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Positive solutions to p-Laplacian fractional differential equations with infinite-point boundary value conditions

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Abstract

This paper is devoted to an investigation of the multiple positive solutions to a class of infinite-point boundary value problems of nonlinear fractional differential equations coupled with the p-Laplacian operators and infinite-point boundary value conditions. By means of the properties of Green's function and fixed point theorems, we establish the suitable criteria to guarantee the existence of positive solutions. Finally, an example is given in order to illustrate the main results.

Keywords: Fractional differential equations; Infinite-point boundary conditions; p-Laplacian operators; Fixed point theorems

1 Introduction

With the advance of technology, researchers are not satisfied with the limitations of the integer order calculus anymore; therefore, fractional calculus—the expansion and extension of the integer one—is brought into the public. There is no doubt the study on nonlinear fractional differential equations has become an issue of focus and is widely being used in many fields, such as physics [1, 2], biology [3], energy [4], chemical engineering [5], pharmacokinetics [6].

Based on practical application, researchers established lots of fractional differential equation models, so how to get the solutions to the equations is a big challenge to them. As is well known, it is difficult to give the exact solution, so giving the properties of the solutions is a significant point.

A great number of researchers are focused on the existence of solutions for linear or nonlinear FODE with different boundary conditions. Take Zhang [7] for example, one studied the existence of minimal non-negative solution for a class of nonlinear fractional integro-differential equations by applying the cone theory and the monotone iterative technique:

$$\begin{cases} D_{0+}^{\alpha} x(t) + f(t, x(t), Tx(t), Sx(t)) = 0, & t \in [0, \infty), \\ x(0) = 0, & D_{0+}^{\alpha-1} x(\infty) = x^*, \end{cases}$$

where

$$(Tx)(t) = \int_0^t k(t, s)x(s) ds, \quad (Sx)(t) = \int_0^{\infty} h(t, s)x(s) ds,$$

and $1 < \alpha \leq 2, D_{0+}^\alpha$ is the Riemann–Liouville derivative, f, k, h are continuous functions.

Wang [8] considered the iterative positive solutions for a class of nonlocal fractional differential equations with nonlocal Hadamard integral and discrete boundary conditions by the monotone iterative method:

$$\begin{cases} {}^H D^\alpha x(t) + \delta(t)f(t, x(t)) = 0, & t \in (1, \infty), \\ x(1) = x'(1) = 0, \\ {}^H D^{\alpha-1}x(\infty) = a {}^H I^\beta x(\xi) + b \sum_{i=1}^{m-2} c_i x(\eta_i), \end{cases}$$

where $2 < \alpha \leq 3, a, b$ are real constants, $c_i (i = 1, 2, \dots, m - 2)$ are positive real constants, $1 < \xi < \eta_1 < \eta_2 < \dots < \eta_{m-2} < +\infty$, and ${}^H D^\alpha$ is the Hadamard fractional derivative.

As for Guo [9], they investigated the existence of solutions for the following infinite-point BVP with Caputo fractional derivative by Avery–Peterson’s fixed point theorem:

$$\begin{cases} {}^C D^\alpha x(t) + f(t, x(t), x'(t)) = 0, & 0 < t < 1, \\ x(0) = x''(0) = 0, & x'(1) = \sum_{j=1}^\infty \zeta_j x(\xi_j), \end{cases}$$

where $2 < \alpha \leq 3, \zeta_j \geq 0, 0 < \xi_1 < \xi_2 < \dots < \xi_i < \dots < 1$, and $\sum_{i=1}^\infty \zeta_i \xi_i < 1$.

Recently, some scientists have become interested in the study of the fractional differential equations with p-Laplacian operators [10–13]. For example, Xiping Liu [12, 13] obtained some new results on the existence of positive solutions for the four-point BVP with mixed fractional derivatives and p-Laplacian operator by a new method of lower and upper solutions which is based on the monotone iterative technique:

$$\begin{cases} D_{0+}^\beta (\phi_p({}^C D_{0+}^\alpha x(t))) = f(t, x(t), {}^C D_{0+}^\alpha x(t)), & t \in (0, 1), \\ {}^C D_{0+}^\alpha x(0) = x'(0) = 0, \\ x(1) = r_1 x(\eta), & {}^C D_{0+}^\alpha x(1) = r_2 {}^C D_{0+}^\alpha x(\xi), \end{cases}$$

where $1 < \alpha, \beta \leq 2, r_1, r_2 \geq 0, f \in C([0, 1] \times [0, +\infty) \times (-\infty, 0], [0, +\infty))$, ${}^C D_{0+}^\alpha$ is the Caputo derivative, D_{0+}^α is the Riemann–Liouville derivative, ϕ_p is the p-Laplacian operator which is defined as $\phi_p(s) = |s|^{p-2}s (p > 1)$, ϕ_q is the inverse function of ϕ_p and $\frac{1}{p} + \frac{1}{q} = 1$.

In this paper, we consider a class of nonlinear fractional differential equations coupled with the p-Laplacian operator and infinite-point boundary value conditions:

$$\begin{cases} -D_{0+}^\beta (\phi_p(D_{0+}^\alpha x(t) - \lambda x(t))) = f(t, x(t)), & t \in (0, 1], \\ \lim_{t \rightarrow 0+} t^{1-\alpha} x(t) = \sum_{i=1}^\infty \mu_i x(\xi_i), \\ \lim_{t \rightarrow 0+} t^{2-\beta} (\phi_p(D_{0+}^\alpha x(t) - \lambda x(t))) = \phi_p(D_{0+}^\alpha x(1) - \lambda x(1)) = 0, \end{cases} \tag{1}$$

where $0 < \alpha \leq 1, 1 < \beta \leq 2, \lambda < 0, \mu_i \geq 0, 0 < \xi_i < 1, i \in \mathbb{R}^+, \sum_{i=1}^\infty \mu_i \xi_i^{\alpha-1} < 1, f \in C([0, 1] \times [0, +\infty), [0, +\infty))$, ϕ_p is the p-Laplacian operator and D^α, D^β are the Riemann–Liouville fractional derivatives.

