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Complete-closed time scales under shifts and related functions

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Abstract

In this paper, we introduce the concept of complete-closed time scales under translational and non-translational shifts and propose new definitions of special functions arising from dynamic equations on time scales including the concepts of almost periodic functions, almost automorphic functions and Stepanov almost automorphic functions. All the functions introduced in the paper are not only effective on periodic time scales under translations but are also valid on irregular time scales like $\overline{q^{\mathbb{Z}}}$, $\overline{(-q)^{\mathbb{Z}}}$ and $\mathbb{N}_{\pm}^{\frac{1}{2}}$, etc.

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1 Introduction

Classical periodic functions, almost periodic functions and almost automorphic functions defined by shifts (see [2, 4, 6, 7, 13, 15, 21–23, 31, 32]) and their applications to dynamic equations were studied in the literature (see [5, 8–11, 16–19, 32]). Using time scales calculus, one can study these functions on periodic time scales under translations since periodic time scales have a very nice closedness property under translations and we find that all periodic time scales under translations have a bounded graininess function μ (see [3, 12, 14, 20, 24–30]). However, there are many irregular time scales that have no translational closedness. For example, consider the time scale $\overline{q^{\mathbb{Z}}} := \overline{\{q^n : q > 1, n \in \mathbb{Z}\}}$, which arises when one considers q-difference equations. This time scale is irregular and it is not a periodic time scale under translations and its graininess function μ is unbounded. Also consider the time scales $-\overline{q^{\mathbb{Z}}} \cup \{1\} = \overline{\{-q^n : q > 1, n \in \mathbb{Z}\}} \cup \{1\}$ and $\overline{(-q)^{\mathbb{Z}}} = \{(-q)^n : q > 1, n \in \mathbb{Z}\} \cup \{0\}$ and $\mathbb{N}_{\pm}^{\frac{1}{2}} = \{\pm\sqrt{n} : n \in \mathbb{N}\}$. Is it possible to consider almost periodic, almost automorphic and generalized problems on these irregular time scales?

We introduce the concept of complete-closed time scales under translational and non-translational shifts (T-CCTSs and S-CCTSs) and propose new definitions of special functions.

In 2013, Adıvar introduced a new concept of periodic time scales and studied some periodic solutions for differential equations on irregular time scales which includes q-difference equations so one could consider periodic problem on $q^{\mathbb{Z}}$ (see [1, 2]). Motivated by the above, based on results of shift operators proposed in [1], we introduce a concept of complete-closed time scales under shifts which is more general than the concept of periodic time scales in [1]. Then we construct almost periodic functions and almost automorphic functions and propose definitions and generalizations including Stepanov almost periodic and Stepanov almost automorphic functions, etc.

2 Complete-closed time scales under shifts (S-CCTSs)

Throughout the paper, we assume that $\delta_\pm$ are shift operators satisfying Definition 3 from [1] and $\tilde{\mathcal{D}}_\pm := \{(s,t) \in [t_0,\infty)_\mathbb{T} \times \mathbb{T}^* : \delta_\pm(s,t) \in \mathbb{T}^*\}$, where $\mathbb{T}^*$ is the largest subset of the time scale $\mathbb{T}$, i.e., $\overline{\mathbb{T}^*} = \mathbb{T}$.

Definition 2.1 ([1]) Let $\mathbb{T}$ be a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0 \in \mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be periodic in shifts $\delta_\pm$ if there exists a $p \in (t_0,\infty)_{\mathbb{T}^*}$ such that $(p,t) \in \tilde{\mathcal{D}}_\mp$ for all $t \in \mathbb{T}^*$. Furthermore, if

$$P := \inf\bigl\{p \in (t_0,\infty)_{\mathbb{T}^*} : (p,t) \in \tilde{\mathcal{D}}_\mp \text{ for all } t \in \mathbb{T}^*\bigr\} \neq t_0, \tag{1}$$

then P is called the period of the time scale $\mathbb{T}$, where $\tilde{\mathcal{D}}_\pm = \{(s,t) \in [t_0,\infty)_\mathbb{T} \times \mathbb{T}^* : \delta_\pm(s,t) \in \mathbb{T}^*\}$.

Now, we give an example to show that Definition 2.1 is not appropriate for certain time scales.

Example 2.1 Let a time scale be

$$\mathbb{T} = -\overline{q^\mathbb{Z}} \cup \{1\} = \overline{\bigl\{-q^n : q > 1, n \in \mathbb{Z}\bigr\}} \cup \{1\}, \tag{2}$$

then $\mathbb{T}^* = \{-q^n : q > 1, n \in \mathbb{Z}\} \cup \{1\}$. Take the initial point $t_0 = 1$ and attach the shift operators $\delta_-(s,t) = -st$, $\quad \delta_+(s,t) = -\frac{t}{s}$. Let $\Pi^- = \{-q^n : q > 1, n \in \mathbb{Z}^+\}$. We obtain $\delta_\pm(s,t) \in \mathbb{T}^*$ for any $s \in \Pi^-$. However, by Definition 2.1, $\mathbb{T}$ cannot be regarded as a periodic time scale under shifts $\delta_\pm$ since there is **no** number $P \in (1,+\infty)_{\mathbb{T}^*}$ satisfying (1). In fact, this time scale is the opposite number set of the time scale $\overline{q^\mathbb{Z}} = \{q^n : q > 1, n \in \mathbb{Z}\} \cup \{0\}$, and the time scale (2) also plays an important role in q-difference equations. Moreover, from (2), it is easy to observe that, for any $t \in \mathbb{T}^*$, we have $-(-q)t \in \mathbb{T}^*$ but $-q \notin [1,+\infty)$.

For convenience, we introduce the notations. Let

$$\mathcal{D}_\pm = \bigl\{(s,t) \in \mathbb{T}^* \times \mathbb{T}^* : \delta_\pm(s,t) \in \mathbb{T}^*\bigr\}.$$

For any $s \in \mathbb{T}^*$, denote

$$\mathbb{T}_*^{\delta_{s^-}} := \delta_-\bigl(s,\mathbb{T}^*\bigr) := \bigl\{\delta_-(s,t) : (s,t) \in \mathcal{D}_-, \forall t \in \mathbb{T}^*\bigr\}, \tag{3}$$

$$\mathbb{T}_*^{\delta_{s^+}} := \delta_+\bigl(s,\mathbb{T}^*\bigr) := \bigl\{\delta_+(s,t) : (s,t) \in \mathcal{D}_+, \forall t \in \mathbb{T}^*\bigr\}. \tag{4}$$

Next, we introduce a concept related with S-CCTSs.

Definition 2.2 Let $\mathbb{T}$ be a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0 \in \mathbb{T}^*$. We say the time scale $\mathbb{T}$ is a bi-direction S-CCTS in shifts $\delta_\pm$ if

$$\Pi^\pm := \bigl\{p \in \mathbb{T}^* : (p,t) \in \mathcal{D}_\pm \text{ for all } t \in \mathbb{T}^*\bigr\} \notin \bigl\{\{t_0\}, \emptyset\bigr\}. \tag{5}$$

Remark 2.2 Note that by using (3) and (4), the set Π in (5) can be written in the equivalent form

$$\Pi^{\pm} = \{p \in \mathbb{T}^* : \mathbb{T}_*^{\delta_{p^{\pm}}} \subseteq \mathbb{T}^*\} \notin \{\{t_0\}, \emptyset\}.$$

Next, from (5), we introduce a concept of *S*-CCTS attached with shift direction. For convenience, we will use the notations

$$\Pi^{+} := \{p \in \mathbb{T}^* : \mathbb{T}_*^{\delta_p} \subseteq \mathbb{T}^*\}, \qquad \Pi^{-} := \{p \in \mathbb{T}^* : \mathbb{T}_*^{\delta_{p^-}} \subseteq \mathbb{T}^*\}.$$

Definition 2.3 Let $\mathbb{T}$ be a *S*-CCTS. Then:

(i) we say S-CCTS is with positive-direction if $\Pi^{+} \notin \{\{t_0\}, \emptyset\}$;
(ii) we say S-CCTS is with negative-direction if $\Pi^{-} \notin \{\{t_0\}, \emptyset\}$;
(iii) we say S-CCTS is with bi-direction if $\Pi^{\pm} \notin \{\{t_0\}, \emptyset\}$.

Remark 2.3 From Definition 2.3, one observes that a bi-direction S-CCTS also comes with a positive-direction and a negative-direction.

Remark 2.4 In (iii) of Definition 2.3, let $\delta_{\pm}(p,t) = t \pm p$, then it follows that $\mathbb{T}$ is a periodic time scale with period p (see [3] and Definitions 2.2–2.3 of [30]).

