [21]. The authors considered the persistence in mean and extinction of species.

As we know, time delays are very important in ecosystem models. They may cause populations to fluctuate. Time delays can reflect natural phenomena more authentically. Kuang [22] has pointed out that ignoring time delays means ignoring the reality. Therefore, it is essential to take into account the influence of time delays in the biological modeling [22–26].

Moreover, in ecological managing, harvesting often appears. Since over-harvesting or unreasonable harvesting may cause a number of detrimental effects including ecological destruction and species extinction, it is important for us to study the optimal harvesting policies for sustainable development.

Finally, it has been recognized that single-species or two-species ecological models cannot describe the natural phenomena accurately, and many vital behaviors can only be exhibited by a system with three or more species [27–30].

Motivated by the above discussions, in this paper, we propose and consider the following three-species stochastic cooperative system with time delays and Lévy jumps:

$$\begin{cases} dx_1(t) = x_1(t^-)(r_1 - c_1 - a_{11}x_1(t^-) + a_{12}x_2(t^- - \tau_{12}) + a_{13}x_3(t^- - \tau_{13})) dt \\ \quad + \sigma_1x_1(t^-) d\omega_1(t) + x_1(t^-) \int_{\mathbb{Z}} \gamma_1(u) \tilde{N}(du, dt), \\ dx_2(t) = x_2(t^-)(r_2 - c_2 + a_{21}x_1(t^- - \tau_{21}) - a_{22}x_2(t^-) + a_{23}x_3(t^- - \tau_{23})) dt \\ \quad + \sigma_2x_2(t^-) d\omega_2(t) + x_2(t^-) \int_{\mathbb{Z}} \gamma_2(u) \tilde{N}(du, dt), \\ dx_3(t) = x_3(t^-)(r_3 - c_3 + a_{31}x_1(t^- - \tau_{31}) + a_{32}x_2(t^- - \tau_{32}) + a_{33}x_3(t^-)) dt \\ \quad + \sigma_3x_3(t^-) d\omega_3(t) + x_3(t^-) \int_{\mathbb{Z}} \gamma_3(u) \tilde{N}(du, dt), \end{cases} \tag{1.2}$$

with initial data

$$x_i(\theta) = \phi_i(\theta) > 0 \quad \text{for } \theta \in [-\tau, 0] \text{ and } i = 1, 2, 3,$$

where  $\tau = \max\{\tau_{12}, \tau_{13}, \tau_{21}, \tau_{23}, \tau_{31}, \tau_{32}\}$ ,  $x_i(t)$  stands for the population size of the  $i$ th species at time  $t$ ,  $x_i(t^-)$  is the left limit of  $x_i(t)$ ,  $r_i > 0$  is the growth rate of  $x_i(t)$ ,  $c_i$  is the harvesting effort of  $x_i(t)$ ,  $a_{ii} > 0$  represents the intra-specific competitive coefficient of  $x_i(t)$ ,  $a_{ij}$  is the interspecific cooperative rate,  $\tau_{ij} > 0$  is the time delay,  $\phi = (\phi_1, \phi_2, \phi_3) \in C([-\tau, 0], R_+^3)$  (the set of all continuous functions from  $[-\tau, 0]$  into  $R_+^3$ ),  $\sigma_i^2$  denotes the intensity of the white noise,  $\omega_i(t)_{t>0}$  is the standard independent Brownian motion defined on a complete probability space  $(\Omega, \mathcal{F}, \mathcal{F}_{t \geq 0}, P)$  ( $i \neq j, i, j = 1, 2, 3$ ). The Lévy jump is the same as before.  $\gamma_i(u)$  is the effect of the Lévy noise on the  $i$ th species.  $\gamma_i(u) > 0$  means that the Lévy jump brings increasing of the species, whereas  $\gamma_i(u) < 0$  represents the decreasing of the species. Hence we always suppose that  $1 + \gamma_i(u) > 0, u \in \mathbb{Z}, i = 1, 2, 3$ .

Our main aims are as follows. Firstly, since time delays, white noises, and Lévy jumps are included in (1.2), it is of great significance to study their effects on dynamics. By use of comparison methods and some inequality techniques, we obtain the stability in mean, extinction of populations, and stochastic permanence.

Secondly, a stochastic model cannot tend to a positive fixed point, *i.e.*, there exists no traditional positive equilibrium state. It is necessary to study the convergence in distribution of solutions. Because of time delays, we cannot apply the traditional method like using the explicit solution by solving the corresponding Fokker–Planck equation. Here we will apply an asymptotic approach to get the convergence.

Lastly, in view of the importance of optimal harvesting policy, by applying the ergodic methods, we get the optimal harvesting effort (OHE)  $C^* = (c_1^*, c_2^*, c_3^*)^T$  such that the expectation of sustainable yield (ESY)  $Y(C) = \lim_{t \rightarrow \infty} \sum_{i=1}^3 \mathbb{E}(c_i x_i(t))$  is the maximum and all species are still persistent.

The rest of this paper is organized as follows. Section 2 begins with some notations, definitions, and some important lemmas which are essential in our discussion. Section 3 focuses on the dynamical behavior of (1.2) including persistence, extinction of species, and stochastic permanence. Section 4 is devoted to the stability in distribution. Section 5 considers the existence of optimal harvesting policy and obtains the maximum of ESY (MESY). Some numerical examples are given in Sect. 6 to demonstrate the obtained theoretical results. The paper concludes with a brief conclusion and discussion.

## 2 Preliminaries

For simplicity of notations, we introduce

$$\begin{aligned} \alpha_1 &= (a_{11}, -a_{12}, -a_{13})^T, \\ \alpha_2 &= (-a_{21}, a_{22}, -a_{23})^T, \\ \alpha_3 &= (-a_{31}, -a_{32}, a_{33})^T, \\ b_i &= \sigma_i^2/2 + \int_{\mathbb{Z}} \gamma_i(u) - \ln(1 + \gamma_i(u)\lambda)(du), \quad i = 1, 2, 3, \\ \beta_i &= r_i - c_i - b_i, \quad i = 1, 2, 3, \\ b &= (b_1, b_2, b_3)^T, \\ A &= \det(\alpha_1, \alpha_2, \alpha_3), \\ A_1 &= \det(R, \alpha_2, \alpha_3), \\ A_2 &= \det(\alpha_1, R, \alpha_3), \\ A_3 &= \det(\alpha_1, \alpha_2, R), \\ \tilde{A}_1 &= \det(b, \alpha_2, \alpha_3), \\ \tilde{A}_2 &= \det(\alpha_1, b, \alpha_3), \\ \tilde{A}_3 &= \det(\alpha_1, \alpha_2, b), \\ \Delta_{ij} &= a_{kk}(r_j - c_j) + a_{jk}(r_k - c_k), \quad i, j, k = 1, 2, 3, i \neq j \neq k, \\ \tilde{\Delta}_{ij} &= a_{kk}b_j + a_{jk}b_k, \quad i, j, k = 1, 2, 3, i \neq j \neq k, \\ R &= (r_1 - c_1, r_2 - c_2, r_3 - c_3)^T, \\ \mathbb{R}_1 &= \det(\alpha_1, R, b), \\ \mathbb{R}_2 &= \det(R, \alpha_2, b), \\ \mathbb{R}_3 &= \det(R, b, \alpha_3). \end{aligned}$$

Moreover, for a  $3 \times 3$  matrix  $C = (c_{ij})$ ,  $C_{ij}$  denotes the complement minor of  $c_{ij}$  in the determinant of  $C$ ,  $i, j = 1, 2, 3$ . For any function  $x(t)$ ,  $t > 0$ , we let

$$\langle x(t) \rangle = t^{-1} \int_0^t x(s) ds, \quad x^* = \limsup_{t \rightarrow \infty} x(t), \quad x_* = \liminf_{t \rightarrow \infty} x(t).$$

Throughout this paper, we always assume that  $A > 0$  and  $A_i > 0$  ( $i = 1, 2, 3$ ). This means that when there are no stochastic perturbations, a positive equilibrium state exists for model 1.2 (1.2). Furthermore, we always assume that  $K$  stands for a generic positive constant whose value may be different at different places.

On the parameters, we make the following assumptions.

**Assumption 2.1**  $\mathbb{R}_1 > 0$ ,  $\mathbb{R}_2 > 0$ , and  $\mathbb{R}_3 > 0$ .

*Remark 2.1* Under Assumption 2.1, an easy computation yields  $\frac{\Delta_{ij}}{\Delta_{ij}} > \frac{A_i}{A_i}$ ,  $i, j = 1, 2, 3, i \neq j$ .

**Assumption 2.2**  $a_{ii} > \sum_{j=1, j \neq i}^3 a_{ij}$  and  $a_{ii} > \sum_{j=1, j \neq i}^3 a_{ji}$ ,  $i, j = 1, 2, 3$ .

**Assumption 2.3** There exists a constant  $K$  such that

$$\int_{\mathbb{Z}} [\ln(1 + \gamma_i(u))]^2 \lambda(du) < K, \quad i = 1, 2, 3.$$

Now we give some definitions and lemmas which will be used in stating and proving our main results.

**Definition 2.1** ([20, 24]) Let  $x(t) = (x_1(t), x_2(t), x_3(t))^T \in R_+^3$  be a solution of system (1.2). Then

- (a) the population  $x(t)$  is said to go to extinction if  $\lim_{t \rightarrow \infty} x(t) = 0$ ;
- (b) the population  $x(t)$  is said to be stable in mean if  $\lim_{t \rightarrow \infty} \langle x(t) \rangle = K$  a.s., where  $K$  is a constant.