To the best of our knowledge, few researchers studied fractional differential equations with p-Laplacian by the Riemann–Liouville derivative; this condition coupled with values

at infinite number of points is not covered in the previous situations, and it is more general for the fractional differential equation models. By means of the properties of Green’s function, Krasnosel’skii fixed point theorem, the corollary of Krasnosel’skii fixed point theorem and Avery–Peterson fixed point theorem, we establish the suitable criteria to warrant the existence of the positive solutions for the BVP (1).

The rest of the paper is organized as follows. In Sect. 2, we list several basic definitions and lemmas to prepare for the further study in this paper. On top of that, the main result is showed in Sect. 3. In Sect. 4, we give an example to illustrate our main results.

2 Preliminaries

As is well known, $C[a, b]$ is the Banach space of all continuous real functions defined on $[a, b]$ with the norm $\|x\| = \sup_{t \in [a, b]} |x(t)|$ and $L^1[a, b]$ is for the Lebesgue integrable real functions defined on $[a, b]$. In this paper, we consider the following space:

$$C_{1-\alpha}[0, 1] = \{x(t) : t^{1-\alpha}x(t) \in C[0, 1], 0 < \alpha \leq 1\},$$

endowed with its norm $\|x\|_{1-\alpha} = \sup_{t \in [0, 1]} t^{1-\alpha}|x(t)|$. Obviously, $C_{1-\alpha}[0, 1]$ is also a Banach space; moreover, $C_{1-\alpha}[0, 1] \subset L^1[0, 1]$.

Definition 2.1 ([14]) Let $\alpha > 0$, the Riemann–Liouville fractional integral of the function f is defined by

$$I_{0^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

Definition 2.2 ([14]) Let $\alpha > 0$, the Riemann–Liouville fractional derivative of the function f is defined by

$$D_{0^+}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

where n is the smallest integer greater than or equal to α .

Definition 2.3 ([15]) The two-parameter Mittag–Leffler function is defined by

$$E_{\alpha, \beta}(\zeta) = \sum_{k=0}^\infty \frac{\zeta^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0, \zeta \in \mathbb{R}.$$

Definition 2.4 ([16]) If the map T satisfies

$$T(ax + (1-a)y) \leq (\geq) aT(x) + (1-a)T(y), \quad x, y \in P, a \in [0, 1],$$

then T is called a non-negative convex (concave) operator on a cone P .

Lemma 2.1 ([17]) Assume $\alpha > 0, u \in C(0, 1) \cap L(0, 1)$, then

$$I_{0^+}^\alpha D_{0^+}^\alpha f(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R}, i = 1, 2, \dots, n$ with n is the smallest integer greater than or equal to α .

Lemma 2.2 ([18]) *Assume $\alpha \in (0, 1], \beta > 0, \lambda$ is a negative constant, then the following conclusions hold:*

- (i) $\lim_{t \rightarrow 0^+} E_{\alpha, \beta}(\lambda t^\alpha) = \frac{1}{\Gamma(\beta)}$,
- (ii) $E_{\alpha, \alpha}(\lambda t^\alpha) > 0$ for $\forall t \in [0, +\infty)$,
- (iii) $E_{\alpha, \alpha}(\lambda t^\alpha)$ is decreasing in t for $\forall t \in (0, +\infty)$,
- (iv) $E_{\alpha, \alpha}(\lambda t^\alpha) \leq E_{\alpha, \alpha+1}(\lambda t^\alpha)$ for $\forall t \in (0, +\infty)$.

Lemma 2.3 *Assume $\sigma(t) \in C(0, 1], 0 < \alpha \leq 1$, then the problem*

$$\begin{cases} D_{0^+}^\alpha x(t) - \lambda x(t) = \sigma(t), & t \in (0, 1], \\ \lim_{t \rightarrow 0^+} t^{1-\alpha} x(t) = \sum_{i=1}^\infty \mu_i x(\xi_i), \end{cases} \tag{2}$$

has an unique solution

$$x(t) = \int_0^1 G(t, s) \sigma(s) ds, \tag{3}$$

where

$$G(t, s) = \begin{cases} \frac{\Gamma(\alpha) E_{\alpha, \alpha}(\lambda t^\alpha) \sum_{s \leq \xi_i} \mu_i (\xi_i - s)^{\alpha-1} E_{\alpha, \alpha}(\lambda (\xi_i - s)^\alpha)}{t^{1-\alpha} (1 - \Gamma(\alpha) \sum_{i=1}^\infty \mu_i \xi_i^{\alpha-1} E_{\alpha, \alpha}(\lambda \xi_i^\alpha))} + \frac{E_{\alpha, \alpha}(\lambda (t - \alpha)^\alpha)}{(t - s)^{1-\alpha}}, & 0 \leq s < t \leq 1, \\ \frac{\Gamma(\alpha) E_{\alpha, \alpha}(\lambda t^\alpha) \sum_{s \leq \xi_i} \mu_i (\xi_i - s)^{\alpha-1} E_{\alpha, \alpha}(\lambda (\xi_i - s)^\alpha)}{t^{1-\alpha} (1 - \Gamma(\alpha) \sum_{i=1}^\infty \mu_i \xi_i^{\alpha-1} E_{\alpha, \alpha}(\lambda \xi_i^\alpha))}, & 0 < t \leq s \leq 1. \end{cases} \tag{4}$$

Proof According to ([19]), if $\lim_{t \rightarrow 0^+} t^{1-\alpha} x(t) = c_0$, we can easily get

$$x(t) = c_0 \Gamma(\alpha) t^{\alpha-1} E_{\alpha, \alpha}(\lambda t^\alpha) + \int_0^t (t - s)^{\alpha-1} E_{\alpha, \alpha}(\lambda (t - \alpha)^\alpha) \sigma(s) ds.$$