Remark 2.5 Note that if $\mathbb{T}$ is a S-CCTS with positive-direction, then $\Pi^{+} \cap \mathbb{T} \in \{\{t_0\}, \emptyset\}$ may hold. For example, consider the time scale $\mathbb{T} = \bigcup_{k=0}^{+\infty}[3k, 3k+1]$, if we take $\delta_{+}(p,t) = t - p$, then $\Pi^{+} = \{3n, n \in \mathbb{Z}^{-}\}$, which indicates that $\Pi^{+} \cap \mathbb{T} = \emptyset$; if we take $\delta_{+}(p,t) = t + p$, then $\Pi^{+} = \{3n, n \in \mathbb{Z}^{+}\}$, which indicates that $\Pi^{+} \subset \mathbb{T}$. Hence, for the time scale attached with shift directions, whether Π^{+} is a subset of $\mathbb{T}$ is determined by the choice of δ_{+}. Similarly, for the negative-direction S-CCTS, there may appear the same situation.

Example 2.6 From Definitions 2.2 and 2.3, we provide some examples of *S*-CCTS.

(1) Let $\mathbb{T} = -\overline{q^{\mathbb{Z}}} \cup \{1\} = \overline{\{-q^n : q > 1, n \in \mathbb{Z}\}} \cup \{1\}$. For such a time scale, take $t_0 = 1$, we attach the shift operators

$$\delta_{-}(s,t) = -st, \qquad \delta_{+}(s,t) = -\frac{t}{s}, \qquad \Pi^{\pm} = \{-q^n : q > 1, n \in \mathbb{Z}^{+}\}.$$

Hence, there exists $-q \in \Pi^{\pm}$ such that $\Pi^{\pm} \notin \{\{1\}, \emptyset\}$. From Definition 2.3, $\mathbb{T}$ is a *S*-CCTS with bi-direction.

(2) Let $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^n : q > 1, n \in \mathbb{Z}\} \cup \{0\}$. For such a time scale, take $t_0 = 1$, we attach the shift operators

$$\delta_{+}(s,t) = st, \qquad \delta_{-}(s,t) = \frac{t}{s}, \qquad \Pi^{\pm} = \{q^n : q > 1, n \in \mathbb{Z}^{+}\}.$$

Hence, there exists $q \in \Pi^{\pm}$ such that $\Pi^{\pm} \notin \{\{1\}, \emptyset\}$. From Definition 2.3, $\mathbb{T}$ is a *S*-CCTS with bi-direction.

(3) Let $\mathbb{T} = \overline{(-q)^{\mathbb{Z}}} = \{(-q)^n : q > 1, n \in \mathbb{Z}\} \cup \{0\}$. We obtain $\Pi^{\pm} = \{(-q)^{2n} : q > 1, n \in \mathbb{Z}^{+}\}$. For such a time scale, for any $t \in \mathbb{T}^*$, take $t_0 = 1$, we attach the shift operators

$$\delta_{+}(s,t) = \begin{cases} st, & t > 0, \\ \frac{t}{s}, & t < 0, \end{cases} \qquad \delta_{-}(s,t) = \begin{cases} \frac{t}{s}, & t > 0, \\ st, & t < 0. \end{cases}$$

Hence, there exists $q^2 \in \Pi^\pm$ such that $\delta_\pm(q^2,t) \in \mathbb{T}^*$ for all $t \in \mathbb{T}^*$, i.e., $\Pi^\pm \notin \{\{1\},\emptyset\}$. From Definition 2.3, $\mathbb{T}$ is a S-CCTS with bi-direction.

(4) Consider $\mathbb{T} = \{q^n : q > 1, n \in \mathbb{Z}\} \cup \{-q^n : q > 1, n \in \mathbb{Z}\} \cup \{0\}$. For such a time scale, for any $t \in \mathbb{T}^*$, take $t_0 = 1$, we attach the shift operators

$$\delta_+(s,t) = \begin{cases} st, & t > 0, \\ \frac{t}{s}, & t < 0, \end{cases} \qquad \delta_-(s,t) = \begin{cases} \frac{t}{s}, & t > 0, \\ st, & t < 0. \end{cases}$$

We obtain $\Pi^\pm = \{q^n : q > 1, n \in \mathbb{Z}^+\}$. Hence, there exists $q \in \Pi^\pm$ such that $\delta_\pm(q,t) \in \mathbb{T}^*$ for all $t \in \mathbb{T}^*$, i.e., $\Pi^\pm \notin \{\{1\},\emptyset\}$. From Definition 2.3, $\mathbb{T}$ is a S-CCTS with bi-direction.

(5) Consider $\mathbb{N}_\pm^{\frac{1}{2}} = \{\pm\sqrt{n}, n \in \mathbb{N}\}$, For such a time scale, for any $t \in \mathbb{T}^*$, take $t_0 = 0$, we attach the shift operators

$$\delta_+(s,t) = \begin{cases} \sqrt{s^2 + t^2}, & t > 0, \\ -\sqrt{t^2 - s^2}, & t < 0, \end{cases} \qquad \delta_-(s,t) = \begin{cases} \sqrt{t^2 - s^2}, & t > 0, \\ -\sqrt{t^2 + s^2}, & t < 0. \end{cases}$$

We obtain $\Pi^\pm = \mathbb{N}^{\frac{1}{2}} = \{\sqrt{n} : n \in \mathbb{N}\}$. Hence, there exists $1 \in \Pi^\pm$ such that $\delta_\pm(1,t) \in \mathbb{T}^*$ for all $t \in \mathbb{T}^*$, i.e., $\Pi^\pm \notin \{\{0\},\emptyset\}$. From Definition 2.3, $\mathbb{T}$ is a S-CCTS with bi-direction.

(6) Let $\mathbb{T}_1 = \{q^n : q > 1, n \in \mathbb{Z}^+\} \cup \{1\}$ and $\mathbb{T}_2 = \{q^n : q > 1, n \in \mathbb{Z}^-\} \cup \{0,1\}$. For these two time scales, take $t_0 = 1$ and

$$\Pi_1^+ = \left\{q^n : q > 1, n \in \mathbb{Z}^+\right\} \subseteq \mathbb{T}_1^*,$$
$$\Pi_2^- = \left\{q^n : q > 1, n \in \mathbb{Z}^-\right\} \subseteq \mathbb{T}_2^*.$$

It is clear that, for any $s_1 \in \Pi_1^+, s_2 \in \Pi_2^-$, we obtain

$$\delta_+(s_1,t_1) = s_1 t_1 \in \mathbb{T}_1^* \quad \text{for all } t_1 \in \mathbb{T}_1^*,$$
$$\delta_-(s_1,t_1) = \frac{t_1}{s_1} \notin \mathbb{T}_1^* \quad \text{for } t_1 = q, s_1 = q^2,$$

and

$$\delta_-(s_2,t_2) = s_2 t_2 \in \mathbb{T}_2^* \quad \text{for all } t_2 \in \mathbb{T}_2^*,$$
$$\delta_+(s_2,t_2) = \frac{t_2}{s_2} \notin \mathbb{T}_2^* \quad \text{for } t_2 = \frac{1}{q}, s_2 = \frac{1}{q^2}.$$

Hence, for the shift operator $\delta_+(s,t) = st$, it follows that $\mathbb{T}_1$ is a positive-direction S-CCTS. For the shift operator $\delta_-(s,t) = st$, we see that $\mathbb{T}_2$ is a negative-direction S-CCTS.

In the literature [30], the authors proposed some periodic time scales attached with translation direction. In fact, they are complete-closed time scales under translations (i.e., T-CCTSs) .

Definition 2.4 ([30]) We say $\mathbb{T}$ is a complete-closed time scale T-CCTS if $\Pi_0 := \{\tau \in \mathbb{R} : \mathbb{T}^\tau \subseteq \mathbb{T}\} \neq \{0\}$. We say Π_0 is the complete-closed translation number set of T-CCTS. Furthermore, we can describe it in detail as follows:

(a) if for any $p > 0$, there exists a number $P > p$ and $P \in \Pi_0$, we say $\mathbb{T}$ is a positive-direction T-CCTS;
(b) if for any $q < 0$, there exists a number $Q < q$ and $Q \in \Pi_0$, we say $\mathbb{T}$ is a negative-direction T-CCTS;
(c) if $\pm\tau \in \Pi_0$, we say $\mathbb{T}$ is a bi-direction T-CCTS;
(d) we say $\mathbb{T}$ is an oriented-direction T-CCTS if $\mathbb{T}$ is a positive-direction T-CCTS or a negative-direction T-CCTS.