**Definition 2.2** ([20]) System (1.2) is said to be stochastically permanent if, for any  $\varepsilon \in (0, 1)$ , there exists a pair of positive constants  $\zeta = \zeta(\varepsilon)$  and  $\chi = \chi(\varepsilon)$  such that, for any initial data, the solution  $x(t)$  of (1.2) has the following property:

$$\liminf_{t \rightarrow \infty} P\{|x(t)| \leq \zeta\} \geq 1 - \varepsilon \quad \text{and} \quad \liminf_{t \rightarrow \infty} P\{|x(t)| \geq \chi\} \geq 1 - \varepsilon.$$

**Lemma 2.1** ([20]) Let  $Z(t) \in C[\Omega \times R_+, R_+]$  and Assumption 2.3 hold.

- (i) If there exist some constants  $T > 0, \lambda_0 > 0, \lambda, \sigma_i$ , and  $\lambda_i$  such that, for all  $T > 0$ ,

$$\ln Z(t) \leq \lambda t - \lambda_0 \int_0^t z(s) ds + \sum_{i=1}^n \sigma_i B_i(t) + \sum_{i=1}^n \lambda_i \int_0^t \int_{\mathbb{Z}} \ln(1 + \gamma_i(v)) \tilde{N}(ds, dv) \quad a.s.,$$

then

$$\begin{cases} \langle Z \rangle^* \leq \lambda / \lambda_0 & a.s. \text{ if } \lambda \geq 0, \\ \lim_{t \rightarrow \infty} Z(t) = 0 & a.s. \text{ if } \lambda < 0. \end{cases}$$

- (ii) If there exist some constants  $T > 0, \lambda_0 > 0, \lambda, \sigma_i$ , and  $\lambda_i$  such that, for all  $T > 0$ ,

$$\ln Z(t) \geq \lambda t - \lambda_0 \int_0^t z(s) ds + \sum_{i=1}^n \sigma_i B_i(t) + \sum_{i=1}^n \lambda_i \int_0^t \int_{\mathbb{Z}} \ln(1 + \gamma_i(v)) \tilde{N}(ds, dv), \quad a.s.,$$

then

$$\langle Z \rangle_* \geq \lambda / \lambda_0 \quad a.s.$$

**Lemma 2.2** If Assumption 2.2 holds, then for any given initial data  $\phi = (\phi_1, \phi_2, \phi_3) \in C([-\tau, 0], R_+^3)$ , system (1.2) has a unique solution  $x(t) = (x_1(t), x_2(t), x_3(t))^T \in R_+^3$  a.s.

*Proof* Define

$$V(x(t)) = \sum_{i=1}^3 x_i(t) + \sum_{i=1}^3 \left( \sum_{j=1, j \neq i}^3 a_{ij} \int_{t-\tau_{ij}}^t x_i(s) ds \right).$$

Applying Itô’s formula to  $V(x)$ , we can verify that  $LV(x) \leq 0$ . The remaining proof is very standard. One can refer to [18, 20] and hence we omit it here.  $\square$

*Remark 2.2* Actually, by Corollary 3.5 of Hu *et al.* [8], the condition  $2a_{ii} > \sum_{j \neq i, j=1}^3 a_{ij}$  can ensure that system (1.2) has a unique positive solution for any initial data  $\phi \in C([-\tau, 0], \mathbb{R}_+^3)$ . But for the proof of the latter global attractivity, we conduct our study always under Assumption 2.2 in this paper.

**Lemma 2.3** *Under Assumptions 2.2 and 2.3, the solution  $x(t)$  of (1.2) satisfies*

$$\limsup_{t \rightarrow \infty} \frac{\ln x_i(t)}{\ln t} \leq 1 \quad a.s.,$$

and there exists a positive constant  $K$  such that

$$\limsup_{t \rightarrow \infty} \mathbb{E}(x_i(t)) \leq K, \quad i = 1, 2, 3.$$

*Proof* The proof is motivated by that of Lemma 4 in [20]. By using Itô’s formula to the function  $e^t V(x)$ , where  $V$  is the function defined in the proof of Lemma 2.2, one can get  $\limsup_{t \rightarrow \infty} \mathbb{E} V(x(t)) \leq K$  and  $\limsup_{t \rightarrow \infty} \mathbb{E}|x(t)| \leq K$ . The rest of the proof is similar to that in [20] and hence is omitted.  $\square$

**Lemma 2.4** *If  $\lim_{t \rightarrow \infty} \langle x(t) \rangle = K$ , where  $K$  is a constant, then*

$$\lim_{t \rightarrow \infty} t^{-1} \int_{t-\tau}^t x(s) ds = 0.$$

*Proof* Obviously, we have

$$\lim_{t \rightarrow \infty} t^{-1} \int_{t-\tau}^t x(s) ds = \lim_{t \rightarrow \infty} t^{-1} \left( \int_0^t x(s) ds - \int_0^{t-\tau} x(s) ds \right) = 0. \quad \square$$

### 3 Dynamical analysis of system (1.2)

Firstly, we give our main results on the stability in mean and extinction of species for model (1.2).

**Theorem 3.1** *Let Assumptions 2.1, 2.2, and 2.3 hold. Then the following statements for system (1.2) are valid.*

(i) *If  $A_i > \tilde{A}_i$  for  $i = 1, 2, 3$ , then*

$$\lim_{t \rightarrow \infty} \langle x_i(t) \rangle = \frac{A_i - \tilde{A}_i}{A} \quad \text{for } i = 1, 2, 3.$$

(ii) *For  $i \in \{1, 2, 3\}$ , if  $A_i < \tilde{A}_i$  and  $\Delta_{ij} > \tilde{\Delta}_{ij}$  for all  $j \neq i$ , then*

$$\lim_{t \rightarrow \infty} x_i(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \langle x_j(t) \rangle = \frac{\Delta_{ij} - \tilde{\Delta}_{ij}}{A_{ii}} \quad \text{for } j \neq i.$$

(iii) For  $i \in \{1, 2, 3\}$ , if  $r_i > b_i$  and  $A_j < \tilde{A}_j$  for all  $j \neq i$ , then

$$\lim_{t \rightarrow \infty} \langle x_i(t) \rangle = \frac{\beta_i}{a_{ii}} \quad \text{and} \quad \lim_{t \rightarrow \infty} x_j(t) = 0 \quad \text{for } j \neq i.$$

(iv) If  $r_i < b_i$  ( $i = 1, 2, 3$ ), then  $\lim_{t \rightarrow \infty} x_i(t) = 0$ , while if  $r_i > b_i$  ( $i = 1, 2, 3$ ), then the conclusions of case (i) also hold.

*Proof* For system (1.2), by utilizing Itô’s formula to functions  $\ln x_i(t)$ ,  $i = 1, 2, 3$ , and then integrating both sides of every equation from 0 to  $t$ , we have

$$t^{-1}[\ln x_1(t) - \ln x_1(0)] = \beta_1 - a_{11}\langle x_1(t) \rangle + a_{12}\langle x_2(t) \rangle + a_{13}\langle x_3(t) \rangle + t^{-1}\vartheta_1(t), \tag{3.1}$$

$$t^{-1}[\ln x_2(t) - \ln x_2(0)] = \beta_2 + a_{21}\langle x_1(t) \rangle - a_{22}\langle x_2(t) \rangle + a_{23}\langle x_3(t) \rangle + t^{-1}\vartheta_2(t), \tag{3.2}$$

$$t^{-1}[\ln x_3(t) - \ln x_3(0)] = \beta_3 + a_{31}\langle x_1(t) \rangle + a_{32}\langle x_2(t) \rangle - a_{33}\langle x_3(t) \rangle + t^{-1}\vartheta_3(t), \tag{3.3}$$

where  $\vartheta_i(t) = \sigma_i \omega_i(t) + \int_0^t \int_{\mathbb{Z}} \ln(1 + \gamma_i(u)) \tilde{N}(ds, du) + \check{\vartheta}_i(t)$  for  $i = 1, 2, 3$  with

$$\check{\vartheta}_i(t) = \sum_{j=1, j \neq i}^3 a_{ij} \left( \int_{-\tau_{ij}}^0 x_i(s) ds - \int_{t-\tau_{ij}}^t x_i(s) ds \right).$$

Now we eliminate  $\langle x_1(t) \rangle$  and  $\langle x_2(t) \rangle$  from (3.1)–(3.3) by the elimination method. Note that we can show  $p = A_{13}/A_{33} > 0$  and  $q = -A_{23}/A_{33} > 0$ . Multiplying both sides of (3.1)–(3.3) respectively by  $p$ ,  $q$ , and 1 and adding the resulting three equalities give us

$$\begin{aligned} & t^{-1}[\ln x_3(t) - \ln x_3(0)] + t^{-1}(p[\ln x_1(t) - \ln x_1(0)] + q[\ln x_2(t) - \ln x_2(0)]) \\ &= \beta_1 p + \beta_2 q + \beta_3 - (a_{13}p + a_{23}q - a_{33})\langle x_3(t) \rangle + t^{-1}(\vartheta_1(t)p + \vartheta_2(t)q + \vartheta_3(t)) \\ &= \frac{A_3 - \tilde{A}_3}{A_{33}} - \frac{A}{A_{33}}\langle x_3(t) \rangle + t^{-1}(\vartheta_1(t)p + \vartheta_2(t)q + \vartheta_3(t)). \end{aligned} \tag{3.4}$$

Similarly, we have

$$\begin{aligned} & t^{-1}[\ln x_1(t) - \ln x_1(0)] + t^{-1}(\tilde{p}[\ln x_2(t) - \ln x_2(0)] + \tilde{q}[\ln x_3(t) - \ln x_3(0)]) \\ &= \frac{A_1 - \tilde{A}_1}{A_{11}} - \frac{A}{A_{11}}\langle x_1(t) \rangle + t^{-1}(\vartheta_1(t)\tilde{p} + \vartheta_2(t)\tilde{q} + \vartheta_3(t)) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} & t^{-1}[\ln x_2(t) - \ln x_2(0)] + t^{-1}(\tilde{p}[\ln x_1(t) - \ln x_1(0)] + \tilde{q}[\ln x_3(t) - \ln x_3(0)]) \\ &= \frac{A_2 - \tilde{A}_2}{A_{22}} - \frac{A}{A_{22}}\langle x_2(t) \rangle + t^{-1}(\vartheta_1(t)\tilde{p} + \vartheta_2(t)\tilde{q} + \vartheta_3(t)), \end{aligned} \tag{3.6}$$

where

$$\tilde{p} = \frac{-A_{21}}{A_{11}} > 0, \quad \tilde{q} = \frac{A_{31}}{A_{11}} > 0, \quad \bar{p} = \frac{-A_{12}}{A_{22}} > 0, \quad \bar{q} = \frac{-A_{32}}{A_{22}} > 0.$$

Moreover, it follows from (3.1)–(3.3) and Lemma 2.1 that, for sufficiently large  $t$ ,

$$t^{-1}[\ln x_1(t) - \ln x_1(0)] \leq \beta_1 + \varepsilon + a_{12}\langle x_2(t) \rangle^* + a_{13}\langle x_3(t) \rangle^* - a_{11}\langle x_1(t) \rangle + t^{-1}(\vartheta_1(t)\tilde{p} + \vartheta_2(t)\tilde{q} + \vartheta_3(t)), \tag{3.7}$$

$$t^{-1}[\ln x_2(t) - \ln x_2(0)] \leq \beta_2 + \varepsilon + a_{21}\langle x_1(t) \rangle^* + a_{23}\langle x_3(t) \rangle^* - a_{22}\langle x_2(t) \rangle + t^{-1}(\vartheta_1(t)\tilde{p} + \vartheta_2(t)\tilde{q} + \vartheta_3(t)), \tag{3.8}$$

$$t^{-1}[\ln x_3(t) - \ln x_3(0)] \leq \beta_3 + \varepsilon + a_{31}\langle x_1(t) \rangle^* + a_{32}\langle x_2(t) \rangle^* - a_{33}\langle x_3(t) \rangle + t^{-1}(\vartheta_1(t)p + \vartheta_2(t)q + \vartheta_3(t)). \tag{3.9}$$