In particular, if we choose $c_0 = \sum_{i=1}^\infty \mu_i x(\xi_i)$, by simple calculation,

$$c_0 = \frac{\sum_{i=1}^\infty \mu_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} E_{\alpha, \alpha}(\lambda (\xi_i - s)^\alpha) \sigma(s) ds}{1 - \Gamma(\alpha) \sum_{i=1}^\infty \mu_i \xi_i^{\alpha-1} E_{\alpha, \alpha}(\lambda \xi_i^\alpha)}.$$

Hence,

$$\begin{aligned} x(t) &= \frac{\Gamma(\alpha) t^{\alpha-1}}{1 - \Gamma(\alpha) \sum_{i=1}^\infty \mu_i \xi_i^{\alpha-1} E_{\alpha, \alpha}(\lambda \xi_i^\alpha)} \\ &\quad \times \sum_{i=1}^\infty \mu_i E_{\alpha, \alpha}(\lambda t^\alpha) \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} E_{\alpha, \alpha}(\lambda (\xi_i - s)^\alpha) \sigma(s) ds \\ &\quad + \int_0^t (t - s)^{\alpha-1} E_{\alpha, \alpha}(\lambda (t - \alpha)^\alpha) \sigma(s) ds \\ &= \int_0^1 \frac{\Gamma(\alpha) E_{\alpha, \alpha}(\lambda t^\alpha) \sum_{s \leq \xi_i} \mu_i (\xi_i - s)^{\alpha-1} E_{\alpha, \alpha}(\lambda (\xi_i - s)^\alpha)}{t^{1-\alpha} (1 - \Gamma(\alpha) \sum_{i=1}^\infty \mu_i \xi_i^{\alpha-1} E_{\alpha, \alpha}(\lambda \xi_i^\alpha))} \sigma(s) ds \\ &\quad + \int_0^t \frac{E_{\alpha, \alpha}(\lambda (t - \alpha)^\alpha)}{(t - s)^{1-\alpha}} \sigma(s) ds \end{aligned}$$

$$= \int_0^1 G(t,s)\sigma(s) ds, \tag{5}$$

where $G(t,s)$ is given by (4).

The proof is finished. □

Lemma 2.4 *Assume $\alpha \in (0, 1], \lambda$ is a negative constant, then*

- (i) $G(t,s) > 0$ for $t \in (0, 1], s \in [0, 1)$,
- (ii) $t^{1-\alpha} G(t,s) \geq \frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)^2 \sum_{s \leq \xi_i} \mu_i \xi_i^{\alpha-1}}{1 - \Gamma(\alpha) \sum_{i=1}^{\infty} \mu_i \xi_i^{\alpha-1} E_{\alpha,\alpha}(\lambda)} \triangleq m$ for $t,s \in [0, 1]$,
- (iii) $\min_{t \in [0,1]} t^{1-\alpha} G(t,s) \geq \rho \max_{t \in [0,1]} t^{1-\alpha} G(t,s)$ for $\rho \in (0, 1)$,
- (iv) $t^{1-\alpha} \int_0^1 G(t,s) ds \leq (\Gamma(\alpha)(1 - \sum_{i=0}^{\infty} \mu_i \xi_i^{\alpha-1}))^{-1} \triangleq n$ for $t,s \in [0, 1]$.

Proof (i) By simple calculation,

$$1 - \Gamma(\alpha) \sum_{i=1}^{\infty} \mu_i \xi_i^{\alpha-1} E_{\alpha,\alpha}(\lambda \xi_i^\alpha) \geq 1 - \sum_{i=1}^{\infty} \mu_i \xi_i^{\alpha-1} > 0.$$

In addition, according to Lemma 2.2, we can easily see that (i) holds.

(ii) Since $(t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) \geq 0$ for all $0 \leq s < t \leq 1$,

$$\begin{aligned} t^{1-\alpha} G(t,s) &\geq \frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha) \sum_{s \leq \xi_i} \mu_i (\xi_i - s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(\xi_i - s)^\alpha)}{1 - \Gamma(\alpha) \sum_{i=1}^{\infty} \mu_i \xi_i^{\alpha-1} E_{\alpha,\alpha}(\lambda \xi_i^\alpha)} \\ &\geq \frac{\Gamma(\alpha)(E_{\alpha,\alpha}(\lambda))^2 \sum_{s \leq \xi_i} \mu_i \xi_i^{\alpha-1}}{1 - \Gamma(\alpha)E_{\alpha,\alpha}(\lambda) \sum_{i=1}^{\infty} \mu_i \xi_i^{\alpha-1}} \\ &\triangleq m > 0. \end{aligned}$$

(iii) As $t^{1-\alpha} G(t,s)$ is increasing in s and decreasing in t

$$\max_{t \in [0,1]} t^{1-\alpha} G(t,s) = \frac{\sum_{s \leq \xi_i} \mu_i (\xi_i - s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(\xi_i - s)^\alpha)}{1 - \Gamma(\alpha) \sum_{i=1}^{\infty} \mu_i \xi_i^{\alpha-1} E_{\alpha,\alpha}(\lambda \xi_i^\alpha)}.$$

According to Lemma 2.4(i), we have

$$\begin{aligned} \min_{t \in [0,1]} t^{1-\alpha} G(t,s) &\geq \Gamma(\alpha)E_{\alpha,\alpha}(\lambda) \max_{t \in [0,1]} t^{1-\alpha} G(t,s) \\ &\triangleq \rho \max_{t \in [0,1]} t^{1-\alpha} G(t,s), \end{aligned}$$

where $\rho = \Gamma(\alpha)E_{\alpha,\alpha}(\lambda)$, so $0 < \rho < 1$.