Example 2.7 Let an oriented-direction T-CCTS be

$$\mathbb{T}_1 = \bigcup_{k=0}^{+\infty} \bigl[k(a+b), k(a+b)+a\bigr], \quad a, b \geq 0, a+b > 0,$$
$$\mathbb{T}_2 = \bigcup_{k=0}^{+\infty} \bigl[k(a+b), k(a+b)+a\bigr], \quad a, b \leq 0, a+b < 0$$

then $\mathbb{T}_1$ is a positive-direction T-CCTS and $\mathbb{T}_2$ is a negative-direction T-CCTS with the translation number $a+b$, but they have no invariance under translations in the sense of Definition 1.1 of [13] because $\inf \mathbb{T}_1 = \sup \mathbb{T}_2 = 0$.

Remark 2.8 We attached the translation direction to the time scales in [30] and now introduce the concept of T-CCTS. We also introduced the concepts of some special functions arising from differential and difference equations on T-CCTS including almost periodic functions and almost automorphic functions. However, these results will be invalid on some irregular time scales like $\overline{(-q)^{\mathbb{Z}}}$, $\overline{q^{\mathbb{Z}}}$ and $\pm\mathbb{N}^{\frac{1}{2}}$, etc.

Remark 2.9 Note that if $\mathbb{T}$ is a periodic time scales under translations and $\Pi^{\pm} \subseteq \mathbb{T}^*$, then the shift operators will satisfy $\delta_{\pm}(\tau, t) = t \pm \tau \in \mathbb{T}$ with the initial point $t_0 = 0$. Hence, if $\Pi^{\pm} \subseteq \mathbb{T}^*$, then T-CCTS is included in S-CCTS.

3 Related functions on S-CCTS

In this section, based on S-CCTS, we introduce the definitions of some special functions involving almost periodic functions, almost automorphic functions and their generalizations so that it is possible to study almost periodic problems, almost automorphic problems and their related general problems of dynamic equations on irregular time scales which include quantum-like time scales and more.

3.1 Almost periodic functions on S-CCTS

In this subsection, we introduce the concepts of almost periodic functions and Δ-almost periodic functions on S-CCTS based on the shift operators $\delta_{\pm}$ of time scales. Throughout the paper, we assume that $\mathbb{X}$ is a Banach space and $D \subseteq \mathbb{X}$ is an open set. In what follows, we introduce the concepts of relatively dense sets attached with directions under S-CCTS.

Definition 3.1 On S-CCTS, the relatively dense sets attached with directions are defined as follows:

(a) Let $\mathbb{T}$ be a S-CCTS with the shift operator δ_+ associated with the initial point $t_0 \in \mathbb{T}^*$. A subset S of $\mathbb{R}$ is called positive-direction relatively dense under the pair $(\mathbb{T}^*, \delta_+)$ if there exists a number $L \in \Pi^+$ $(L > t_0)$ such that $[a, \delta_+(L,a)]_{\mathbb{T}^*} \cap S \neq \emptyset$ for all $a \in \mathbb{T}^*$. The number $|L|$ is called the inclusion length with respect to the pair $(\mathbb{T}^*, \delta_+)$.

(b) Let $\mathbb{T}$ be a S-CCTS with the shift operator δ_- associated with the initial point $t_0 \in \mathbb{T}^*$. A subset S of $\mathbb{R}$ is called positive-direction relatively dense under the pair $(\mathbb{T}^*, \delta_-)$ if there exists a number $L \in \Pi^-$ $(L < t_0)$ such that $[\delta_-(L,a), a]_{\mathbb{T}^*} \cap S \neq \emptyset$ for all $a \in \mathbb{T}^*$. The number $|L|$ is called the inclusion length with respect to the pair $(\mathbb{T}^*, \delta_-)$.

(c) Let $\Pi^{\pm} \notin \{\{t_0\}, \emptyset\}$. A subset S of $\mathbb{R}$ is called bi-direction relatively dense under the pair $(\mathbb{T}^*, \delta_{\pm})$ if there exists a number $L \in \Pi^{\pm}$ $(L > t_0)$ such that $[a, \delta_+(L,a)]_{\mathbb{T}^*} \cap S \neq \emptyset$ and $[\delta_-(L,a), a]_{\mathbb{T}^*} \cap S \neq \emptyset$ for all $a \in \mathbb{T}^*$. The number $|L|$ is called the inclusion length with respect to the pair $(\mathbb{T}^*, \delta_{\pm})$.

Remark 3.1 From cases (a)–(c), let $\mathbb{T}_1 = \bigcup_{k=0}^{+\infty}[2k, 2k+1]$, $\mathbb{T}_2 = \bigcup_{k=-\infty}^{0}[2k-1, 2k]$ and $\mathbb{T}_3 = \bigcup_{k=-\infty}^{+\infty}[2k, 2k+1]$, then $\Pi_1^+ = \{2n : n \in \mathbb{Z}^+\}$, $\Pi_2^- = \{2n, n \in \mathbb{Z}^-\}$ and $\Pi_3^{\pm} = \{2n : n \in \mathbb{Z}^+\}$. It is obvious that $\delta_+(p,t) = t + p$ for $t \in \mathbb{T}_1, p \in \Pi_1^+$; $\delta_-(p,t) = t + p$ for $t \in \mathbb{T}_2, p \in \Pi_2^-$; and $\delta_{\pm}(p,t) = t \pm p$ for $t \in \mathbb{T}_3, p \in \Pi_3^{\pm}$. If $S \subset \mathbb{R}$ is relatively dense attached with directions according to Definition 3.1, then in case (a), we can take $L = 2 \in \Pi_1^+$ such that $[a, a+2]_{\mathbb{T}_1} \cap S \neq \emptyset$; in case (b), there exists $L = -2 \in \Pi_2^-$ such that $[a-2, a]_{\mathbb{T}_2} \cap S \neq \emptyset$; similarly, in case (c), there exists $L = 2 \in \Pi_3^+$ such that $[a, a+2]_{\mathbb{T}_3} \cap S \neq \emptyset$ and $[a-2, a]_{\mathbb{T}_3} \cap S \neq \emptyset$.

To introduce the concept of almost periodic functions under S-CCTS conveniently, we introduce the following definition.

Definition 3.2 Let $\mathbb{T}$ be a S-CCTS, and we say $(\mathbb{T}, \bar{\delta}, \bar{\Pi})$ is compatible,

(i) if $\mathbb{T}$ is a positive-direction S-CCTS, then $\bar{\delta}(s,t) = \delta_+(s,t)$ and $\bar{\Pi} = \Pi^+$;

(ii) if $\mathbb{T}$ is a negative-direction S-CCTS, then $\bar{\delta}(s,t) = \delta_-(s,t)$ and $\bar{\Pi} = \Pi^-$;

(iii) if $\mathbb{T}$ is a bi-direction S-CCTS, then

$$\bar{\delta}(s,t) = \begin{cases} \delta_+(s,t), & s \in \Pi^+, \\ \delta_-(s,t), & s \in \Pi^-, \end{cases} \quad \text{and} \quad \bar{\Pi} = \Pi^{\pm}.$$

Remark 3.2 Because of the Remark 2.5, for convenience, we always choose a suitable $\bar{\Pi}$ satisfying $\bar{\Pi} \subset \mathbb{T}^*$ such that $\bar{\delta}(s,t) \in \mathbb{T}$, where $(s,t) \in \bar{\Pi} \times \mathbb{T}^*$.

Based on Definition 3.2, we introduce the concept of almost periodic functions on S-CCTS as follows:

Definition 3.3 Let $\mathbb{T}$ be a S-CCTS under shifts and $(\mathbb{T}, \bar{\delta}, \bar{\Pi})$ be compatible. A function $f \in C(\mathbb{T} \times D, \mathbb{X})$ is called an almost periodic function with shift operators in $t \in \mathbb{T}$ uniformly for $x \in D$ if the ε-shift set of f

$$E\{\varepsilon, f, S\} = \left\{\tau \in \bar{\Pi} : \left\| f\left(\bar{\delta}(\tau,t), x\right) - f(t,x)\right\| < \varepsilon, \text{for all } t \in \mathbb{T}^* \text{ and } x \in S\right\}$$

is a relatively dense set with respect to the pair $(\bar{\Pi},\bar{\delta})$ for all $\varepsilon > 0$ and for each compact subset S of D; that is, for any given $\varepsilon > 0$ and each compact subset S of D, there exists a constant $l(\varepsilon,S) > 0$ such that each interval of length $l(\varepsilon,S)$ contains a $\tau(\varepsilon,S) \in E\{\varepsilon,f,S\}$ such that

$$\left\|f\left(\bar{\delta}(\tau,t),x\right) - f(t,x)\right\| < \varepsilon, \quad \text{for all } t \in \mathbb{T}^* \text{ and } x \in S.$$

Now τ is called the ε-shift number of f and $l(\varepsilon,S)$ is called the inclusion length of $E\{\varepsilon,f,S\}$.