(i) First assume that  $A_3 > \tilde{A}_3$ . Then applying Lemma 2.3 to equality (3.4) yields that, for arbitrary  $\varepsilon > 0$ , there exists  $T > 0$  such that, for all  $t > T$ ,

$$t^{-1}(p[\ln x_1(t) - \ln x_1(0)] + q[\ln x_2(t) - \ln x_2(0)]) \leq pt^{-1} \ln x_1(t) + qt^{-1} \ln x_2(t) + \varepsilon \leq \varepsilon.$$

Substituting it into (3.4) gives

$$t^{-1}[\ln x_3(t) - \ln x_3(0)] \geq \frac{A_3 - \tilde{A}_3}{A_{33}} - \varepsilon - \frac{A}{A_{33}}\langle x_3(t) \rangle + t^{-1}(\vartheta_1(t)p + \vartheta_2(t)q + \vartheta_3(t)).$$

Since  $A_3 > \tilde{A}_3$ , we can choose  $\varepsilon > 0$  small enough such that  $A_3 - \tilde{A}_3 - \varepsilon > 0$ . Then Lemma 2.1 implies that

$$\langle x_3(t) \rangle_* \geq \frac{A_3 - \tilde{A}_3}{A}.$$

Similarly, if  $A_1 > \tilde{A}_1$  and  $A_2 > \tilde{A}_2$ , then we can get  $\langle x_1(t) \rangle_* \geq \frac{A_1 - \tilde{A}_1}{A}$  and  $\langle x_2(t) \rangle_* \geq \frac{A_2 - \tilde{A}_2}{A}$ , respectively.

We claim that

$$\begin{aligned} \beta_1 + \varepsilon + a_{12}\langle x_2(t) \rangle^* + a_{13}\langle x_3(t) \rangle^* &> 0, \\ \beta_2 + \varepsilon + a_{21}\langle x_1(t) \rangle^* + a_{23}\langle x_3(t) \rangle^* &> 0, \\ \beta_3 + \varepsilon + a_{31}\langle x_1(t) \rangle^* + a_{32}\langle x_2(t) \rangle^* &> 0. \end{aligned}$$

Otherwise, by Lemma 2.1, we can obtain that  $\langle x_1(t) \rangle_* = \langle x_2(t) \rangle_* = \langle x_3(t) \rangle_* = 0$ , which contradicts with the above just obtained results. This proves the claim. Again applying Lemma 2.1 to (3.7)–(3.9) gives us

$$\begin{aligned} \langle x_1(t) \rangle^* &\leq \frac{\beta_1 + \varepsilon + a_{12}\langle x_2(t) \rangle^* + a_{13}\langle x_3(t) \rangle^*}{a_{11}}, \\ \langle x_2(t) \rangle^* &\leq \frac{\beta_2 + \varepsilon + a_{21}\langle x_1(t) \rangle^* + a_{23}\langle x_3(t) \rangle^*}{a_{22}}, \\ \langle x_3(t) \rangle^* &\leq \frac{\beta_3 + \varepsilon + a_{31}\langle x_1(t) \rangle^* + a_{32}\langle x_2(t) \rangle^*}{a_{33}}. \end{aligned}$$



Letting  $\varepsilon \rightarrow 0$  and solving the resultants produce

$$\langle x_1(t) \rangle^* \leq \frac{A_1 - \tilde{A}_1}{A}, \quad \langle x_2(t) \rangle^* \leq \frac{A_2 - \tilde{A}_2}{A}, \quad \langle x_3(t) \rangle^* \leq \frac{A_3 - \tilde{A}_3}{A}.$$

To summarize, we have shown

$$\lim_{t \rightarrow \infty} \langle x_1(t) \rangle = \frac{A_1 - \tilde{A}_1}{A}, \quad \lim_{t \rightarrow \infty} \langle x_2(t) \rangle = \frac{A_2 - \tilde{A}_2}{A}, \quad \lim_{t \rightarrow \infty} \langle x_3(t) \rangle = \frac{A_3 - \tilde{A}_3}{A}.$$

(ii) We only consider the case where  $A_1 < \tilde{A}_1$ ,  $\Delta_{12} > \tilde{\Delta}_{12}$ , and  $\Delta_{13} > \tilde{\Delta}_{13}$  as the other two cases can be dealt with similarly. In this case, first by case (i),  $\lim_{t \rightarrow \infty} x_1(t) = 0$ . Next, from (3.2) and (3.3), we have

$$t^{-1}[\ln x_2(t) - \ln x_2(0)] = \beta_2 - a_{22}\langle x_2(t) \rangle + a_{23}\langle x_3(t) \rangle + t^{-1}\vartheta_2(t), \tag{3.10}$$

$$t^{-1}[\ln x_3(t) - \ln x_3(0)] = \beta_3 + a_{32}\langle x_2(t) \rangle - a_{33}\langle x_3(t) \rangle + t^{-1}\vartheta_3(t). \tag{3.11}$$

Then adding (3.10) multiplied by  $a_{33}$  and (3.11) multiplied by  $a_{23}$  yields

$$\begin{aligned} & t^{-1}a_{33}[\ln x_2(t) - \ln x_2(0)] + t^{-1}a_{23}[\ln x_3(t) - \ln x_3(0)] \\ &= \beta_2 a_{33} + \beta_3 a_{23} - (a_{22}a_{33} - a_{32}a_{23})\langle x_2(t) \rangle + t^{-1}(a_{33}\vartheta_2(t) + a_{23}\vartheta_3(t)). \end{aligned}$$

According to Lemma 2.3, for arbitrarily small  $\varepsilon > 0$ , there exists  $T > 0$  such that  $t^{-1}a_{23}[\ln x_3(t) - \ln x_3(0)] \leq \varepsilon$  for any  $t > T$ . It follows that

$$\begin{aligned} & t^{-1}a_{33}[\ln x_2(t) - \ln x_2(0)] \\ & \geq \beta_2 a_{33} + \beta_3 a_{23} - \varepsilon - (a_{22}a_{33} - a_{32}a_{23})\langle x_2(t) \rangle + t^{-1}(a_{33}\vartheta_2(t) + a_{23}\vartheta_3(t)). \end{aligned} \tag{3.12}$$

Since  $\tilde{\Delta}_{12} < \Delta_{12}$ ,  $\beta_2 a_{33} + \beta_3 a_{23} - \varepsilon > 0$  for  $\varepsilon > 0$  small enough. By Lemma 2.1 again and letting  $\varepsilon \rightarrow 0$ , it follows from (3.12) that

$$\langle x_2(t) \rangle_* \geq \frac{\Delta_{12} - \tilde{\Delta}_{12}}{A_{11}}.$$

In the same manner, we can obtain

$$\langle x_3(t) \rangle_* \geq \frac{\Delta_{13} - \tilde{\Delta}_{13}}{A_{11}}.$$

Moreover, by (3.10) and (3.11), we have

$$t^{-1}[\ln x_2(t) - \ln x_2(0)] \leq \beta_2 - a_{22}\langle x_2(t) \rangle + a_{23}\langle x_3(t) \rangle^* + t^{-1}\vartheta_2(t) \tag{3.13}$$

and

$$t^{-1}[\ln x_3(t) - \ln x_3(0)] \leq \beta_3 + a_{32}\langle x_2(t) \rangle^* - a_{33}\langle x_3(t) \rangle + t^{-1}\vartheta_3(t). \tag{3.14}$$

Also, by Lemma 2.1 again, we can get from (3.13) and (3.14) that

$$\langle x_2(t) \rangle^* \leq \frac{\beta_2 + a_{23} \langle x_3(t) \rangle^*}{a_{22}}, \quad \langle x_3(t) \rangle^* \leq \frac{\beta_3 + a_{32} \langle x_2(t) \rangle^*}{a_{33}}.$$

Solving these two inequalities produces

$$\langle x_2(t) \rangle^* \leq \frac{\Delta_{12} - \tilde{\Delta}_{12}}{A_{11}}, \quad \langle x_3(t) \rangle^* \leq \frac{\Delta_{13} - \tilde{\Delta}_{13}}{A_{11}}.$$

To summarize, we have shown

$$\lim_{t \rightarrow \infty} \langle x_2(t) \rangle = \frac{\beta_2 a_{33} + \beta_3 a_{23}}{a_{22} a_{33} - a_{23} a_{32}} = \frac{\Delta_{12} - \tilde{\Delta}_{12}}{A_{11}}$$

and

$$\lim_{t \rightarrow \infty} \langle x_3(t) \rangle = \frac{\beta_2 a_{32} + \beta_3 a_{22}}{a_{22} a_{33} - a_{23} a_{32}} = \frac{\Delta_{13} - \tilde{\Delta}_{13}}{A_{11}}.$$

(iii) Again, we only consider the case where  $A_1 < \tilde{A}_1$ ,  $A_2 < \tilde{A}_2$ , and  $r_3 > b_3$  as the other two cases can be dealt with similarly. When  $A_1 < \tilde{A}_1$  and  $A_2 < \tilde{A}_2$ , we have  $\lim_{t \rightarrow \infty} x_1(t) = 0$  and  $\lim_{t \rightarrow \infty} x_2(t) = 0$  respectively by (3.5) and (3.6). Then (3.3) becomes

$$t^{-1} [\ln x_3(t) - \ln x_3(0)] = \beta_3 - a_{33} \langle x_3(t) \rangle + t^{-1} \vartheta_3(t).$$

In view of  $\beta_3 > 0$ , by employing Lemma 2.1, we have  $\lim_{t \rightarrow \infty} \langle x_3(t) \rangle = \beta_3 / a_{33}$ .

(iv) Obviously, when  $r_i - b_i < 0$ , by an easy computation, we have  $A_i - \tilde{A}_i < 0$ . By Lemma 2.1, it follows from (3.1)–(3.3) that  $\lim_{t \rightarrow \infty} x_1(t) = \lim_{t \rightarrow \infty} x_2(t) = \lim_{t \rightarrow \infty} x_3(t) = 0$ , which means that all species go extinct. When  $r_i - b_i > 0$ , by a similar computation, we have  $A_i - \tilde{A}_i > 0$ ,  $i = 1, 2, 3$ . This completes the proof.  $\square$

Next we consider the stochastic permanence of system (1.2).