(iv) We divide this proof into two parts:

Part 1: For $0 \leq s \leq t \leq 1$,

$$\begin{aligned} t^{1-\alpha} \int_0^1 G(t,s) ds &= \int_0^t \frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha) \sum_{s \leq \xi_i} \mu_i (\xi_i - s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(\xi_i - s)^\alpha)}{1 - \Gamma(\alpha) \sum_{i=1}^{\infty} \mu_i \xi_i^{\alpha-1} E_{\alpha,\alpha}(\lambda \xi_i^\alpha)} ds \\ &\quad + \int_0^t t^{1-\alpha} (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) ds \\ &\leq \frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha) \sum_{i=1}^{\infty} \mu_i \xi_i^{\alpha-1}}{1 - \Gamma(\alpha) \sum_{i=1}^{\infty} \mu_i \xi_i^{\alpha-1} E_{\alpha,\alpha}(\lambda \xi_i^\alpha)} + t E_{\alpha,\alpha+1}(\lambda t^\alpha) \end{aligned}$$

$$\begin{aligned} &\leq \frac{\sum_{i=1}^{\infty} \mu_i \xi_i^{\alpha-1}}{\Gamma(\alpha)(1 - \sum_{i=1}^{\infty} \mu_i \xi_i^{\alpha-1})} + \frac{1}{\Gamma(\alpha)} \\ &= \frac{1}{\Gamma(\alpha)(1 - \sum_{i=1}^{\infty} \mu_i \xi_i^{\alpha-1})} \triangleq n. \end{aligned}$$

Part 2: For $0 \leq t \leq s \leq 1$,

$$\begin{aligned} t^{1-\alpha} \int_0^1 G(t,s) ds &= \int_0^1 \frac{\Gamma(\alpha) E_{\alpha,\alpha}(\lambda t^\alpha) \sum_{s \leq \xi_i} \mu_i (\xi_i - s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (\xi_i - s)^\alpha)}{1 - \Gamma(\alpha) \sum_{i=1}^{\infty} \mu_i \xi_i^{\alpha-1} E_{\alpha,\alpha}(\lambda \xi_i^\alpha)} ds \\ &\leq \frac{\Gamma(\alpha) E_{\alpha,\alpha}(\lambda t^\alpha) \sum_{i=1}^{\infty} \mu_i \xi_i^{\alpha-1}}{1 - \Gamma(\alpha) \sum_{i=1}^{\infty} \mu_i \xi_i^{\alpha-1} E_{\alpha,\alpha}(\lambda \xi_i^\alpha)} \\ &\leq \frac{\sum_{i=1}^{\infty} \mu_i \xi_i^{\alpha-1}}{\Gamma(\alpha)(1 - \sum_{i=1}^{\infty} \mu_i \xi_i^{\alpha-1})} \\ &< \frac{1}{\Gamma(\alpha)(1 - \sum_{i=1}^{\infty} \mu_i \xi_i^{\alpha-1})} \triangleq n. \end{aligned}$$

The proof is completed. □

Lemma 2.5 Assume $h(t) \in C(0, 1]$, $1 < \beta \leq 2$, then the problem

$$\begin{cases} -D_{0+}^\beta u(t) = h(t), & t \in (0, 1], \\ \lim_{t \rightarrow 0+} t^{2-\beta} u(t) = u(1) = 0, \end{cases} \tag{6}$$

has an unique solution

$$u(t) = \int_0^1 H(t,s)h(s) ds, \tag{7}$$

where

$$H(t,s) = \frac{1}{\Gamma(\beta)} \begin{cases} t^{\beta-1}(1-s)^{\beta-1} - (t-s)^{\beta-1}, & 0 \leq s < t \leq 1, \\ t^{\beta-1}(1-s)^{\beta-1}, & 0 < t \leq s \leq 1. \end{cases} \tag{8}$$

Proof According to Lemma 2.1, we can easily get

$$u(t) = -I_{0+}^\beta h(t) + c_1 t^{\beta-1} + c_2 t^{\beta-2},$$

by the condition (6), we obtain

$$c_1 = I_{0+}^\beta h(1), \quad c_2 = 0.$$

Hence,

$$u(t) = \frac{t^{\beta-1}}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s) ds - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h(s) ds = \int_0^1 H(t,s)h(s) ds,$$

where $H(t,s)$ is given by (8).

The proof is finished. □

Lemma 2.6 ([20]) *Assume $\beta \in (1, 2]$, then*

- (i) $H(t, s) > 0$ for $t \in (0, 1], s \in [0, 1)$,
- (ii) $H(t, s) = H(1 - s, 1 - t)$ for $t, s \in (0, 1)$,
- (iii) $t^{\beta-1}(1 - t)s(1 - s)^{\beta-1} \leq \Gamma(\beta)H(t, s) \leq (\beta - 1)s(1 - s)^{\beta-1}$ for $t \in (0, 1], s \in [0, 1)$,
- (iv) $t^{\beta-1}(1 - t)s(1 - s)^{\beta-1} \leq \Gamma(\beta)H(t, s) \leq (\beta - 1)(1 - t)t^{\beta-1}$ for $t \in (0, 1], s \in [0, 1)$.

Lemma 2.7 *Suppose $h(t) = f(t, x(t)), \sigma(t) = \phi_q(u(t))$ hold, then we obtain the unique solution of BVP (1):*

$$x(t) = \int_0^1 G(t, s)\phi_q\left(\int_0^1 H(s, \tau)f(\tau, x(\tau)) d\tau\right) ds. \tag{9}$$

3 Main results

In this section, we establish the related condition to warrant the existence of the multiple positive solutions for BVP (1) by means of the properties of Green’s function we proved in the last section and the well-known Krasnosel’skii fixed point theorem, the corollary of Krasnosel’skii fixed point theorem and the Avery–Peterson fixed point theorem.

Let $E = \{x(t) : x(t) \in C_{1-\alpha}[0, 1]\}$. Define $P \subset E$,

$$P = \left\{x(t) \in E : x(t) \geq 0, \min_{t \in [0,1]} t^{1-\alpha}x(t) \geq \rho \max_{t \in [0,1]} t^{1-\alpha}x(t)\right\},$$

where ρ is defined in Lemma 2.4.

Suppose $T : P \rightarrow E$ is the operator defined by

$$Tx(t) := \int_0^1 G(t, s)\phi_q\left(\int_0^1 H(s, \tau)f(\tau, x(\tau)) d\tau\right) ds.$$

Throughout this section, suppose the following assumption holds:

$$(H_0) |f(t, x)| \leq M.$$

Lemma 3.1 *$T : P \rightarrow P$ is a completely continuous operator.*

Proof Step 1: We show that $T(P) \subseteq P$.