Remark 3.3 From Definitions 3.2 and 3.3, we obtain:

(1) if $\mathbb{T}$ is a positive-direction S-CCTS, we say f is a positive-direction almost periodic function;
(2) if $\mathbb{T}$ is a negative-direction S-CCTS, we say f is a negative-direction almost periodic function;
(3) if $\mathbb{T}$ is a bi-direction S-CCTS, we say f is a bi-direction almost periodic function.

Remark 3.4 We can easily obtain the classical almost periodic functions on $\mathbb{R}$ and $\mathbb{Z}$ from Definition 3.3 by letting $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, where $\bar{\delta}(\tau,t) = \delta_{\pm}(\tau,t) = t \pm \tau$. Note that Definition 3.3 is also suitable for irregular time scales like $\overline{q^{\mathbb{Z}}}$, $\overline{(-q)^{\mathbb{Z}}}$ and $\mathbb{N}_{\pm}^{\frac{1}{2}}$, etc.

Remark 3.5 We demonstrate some concrete examples of Definition 3.3 under different irregular time scales.

(i) Let $\mathbb{T} = \{q^n : q > 1, n \in \mathbb{Z}^-\} \cup \{0,1\}$, we take $\Pi^- = \{q^n : q > 1, n \in \mathbb{Z}^-\} \subset \mathbb{T}^*$ and $\bar{\delta}(s,t) = \delta_-(s,t) = st$, then Definition 3.3 turns into the following.
A function $f \in C(\mathbb{T} \times D,\mathbb{X})$ is called a negative almost periodic function with shift operator in $t \in \mathbb{T}$ uniformly for $x \in D$ if the ε-shift set of f

$$E\{\varepsilon,f,S\} = \left\{\tau \in \Pi^- : \left\|f(\tau t,x) - f(t,x)\right\| < \varepsilon, \text{for all } t \in \mathbb{T}^* \text{ and } x \in S\right\}$$

is a relatively dense set with respect to the pair (Π^-,δ_-) for all $\varepsilon > 0$ and for each compact subset S of D.

(ii) Let $\mathbb{T} = \{\sqrt{n} : n \in \mathbb{Z}^+\}$, we take $\Pi^+ = \mathbb{T}^* = \mathbb{T}$ and $\bar{\delta}(s,t) = \delta_+(s,t) = \sqrt{s^2 + t^2}$, then Definition 3.3 turns into the following.
A function $f \in C(\mathbb{T} \times D,\mathbb{X})$ is called a positive almost periodic function with shift operator in $t \in \mathbb{T}$ uniformly for $x \in D$ if the ε-shift set of f

$$E\{\varepsilon,f,S\} = \left\{\tau \in \Pi^+ : \left\|f\left(\sqrt{\tau^2 + t^2},x\right) - f(t,x)\right\| < \varepsilon, \text{for all } t \in \mathbb{T}^* \text{ and } x \in S\right\}$$

is a relatively dense set with respect to the pair (Π^+,δ_+) for all $\varepsilon > 0$ and for each compact subset S of D.

Now, we will construct some examples of almost periodic functions on S-CCTS through periodic functions on S-CCTS.

Example 3.6 Let $\mathbb{T} = \mathbb{R}$ and $\Pi = (0,+\infty)$, and we define the following shift operators:

$$\delta_+(\tau,t) = \begin{cases} \tau t, & \text{if } t \geq, \\ t/\tau, & \text{if } t < 0, \end{cases} \quad \text{for } \tau \in [1,+\infty),$$

$$\delta_-(\tau,t) = \begin{cases} t/\tau, & \text{if } t \geq 0, \\ \tau t, & \text{if } t < 0, \end{cases} \quad \text{for } \tau \in [1,+\infty).$$

Under the shifts $\delta_\pm$, the function

$$f_\tau(t) = \cos\left(\frac{\ln|t|}{\ln(1/\sqrt{\tau})}\pi\right), \quad \tau > 1 \text{ and } t \in \mathbb{T}^* = \mathbb{R}\backslash\{0\}$$

is periodic under shifts with the period $\tau = P^2$, $P > 1$ since

$$\begin{aligned} &f_\tau\big(\delta_\pm(\tau,t)\big) \\ &\quad = \begin{cases} f_\tau(tP^{\pm 2}), & \text{if } t \geq 0, \\ f_\tau(t/P^{\pm 2}), & \text{if } t < 0, \end{cases} \\ &\quad = \cos\left(\frac{\ln|t| \pm 2\ln(1/P)}{\ln(1/P)}\pi\right) \\ &\quad = \cos\left(\frac{\ln|t|}{\ln(1/P)}\pi \pm 2\pi\right) \\ &\quad = \cos\left(\frac{\ln|t|}{\ln(1/P)}\pi\right) = f_\tau(t). \end{aligned}$$

Example 3.7 Based on the time scale in Example 3.6, consider the function

$$\tilde{F}(t) = \cos\left(\frac{\ln|\sqrt{2}t|}{\ln(1/P_1)}\pi\right) + \cos\left(\frac{\ln|\sqrt{3}t|}{\ln(1/P_2)}\pi\right),$$

where $P_1 \neq P_2, P_1, P_2 > 1$ and $t \in \mathbb{T}^* = \mathbb{R}\backslash\{0\}$. One can observe that $\tilde{F}(t)$ is almost periodic under shifts $\delta_\pm$. From Example 3.6, let

$$f_{P_1^2}(\sqrt{2}t) = \cos\left(\frac{\ln|\sqrt{2}t|}{\ln(1/P_1)}\pi\right), \qquad f_{P_2^2}(\sqrt{3}t) = \cos\left(\frac{\ln|\sqrt{3}t|}{\ln(1/P_2)}\pi\right),$$

we obtain $\tilde{F}(t) = f_{P_1^2}(\sqrt{2}t) + f_{P_2^2}(\sqrt{3}t)$, and note that $f_{P_1^2}$ and $f_{P_2^2}$ are periodic with different periods P_1^2, P_2^2, respectively (see Fig. 1).

We can extend Definition 3.3 to Δ-almost periodic functions.

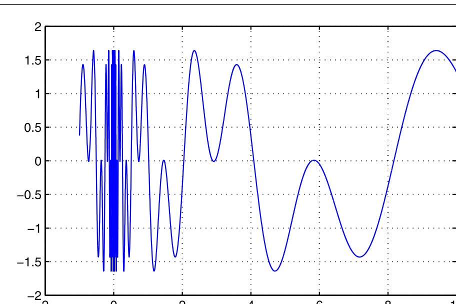

Figure 1 Graph of $\tilde{F}(t) = \cos(\frac{\ln|\sqrt{2}t|}{\ln(1/P_1)}\pi) + \cos(\frac{\ln|\sqrt{3}t|}{\ln(1/P_2)}\pi)$ with $P_1 = 2$, $P_2 = \sqrt[3]{2}$

Definition 3.4 Let $\mathbb{T}$ be a S-CCTS under shifts and $(\mathbb{T},\bar{\delta},\bar{\Pi})$ be compatible, the shift $\bar{\delta}(\tau,t)$ is Δ-differentiable in its second argument with *rd*-continuous bounded derivatives $\bar{\delta}^{\Delta}(\tau,t)$ for all $t\in\mathbb{T}^*$. A function $f\in C(\mathbb{T}\times D,\mathbb{X})$ is called a Δ-almost periodic function with shift operators in $t\in\mathbb{T}$ uniformly for $x\in D$ if the ε-shift set of f

$$E\{\varepsilon,f,S\}=\bigl\{\tau\in\bar{\Pi}:\bigl\|f\bigl(\bar{\delta}(\tau,t),x\bigr)\bar{\delta}^{\Delta}(\tau,t)-f(t,x)\bigr\|<\varepsilon,\text{for all } t\in\mathbb{T}^*\text{ and } x\in S\bigr\}$$

is a relatively dense set with respect to the pair $(\bar{\Pi},\bar{\delta})$ for all $\varepsilon>0$ and for each compact subset S of D; that is, for any given $\varepsilon>0$ and each compact subset S of D, there exists a constant $l(\varepsilon,S)>0$ such that each interval of length $l(\varepsilon,S)$ contains a $\tau(\varepsilon,S)\in E\{\varepsilon,f,S\}$ such that

$$\bigl\|f\bigl(\bar{\delta}(\tau,t),x\bigr)\bar{\delta}^{\Delta}(\tau,t)-f(t,x)\bigr\|<\varepsilon,\quad\text{for all } t\in\mathbb{T}^*\text{ and } x\in S.$$

Now τ is called the ε-shift number of f and $l(\varepsilon,S)$ is called the inclusion length of $E\{\varepsilon,f,S\}$.