**Theorem 3.2** *Then model (1.2) is stochastically permanent provided that the following assumption holds.*

**Assumption 3.1**  $\min_{i=1,2,3} r_i - \max_{i=1,2,3} \sigma_i^2 - \int_{\mathbb{Z}} \max_{i=1,2,3} (\gamma_i - \ln(1 + \gamma_i(u))) \lambda(du) > 0$ .

*Proof* Define

$$V_1(x) = \frac{1}{x_1(t) + x_2(t) + x_3(t)}.$$

With the help of Itô’s formula, we have

$$\begin{aligned} dV_1(x) = & -V_1(x)^2 \left[ \sum_{i=1}^3 x_i(t) \left( r_i - c_i - a_{ii} x_i(t) + \sum_{j=1, j \neq i}^3 a_{ij} x_j(t - \tau_{ij}) \right) \right] + V_1^3(x) \sum_{i=1}^3 \sigma_i^2 x_i^2(t) \\ & + \int_{\mathbb{Z}} \left( \frac{1}{\sum_{i=1}^3 x_i(t)(1 + \gamma_i(u))} - V_1(x) - V_1^2(x) \sum_{i=1}^3 x_i(t) \gamma_i(u) \right) \lambda(du) dt \end{aligned}$$

$$-V_1^2(x) \sum_{i=1}^3 \sigma_i x_i(t) d\omega_i(t) + \int_{\mathbb{Z}} \left( \frac{1}{\sum_{i=1}^3 x_i(t)(1 + \gamma_i(u))} - V_1(x) \right) \tilde{N}(du, dt).$$

Then, for an arbitrarily small positive constant  $\iota > 0$ ,

$$\begin{aligned} dV_1^\iota(x) &= \iota V_1^{\iota-1}(x) \left\{ -V_1(x)^2 \left[ \sum_{i=1}^3 x_i(t) \left( r_i - c_i - a_{ii}x_i(t) + \sum_{j=1, j \neq i}^3 a_{ij}x_j(t - \tau_{ij}) \right) \right] \right. \\ &\quad + V_1^3(x) \sum_{i=1}^3 \sigma_i^2 x_i^2(t) \\ &\quad + \left. \int_{\mathbb{Z}} \left( \frac{1}{\sum_{i=1}^3 x_i(t)(1 + \gamma_i(u))} - V_1(x) + V_1^2(x) \left( \sum_{i=1}^3 x_i(t)\gamma_i(u) \right) \right) \lambda(du) \right\} dt \\ &\quad + \frac{\iota(\iota - 1)}{2} V_1^{\iota+2}(x) \sum_{i=1}^3 \sigma_i^2 x_i^2(t) dt - \iota V_1^{\iota+1}(x) \sum_{i=1}^3 \sigma_i x_i(t) d\omega_i(t) \\ &\quad + \int_{\mathbb{Z}} \left[ \frac{1}{\sum_{i=1}^3 [x_i(t)(1 + \gamma_i)]^\iota} - V_1^\iota(x) \right. \\ &\quad \left. - \iota V_1^{\iota-1}(x) \left( \frac{1}{\sum_{i=1}^3 x_i(t)(1 + \gamma_i(u))} - V_1(x) \right) \right] \lambda(du) dt \\ &\quad + \int_{\mathbb{Z}} \left( \frac{1}{\sum_{i=1}^3 [x_i(t)(1 + \gamma_i)]^\iota} - V_1^\iota(x) \right) \tilde{N}(du, dt) \\ &\leq \iota V_1^{\iota-2}(x) \left\{ -V_1^3(x) \left( \sum_{i=1}^3 (r_i - a_{ii}x_i(t))x_i(t) - V_1(x) \sum_{i=1}^3 \sigma_i^2 x_i^2(t) \right) \right. \\ &\quad - \frac{\iota - 1}{2} V_1(x) \sum_{i=1}^3 \sigma_i^2 x_i^2(t) \\ &\quad - V_1(x) \int_{\mathbb{Z}} \left( \frac{1}{\sum_{i=1}^3 x_i(t)(1 + \gamma_i(u))} - V_1(x) - V_1^2(x) \sum_{i=1}^3 x_i(t)r_i(u) \right) \lambda(du) \Big\} dt \\ &\quad - \iota V_1^{\iota+1}(x) \sum_{i=1}^3 \sigma_i x_i(t) d\omega_i(t) + \int_{\mathbb{Z}} \left( \frac{1}{\sum_{i=1}^3 [x_i(t)(1 + \gamma_i)]^\iota} - V_1^\iota(x) \right) \tilde{N}(du, dt) \\ &\quad + \int_{\mathbb{Z}} \left( \frac{1}{\sum_{i=1}^3 x_i(t)(1 + \gamma_i(u))^\iota} - V_1^\iota(x) \right. \\ &\quad \left. - \iota V_1^{\iota-1}(x) \left( \frac{1}{\sum_{i=1}^3 x_i(t)(1 + \gamma_i)} - V_1(x) \right) \right) \lambda(du) dt \\ &\leq \iota V_1^{\iota-2}(x) G(x) dt - \iota V_1^{\iota+1}(x) \sum_{i=1}^3 \sigma_i x_i(t) d\omega_i(t) \\ &\quad + \int_{\mathbb{Z}} \left( \frac{1}{\sum_{i=1}^3 [x_i(t)(1 + \gamma_i(u))]^\iota} - V_1^\iota(x) \right) \tilde{N}(du, dt), \end{aligned}$$

where

$$G(x) = -V_1^2(x) \left[ \min_{i=1,2,3} r_i - \frac{1 + \iota}{2} \max_{i=1,2,3} \sigma_i^2 - \int_{\mathbb{Z}} \max_{i=1,2,3} \gamma_i(u) \lambda(du) \right]$$

$$- \int_{\mathbb{Z}} \left( \frac{1}{\iota \min_{i=1,2,3}(1 + \gamma_i(u)^\iota)} - \frac{1}{\iota} \right) \lambda(du) \Big] + \max_{i=1,2,3} a_{ii} V_1(x).$$

Note that, by the definition of derivative, we can derive that

$$\begin{aligned} & \lim_{t \rightarrow \infty} \left( \iota \max_{i=1,2,3} (\sigma_i^2) / 2 + \int_{\mathbb{Z}} \left( \frac{1}{\iota (\min_{i=1,2,3}(1 + \gamma_i(u))^\iota)} - \frac{1}{\iota} \right) \lambda(du) dt \right) \\ &= \int_{\mathbb{Z}} \ln \frac{1}{\min_{i=1,2,3}(1 + \gamma_i(u))} \lambda(du) \\ &= - \int_{\mathbb{Z}} \ln \left[ \min_{i=1,2,3} (1 + \gamma_i(u)) \right] \lambda(du). \end{aligned}$$

By Assumption 3.1, we can choose  $\iota$  small enough such that

$$\min_{i=1,2,3} r_i - \frac{1 + \iota}{2} \max_{i=1,2,3} \sigma_i^2 - \int_{\mathbb{Z}} \max_{i=1,2,3} \gamma_i(u) \lambda(du) - \int_{\mathbb{Z}} \left( \frac{1}{\iota \min_{i=1,2,3}(1 + \gamma_i(u)^\iota)} - \frac{1}{\iota} \right) \lambda(du) > 0.$$

Hence  $G(x)$  is bounded above. Then let  $\eta > 0$  such that

$$\begin{aligned} \frac{\eta}{\iota} &< \min_{i=1,2,3} r_i - \frac{1 + \iota}{2} \max_{i=1,2,3} \sigma_i^2 - \int_{\mathbb{Z}} \max_{i=1,2,3} \gamma_i(u) \lambda(du) \\ &- \int_{\mathbb{Z}} \left( \frac{1}{\iota \min_{i=1,2,3}(1 + \gamma_i(u)^\iota)} - \frac{1}{\iota} \right) \lambda(du). \end{aligned} \tag{3.15}$$

Define  $V_2(x) = e^{\eta t} V_1^\iota(x)$ . Then

$$\begin{aligned} dV_2(x) &\leq \eta e^{\eta t} V_1^\iota dt + e^{\eta t} \iota V_1^{\iota-2} G(x) dt - \iota e^{\eta t} V_1^{\iota+1} \sum_{i=1}^3 \sigma_i x_i(t) d\omega_i(t) \\ &+ e^{\eta t} \int_{\mathbb{Z}} \left( \frac{1}{\sum_{i=1}^3 [x_i(t)(1 + \gamma_i)]^\iota} - V_1^\iota(x) \right) \tilde{N}(du, dt) \\ &= e^{\eta t} H(x(t)) dt - \iota e^{\eta t} V_1^{\iota+1} \sum_{i=1}^3 \sigma_i x_i(t) d\omega_i(t) \\ &+ e^{\eta t} \int_{\mathbb{Z}} \left( \frac{1}{\sum_{i=1}^3 [x_i(t)(1 + \gamma_i)]^\iota} - V_1^\iota(x) \right) \tilde{N}(du, dt), \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} H(x(t)) &= \iota V_1^{\iota-2}(x) \left\{ -V_1^2(x) \left[ \min_{i=1,2,3} r_i - \frac{1 + \iota}{2} \max_{i=1,2,3} \sigma_i^2 - \max_{i=1,2,3} \gamma_i(u) \lambda(du) \right. \right. \\ &\quad \left. \left. - \int_{\mathbb{Z}} \left( \frac{1}{\iota \min_{i=1,2,3}(1 + \gamma_i(u)^\iota)} - \frac{1}{\iota} \right) \lambda(du) - \eta / \iota \right] \right\} + \max_{i=1,2,3} a_{ii} V_1(x). \end{aligned}$$

By (3.15), we know that  $H(x(t))$  is bounded above, that is,  $\sup_{x \in \mathbb{R}_+^3} H(x) = K$  for some constant  $K$ . Then integrate both sides of inequality (3.16) from 0 to  $t$  and take the expectations to get

$$\mathbb{E}(V_2(x)) = \mathbb{E}(e^{\eta t} V_1^\iota(x)) \leq V_1^\iota(0) + \frac{K(e^{\eta t} - 1)}{\eta},$$

which gives

$$\lim_{t \rightarrow \infty} \mathbb{E}V_1^t(t) \leq \frac{K}{\eta} \doteq \check{\mathcal{H}}.$$

Clearly

$$(x_1(t) + x_2(t) + x_3(t))^t \leq \left(3 \max_{i=1,2,3} x_i(t)\right)^t \leq 3 \left(\max_{i=1,2,3} x_i^2(t)\right)^{t/2} \leq 3^t |x(t)|^t.$$

For any given  $\varepsilon > 0$ , let  $\epsilon = (\frac{\varepsilon}{\check{H}})^{1/t}$ . By Chebyshev’s inequality, we have

$$P(|x(t)| < \epsilon) = P\left(\frac{1}{|x(t)|^t} > \frac{1}{\epsilon^t}\right) \leq \varepsilon,$$

which means  $\limsup_{t \rightarrow \infty} P\{|x(t)| \geq \epsilon\} \geq 1 - \varepsilon$ .

