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Ground state solutions of Kirchhoff-type fractional Dirichlet problem with p-Laplacian

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Abstract
We consider the Kirchhoff-type p-Laplacian Dirichlet problem containing the left and right fractional derivative operators. By using the Nehari method in critical point theory, we obtain the existence theorem of ground state solutions for such Dirichlet problem.

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Keywords: Kirchhoff-type equation; Fractional p-Laplacian; Dirichlet problem; Ground state solution; Nehari manifold

1 Introduction

In the present paper, we discuss the existence of ground state solutions for the Kirchhoff-type fractional Dirichlet problem with p-Laplacian of the form

$$\begin{cases} (a + b \int_0^T |{}_0D_t^\alpha u(t)|^p\,dt)^{p-1}\, {}_tD_T^\alpha \phi_p({}_0D_t^\alpha u(t)) = f(t,u(t)), & t \in (0,T), \\ u(0) = u(T) = 0, \end{cases} \tag{1}$$

where $a, b > 0$, $p > 1$ are constants, ${}_0D_t^\alpha$ and ${}_tD_T^\alpha$ are the left and right Riemann–Liouville fractional derivatives of order $\alpha \in (1/p, 1]$, respectively, $\phi_p : \mathbb{R} \to \mathbb{R}$ is the p-Laplacian defined by

$$\phi_p(s) = |s|^{p-2}s \quad (s \neq 0), \qquad \phi_p(0) = 0,$$

and $f \in C^1([0,T] \times \mathbb{R}, \mathbb{R})$.

The Kirchhoff equation [21] is an extension of the wave equation which comes from the free vibrations of elastic strings and takes into account the changes in length of the string produced by transverse vibrations. In addition, the fractional order models are more appropriate than the integer order models in real world owing to the fact that the fractional derivatives offer a wonderful tool to describe the memory and hereditary properties of a great deal of processes and materials [12, 15, 16, 22, 25]. Moreover, the p-Laplacian [23] often appears in non-Newtonian fluid theory, nonlinear elastic mechanics, and so on.

Notice that, when $a = 1$, $b = 0$, and $p = 2$, the left-hand side of equation of BVP (1), which is nonlinear and nonlocal, reduces to the linear operator ${}_tD_T^\alpha {}_0D_t^\alpha$, and further reduces to the local operator $-d^2/dt^2$ when $\alpha = 1$.

In recent years, there have been many authors to study the fractional boundary value problems (BVPs for short) [1, 3, 4, 7, 11, 17] and the Kirchhoff equations [2, 6, 8, 10, 24, 26], and to obtain numerous important results. In addition, the models containing left and right fractional derivatives have been recently gaining more attention [5, 9, 13, 14, 18, 19, 28] because of the applications in physical phenomena exhibiting anomalous diffusion.

Motivated by the above works, in this paper, we discuss the existence of nontrivial ground state solutions for BVP (1). The main tool used here is the Nehari method.

For the nonlinearity f, we make the following assumptions throughout this paper.

(H_1) The mapping $x \to f(t,x)/|x|^{p^2-1}$ is strictly increasing on $\mathbb{R} \setminus \{0\}$ for $\forall t \in [0,T]$.

(H_2) $f(t,x) = o(|x|^{p-1})$ as $|x| \to 0$ uniformly for $\forall t \in [0,T]$.

(H_3) There exist two constants $\mu > p^2$, $R > 0$ such that

$$0 < \mu F(t,x) \le xf(t,x), \quad \forall t \in [0,T], x \in \mathbb{R} \text{ with } |x| \ge R,$$

where $F(t,x) = \int_0^x f(t,s)\,ds$.

Now we state our main result.

Theorem 1.1 *Let* (H_1)–(H_3) *be satisfied. Then BVP* (1) *possesses at least one nontrivial ground state solution.*

The rest of this paper is organized as follows. Some preliminary results are presented in Sect. 2. Section 3 is devoted to proving Theorem 1.1.

2 Preliminaries

In this section, we present some basic definitions and notations of the fractional calculus [20, 27]. Moreover we introduce a fractional Sobolev space and some properties of this space [19].