Remark 3.8 From Definitions 3.2 and 3.4, we obtain:

(1) if $\mathbb{T}$ is a positive-direction S-CCTS, we say f is a positive-direction Δ-almost periodic function;
(2) if $\mathbb{T}$ is a negative-direction S-CCTS, we say f is a negative-direction Δ-almost periodic function;
(3) if $\mathbb{T}$ is a bi-direction S-CCTS, we say f is a bi-direction Δ-almost periodic function.

In what follows, we will construct some examples of Δ-almost periodic functions on S-CCTS through Δ-periodic functions on S-CCTS.

Example 3.9 For any $a\in\mathbb{R}\backslash\{0\}$, the real valued function $f(t)=a/t$ defined on $(\sqrt{5})^{\mathbb{Z}}=\{(\sqrt{5})^n,n\in\mathbb{Z}\}$ is Δ-periodic under the shifts $\delta_{\pm}$ with the period $\tau=\sqrt{5}$ since

$$f\bigl(\delta_{\pm}(\sqrt{5},t)\bigr)\delta_{\pm}^{\Delta}(\sqrt{5},t)=\frac{a}{(\sqrt{5})^{\pm1}t}(\sqrt{5})^{\pm1}=\frac{a}{t}=f(t).$$

Example 3.10 On the time scale $5^{\mathbb{Z}}=\{5^n,n\in\mathbb{Z}\}$, let $a,b\in\mathbb{R}\backslash\{0\}$, $a\neq b$ and

$$g_1(t)=\frac{a}{t},\qquad g_2(t)=\frac{b}{(-1)^{\log_{\sqrt{5}}t}t},\qquad \tilde{G}(t)=g_1(t)+g_2(t)=\frac{a}{t}+\frac{b}{(-1)^{\log_{\sqrt{5}}t}t}.$$

From Example 3.9, one observes that $g_1(\delta_{\pm}(\sqrt{5},t))\delta_{\pm}^{\Delta}(\sqrt{5},t)=g_1(t)$ and we note that

$$\begin{aligned}g_2\bigl(\delta_{\pm}(5,t)\bigr)\delta_{\pm}^{\Delta}(5,t)&=\frac{b}{(-1)^{\log_{\sqrt{5}}(\sqrt{5})^{\pm2}t}\cdot(\sqrt{5})^{\pm2}t}\cdot(\sqrt{5})^{\pm2}\\&=\frac{b}{(-1)^{\pm2+\log_{\sqrt{5}}t}\cdot t}\\&=\frac{b}{(-1)^{\log_{\sqrt{5}}t}t}=g_2(t).\end{aligned}$$

Hence, $\tilde{G}(t)$ is a Δ-almost periodic function under the shifts $\delta_{\pm}$. Note that the periods of g_1 and g_2 are completely different.

Remark 3.11 We demonstrate a few concrete examples of Definition 3.4 under different irregular time scales:

Let $\mathbb{T} = \{(-q)^n : q > 1, n \in \mathbb{Z}\} \cup \{0\}$, then $\Pi^{\pm} = \{(-q)^{2n} : q > 1, n \in \mathbb{Z}^+\}$ and $\mathbb{T}^* = \{(-q)^n : q > 1, n \in \mathbb{Z}\}$. Let

$$\delta_+(\tau,t) = \begin{cases} st, & t > 0, \\ \frac{t}{s}, & t < 0, \end{cases} \qquad \delta_-(\tau,t) = \begin{cases} \frac{t}{s}, & t > 0, \\ st, & t < 0. \end{cases}$$

A function $f \in C(\mathbb{T} \times D, \mathbb{X})$ is called a positive-direction Δ-almost periodic function with shift operators in $t \in \mathbb{T}$ uniformly for $x \in D$ if the ε-shift set of f

$$E\{\varepsilon, f, S\} = \bigl\{\tau \in \Pi^{\pm} : \bigl\| f\bigl(\delta_+(\tau,t), x\bigr)\delta_+^{\Delta}(\tau,t) - f(t,x)\bigr\| < \varepsilon, \text{for all } t \in \mathbb{T}^* \text{ and } x \in S\bigr\}$$

is a relatively dense set with respect to the pair $(\Pi^{\pm}, \delta_+)$ for all $\varepsilon > 0$ and for each compact subset S of D.

3.2 Stepanov almost periodic functions on S-CCTS

In this subsection, based on Sect. 3.1, we introduce the concept of Stepanov almost periodic functions (S_L^p-almost periodic function) on S-CCTS.

For $h > 0$, denote

$$\mathbb{C}_h := \left\{z \in \mathbb{C} : z \neq -\frac{1}{h}\right\}, \qquad \mathbb{R}_h := \left\{z \in \mathbb{C}_h : z \in \mathbb{R} \text{ and } z > -\frac{1}{h}\right\}.$$

Let $-\frac{\pi}{h} < \omega \leq \frac{\pi}{h}$ and denote the Hilger purely imaginary number $\mathring{\imath}\omega = \frac{e^{i\omega h - 1}}{h}$.

Throughout this subsection, we assume that $(\mathbb{T}, \bar{\delta}, \bar{\Pi})$ is compatible and $L^p_{\mathrm{loc}}(\mathbb{T}, \mathbb{X})$ denotes the set of all pth power local-integrable functions in $\mathbb{X}$ ($p \geq 1$), i.e.,

$$L^p_{\mathrm{loc}}(\mathbb{T}, \mathbb{X}) := \left\{f : \int_{\mathbb{T}_1} \bigl\|f(t)\bigr\|^p \Delta t < \infty, \text{where } \mathbb{T}_1 \text{ is a closed subset of } \mathbb{T}\right\}.$$

For $L^p_{\mathrm{loc}}(\mathbb{T}, \mathbb{C}^n)$, from the theory of Stepanov almost periodic functions introduced by V.V. Stepanov (see [22]), we can introduce a class of functions S_L^p that are measurable and summable together with their pth power ($p \geq 1$) on every finite interval $[\widehat{t, \bar{\delta}(L,t)}]_{\mathbb{T}^*}$ and that can be approximated in the metric of the Stepanov space by finite sums $\sum_{n=1}^N a_n e^{\mathring{\imath}\lambda_n t}$, where $a_n \in \mathbb{C}_h$ are Hilger complex coefficients and $-\frac{\pi}{h} < \lambda_n \leq \frac{\pi}{h}$. The distance in the Stepanov space is defined by the formula

$$D_{S_L^p}[f,g] = \sup_{t \in \mathbb{T}^*} \left[\frac{1}{|L|} \int_{[\widehat{t, \bar{\delta}(L,t)}]_{\mathbb{T}^*}} \bigl|f(s) - g(s)\bigr|^p \Delta s\right]^{\frac{1}{p}},$$

where

$$\bigl[\widehat{t, \bar{\delta}(L,t)}\bigr]_{\mathbb{T}^*} = \begin{cases} [t, \delta_+(L,t)]_{\mathbb{T}^*} & \text{if } L \in \Pi^+, \\ [\delta_-(L,t), t]_{\mathbb{T}^*} & \text{if } L \in \Pi^-, \end{cases} \tag{6}$$

and $f, g : \mathbb{T} \to \mathbb{R}^n$.

Now, based on the above idea, we consider a Stepanov space on time scales whose distance is defined by the following:

$$D_{S_L^p}[f,g] = \sup_{t\in\mathbb{T}^*}\left[\frac{1}{|L|}\int_{\widehat{[t,\bar{\delta}(L,t)]}_{\mathbb{T}^*}} \|f(s)-g(s)\|^p\,\Delta s\right]^{\frac{1}{p}}, \tag{7}$$

where $f,g:\mathbb{T}\to\mathbb{X}$ and $\mathbb{X}$ is a Banach space. From (7), we can obtain the following definition.

Definition 3.5 We say $f\in L^p_{\text{loc}}(\mathbb{T},\mathbb{X})$ is S_L^p-bounded if

$$\|f\|_{S_L^p} = \sup_{t\in\mathbb{T}^*}\left[\frac{1}{|L|}\int_{\widehat{[t,\bar{\delta}(L,t)]}_{\mathbb{T}^*}} \|f(s)\|^p\,\Delta s\right]^{\frac{1}{p}} < \infty,$$

where $\widehat{[t,\bar{\delta}(L,t)]}_{\mathbb{T}^*}$ is defined in (6). Denote $BS_L^p(\mathbb{T},\mathbb{X})$ the set of all S_L^p-bounded functions from $\mathbb{T}$ to $\mathbb{X}$.