Next we prove that there exists a constant  $M > 0$  such that  $P(|x| \leq M) \geq 1 - \varepsilon$  for  $\varepsilon > 0$  small enough. Define

$$U(x) = \sum_{i=1}^3 x_i(t) + \sum_{j=1, j \neq i}^3 a_{ij} \int_{t-\tau_{ij}}^t x_i(s) ds.$$

By using Itô’s formula to the function  $e^t U(t)$ , we get

$$\begin{aligned} d(e^t U(x)) &= e^t U(x) dt + e^t dU \\ &= e^t \left( \sum_{i=1}^3 x_i(t) + \sum_{i=1}^3 \sum_{j=1, j \neq i}^3 x_i(r_i - c_i - a_{ii}x_i(t) + a_{ij}x_j(t - \tau_{ij}(t))) dt \right. \\ &\quad \left. + e^t \sum_{i=1}^3 \left( \sigma_i x_i(t) d\omega_i(t) + x_i(t) \int_{\mathbb{Z}} \gamma_i(u) \tilde{N}(du, dt) \right) \right) \\ &\leq e^t \left( \sum_{i=1}^3 x_i(t) + a_{12} \frac{x_1^2(t) + x_2^2(t)}{2} + a_{13} \frac{x_1^2(t) + x_3^2(t)}{2} + a_{21} \frac{x_1^2(t) + x_2^2(t)}{2} \right. \\ &\quad \left. + a_{23} \frac{x_3^2(t) + x_2^2(t)}{2} + a_{31} \frac{x_1^2(t) + x_3^2(t)}{2} + a_{32} \frac{x_3^2(t) + x_2^2(t)}{2} \right) dt \\ &\quad + e^t \sum_{i=1}^3 \left( \sigma_i x_i(t) d\omega_i(t) + x_i(t) \int_{\mathbb{Z}} \gamma_i(u) \tilde{N}(du, dt) \right) \\ &\leq Ke^t dt + e^t \sum_{i=1}^3 \left( \sigma_i x_i(t) d\omega_i(t) + x_i(t) \int_{\mathbb{Z}} \gamma_i(u) \tilde{N}(du, dt) \right). \end{aligned}$$

Computing the expectation  $\mathbb{E}(e^t U(x(t)))$  gives

$$\limsup_{t \rightarrow \infty} \mathbb{E}(U(x(t))) \leq K \quad \text{and} \quad \limsup_{t \rightarrow \infty} \mathbb{E}(x(t)) \leq K.$$

By use of Chebyshev’s inequality again, we can obtain that  $P(|x| \leq M) \geq 1 - \varepsilon$  for  $\varepsilon > 0$  small enough. This completes the proof.  $\square$

*Remark 3.1* It is clear from Assumption 3.1 that the Lévy noise is harmful to the permanence of system (1.2). Furthermore, for the case of one species, our obtained result is in accordance with that of Liu and Wang [20].

#### 4 Stability in distribution

**Theorem 4.1** *Model (1.2) is stable in distribution if Assumption 2.2 holds.*

*Proof* Firstly, let  $x(t) = x(t, \phi)$  and  $\check{x}(t) = x(t, \check{\phi})$  be any two solutions of model (1.2) with initial data  $\phi \in C([-\tau, 0], R_+^3)$  and  $\check{\phi} \in C([-\tau, 0], R_+^3)$ , respectively. For  $i = 1, 2, 3$ , we denote  $\bar{x}_i(t) = x_i(t) - \check{x}_i(t)$  and prove  $\lim_{t \rightarrow \infty} \mathbb{E}|\bar{x}_i(t)| = 0$ , that is, system (1.2) is globally attractive.

Define

$$V(t) = \sum_{i=1}^3 |\ln x_i(t) - \ln \check{x}_i(t)| + \sum_{i=1}^3 \sum_{j=1, j \neq i}^3 a_{ij} \int_{t-\tau_{ij}}^t |\bar{x}_j(t)|.$$

By computing the upper right derivative of  $V(t)$ , we have

$$\begin{aligned} D^+ V(t) &= \sum_{i=1}^3 \operatorname{sgn} \bar{x}_i(t) \left( \sum_{j=1, j \neq i}^3 (a_{ij} \bar{x}_j(t - \tau_{ij}) - a_{ii} \bar{x}_i(t)) \right) dt \\ &\quad + \sum_{i=1}^3 \sum_{j=1, j \neq i}^3 a_{ij} |\bar{x}_j(t)| dt - \sum_{i=1}^3 \sum_{j=1, j \neq i}^3 a_{ij} |\bar{x}_j(t - \tau_{ij})| dt \\ &\leq - \sum_{i=1}^3 a_{ii} \bar{x}_i(t) dt + \sum_{j=1, j \neq i}^3 a_{ij} \bar{x}_j(t) dt. \end{aligned}$$

Integrating both sides of the above inequality from 0 to  $t$  and taking expectations give us

$$\begin{aligned} \mathbb{E}(V(t)) &\leq V(0) - (a_{11} - a_{12} - a_{13}) \int_0^t \mathbb{E} \bar{x}_1(s) ds \\ &\quad - (a_{22} - a_{21} - a_{23}) \int_0^t \mathbb{E} \bar{x}_2(s) ds \\ &\quad - (a_{33} - a_{31} - a_{32}) \int_0^t \mathbb{E} \bar{x}_3(s) ds, \end{aligned}$$

which implies that

$$\begin{aligned} &(a_{11} - a_{12} - a_{13}) \int_0^t \mathbb{E} \bar{x}_1(s) ds \\ &\quad + (a_{22} - a_{21} - a_{23}) \int_0^t \mathbb{E} \bar{x}_2(s) ds \\ &\quad + (a_{33} - a_{31} - a_{32}) \int_0^t \mathbb{E} \bar{x}_3(s) ds \\ &\leq V(0) \\ &< \infty. \end{aligned}$$

By Assumption 2.2, we have

$$\mathbb{E}|\bar{x}_i(t)| \in L^1[0, \infty).$$

Moreover, by (1.2), we obtain

$$\begin{aligned} \mathbb{E}x_1(t) &= x_1(0) + \int_0^t [\mathbb{E}\beta_1 x_1(s) - a_{11}\mathbb{E}(x_1(s))^2 \\ &\quad - a_{12}\mathbb{E}x_1(s)x_2(s - \tau_{12}) - a_{13}\mathbb{E}x_1(s)x_3(s - \tau_{13})] ds. \end{aligned}$$

It implies that  $\mathbb{E}x_1(t)$  is differentiable. This, combined with Lemma 2.3, gives

$$\begin{aligned} \frac{d\mathbb{E}x_1(t)}{dt} &= \mathbb{E}\beta_1 x_1(s) - a_{11}\mathbb{E}(x_1(s))^2 - a_{12}\mathbb{E}x_1(s)x_2(s - \tau_{12}) - a_{13}\mathbb{E}x_1(s)x_3(s - \tau_{13}) \\ &\leq \mathbb{E}\beta_1 x_1(s) \\ &\leq \beta_1 K. \end{aligned}$$

Hence  $\mathbb{E}x_1(t)$  is uniformly continuous. Similarly,  $\mathbb{E}x_2(t)$  and  $\mathbb{E}x_3(t)$  are also uniformly continuous. Now, it follows from Barbalat lemma [31] that

$$\lim_{t \rightarrow \infty} \mathbb{E}|\bar{x}_i(t)| = 0, \quad i = 1, 2, 3.$$

Secondly, denote by  $p(t, \phi, dy)$  the transition probability of the process  $x(t, \phi)$  and denote by  $P(t, \phi, R_+^3)$  the probability of event  $x(t, \phi) \in R_+^3$  with the initial data  $\phi \in C([-\tau, 0]; R_+^3)$ . Let  $\mathbb{P}(C([-\tau, 0]; R_+^3))$  be the space of all probability measure on  $C([-\tau, 0]; R_+^3)$ . Then it follows from Lemma 2.3 and Chebyshev's inequality that the family  $\{p(t, \phi, dy)\}$  is tight, that is, for any arbitrarily given  $\varepsilon > 0$ , there exists a compact subset  $\mathcal{K} \subseteq R_+^3$  such that  $P(t, \phi, \mathcal{K}) \geq 1 - \varepsilon$ . We prove that the series  $\{p(x(t, \phi, R_+^3))\}$  is Cauchy in  $\mathbb{P}(C([-\tau, 0]; R_+^3))$ .