Definition 2.1 For $\gamma > 0$, the left and right Riemann–Liouville fractional integrals of order γ of a function $u : [a,b] \to \mathbb{R}$ are given by

$$
\begin{aligned}
{}_aI_t^{\gamma} u(t) &= \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} u(s)\,ds, \\
{}_tI_b^{\gamma} u(t) &= \frac{1}{\Gamma(\gamma)} \int_t^b (s-t)^{\gamma-1} u(s)\,ds,
\end{aligned}
$$

provided that the right-hand side integrals are pointwise defined on $[a,b]$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 For $n-1 \le \gamma < n$ ($n \in \mathbb{N}$), the left and right Riemann–Liouville fractional derivatives of order γ of a function $u : [a,b] \to \mathbb{R}$ are given by

$$
\begin{aligned}
{}_aD_t^{\gamma} u(t) &= \frac{d^n}{dt^n} {}_aI_t^{n-\gamma} u(t), \\
{}_tD_b^{\gamma} u(t) &= (-1)^n \frac{d^n}{dt^n} {}_tI_b^{n-\gamma} u(t).
\end{aligned}
$$

Remark 2.3 When $\gamma = 1$, one can obtain from Definitions 2.1 and 2.2 that

$$ {}_aD_t^1 u(t) = u'(t), \qquad {}_tD_b^1 u(t) = -u'(t), $$

where u' is the usual first-order derivative of u.

Definition 2.4 For $0 < \alpha \leq 1$ and $1 < p < \infty$, the fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^\infty((0,T),\mathbb{R})$ with respect to the following norm:

$$ \|u\|_{E^{\alpha,p}} = \left(\|u\|_{L^p}^p + \left\|{}_0D_t^\alpha u\right\|_{L^p}^p\right)^{\frac{1}{p}}, $$

where $\|u\|_{L^p} = (\int_0^T |u(t)|^p \, dt)^{1/p}$ is the norm of $L^p((0,T),\mathbb{R})$.

Remark 2.5 It is obvious that, for $u \in E_0^{\alpha,p}$, one has

$$ u, {}_0D_t^\alpha u \in L^p\big((0,T),\mathbb{R}\big), \qquad u(0) = u(T) = 0. $$

Lemma 2.6 (see [19]) *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.*

Lemma 2.7 (see [19]) *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For $u \in E_0^{\alpha,p}$, one has*

$$ \|u\|_{L^p} \leq C_p \left\|{}_0D_t^\alpha u\right\|_{L^p}, \tag{2} $$

where

$$ C_p = \frac{T^\alpha}{\Gamma(\alpha+1)} > 0 $$

is a constant. Moreover, if $\alpha > 1/p$, then

$$ \|u\|_\infty \leq C_\infty \left\|{}_0D_t^\alpha u\right\|_{L^p}, \tag{3} $$

where $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$ is the norm of $C([0,T],\mathbb{R})$ and

$$ C_\infty = \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}} > 0, \qquad q = \frac{p}{p-1} > 1 $$

are two constants.

Remark 2.8 By (2), we can consider the space $E_0^{\alpha,p}$ with the norm

$$ \|u\|_{E^{\alpha,p}} = \left\|{}_0D_t^\alpha u\right\|_{L^p} \tag{4} $$

in what follows.

Lemma 2.9 (see [19]) *Let $1/p < \alpha \leq 1$ and $1 < p < \infty$. The imbedding of $E_0^{\alpha,p}$ in $C([0,T],\mathbb{R})$ is compact.*

3 Ground state solutions of BVP (1)

The purpose of this section is to prove our main result via the Nehari method. To this end, we are going to set up the corresponding variational framework of BVP (1).

Define the functional $I : E_0^{\alpha,p} \to \mathbb{R}$ by

$$\begin{aligned} I(u) &= \frac{1}{bp^2}\left(a + b\int_0^T \left|{}_0D_t^\alpha u(t)\right|^p dt\right)^p - \int {}_0^T F\big(t, u(t)\big)\,dt - \frac{a^p}{bp^2} \\ &= \frac{1}{bp^2}\big(a + b\|u\|_{E^{\alpha,p}}^p\big)^p - \int_0^T F\big(t, u(t)\big)\,dt - \frac{a^p}{bp^2}. \end{aligned}$$

Then there is one-to-one correspondence between the critical points of energy functional I and the weak solutions of BVP (1). It is easy to check from (3), (4), and $f \in C^1([0,T] \times \mathbb{R}, \mathbb{R})$ that the functional I is well defined on $E_0^{\alpha,p}$ and is second-order continuously Fréchet differentiable, that is, $I \in C^2(E_0^{\alpha,p}, \mathbb{R})$. Furthermore, we have

$$\begin{aligned} \big\langle I'(u), v\big\rangle = {}& \big(a + b\|u\|_{E^{\alpha,p}}^p\big)^{p-1}\int_0^T \phi_p\big({}_0D_t^\alpha u(t)\big){}_0D_t^\alpha v(t)\,dt \\ & - \int_0^T f\big(t, u(t)\big)v(t)\,dt, \quad \forall u, v \in E_0^{\alpha,p}, \end{aligned}$$

which yields

$$\big\langle I'(u), u\big\rangle = \big(a + b\|u\|_{E^{\alpha,p}}^p\big)^{p-1}\|u\|_{E^{\alpha,p}}^p - \int {}_0^T f\big(t, u(t)\big)u(t)\,dt.$$