For $\forall x \in P$, since the continuity of $G(t, s), H(t, s), f(t, x(t))$, it is easy to get $t^{1-\alpha}Tx(t) \in C[0, 1]$. According to Lemma 2.4 and Lemma 2.6, we obtain $G(t, s), H(t, s) \geq 0$, thanks to $f(t, x(t))$ being non-negative, therefore, $Tx(t) \geq 0$. Furthermore,

$$\begin{aligned} \min_{t \in [0,1]} t^{1-\alpha}Tx(t) &= \min_{t \in [0,1]} t^{1-\alpha} \int_0^1 G(t, s)\phi_q\left(\int_0^1 H(s, \tau)f(\tau, x(\tau)) d\tau\right) ds \\ &= \int_0^1 \min_{t \in [0,1]} t^{1-\alpha}G(t, s)\phi_q\left(\int_0^1 H(s, \tau)f(\tau, x(\tau)) d\tau\right) ds \\ &\geq \int_0^1 \rho \max_{t \in [0,1]} t^{1-\alpha}G(t, s)\phi_q\left(\int_0^1 H(s, \tau)f(\tau, x(\tau)) d\tau\right) ds \\ &= \rho \max_{t \in [0,1]} Tx(t). \end{aligned}$$

Hence, $T(P) \subseteq P$.

Step 2: We will prove that T is uniformly bounded. We have

$$\begin{aligned} \|Tx(t)\|_{1-\alpha} &= \left| t^{1-\alpha} \int_0^1 G(t,s)\phi_q \left(\int_0^1 H(s,\tau)f(\tau,x(\tau)) d\tau \right) ds \right| \\ &\leq \int_0^1 |t^{1-\alpha} G(t,s)|\phi_q \left(\int_0^1 |H(s,\tau)||f(\tau,x(\tau))| d\tau \right) ds \\ &< n\phi_q \left(\frac{(\beta-1)M}{\Gamma(\beta)} \right) \\ &= n\phi_q \left(\frac{M}{\Gamma(\beta-1)} \right) \triangleq M_1. \end{aligned}$$

Therefore, $\|Tx(t)\|_{1-\alpha} < M_1$, which means T is uniformly bounded.

Step 3: We will give the proof of the equicontinuity of T .

Obviously, $t^{1-\alpha}G(t,s)$ is uniformly continuous in the interval $[0, 1] \times [0, 1]$. Therefore, for all $\varepsilon > 0$, there exists $\delta > 0$, without loss of generality, we choose $t_1, t_2 \in [0, 1]$, when $|t_1 - t_2| < \delta$, we have

$$|G(t_1,s) - G(t_2,s)| < \frac{\varepsilon}{\phi_q(M) \int_0^1 \phi_q \left(\int_0^1 H(s,\tau) d\tau \right) ds}.$$

Thus,

$$\begin{aligned} &\|Tx(t_1) - Tx(t_2)\|_{1-\alpha} \\ &= \left| t_1^{1-\alpha} \int_0^1 G(t_1,s)\phi_q \left(\int_0^1 H(s,\tau)f(\tau,x(\tau)) d\tau \right) ds \right. \\ &\quad \left. - t_2^{1-\alpha} \int_0^1 G(t_2,s)\phi_q \left(\int_0^1 H(s,\tau)f(\tau,x(\tau)) d\tau \right) ds \right| \\ &\leq \int_0^1 |t_1^{1-\alpha}G(t_1,s) - t_2^{1-\alpha}G(t_2,s)|\phi_q \left(\int_0^1 |H(s,\tau)||f(\tau,x(\tau))| d\tau \right) ds \\ &< \frac{\varepsilon}{\phi_q(M) \int_0^1 \phi_q \left(\int_0^1 H(s,\tau) d\tau \right) ds} \phi_q(M) \int_0^1 \phi_q \left(\int_0^1 H(s,\tau) d\tau \right) ds \\ &= \varepsilon. \end{aligned}$$

So, T is equicontinuous.

According to the Arzelà–Ascoli theorem, T is a completely continuous operator.

The proof is finished. □

Define $P_k = \{x \in P : \|x\|_{1-\alpha} < r_k\}$ and then we define the related boundary $\partial P_k = \{x \in P : \|x\|_{1-\alpha} = r_k\}$ where k is the positive integer.

Theorem 3.1 ([21] Krasnosel'skii fixed point theorem) *Suppose E is a Banach space, and $P \subset E$ is a cone. Assume P_1, P_2 are two open and bounded subsets of E coupled with $0 \in P_1, \overline{P_1} \subset P_2$ and suppose $T : P \cap (\overline{P_2} \setminus P_1) \rightarrow P$ is a completely continuous operator such that one of the following conditions holds:*

- (L₁) $\|Tx\| \geq \|x\|$, if $x \in P \cap \partial P_1$, and $\|Tx\| \leq \|x\|$, if $x \in P \cap \partial P_2$,

(L₂) $\|Tx\| \leq \|x\|$, if $x \in P \cap \partial P_1$, and $\|Tx\| \geq \|x\|$, if $x \in P \cap \partial P_2$.

Then T has at least one fixed point in $P \cap (\overline{P_2} \setminus P_1)$.

Theorem 3.2 *If there exist $0 < r_1 < r_2, N_1 \geq (\rho m \int_0^1 \phi_q(\int_0^1 H(s, \tau) d\tau) ds)^{-1}, 0 \leq N_2 \leq (n \int_0^1 \phi_q(\int_0^1 H(s, \tau) d\tau) ds)^{-1}$ and $N_1 r_1 < N_2 r_2$ such that the following conditions hold:*

(L₃) $f(t, x) \geq N_1 r_1, (t, x) \in [0, 1] \times [0, r_1]$,

(L₄) $f(t, x) \leq N_2 r_2, (t, x) \in [0, 1] \times [0, r_2]$,

then T has at least one fixed point in $P \cap (\overline{P_2} \setminus P_1)$. In addition, $r_1 \leq \|x\|_{1-\alpha} \leq r_2$.