For any  $P_1, P_2 \in \mathbb{P}(C([-\tau, 0]; R_+^3))$ , we define the metric of  $P_1$  and  $P_2$  by

$$d_F(P_1, P_2) = \sup_{f \in F} \left| \int_{R_+^3} f(x)P_1(dx) - \int_{R_+^3} f(x)P_2(dx) \right|,$$

where

$$F = \{f : C([-\tau, 0]; R_+^3) \rightarrow R \mid |f(x_1) - f(x_2)| \leq \|x_1 - x_2\|, f(\cdot) \leq 1\}.$$

For any  $f \in F$  and  $s, t > 0$ , we have

$$\begin{aligned} &|\mathbb{E}f(x(t + s, \phi)) - \mathbb{E}f(x(t, \phi))| \\ &= |\mathbb{E}[\mathbb{E}f(x(t + s, \phi)|\mathcal{F}_s)] - \mathbb{E}f(x(t, \phi))| \\ &= \left| \int_{R_+^3} \mathbb{E}f(x(t, \xi))p(s, \phi, d\xi) - \mathbb{E}f(x(t, \phi)) \right| \\ &\leq \int_{R_+^3} |\mathbb{E}f(x(t, \xi)) - \mathbb{E}f(x(t, \phi))| p(s, \phi, d\xi) \end{aligned}$$

$$\begin{aligned}
 &= \int_{\bar{U}_K} |\mathbb{E}f(x(t, \xi)) - \mathbb{E}f(x(t, \phi))| p(s, \phi, d\xi) \\
 &\quad + \int_{R_+^3/\bar{U}_K} |\mathbb{E}f(x(t, \xi)) - \mathbb{E}f(x(t, \phi))| p(s, \phi, d\xi),
 \end{aligned}$$

where  $\bar{U}_K = \{x \in R_+^3 : |x| \leq K\}$ . Because of the tightness of  $\{p(t, \phi, d\gamma)\}$ , we can find sufficiently large  $K$  such that  $P(s, \phi, R_+^3/\bar{U}_K) < \varepsilon$  for any  $s > 0$ . Then

$$\int_{R_+^3/\bar{U}_K} |\mathbb{E}f(x(t, \xi)) - \mathbb{E}f(x(t, \phi))| p(s, \phi, d\xi) \leq 2P(s, \phi, R_+^3/\bar{U}_K) \leq 2\varepsilon.$$

By the global attractivity, for arbitrarily given  $\varepsilon > 0$  and sufficiently large  $t$ , we have

$$\begin{aligned}
 &\int_{\bar{U}_K} |\mathbb{E}f(x(t, \xi)) - \mathbb{E}f(x(t, \phi))| p(s, \phi, d\xi) \\
 &\leq \int_{\bar{U}_K} \mathbb{E}[\|x(t, \xi) - x(t, \phi)\|] p(s, \phi, d\xi) \\
 &\leq \int_{\bar{U}_K} \varepsilon p(s, \phi, d\xi) \\
 &= \varepsilon P(s, \phi, \bar{U}_K) \\
 &\leq \varepsilon.
 \end{aligned}$$

Consequently, for any  $s > 0$  and sufficiently large  $t$ ,

$$d_F(p(t + s, \phi, \cdot), p(t, \phi, \cdot)) = \sup_{f \in F} |\mathbb{E}f(x(t + s, \phi)) - \mathbb{E}f(x(t, \phi))| \leq 3\varepsilon.$$

Hence, for any initial data  $\phi \in C([-\tau, 0]; R_+^3)$ ,  $\{p(t, \phi, \cdot) : t \geq 0\}$  is Cauchy in  $\mathbb{P}(C([-\tau, 0]; R_+^3))$  with respect to the metric  $d_F$  and there exists a probability measure  $\nu(\cdot)$  such that, for fixed initial data  $\psi \in C([-\tau, 0]; R_+^3)$ ,

$$\lim_{t \rightarrow \infty} d_F(p(t, \psi, \cdot), \nu(\cdot)) = 0.$$

Finally, by the triangle inequality,

$$d_F(p(t, \phi, \cdot), \nu(\cdot)) \leq d_F(p(t, \phi, \cdot), p(t, \psi, \cdot)) + d_F(p(t, \psi, \cdot), \nu(\cdot)).$$

In view of the global attractivity, we can deduce that

$$\begin{aligned}
 d_F(p(t, \phi, \cdot), p(t, \psi, \cdot)) &= \sup_{f \in F} |\mathbb{E}f(x(t, \phi)) - \mathbb{E}f(x(t, \psi))| \\
 &\leq \sup_{f \in F} \mathbb{E}(f(x(t, \phi)) - f(x(t, \psi))) \\
 &\leq \mathbb{E}\|x(t, \phi) - x(t, \psi)\| \\
 &\rightarrow 0.
 \end{aligned}$$

Therefore, for any  $\phi \in C([-\tau, 0], R_+^3)$ , we have  $\lim_{t \rightarrow \infty} d_F(p(t, \phi, \cdot), \nu(\cdot)) = 0$ , which is the required assertion. □



### 5 Optimal harvesting effort

Denote

$$L = (r_1 - \sigma_1^2/2 - \kappa_1, r_2 - \sigma_2^2/2 - \kappa_2, r_3 - \sigma_3^2/2 - \kappa_3)^T,$$

$$G = (g_1, g_2, g_3)^T = [AA^{-1} + I]^{-1}L,$$

where  $\kappa_i = \int_{\mathbb{Z}} (\gamma_i(u) - \ln(1 + \gamma_i(u)))\lambda(du)$ ,  $i = 1, 2, 3$ . As for the OHE of (1.2), we have the following result.

**Theorem 5.1** *Under Assumptions 2.1–2.3, the following statements are true.*

- (i) *Suppose that  $\beta_1|_{C=G} > 0, \beta_2|_{C=G} > 0, \beta_3|_{C=G} > 0, A_1|_{C=G} > \tilde{A}_1|_{C=G}, A_2|_{C=G} > \tilde{A}_2|_{C=G}, A_3|_{C=G} > \tilde{A}_3|_{C=G}$ . Then the OHE is  $C^* = G$  and the MESY is  $Y^* = G^T A^{-1}(L - G)$ .*
- (ii) *If conditions of (i) do not hold, then the OHE does not exist.*

*Proof* Let

$$\mathcal{E} = \{C = (c_1, c_2, c_3)^T \in \mathbb{R}^3 | \beta_i > 0, A_i > \tilde{A}_i, c_i \geq 0, i = 1, 2, 3\}.$$

If  $C^*$  exists, then  $C^* \in \mathcal{E}$ .

Firstly we prove (i). Obviously  $G \in \mathcal{E}$  and, for any  $C \in \mathcal{E}$ , we have

$$\lim_{t \rightarrow \infty} t^{-1} \int_0^t C^T x(s) ds = \sum_{i=1}^3 c_i \lim_{t \rightarrow \infty} \int_0^t x_i(s) ds = C^T A^{-1}(L - C) \quad \text{a.s.}$$

By Theorem 4.1, (1.2) only has an invariant measure  $\nu(\cdot)$ . By Corollary 3.4.3 of Prato and Zabczyk [32],  $\nu(\cdot)$  is strong mixing. By Theorem 3.2.6 in [32],  $\nu(\cdot)$  is ergodic. Hence  $\lim_{t \rightarrow \infty} t^{-1} \int_0^t C^T x(s) ds = \int_{\mathbb{R}_+^3} C^T \nu(dx)$ .

Let  $\varrho(x)$  be the stationary probability density of (1.2). Then

$$Y(C) = \lim_{t \rightarrow \infty} \sum_{i=1}^3 \mathbb{E}(c_i x_i(t)) = \lim_{t \rightarrow \infty} \mathbb{E}(C^T x(t)) = \int_{\mathbb{R}_+^3} C^T x \varrho(x) dx.$$

Since the invariant measure of (1.2) is unique, by one-to-one correspondence between  $\varrho(x)$  and its invariant measure, we have

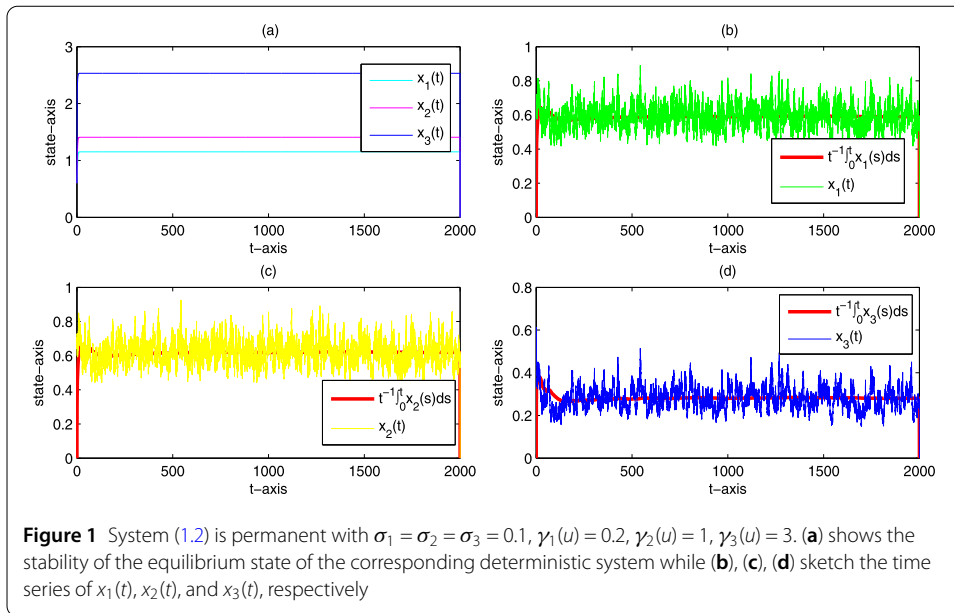
$$\int_{\mathbb{R}_+^3} C^T x \varrho(x) dx = \int_{\mathbb{R}_+^3} C^T x \nu(dx).$$

Therefore, we get  $Y(C) = C^T A^{-1}(L - C)$ .

Let  $G = (g_1, g_2, g_3)^T$  be the unique solution of the following equation:

$$\frac{dY(C)}{dC} = \frac{dC^T}{dC} A^{-1}(L - C) + \frac{d(L - C)^T (A^{-1})^T}{dC} C = A^{-1}L - [A^{-1} + (A^{-1})^T]C = 0. \quad (5.1)$$

Then  $G = [A(A^{-1})^T + I]^{-1}L$ . Further,  $\frac{d(Y(C)/dC)}{dC^T} = -[A^{-1} + (A^{-1})^T]$  is negative definite. Consequently,  $G$  is the unique extreme point of  $Y(C)$ . Hence if  $G \in \mathcal{E}$ , then  $C^* = G$  and the MESY is  $Y^* = G^T A^{-1}(L - G)$ .