Now let us define

$$\mathcal{N} = \big\{u \in E_0^{\alpha,p} \setminus \{0\} | G(u) = 0\big\},$$

where

$$G(u) = \big\langle I'(u), u\big\rangle.$$

Thus we know that any non-zero critical point of I must be on $\mathcal{N}$. In the following, for simplicity, let

$$M_u = a + b\|u\|_{E^{\alpha,p}}^p.$$

From (H_1), one has

$$f_2'(t,x)x^2 \ge \big(p^2 - 1\big)f(t,x)x, \quad \forall (t,x) \in [0,T] \times \big(\mathbb{R} \setminus \{0\}\big), \tag{5}$$

where $f_2'(t,x) = \frac{\partial f(t,x)}{\partial x}$. Then, for $u \in \mathcal{N}$, we have

$$\begin{aligned} \big\langle G'(u), u\big\rangle = {}& bp(p-1)M_u^{p-2}\|u\|_{E^{\alpha,p}}^{2p} + pM_u^{p-1}\|u\|_{E^{\alpha,p}}^p \\ & - \int_0^T f_2'\big(t, u(t)\big)u^2(t)\,dt - \int_0^T f\big(t, u(t)\big)u(t)\,dt \end{aligned}$$

$$\begin{aligned}&\leq M_u^{p-2}\|u\|_{E^{\alpha,p}}^p\big(bp^2\|u\|_{E^{\alpha,p}}^p+ap\big)-p^2\int_0^T f\big(t,u(t)\big)u(t)\,dt\\&=a\big(p-p^2\big)M_u^{p-2}\|u\|_{E^{\alpha,p}}^p<0,\end{aligned}\tag{6}$$

which means that $\mathcal{N}$ has a C^1 structure and is a manifold.

Lemma 3.1 *Assume that* (H_1) *holds. If* $u\in\mathcal{N}$ *is a critical point of* $I|_{\mathcal{N}}$, *then* $I'(u)=0$, *that is,* $\mathcal{N}$ *is a natural constraint for* I.

Proof If $u\in\mathcal{N}$ is a critical point of $I|_{\mathcal{N}}$, then there exists a Lagrange multiplier $\lambda\in\mathbb{R}$ such that

$$I'(u)=\lambda G'(u).$$

Then we get

$$\big\langle I'(u),u\big\rangle=\lambda\big\langle G'(u),u\big\rangle=0,$$

which together with (6) yields $\lambda=0$. So we have $I'(u)=0$. □

In order to discuss the critical points of $I|_{\mathcal{N}}$, we need to investigate the structure of $\mathcal{N}$.

Lemma 3.2 *Assume that* (H_1)–(H_3) *hold. For each* $u\in E_0^{\alpha,p}\setminus\{0\}$, *there is unique* $s=s(u)\in\mathbb{R}^+$ *such that* $su\in\mathcal{N}$.

Proof First, we claim that there exist constants $\rho,\sigma>0$ such that

$$I(u)>0,\quad \forall u\in B_\rho(0)\setminus\{0\},\qquad I(u)\geq\sigma,\quad \forall u\in\partial B_\rho(0),\tag{7}$$

where $B_\rho(0)$ is an open ball in $E_0^{\alpha,p}$ with the radius ρ and centered at 0, and $\partial B_\rho(0)$ denotes its boundary. That is, by $I(0)=0$, 0 is a strict local minimizer of I. In fact, from (H_2), there are two constants $0<\varepsilon<1$, $\delta>0$ such that

$$F(t,x)\leq\frac{(1-\varepsilon)a^{p-1}}{pC_p^p}|x|^p,\quad \forall(t,x)\in[0,T]\times[-\delta,\delta],\tag{8}$$

where $C_p>0$ is a constant defined in (2). Let $\rho=\delta/C_\infty$ and $\sigma=\varepsilon a^{p-1}\rho^p/p$, where $C_\infty>0$ is a constant defined in (3). Then, by (3) and (4), one has

$$\|u\|_\infty\leq C_\infty\|u\|_{E^{\alpha,p}}\leq\delta,\quad \forall u\in\overline{B_\rho(0)},$$

which together with (2), (4), and (8) yields

$$\begin{aligned}I(u)&=\frac{1}{bp^2}M_u^p-\int_0^T F\big(t,u(t)\big)\,dt-\frac{a^p}{bp^2}\\&\geq\frac{a^{p-1}}{p}\|u\|_{E^{\alpha,p}}^p-\frac{(1-\varepsilon)a^{p-1}}{pC_p^p}\int_0^T\big|u(t)\big|^p\,dt\end{aligned}$$

$$\begin{aligned}
&\geq \frac{a^{p-1}}{p}\|u\|_{E^{\alpha,p}}^{p} - \frac{(1-\varepsilon)a^{p-1}}{p}\|u\|_{E^{\alpha,p}}^{p} \\
&= \frac{\varepsilon a^{p-1}}{p}\|u\|_{E^{\alpha,p}}^{p} = \sigma, \quad \forall u \in \partial B_{\rho}(0).
\end{aligned}$$