Proof Step 1: Let $P_1 = \{x \in P : \|x\|_{1-\alpha} < r_1\}$, by assumption (L₃), for all $t \in [0, 1]$,

$$\begin{aligned} \|Tx\|_{1-\alpha} &= t^{1-\alpha} \int_0^1 G(t, s) \phi_q \left(\int_0^1 H(s, \tau) f(\tau, x(\tau)) d\tau \right) ds \\ &\geq N_1 r_1 \int_0^1 \rho m \phi_q \left(\int_0^1 H(s, \tau) d\tau \right) ds \\ &\geq r_1 = \|x\|_{1-\alpha}. \end{aligned}$$

Then we can get $\|Tx\|_{1-\alpha} \geq \|x\|_{1-\alpha} (\forall x \in P \cap \partial P_1)$.

Step 2: Let $P_2 = \{x \in P : \|x\|_{1-\alpha} < r_2\}$, by assumption (L₄), for all $t \in [0, 1]$,

$$\begin{aligned} \|Tx\|_{1-\alpha} &= t^{1-\alpha} \int_0^1 G(t, s) \phi_q \left(\int_0^1 H(s, \tau) f(\tau, x(\tau)) d\tau \right) ds \\ &\leq N_2 r_2 \int_0^1 n \phi_q \left(\int_0^1 H(s, \tau) d\tau \right) ds \\ &\leq r_2 = \|x\|_{1-\alpha}. \end{aligned}$$

It is easy to obtain $\|Tx\|_{1-\alpha} \leq \|x\|_{1-\alpha} (\forall x \in P \cap \partial P_2)$.

To conclude, the above proof satisfies the condition (L₁) of Theorem 3.1, that is to say, T has at least one fixed point in $P \cap (\overline{P_2} \setminus P_1)$. In addition, $r_1 \leq \|x\|_{1-\alpha} \leq r_2$.

The proof is finished. □

Theorem 3.3 ([22]) *Suppose E is a Banach space, and $P \subset E$ is a cone. Assume P_3, P_4 are two open and bounded subsets of E and $T : \overline{P_3} \rightarrow P$ is a completely continuous operator. If $\|Tx\|_{1-\alpha} < r_3$, for $x \in \partial P_3$ and $\|Tx\|_{1-\alpha} \geq r_4$, for $x \in \partial P_4$ hold, then T has at least two fixed points in $\overline{P_3}$, and*

$$0 \leq \|x_1\|_{1-\alpha} < r_3 < \|x_2\|_{1-\alpha} \leq r_4.$$

Theorem 3.4 *If there exist $0 < r_3 < r_4, 0 \leq N_3 < (\rho m \int_0^1 \phi_q(\int_0^1 H(s, \tau) d\tau) ds)^{-1}, N_4 \geq (n \int_0^1 \phi_q(\int_0^1 H(s, \tau) d\tau) ds)^{-1}$ such that the following conditions hold:*

(L₅) $f(t, x) < N_3 r_3, x \in \partial P_3$,

(L₆) $f(t, x) \geq N_4 r_4, x \in \partial P_4$,

then T has at least two fixed points in $\overline{P_3}$, and

$$0 \leq \|x_1\|_{1-\alpha} < r_3 < \|x_2\|_{1-\alpha} \leq r_4.$$

Proof On the one hand, $P_3 = \{x \in P : \|x\|_{1-\alpha} < r_3\}$, by assumption (L₅), for all $t \in [0, 1]$,

$$\begin{aligned} \|Tx\|_{1-\alpha} &= t^{1-\alpha} \int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau)f(\tau,x(\tau))d\tau\right)ds \\ &\leq N_3r_3 \int_0^1 n\phi_q\left(\int_0^1 H(s,\tau)d\tau\right)ds \\ &< r_3. \end{aligned}$$

Then we can get $\|Tx\|_{1-\alpha} < r_3$ ($\forall x \in \partial P_3$).

On the other hand, $P_4 = \{x \in P : \|x\|_{1-\alpha} < r_4\}$, by assumption (L₆), for all $t \in [0, 1]$,

$$\begin{aligned} \|Tx\|_{1-\alpha} &= t^{1-\alpha} \int_0^1 G(t,s)\phi_q\left(\int_0^1 H(s,\tau)f(\tau,x(\tau))d\tau\right)ds \\ &\geq N_4r_4 \int_0^1 \rho m\phi_q\left(\int_0^1 H(s,\tau)d\tau\right)ds \\ &\geq r_4. \end{aligned}$$

Obviously, $\|Tx\|_{1-\alpha} \geq r_4$ ($\forall x \in \partial P_4$) holds.

To conclude, the above proof satisfies the condition of Corollary 3.3, that is to say, T has at least two fixed points in $\overline{P_3}$, which satisfy

$$0 \leq \|x_1\|_{1-\alpha} < r_3 < \|x_2\|_{1-\alpha} \leq r_4.$$

The proof is finished. □

Suppose φ, ψ are non-negative continuous convex functionals on P , θ is a non-negative continuous concave functional on P , and η is a non-negative continuous functional on P . Define the following convex set:

$$\begin{aligned} P(\varphi; d) &= \{x \in P : \varphi(x) < d\}, \\ P(\varphi, \theta; b, d) &= \{x \in P : b \leq \theta(x), \varphi(x) \leq d\}, \\ P(\varphi, \psi, \theta; b, c, d) &= \{x \in P : b \leq \theta(x), \psi(x) \leq c, \varphi(x) \leq d\}, \end{aligned}$$

and a closed set

$$P(\varphi, \eta; a, d) = \{x \in P : a \leq \theta(x), \varphi(x) \leq d\},$$

where a, b, c, d are positive constants.