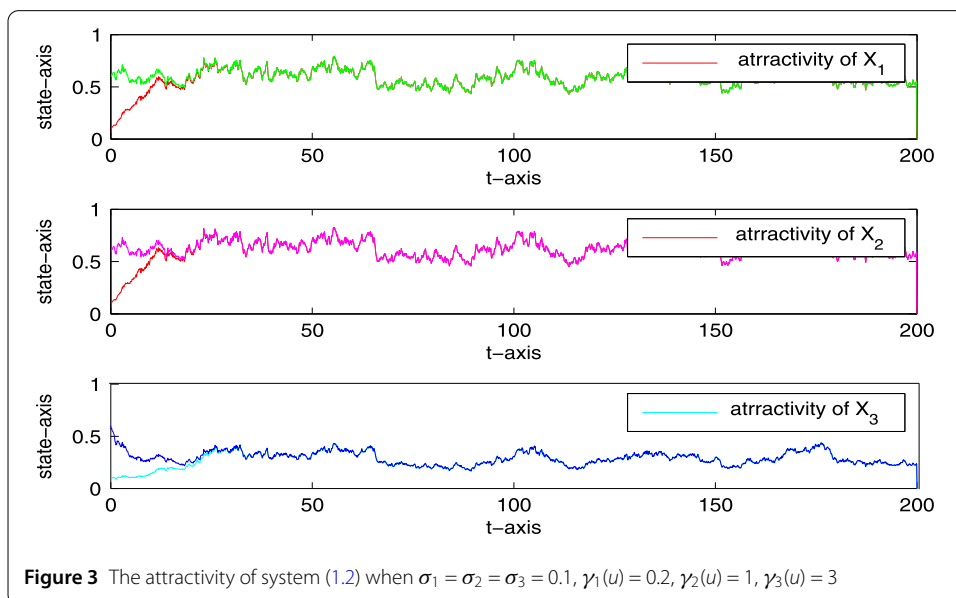
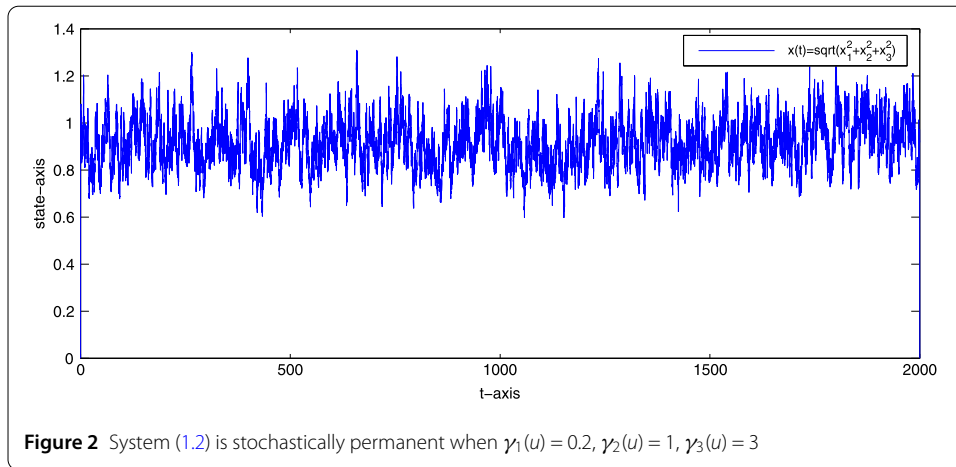
(ii) By contradiction, suppose that there exists an OHE  $\tilde{C}^* = (\tilde{C}_1^*, \tilde{C}_2^*, \tilde{C}_3^*)^T$ . Then  $\tilde{C}^* \in \mathcal{E}$ , which implies that  $\beta_1|_{C=\tilde{C}^*} > 0$ ,  $\beta_2|_{C=\tilde{C}^*} > 0$ ,  $\beta_3|_{C=\tilde{C}^*} > 0$ ,  $A_1|_{C=\tilde{C}^*} > \tilde{A}_1|_{C=\tilde{C}^*}$ ,  $A_2|_{C=\tilde{C}^*} > \tilde{A}_2|_{C=\tilde{C}^*}$ ,  $A_3|_{C=\tilde{C}^*} > \tilde{A}_3|_{C=\tilde{C}^*}$ ,  $g_1, g_2, g_3 \geq 0$ . Further,  $\tilde{C}^*$  is a solution of (5.1). In view of the uniqueness of its solutions,  $\tilde{C}^* = G$ , which means that  $\beta_1|_{C=G} > 0$ ,  $\beta_2|_{C=G} > 0$ ,  $\beta_3|_{C=G} > 0$ ,  $A_1|_{C=G} > \tilde{A}_1|_{C=G}$ ,  $A_2|_{C=G} > \tilde{A}_2|_{C=G}$ ,  $A_3|_{C=G} > \tilde{A}_3|_{C=G}$ . Clearly, they contradict with the assumption. The proof is completed.  $\square$

### 6 Numerical simulations

In this section, we give some numerical simulations to demonstrate our theoretical results. For this purpose, we choose  $\alpha_1 = (0.8, -0.1, -0.15)^T$ ,  $\alpha_2 = (-0.15, 0.8, -0.1)^T$ ,  $\alpha_3 = (-0.1, -0.15, 0.8)^T$ ,  $r_1 = 0.5$ ,  $r_2 = 0.8$ ,  $r_3 = 1.8$ ,  $c_1 = c_2 = c_3 = 0.1$ ,  $\sigma_1 = \sigma_2 = \sigma_3 = 0.1$ ,  $\gamma_1(u) = 0.2$ ,  $\gamma_2(u) = 1$ ,  $\gamma_3(u) = 3$ ,  $\mathbb{Z} = (0, \infty)$ ,  $\lambda(\mathbb{Z}) = 1$ . Then  $x_1^* = 1.0927$ ,  $x_2^* = 1.4832$ ,  $x_3^* = 2.5155$ ,  $r_1 - c_1 = 0.4$ ,  $r_2 - c_2 = 0.7$ ,  $r_3 - c_3 = 1.7$ ,  $b_1 = 0.0227$ ,  $b_2 = 0.3119$ ,  $b_3 = 1.6187$ ,  $A = 0.4716$ ,  $A_1 = 0.5153$ ,  $A_2 = 0.6995$ ,  $A_3 = 1.1863$ ,  $\tilde{A}_1 = 0.2207$ ,  $\tilde{A}_2 = 0.4077$ ,  $\tilde{A}_3 = 1.0466$ ,  $A_1 - \tilde{A}_1 = 0.2946$ ,  $A_2 - \tilde{A}_2 = 0.2918$ ,  $A_3 - \tilde{A}_3 = 0.1397$ ,  $\beta_1 = 0.4723$ ,  $\beta_2 = 0.4881$ ,  $\beta_3 = 0.1813$ ,  $\frac{A_1 - \tilde{A}_1}{A} = 0.6246$ ,  $\frac{A_2 - \tilde{A}_2}{A} = 0.6187$ ,  $\frac{A_3 - \tilde{A}_3}{A} = 0.2961$ . By Theorem 3.1, we have  $\lim_{t \rightarrow \infty} \langle x_1(t) \rangle = 0.6246$ ,  $\lim_{t \rightarrow \infty} \langle x_2(t) \rangle = 0.6187$ ,  $\lim_{t \rightarrow \infty} \langle x_3(t) \rangle = 0.2961$  and hence (1.2) is permanent, Fig. 1 confirms it. Figure 1(a) is the equilibrium state of the corresponding deterministic system. Figure 1(b), (c), (d) are the time series of  $x_1(t)$ ,  $x_2(t)$ , and  $x_3(t)$ , respectively. The stochastic permanence of system (1.2) is plotted in Fig. 2. The attractivity of solutions is plotted in Fig. 3, while the distributions of  $x_1(t)$ ,  $x_2(t)$ , and  $x_3(t)$  are plotted in Fig. 4.

Furthermore, it is easy to verify that  $G = ((A(A^{-1})^T + I)^{-1})L = (0.2345; 0.2480; 0.0908)^T$ . According to Theorem 5.1, the OHE is  $C^*$  and the MESY is  $Y^* = G^T A^{-1}(L - G) = 0.2156$ . Actually, if we take  $G_1 = (0.25, 0.28, 0.01)^T$  and  $G_2 = (0.1, 0.01, 0.05)^T$  as two examples, by computation, we have  $Y_* = G_1^T A^{-1}(L - G_1) = 0.2069$  and  $Y^- = G_2^T A^{-1}(L - G_2) = 0.0911$ , which are both less than the MESY  $Y^* = 0.2165$ . The numerical result is shown in Fig. 5.

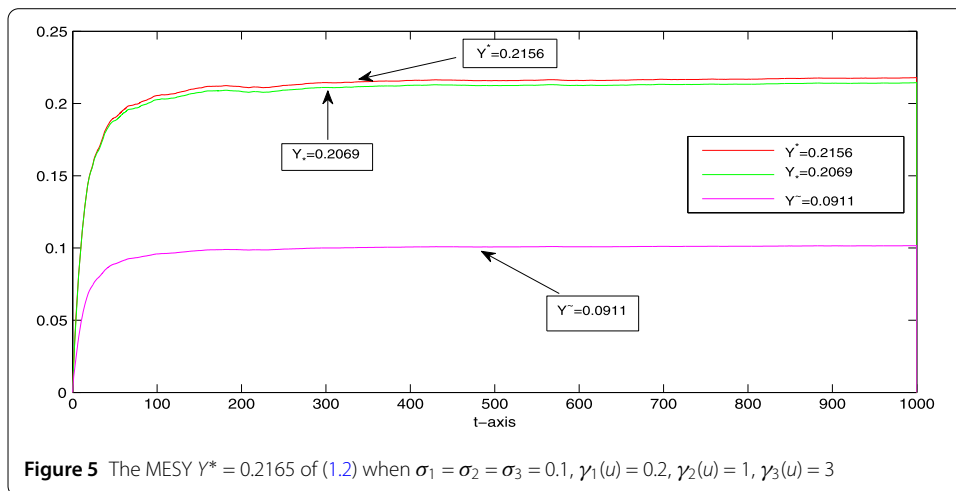
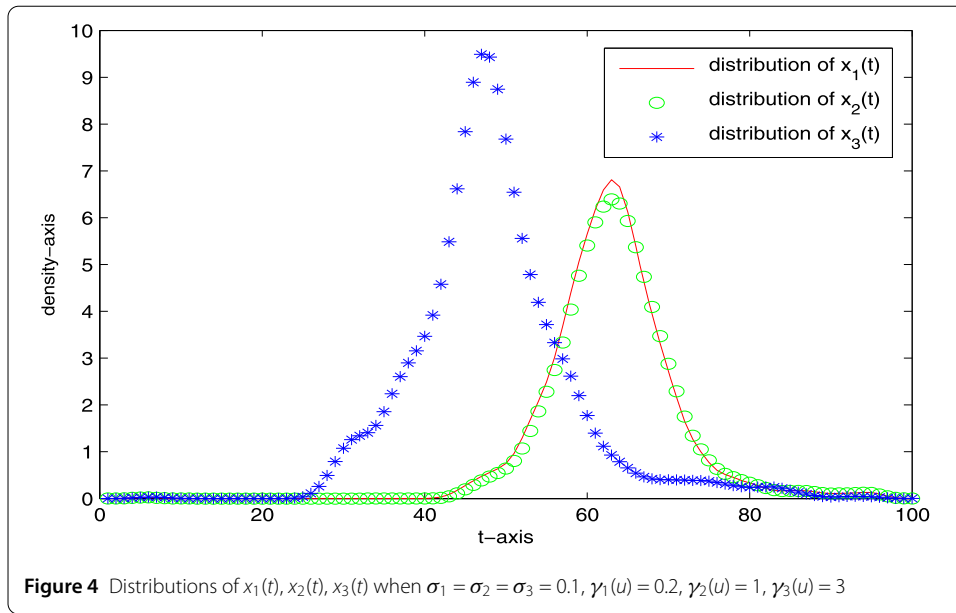
Next we consider the effects of Lévy jumps by letting all parameters be the same as above except the parameters  $\gamma_i(u)$ ,  $i = 1, 2, 3$ .