Remark 3.12 According to Definition 3.5, let $\mathbb{T}=\{\sqrt{n}:n\in\mathbb{N}\}$, then we have $L=1$ and $\bar{\delta}(L,t)=\delta_+(L,t)=\sqrt{t^2+1}$, we say $f\in L^p_{\text{loc}}(\mathbb{T},\mathbb{X})$ is S_L^p-bounded if

$$\|f\|_{S_L^p} = \sup_{t\in\mathbb{T}}\left[\int_t^{\sqrt{t^2+1}} \|f(s)\|^p\,\Delta s\right]^{\frac{1}{p}} < \infty.$$

Now, we introduce the concept of S_L^p-almost periodic functions on S-CCTS.

Definition 3.6 A function $f\in L^p_{\text{loc}}(\mathbb{T}\times D,\mathbb{X})$ is called a S_L^p-almost periodic function with shift operators in $t\in\mathbb{T}$ uniformly for $x\in D$ if the ε-shift set of f

$$E\{\varepsilon,f,S\} = \left\{\tau\in\bar{\Pi}: \left\|f\left(\bar{\delta}(\tau,t),x\right)-f(t,x)\right\|_{S_L^p}<\varepsilon, \text{for all } t\in\mathbb{T}^* \text{ and } x\in S\right\}$$

is a relatively dense set with respect to the pair $(\bar{\Pi},\bar{\delta})$ for all $\varepsilon>0$ and for each compact subset S of D; that is, for any given $\varepsilon>0$ and each compact subset S of D, there exists a constant $l(\varepsilon,S)>0$ such that each interval of length $l(\varepsilon,S)$ contains a $\tau(\varepsilon,S)\in E\{\varepsilon,f,S\}$ such that

$$\left\|f\left(\bar{\delta}(\tau,t),x\right)-f(t,x)\right\|_{S_L^p}<\varepsilon, \quad \text{for all } t\in\mathbb{T}^* \text{ and } x\in S.$$

Now τ is called the ε-shift number of f and $l(\varepsilon,S)$ is called the inclusion length of $E\{\varepsilon,f,S\}$, where

$$\left\|f\left(\bar{\delta}(\tau,t),x\right)-f(t,x)\right\|_{S_L^p} = \sup_{t\in\mathbb{T}^*}\left[\frac{1}{|L|}\int_{\widehat{[t,\bar{\delta}(L,t)]}_{\mathbb{T}^*}} \left\|f\left(\bar{\delta}(\tau,s),x\right)-f(s,x)\right\|^p\,\Delta s\right]^{\frac{1}{p}}.$$

Remark 3.13 From Definitions 3.2 and 3.6, we obtain:

(1) if $\mathbb{T}$ is a positive-direction S-CCTS, we say f is a positive-direction S_L^p-almost periodic function;

(2) if $\mathbb{T}$ is a negative-direction S-CCTS, we say f is a negative-direction S_L^p-almost periodic function;
(3) if $\mathbb{T}$ is a bi-direction S-CCTS, we say f is a bi-direction S_L^p-almost periodic function.

Remark 3.14 Note that Definition 3.6 is a general definition of an S_L^p-almost periodic function on any arbitrary closed subset of $\mathbb{R}$. One can obtain the classical S_L^p-almost periodic functions easily on $\mathbb{R}$ and $\mathbb{Z}$ from Definition 3.6 by letting $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, where $|L| = 1$, $\bar{\delta}(\tau,t) = \delta_{\pm}(\tau,t) = t \pm \tau$. We emphasize that Definition 3.6 is also suitable for irregular time scales like $\overline{q^{\mathbb{Z}}}$, $\overline{(-q)^{\mathbb{Z}}}$ and $\mathbb{N}_{\pm}^{\frac{1}{2}}$, etc. Therefore, it is new to consider S_L^p-almost periodic problems for dynamic equations on time scales under Definition 3.6.

Remark 3.15 Let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, then we have $\Pi^{\pm} = \{q^n : q > 1, n \in \mathbb{Z}^+\}$ and $\delta_+(\tau,t) = \tau t$, $\delta_-(\tau,t) = t/\tau$, $\tau \in \Pi^{\pm}$. Then a function $f \in L^p_{\mathrm{loc}}(\mathbb{T} \times D, \mathbb{X})$ is called a positive-direction S_L^p-almost periodic function with shift operators in $t \in \mathbb{T}$ uniformly for $x \in D$ if the ε-shift set of f

$$E\{\varepsilon,f,S\} = \bigl\{\tau \in \Pi^{\pm} : \bigl\|f(\tau t,x) - f(t,x)\bigr\|_{S_L^p} < \varepsilon, \text{for all } t \in \mathbb{T}^* \text{ and } x \in S\bigr\}$$

is a relatively dense set with respect to the pair $(\Pi^{\pm}, \delta_+)$ for all $\varepsilon > 0$ and for each compact subset S of D, where

$$\bigl\|f(\tau t,x) - f(t,x)\bigr\|_{S_L^p} = \sup_{t\in\mathbb{T}^*}\left[\frac{1}{q}\int_t^{qt}\bigl\|f(\tau s,x) - f(s,x)\bigr\|^p \Delta s\right]^{\frac{1}{p}}.$$

We can extend Definition 3.6 to Δ-S_L^p-almost periodic functions.

Definition 3.7 Let $\mathbb{T}$ be a S-CCTS under shifts and $(\mathbb{T}, \bar{\delta}, \bar{\Pi})$ be compatible, the shift $\bar{\delta}(\tau,t)$ is Δ-differentiable in its second argument with rd-continuous bounded derivatives $\bar{\delta}^{\Delta}(\tau,t)$ for all $t \in \mathbb{T}^*$. A function $f \in C(\mathbb{T} \times D, \mathbb{X})$ is called a Δ-S_L^p-almost periodic function with shift operators in $t \in \mathbb{T}$ uniformly for $x \in D$ if the ε-shift set of f

$$E\{\varepsilon,f,S\} = \bigl\{\tau \in \bar{\Pi} : \bigl\|f\bigl(\bar{\delta}(\tau,t),x\bigr)\bar{\delta}^{\Delta}(\tau,t) - f(t,x)\bigr\|_{S_L^p} < \varepsilon, \text{for all } t \in \mathbb{T}^* \text{ and } x \in S\bigr\}$$

is a relatively dense set with respect to the pair $(\bar{\Pi}, \bar{\delta})$ for all $\varepsilon > 0$ and for each compact subset S of D; that is, for any given $\varepsilon > 0$ and each compact subset S of D, there exists a constant $l(\varepsilon,S) > 0$ such that each interval of length $l(\varepsilon,S)$ contains a $\tau(\varepsilon,S) \in E\{\varepsilon,f,S\}$ such that

$$\bigl\|f\bigl(\bar{\delta}(\tau,t),x\bigr)\bar{\delta}^{\Delta}(\tau,t) - f(t,x)\bigr\|_{S_L^p} < \varepsilon, \quad \text{for all } t \in \mathbb{T}^* \text{ and } x \in S.$$

Now τ is called the ε-shift number of f and $l(\varepsilon,S)$ is called the inclusion length of $E\{\varepsilon,f,S\}$.

Remark 3.16 From Definitions 3.2 and 3.7, we obtain:
(1) if $\mathbb{T}$ is a positive-direction S-CCTS, we say f is a positive-direction Δ-S_L^p-almost periodic function;
(2) if $\mathbb{T}$ is a negative-direction S-CCTS, we say f is a negative-direction Δ-S_L^p-almost periodic function;
(3) if $\mathbb{T}$ is a bi-direction S-CCTS, we say f is a bi-direction Δ-S_L^p-almost periodic function.

3.3 Almost automorphic functions on S-CCTS

In this subsection, we introduce the concepts of almost automorphic functions and Δ-almost automorphic functions on S-CCTS.

Definition 3.8 Let $\mathbb{T}$ be a S-CCTS under shifts and $(\mathbb{T}, \bar{\delta}, \bar{\Pi})$ be compatible.

(i) Let $f : \mathbb{T} \to \mathbb{X}$ be a bounded continuous function. We say that f is almost automorphic if, from every sequence $\{s_n\} \subset \bar{\Pi}$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$g(t) = \lim_{n\to\infty} f\big(\bar{\delta}(\tau_n, t)\big), \tag{8}$$

is well defined for each $t \in \mathbb{T}^*$. Denote by $AA_{\bar{\delta}}(\mathbb{T}, \mathbb{X})$ the set of all such functions.

(ii) A continuous function $f : \mathbb{T} \times \mathbb{X} \to \mathbb{X}$ is said to be almost automorphic if $f(t, x)$ is almost automorphic in $t \in \mathbb{T}^*$ uniformly in $x \in B$, where B is any bounded subset of $\mathbb{X}$. Denote by $AA_{\bar{\delta}}(\mathbb{T} \times \mathbb{X}, \mathbb{X})$ the set of all such functions.