Theorem 3.5 ([23] Avery–Peterson fixed point theorem) *Suppose P is a cone on the real Banach space E . Suppose φ, ψ are non-negative continuous convex functionals on P , θ is a non-negative continuous concave functional on P , and η is a non-negative continuous functional on P satisfying $\eta(\rho x) \leq \rho\eta(x)$ for $0 \leq \rho \leq 1$, for positive constants k and d , we have*

$$\theta(x) \leq \eta(x), \quad \|x\| \leq k\varphi(x), \quad \forall x \in \overline{P(\varphi; d)}. \tag{10}$$

Suppose $T : \overline{P(\varphi; d)} \rightarrow \overline{P(\varphi; d)}$ is a completely continuous operator and there exist $a, b, c > 0$ with $a < b$ such that the following conditions hold:

$$(H_1) \{x \in P(\varphi, \psi, \theta; b, c, d) : \theta(x) > b\} \neq \emptyset \text{ and } \theta(x) > b \text{ for } x \in P(\varphi, \psi, \theta; b, c, d),$$

$$(H_2) \theta(Tx) > b \text{ for } x \in P(\varphi, \theta; b, d) \text{ with } \eta(Tx) > c,$$

$$(H_3) 0 \notin P(\varphi, \eta; a, d) \text{ and } \eta(Tx) < a, \text{ for } P(\varphi, \eta; a, d) \text{ with } \eta(x) = a.$$

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\varphi; d)}$ such that

$$\varphi(x_i) \leq d, \quad \theta(x_1) > b, \quad \theta(x_2) > a, \quad \eta(x_2) < b, \quad \eta(x_3) < a,$$

where $i = 1, 2, 3$.

Define the following functionals:

$$\varphi(x) = \|x\|_{1-\alpha}, \quad \psi(x) = \eta(x) = \max_{t \in [0,1]} |t^{1-\alpha} x(t)|, \quad \theta(x) = \min_{t \in [0,1]} |t^{1-\alpha} x(t)|,$$

$$N = \rho m \int_0^1 \phi_q \left(\int_0^1 H(s, \tau) f(\tau, x(\tau)) d\tau \right) ds,$$

therefore, $\varphi, \psi, \eta, \theta$ satisfy the condition as mentioned above, and

$$\rho \eta(x) \leq \theta(x) \leq \psi(x) = \eta(x), \quad \|x\|_{1-\alpha} \leq \varphi(x).$$

Theorem 3.6 Suppose (H_0) holds, there exist positive constants a, b, c, d with $a < b < c = b\rho^{-1}$,

$$(H_4) f(t, x) \leq \Gamma(\beta - 1)\phi_p(dn^{-1}) \text{ for } (t, x) \in [0, 1] \times [0, d],$$

$$(H_5) f(t, x) > \phi_p(bN^{-1}) \text{ for } (t, x) \in [0, 1] \times [b, c],$$

$$(H_6) f(t, x) < \Gamma(\beta - 1)\phi_p(an^{-1}) \text{ for } (t, x) \in [0, 1] \times [0, a],$$

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\varphi; d)}$ such that

$$\varphi(x_i) \leq d, \quad \theta(x_1) > b, \quad \theta(x_2) > a, \quad \eta(x_2) < b, \quad \eta(x_3) < a,$$

where $i = 1, 2, 3$.

Proof For $x(t) \in \overline{P(\varphi; d)}$, we have

$$\varphi(x) = \|x\|_{1-\alpha} \leq d.$$

According to (H_4) , we have

$$\begin{aligned} \|Tx(t)\| &= \left| t^{1-\alpha} \int_0^1 G(t, s) \phi_q \left(\int_0^1 H(s, \tau) f(\tau, x(\tau)) d\tau \right) ds \right| \\ &\leq \int_0^1 |t^{1-\alpha} G(t, s)| \phi_q \left(\int_0^1 |H(s, \tau)| |f(\tau, x(\tau))| d\tau \right) ds \\ &< n \phi_q \left(\frac{\Gamma(\beta - 1) \phi_q \left(\frac{d}{n} \right)}{\Gamma(\beta - 1)} \right) \\ &= d. \end{aligned}$$

Hence, $\varphi(Tx) = \|Tx\|_{1-\alpha} \leq d$, that is to say, $T : \overline{P(\varphi; d)} \rightarrow \overline{P(\varphi; d)}$ is a completely continuous operator.

Let $x(t) = \frac{b}{\rho}$, $x(t) \in P(\varphi, \psi, \theta; b, c, d)$ and $\theta(\frac{b}{\rho}) > b$, and so $\{x(t) \in P(\varphi, \psi, \theta; b, c, d) : \theta(\frac{b}{\rho}) > b\} \neq \emptyset$.

For $x(t) \in P(\varphi, \psi, \theta; b, c, d)$, we obtain $b \leq x(t) \leq c = \frac{b}{\rho}$, then

$$\begin{aligned} \theta(Tx) &= \min_{t \in [0,1]} |t^{1-\alpha} Tx(t)| \\ &= \min_{t \in [0,1]} t^{1-\alpha} \int_0^1 G(t,s) \phi_q \left(\int_0^1 H(s,\tau) f(\tau, x(\tau)) d\tau \right) ds \\ &\geq \int_0^1 \rho \max_{t \in [0,1]} t^{1-\alpha} G(t,s) \phi_q \left(\int_0^1 H(s,\tau) f(\tau, x(\tau)) d\tau \right) ds \\ &> \rho m \int_0^1 \phi_q \left(\int_0^1 H(s,\tau) f(\tau, x(\tau)) d\tau \right) ds \frac{b}{N} \\ &= b. \end{aligned}$$

Hence, $\theta(Tx) > b$ for $x \in P(\varphi, \psi, \theta; b, c, d)$. That is to say, (H_1) holds.

Let $x \in P(\varphi, \theta; b, d)$ with $\psi(Tx) > c = \frac{b}{\rho}$, we obtain

$$\theta(Tx) \geq \rho \psi(Tx) > \rho c = \rho \frac{b}{\rho} = b.$$

Therefore, (H_2) is satisfied.