Firstly, let  $\gamma_1(u) = 1.4$  with  $\gamma_2(u)$  and  $\gamma_3(u)$  being as before. Then  $b_1 = 0.5295, \tilde{A}_1 = 0.5374, \tilde{A}_2 = 0.4596, \tilde{A}_3 = 1.1125; A_1 - \tilde{A}_1 = -0.0211; (A_2 - \tilde{A}_2)/A = 0.5087; (A_3 - \tilde{A}_3)/A = 0.1565$ . Consequently, by Theorem 3.1,  $x_1$  is extinct. Further,  $\Delta_2 - \tilde{\Delta}_2 = 0.3186, \Delta_3 - \tilde{\Delta}_3 = 0.1233, A_{11} = 0.6250, \frac{\Delta_2 - \tilde{\Delta}_2}{A_{11}} = 0.5098, \frac{\Delta_3 - \tilde{\Delta}_3}{A_{11}} = 0.1973$ . It follows that  $\lim_{t \rightarrow \infty} \langle x_2(t) \rangle = 0.5098$  and  $\lim_{t \rightarrow \infty} \langle x_3(t) \rangle = 0.1973$ , which is illustrated in Fig. 6.

Secondly, we take  $\gamma_1(u) = 2.5, \gamma_3(u) = 3.5$ , and  $\gamma_2(u)$  as before. In this case,  $b_1 = 1.2522, b_3 = 2.0009, \tilde{A}_1 = 1.2083, \tilde{A}_2 = 0.5834, \tilde{A}_3 = 1.4453; A_1 - \tilde{A}_1 < 0; A_3 - \tilde{A}_3 < 0; \frac{A_2 - \tilde{A}_2}{A} = 0.2439 > 0, \frac{r_2 - c_2 - b_2}{0.8} = 0.4851$ . By Theorem 3.1, we conclude that both  $x_1$  and  $x_3$  are extinct and  $\lim_{t \rightarrow \infty} \langle x_2(t) \rangle = 0.4851$ , which is shown in Fig. 7.

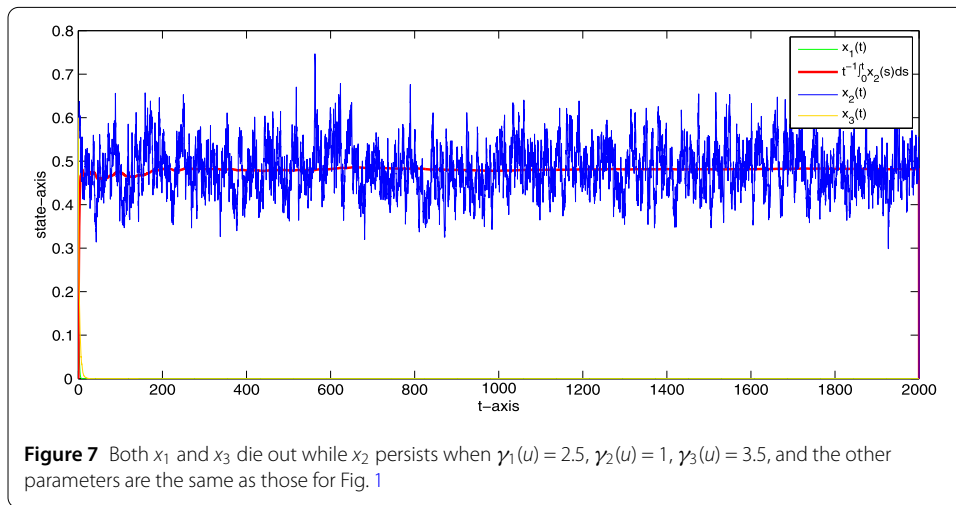
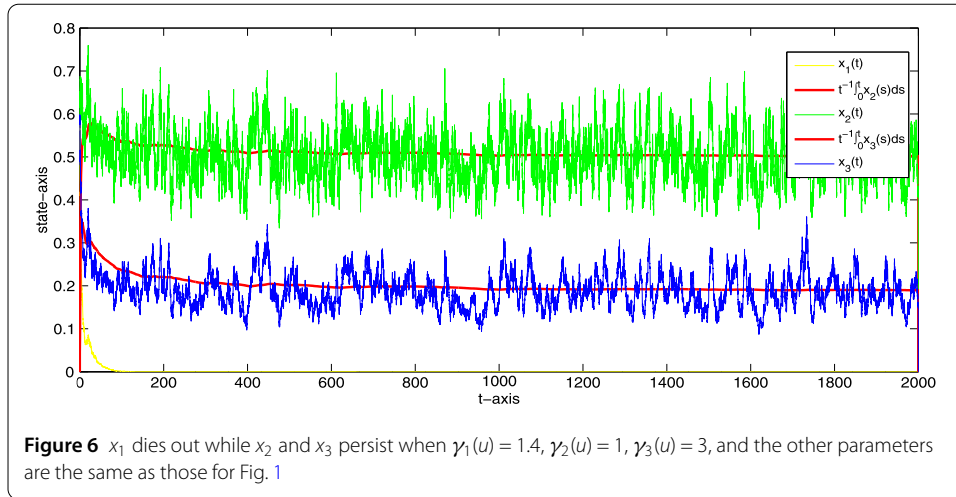
Finally, with  $\gamma_1(u) = 2.5, \gamma_2(u) = 2$ , and  $\gamma_3(u) = 3.5$ , we have  $b_1 = 1.2522, b_2 = 0.9064, b_3 = 2.0009, \tilde{A}_1 = 1.1055, \tilde{A}_2 = 0.9550, \tilde{A}_3 = 1.5063$ . It follows that  $A_1 - \tilde{A}_1 < 0, A_2 - \tilde{A}_2 < 0, A_3 - \tilde{A}_3 < 0$ . Theorem 3.1 tells us that all species are extinct (see Fig. 8).



### 7 Discussion and conclusion

In this paper, we consider a three-species stochastic cooperative system with time delays and Lévy noise. Theorem 3.1 gives some sufficient conditions on the stability in mean and extinction of each population. Theorem 3.2 gives sufficient conditions on permanence. The stability in distribution of each species is covered in Theorem 4.1. Finally, by use of ergodic method, the optimal harvesting policy is established in Theorem 5.1.

Theorem 3.1 and Theorem 3.2 imply that Lévy jumps have key influence on the extinction, stability in mean, and the stochastic permanence of (1.2). In general, the impacts of Lévy jumps on optimal harvesting are complicated since they are not constant usually. For simplicity and convenience, we suppose that  $\gamma_i(u) = \gamma_i$  ( $i = 1, 2, 3$ ) is a constant. Then, by computing the value of  $\frac{dA_i}{d\gamma_i}$ , we find that the permanence of (1.2) is dependent on all  $\gamma_i$ 's.



For example, if  $\lim_{t \rightarrow \infty} x_1(t) = 0$ , then

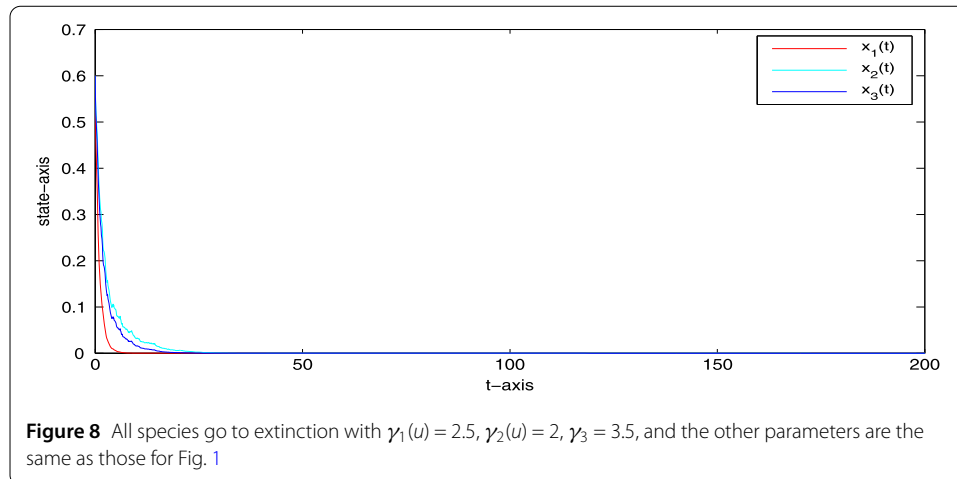
$$\frac{d(\Delta_{12}/\tilde{\Delta}_{12})}{d\gamma_2} \begin{cases} > 0 & \text{if } -1 < \gamma_2 < 0, \\ < 0 & \text{if } \gamma_2 > 0, \end{cases}$$

and

$$\frac{d(\Delta_{12}/\tilde{\Delta}_{12})}{d\gamma_3} \begin{cases} > 0 & \text{if } -1 < \gamma_3 < 0, \\ < 0 & \text{if } \gamma_3 > 0. \end{cases}$$

It follows that if  $\gamma_2 > 0$  or  $\gamma_3 > 0$ , then species  $x_2$  will tend to extinction with the increasing of  $\gamma_2$  or  $\gamma_3$ , while if  $-1 < \gamma_2 < 0$  or  $-1 < \gamma_3 < 0$ , then species  $x_2$  grows and becomes stable with the increasing of  $\gamma_2$  or  $\gamma_3$ .

Actually, time delays are considered in our research. They bring much difficulty for us, and we successfully defined some complex functionals to overcome it. Though any concrete effects of time delays are not reflected in our obtained results, we believe that it is very interesting to further explore them. This will be our future work.



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### Competing interests

The authors declare that they have no competing interests in this paper.

### Authors' contributions

YS carried out all studies and drafted the manuscript. YC conceived of the study and participated in its design and coordination and helped to draft the manuscript. BD participated in the design of the study and performed the simulation analysis. All authors read and approved the final manuscript.

### Author details

<sup>1</sup>College of Science, Guilin University of Technology, Guilin, China. <sup>2</sup>Department of Mathematics, Wilfrid Laurier University, Waterloo, Canada. <sup>3</sup>Department of Mathematics, Central South University of China, Changsha, China.

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