Remark 3.17 From Definitions 3.2 and 3.8, we obtain:

(1) if $\mathbb{T}$ is a positive-direction S-CCTS, we say f is a positive-direction almost automorphic function;

(2) if $\mathbb{T}$ is a negative-direction S-CCTS, we say f is a negative-direction almost automorphic function;

(3) if $\mathbb{T}$ is a bi-direction S-CCTS, we say f is a bi-direction almost automorphic function.

Remark 3.18 In (3) from Remark 3.17, if $\mathbb{T}$ is a bi-direction S-CCTS, then, by (8), from every sequence $\{s_n\} \subset \Pi^{\pm}$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$g(t) = \lim_{n\to\infty} f\big(\delta_+(\tau_n, t)\big) \tag{9}$$

is well defined for each $t \in \mathbb{T}^*$. We can also have

$$\lim_{n\to\infty} g\big(\delta_-(\tau_n, t)\big) = f(t)$$

is also well defined for each $t \in \mathbb{T}^*$.

In fact, if $\Pi^{\pm} \notin \{\{t_0\}, \emptyset\}$, according to (9), we have

$$\lim_{n\to\infty} \big[g(t) - f\big(\delta_+(\tau_n, t)\big)\big] = 0 \tag{10}$$

is well defined for each $t \in \mathbb{T}^*$, then, for each $s \in \mathbb{T}^*$, let $t = \delta_-(\tau_n, s) \in \mathbb{T}^*(n \in \mathbb{N})$, we can obtain

$$\lim_{n\to\infty} \big[g\big(\delta_-(\tau_n, s)\big) - f(s)\big] = 0 \tag{11}$$

is also well defined for each $s \in \mathbb{T}^*$. In fact from (10), for any $\varepsilon > 0$, there exists $N_1 > 0$ such that $n > N_1$ implies $|g(t) - f(\delta_+(\tau_{N_1}, t))| < \varepsilon$ for each $t \in \mathbb{T}^*$. Let $s = \delta_+(\tau_{N_1}, t)$, and we obtain $|g(\delta_-(\tau_{N_1}, s)) - f(s)| < \varepsilon$. For $\Pi \notin \{\{t_0\}, \emptyset\}$, we have

$$\big\{s = \delta_+(\tau_{N_1}, t) : t \in \mathbb{T}^*, \tau_{N_1} \in \Pi^+\big\} = \big\{t : t \in \mathbb{T}^*\big\},$$

which shows that $|g(\delta_-(\tau_{N_1}, s)) - f(s)| < \varepsilon$ for each $s \in \mathbb{T}^*$, i.e., (11) holds.

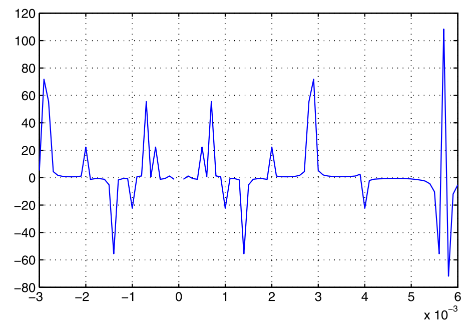

Figure 2 Graph of $\hat{F}(t)=\frac{1}{\cos(\frac{\ln|\sqrt{2}t|}{\ln(1/P_1)}\pi)+\cos(\frac{\ln|\sqrt{3}t|}{\ln(1/P_2)}\pi)}$ with $P_1=2, P_2=\sqrt[3]{2}$

Example 3.19 Recall Example 3.7, and consider the function

$$\hat{F}(t)=1\Big/\left[\cos\left(\frac{\ln|\sqrt{2}t|}{\ln(1/P_1)}\pi\right)+\cos\left(\frac{\ln|\sqrt{3}t|}{\ln(1/P_2)}\pi\right)\right],$$

where $P_1\neq P_2, P_1, P_2>1$ and $t\in\mathbb{T}^*=\mathbb{R}\backslash\{0\}$. One can observe that $\hat{F}(t)$ is almost automorphic under the shift operators. From Example 3.7, we obtain $\hat{F}(t)=\frac{1}{f_{P_1^2}(\sqrt{2}t)+f_{P_2^2}(\sqrt{3}t)}$ (see Fig. 2).

Remark 3.20 Let $\mathbb{T}=\{\pm\sqrt{n}:n\in\mathbb{N}\}$, then $\Pi^{\pm}=\{\sqrt{n}:n\in\mathbb{Z}^+\}$, according to Definition 3.8, we say that f is a bi-direction almost automorphic if, from every sequence $\{s_n\}\subset\Pi^{\pm}$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$g(t)=\lim_{n\to\infty}f\left(\sqrt{t^2+\tau_n^2}\right)$$

and

$$\lim_{n\to\infty}g\left(\sqrt{t^2-\tau_n^2}\right)=f(t)$$

are well defined for each $t\in\mathbb{T}^*$.

We can extend Definition 3.8 to Δ-almost automorphic functions.

Definition 3.9 Let $\mathbb{T}$ be a S-CCTS under shifts and $(\mathbb{T},\bar{\delta},\bar{\Pi})$ be compatible.

(i) Let $f:\mathbb{T}\to\mathbb{X}$ be a bounded continuous function and the shift $\bar{\delta}(\tau,t)$ is Δ-differentiable with rd-continuous bounded derivatives $\bar{\delta}^{\Delta}(\tau,t)$ for all $t\in\mathbb{T}^*$. We say that f is Δ-almost automorphic under shifts if, from every sequence $\{s_n\}\subset\bar{\Pi}$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$g(t)=\lim_{n\to\infty}f\big(\bar{\delta}(\tau_n,t)\big)\bar{\delta}^{\Delta}(\tau_n,t),$$

is well defined for each $t\in\mathbb{T}^*$. Denote by $\Delta-AA_{\bar{\delta}}(\mathbb{T},\mathbb{X})$ the set of all such functions.

(ii) A continuous function $f:\mathbb{T}\times D\to\mathbb{X}$ is said to be Δ-almost automorphic if $f(t,x)$ is Δ-almost automorphic in $t\in\mathbb{T}^*$ uniformly in $x\in D$. Denote by $\Delta-AA_{\bar{\delta}}(\mathbb{T}\times D,\mathbb{X})$ the set of all such functions.

Remark 3.21 From Definitions 3.2 and 3.9, we obtain:

(1) if $\mathbb{T}$ is a positive-direction S-CCTS, we say f is a positive-direction Δ-almost automorphic function;
(2) if $\mathbb{T}$ is a negative-direction S-CCTS, we say f is a negative-direction Δ-almost automorphic function;
(3) if $\mathbb{T}$ is a bi-direction S-CCTS, we say f is a bi-direction Δ-almost automorphic function.

Remark 3.22 By (3) of Remark 3.21 and a similar discussion in Remark 3.18, we can obtain the following definition.

Let $\mathbb{T}$ be a bi-direction S-CCTS under shifts $\delta_{\pm}$. Assume $f : \mathbb{T} \to \mathbb{X}$ is a bounded continuous function and the shifts $\delta_{\pm}(\tau, t)$ are Δ-differentiable with *rd*-continuous bounded derivatives $\delta_{\pm}^{\Delta}(\tau, t)$ for all $t \in \mathbb{T}^*$. We say that f is bi-direction Δ-almost automorphic if, from every sequence $\{s_n\} \subset \Pi^{\pm}$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$g(t) = \lim_{n\to\infty} f\big(\delta_+(\tau_n, t)\big)\delta_+^{\Delta}(\tau_n, t),$$

is well defined for each $t \in \mathbb{T}^*$ and

$$\lim_{n\to\infty} g\big(\delta_-(\tau_n, t)\big)\delta_-^{\Delta}(\tau_n, t) = f(t)$$

for each $t \in \mathbb{T}^*$.

Example 3.23 Recall Example 3.10, and on the time scale $(\sqrt{5})^{\mathbb{Z}} = \{(\sqrt{5})^n, n \in \mathbb{Z}\}$, consider the following function:

$$\hat{G}(t) = 1 \Big/ \left[\frac{a}{t} + \frac{b}{(-1)^{\log_{\sqrt{5}} t} t}\right], \quad a, b \in \mathbb{R}\backslash\{0\}, a \neq b.$$

One can observe that $\hat{G}(t)$ is Δ-almost automorphic under the shifts $\delta_{\pm}$. From Example 3.10, we obtain $\hat{G}(t) = \frac{1}{G(t)}$.

3.4 Stepanov almost automorphic functions on S-CCTS

In this subsection, by considering the Stepanov space on time scales with the distance (7), we introduce the concept of Stepanov almost automorphic functions (S_L^p-almost automorphic functions) on S-CCTS.

Definition 3.10 Let $\mathbb{T}$ be a S-CCTS under shifts and $(\mathbb{T}, \bar{\delta}, \bar{\Pi})$ be compatible.