By $\theta(0) = 0 < a$, we have $0 \notin P(\varphi, \eta; a, d)$.

Let $x \in P(\varphi, \eta; a, d)$ and $\eta(x) = a$, we obtain $\varphi(x) \leq d$, according to (H_6) ,

$$\begin{aligned} \eta(Tx) &= \max_{t \in [0,1]} |t^{1-\alpha} Tx(t)| \\ &= \max_{t \in [0,1]} t^{1-\alpha} \int_0^1 G(t,s) \phi_q \left(\int_0^1 H(s,\tau) f(\tau, x(\tau)) d\tau \right) ds \\ &\leq n \int_0^1 \phi_q \left(\int_0^1 H(s,\tau) f(\tau, x(\tau)) d\tau \right) ds \\ &< n \phi_q \left(\frac{\Gamma(\beta - 1) \phi_q(\frac{a}{n})}{\Gamma(\beta - 1)} \right) \\ &= a. \end{aligned}$$

Hence, the condition (H_3) holds.

The proof is finished. □

4 Illustrative example

Consider the following nonlinear fractional differential equations:

$$\begin{cases} -D_{0^+}^{1.7}(\phi_{3.5}(D_{0^+}^{0.5}x(t) - 3x(t))) = f(t, x(t)), & t \in (0, 1], \\ \lim_{t \rightarrow 0^+} t^{0.5}x(t) = \sum_{k=1}^{\infty} \frac{1}{2^{2k(k-1)}} x(\frac{1}{2^{2k}}), \\ \lim_{t \rightarrow 0^+} t^{0.3}(\phi_{3.5}(D_{0^+}^{0.5}x(t) - 3x(t))) = \phi_{3.5}(D_{0^+}^{0.5}x(1) - 3x(1)) = 0. \end{cases} \tag{11}$$

Obviously, we choose $\alpha = 0.5, \beta = 1.7, p = 3.5, \lambda = -3, \mu_k = \frac{1}{2^{2k(k-1)!}, \xi_k = \frac{1}{2^{2k}}, \sum_{k=1}^{\infty} \mu_k \xi_k^{\alpha-1} = \frac{e^{0.5}}{2} < 1$. By simple calculation, we obtain $\rho \approx 0.04819, m \approx 0.0011246, n \approx 10.091$.

Part 1: In order to satisfy Theorem 3.1, by matlab, we get $N_1 = 3.1 \times 10^4, N_2 = 0.5, r_1 = 0.2, r_2 = 4 \times 10^4$, so we know that the BVP (11) has at least one positive solution x_1 satisfy $0.2 \leq \|x_1\|_{1-\alpha} \leq 4 \times 10^4$, where

$$f(t, x) = 7 \times 10^3 e^t + \sin x, \quad (t, x) \in [0, 1] \times [0, 4 \times 10^4],$$

where $f(t, x)$ satisfies

- (i) $f(t, x) \geq N_1 r_1 = 6.2 \times 10^3$ for $(t, x) \in [0, 1] \times [0, 0.2]$,
- (ii) $f(t, x) > N_2 r_2 = 2 \times 10^4$ for $(t, x) \in [0, 1] \times [0, 4 \times 10^4]$.

Part 2: In order to make Theorem 3.3 hold, we choose $N_3 = 500, N_4 = 1, r_3 = 2, r_4 = 50$, then the BVP (11) has at least two positive solutions x_1, x_2 , and

$$0 \leq \|x_1\|_{1-\alpha} < 2 < \|x_2\|_{1-\alpha} \leq 50,$$

where

$$f(t, x) = (t + 5)^3 + \frac{19}{4} x^2, \quad (t, x) \in [0, 1] \times [0, 100],$$

where $f(t, x)$ satisfies

- (i) $f(t, x) < N_3 r_3 = 1000$ for $(t, x) \in [0, 1] \times [0, 2]$,
- (ii) $f(t, x) > N_4 r_4 = 50$ for $(t, x) \in [0, 1] \times [0, 50]$.

Part 3: In order to make Theorem 3.5 hold, we choose $a = 0.75, b = 2, c = b\rho^{-1} = 41.51, d = 0.5$, then we know the BVP (11) has at least three positive solutions x_1, x_2, x_3 satisfying

$$\|x\|_{0.5}(x_i) \leq 0.5 \quad (i = 1, 2, 3), \quad \min_{t \in [0,1]} |t^{0.5} x_1| > 2, \quad \min_{t \in [0,1]} |t^{0.5} x_2| > 0.75,$$

$$\max_{t \in [0,1]} |t^{0.5} x_2| < 2, \quad \max_{t \in [0,1]} |t^{0.5} x_3| < 0.75,$$

where

$$f(t, x) = \frac{1}{10^3} \sqrt{t} + x^{13}, \quad (t, x) \in [0, 1] \times [0, 200],$$

where $f(t, x)$ satisfies

- (i) $f(t, x) < \Gamma(\beta - 1)\phi_p(dn^{-1}) = 0.617$ for $(t, x) \in [0, 1] \times [0, 0.5]$,
- (ii) $f(t, x) > \phi_p(bN^{-1}) = 3189.505$ for $(t, x) \in [0, 1] \times [2, 41.51]$,
- (iii) $f(t, x) < \Gamma(\beta - 1)\phi_p(an^{-1}) = 0.0342$ for $(t, x) \in [0, 1] \times [0, 0.75]$.

5 Conclusions

In this paper, we consider a class of nonlinear fractional differential equations coupled with the p-Laplacian operator and infinite-point boundary value conditions by means of the properties of Green’s function, Krasnosel’skii fixed point theorem, the corollary of Krasnosel’skii fixed point theorem and the Avery–Peterson fixed point theorem. On top of that, we establish the suitable criteria to warrant the existence of the positive solutions

for the BVP we learned. To the best of our knowledge, few researchers studied fractional differential equations with p -Laplacian by the Riemann–Liouville derivative, and this condition coupled with values at an infinite number of points is not covered in the previous situations; last but not least, it is more general for the fractional differential equation models.

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