(i) We say $f \in L_{\text{loc}}^p(\mathbb{T}, \mathbb{X})$ S_L^p-almost automorphic function under shifts operators if, from every sequence $\{s_n\} \subset \bar{\Pi}$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty}\left(\frac{1}{|L|}\int_{\widehat{[t,\bar{\delta}(L,t)]}_{\mathbb{T}^*}} \big\|f\big(\bar{\delta}(\tau_n, s)\big) - g(s)\big\|^p \Delta s\right)^{\frac{1}{p}} = 0, \tag{12}$$

is well defined for each $t \in \mathbb{T}^*$. Denote by $S_L^p AA_{\bar{\delta}}(\mathbb{T}, \mathbb{X})$ the set of all such functions.

(ii) A continuous function $f \in L_{\text{loc}}^p(\mathbb{T} \times D, \mathbb{X})$ is said to be almost automorphic if $f(t, x)$ is almost automorphic in $t \in \mathbb{T}^*$ uniformly in $x \in D$. Denote by $S_L^p AA_{\bar{\delta}}(\mathbb{T} \times D, \mathbb{X})$ the set of all such functions.

Remark 3.24 From Definitions 3.2 and 3.10, we obtain:

(1) if $\mathbb{T}$ is a positive-direction S-CCTS, we say f is a positive-direction S_L^p-almost automorphic function;
(2) if $\mathbb{T}$ is a negative-direction S-CCTS, we say f is a negative-direction S_L^p-almost automorphic function;
(3) if $\mathbb{T}$ is a bi-direction S-CCTS, we say f is a bi-direction S_L^p-almost automorphic function.

Remark 3.25 By (3) of Remark 3.24 and a similar discussion in Remark 3.18, we can obtain the following definition.

Let $\mathbb{T}$ be a bi-direction S-CCTS under shifts $\delta_\pm$ and $f \in L^p_{\mathrm{loc}}(\mathbb{T},\mathbb{X})$. We say that f is bi-direction S_L^p-almost automorphic if, from every sequence $\{s_n\} \subset \Pi^\pm$, we can extract a subsequence $\{\tau_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty}\left(\frac{1}{|L|}\int_t^{\delta_+(L,t)}\left\|f\bigl(\delta_+(\tau_n,s)\bigr)-g(s)\right\|^p\Delta s\right)^{\frac{1}{p}}=0$$

is well defined for each $t \in \mathbb{T}^*$ and

$$\lim_{n\to\infty}\left(\frac{1}{|L|}\int_{\delta_-(L,t)}^{t}\left\|g\bigl(\delta_-(\tau_n,s)\bigr)-f(s)\right\|^p\Delta s\right)^{\frac{1}{p}}=0$$

for each $t \in \mathbb{T}^*$.

Remark 3.26 Let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, then $\Pi^\pm = \{q^n, q > 1, n \in \mathbb{Z}^+\}$. We say that f is bi-direction S_L^p-almost automorphic if, from every sequence $\{s_n\} \subset \Pi^\pm$, we can extract a subsequence $\{\tau_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty}\left(\frac{1}{q}\int_t^{qt}\left\|f(\tau_n s)-g(s)\right\|^p\Delta s\right)^{\frac{1}{p}}=0$$

is well defined for each $t \in \mathbb{T}^*$ and

$$\lim_{n\to\infty}\left(\frac{1}{q}\int_{\frac{t}{q}}^{t}\left\|g(s/\tau_n)-f(s)\right\|^p\Delta s\right)^{\frac{1}{p}}=0$$

for each $t \in \mathbb{T}^*$.

We can extend Definition 3.10 to Δ-S_L^p-almost automorphic functions.

Definition 3.11 Let $\mathbb{T}$ be a S-CCTS under shifts and $(\mathbb{T},\bar{\delta},\bar{\Pi})$ be compatible.

(i) Let $f \in L^p_{\mathrm{loc}}(\mathbb{T},\mathbb{X})$ and the shift $\bar{\delta}(\tau,t)$ be Δ-differentiable with *rd*-continuous bounded derivatives $\bar{\delta}^\Delta(\tau,t)$ for all $t \in \mathbb{T}^*$. We say that f is Δ-S_L^p-almost automorphic under shifts if, from every sequence $\{s_n\} \subset \bar{\Pi}$, we can extract a subsequence $\{\tau_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty}\left(\frac{1}{|L|}\int_{\widehat{[t,\bar{\delta}(L,t))}_{\mathbb{T}^*}}\left\|f\bigl(\bar{\delta}(\tau_n,s)\bigr)\bar{\delta}^\Delta(\tau_n,s)-g(s)\right\|^p\Delta s\right)^{\frac{1}{p}}=0$$

is well defined for each $t \in \mathbb{T}^*$. Denote by $\Delta - S_L^p AA_{\tilde{\delta}}(\mathbb{T},\mathbb{X})$ the set of all such functions.

(ii) A continuous function $f \in L_{\text{loc}}^p(\mathbb{T}\times D,\mathbb{X})$ is said to be Δ-S_L^p-almost automorphic if $f(t,x)$ is Δ-S_L^p-almost automorphic in $t \in \mathbb{T}^*$ uniformly in $x \in D$. Denote by $\Delta - S_L^p AA_{\tilde{\delta}}(\mathbb{T}\times D,\mathbb{X})$ the set of all such functions.

Remark 3.27 From Definitions 3.2 and 3.11, we obtain:

(1) if $\mathbb{T}$ is a positive-direction S-CCTS, we say f is a positive-direction Δ-S_L^p-almost automorphic function;

(2) if $\mathbb{T}$ is a negative-direction S-CCTS, we say f is a negative-direction Δ-S_L^p-almost automorphic function;

(3) if $\mathbb{T}$ is a bi-direction S-CCTS, we say f is a bi-direction Δ-S_L^p-almost automorphic function.

Remark 3.28 By (3) from Remark 3.24 and a similar discussion as in Remark 3.18, we can obtain the following definition.

Let $\mathbb{T}$ be a bi-direction S-CCTS under shifts $\delta_\pm$. Assume $f \in L_{\text{loc}}^p(\mathbb{T},\mathbb{X})$ and the shift $\delta_\pm(\tau,t)$ is Δ-differentiable with *rd*-continuous bounded derivatives $\delta_\pm^\Delta(\tau,t)$ for all $t \in \mathbb{T}^*$. We say that f is bi-direction Δ-S_L^p-almost automorphic if, from every sequence $\{s_n\} \subset \Pi^\pm$, we can extract a subsequence $\{\tau_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty}\left(\frac{1}{|L|}\int_t^{\delta_+(L,t)}\left\|f\left(\delta_+(\tau_n,s)\right)\delta_+^\Delta(\tau_n,s)-g(s)\right\|^p\Delta s\right)^{\frac{1}{p}}=0$$

is well defined for each $t \in \mathbb{T}^*$ and

$$\lim_{n\to\infty}\left(\frac{1}{|L|}\int_{\delta_-(L,t)}^{t}\left\|g\left(\delta_-(\tau_n,s)\right)\delta_-^\Delta(\tau_n,s)-f(s)\right\|^p\Delta s\right)^{\frac{1}{p}}=0$$

for each $t \in \mathbb{T}^*$.

Remark 3.29 Let $\mathbb{T}=\overline{q^{\mathbb{Z}}}$, then $\Pi^\pm=\{q^n, q>1, n\in\mathbb{Z}^+\}$. We say that f is bi-direction Δ-S_L^p-almost automorphic if, from every sequence $\{s_n\}\subset\Pi^\pm$, we can extract a subsequence $\{\tau_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty}\left(\frac{1}{q}\int_t^{qt}\left\|f(\tau_n s)\tau_n-g(s)\right\|^p\Delta s\right)^{\frac{1}{p}}=0$$

is well defined for each $t \in \mathbb{T}^*$ and

$$\lim_{n\to\infty}\left(\frac{1}{q}\int_{\frac{t}{q}}^{t}\left\|g(s/\tau_n)(1/\tau_n)-f(s)\right\|^p\Delta s\right)^{\frac{1}{p}}=0$$

for each $t \in \mathbb{T}^*$.

4 Conclusion

In the paper, we introduced the concept of S-CCTS to guarantee the closedness of time scales under translational and non-translational shifts. Based on this, we introduced several concepts of some special functions arising from dynamic equations including almost periodic functions, almost automorphic functions and their generalizations in the Stepanov sense. The concepts are suitable for irregular time scales like $q^{\mathbb{Z}}$, $(-q)^{\mathbb{Z}}$ and $\pm\mathbb{N}^{\frac{1}{2}}$, etc. Properties of these functions and their application to dynamic equations will be considered in future work.

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Competing interests
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Authors' contributions
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