Second, we claim that $I(\xi u) \to -\infty$ as $\xi \to \infty$. In fact, from (H_3), a simple argument can show that there are two constants $c_1, c_2 > 0$ such that

$$F(t,x) \geq c_1|x|^{\mu} - c_2, \quad \forall (t,x) \in [0,T] \times \mathbb{R}.$$

Thus, for each $u \in E_0^{\alpha,p} \setminus \{0\}$, $\xi \in \mathbb{R}^+$, we obtain from $\mu > p^2$ that

$$\begin{aligned}
I(\xi u) &= \frac{1}{bp^2}M_{\xi u}^{p} - \int_0^T F\bigl(t, \xi u(t)\bigr)\,dt - \frac{a^p}{bp^2} \\
&\leq \frac{1}{bp^2}M_{\xi u}^{p} - c_1\int_0^T \bigl|\xi u(t)\bigr|^{\mu}\,dt + c_2T - \frac{a^p}{bp^2} \\
&= \frac{1}{bp^2}\bigl(a + b\xi^p\|u\|_{E^{\alpha,p}}^{p}\bigr)^p - c_1\xi^{\mu}\|u\|_{L^{\mu}}^{\mu} + c_2T - \frac{a^p}{bp^2} \\
&\to -\infty \quad \text{as } \xi \to \infty.
\end{aligned}$$

Let

$$g_u(s) = I(su), \quad \forall s \in \mathbb{R}^+.$$

Then, from what we have proved, g_u has at least one maximum point $s(u)$ with maximum value greater than $\sigma > 0$. Next, we prove that g_u has a unique critical point for $s \in \mathbb{R}^+$, which then must be the global maximum point. Considering a critical point of g_u, one has

$$\begin{aligned}
g_u'(s) &= \bigl\langle I'(su), u\bigr\rangle \\
&= \|u\|_{E^{\alpha,p}}^{p}M_{su}^{p-1}s^{p-1} - \int_0^T f\bigl(t, su(t)\bigr)u(t)\,dt \\
&= 0,
\end{aligned}$$

which together with (5) yields

$$\begin{aligned}
g_u''(s) &= bp(p-1)\|u\|_{E^{\alpha,p}}^{2p}M_{su}^{p-2}s^{2p-2} \\
&\quad + (p-1)\|u\|_{E^{\alpha,p}}^{p}M_{su}^{p-1}s^{p-2} - \int_0^T f_2'\bigl(t, su(t)\bigr)u^2(t)\,dt \\
&< \|u\|_{E^{\alpha,p}}^{p}M_{su}^{p-2}\bigl(b\bigl(p^2-1\bigr)\|u\|_{E^{\alpha,p}}^{p}s^{2p-2} + a(p-1)s^{p-2}\bigr) \\
&\quad - \frac{p^2-1}{s}\int_0^T f\bigl(t, su(t)\bigr)u(t)\,dt \\
&= \|u\|_{E^{\alpha,p}}^{p}M_{su}^{p-2}\bigl(b\bigl(p^2-1\bigr)\|u\|_{E^{\alpha,p}}^{p}s^{2p-2} + a(p-1)s^{p-2}\bigr) \\
&\quad - \bigl(p^2-1\bigr)\|u\|_{E^{\alpha,p}}^{p}M_{su}^{p-1}s^{p-2} \\
&= a\|u\|_{E^{\alpha,p}}^{p}M_{su}^{p-2}\bigl(p-p^2\bigr)s^{p-2} \leq 0.
\end{aligned} \tag{9}$$

Hence, if s is a critical point of g_u, then it must be a strict local maximum point. This ensures the uniqueness of a critical point of g_u. Finally, from

$$g_u'(s) = \frac{1}{s}\langle I'(su), su\rangle, \quad \forall t \in \mathbb{R}^+, \tag{10}$$

we obtain that, if s is a critical point of g_u, then $su \in \mathcal{N}$. □

Let us define

$$m = \inf_{\mathcal{N}} I.$$

Then we get from (7) that

$$m \geq \inf_{\partial B_\rho(0)} I \geq \sigma > 0.$$

Lemma 3.3 *Assume that* (H_1)–(H_3) *hold. Then there exists* $u^* \in \mathcal{N}$ *such that* $I(u^*) = m$.

Proof By Lemma 2.9, we obtain that the functional

$$u \to \int_0^T F(t, u(t))\,dt, \quad \forall u \in E_0^{\alpha,p}$$

is weakly continuous. Thus, as the sum of a convex continuous functional and a weakly continuous one, I is weakly lower semi-continuous on $E_0^{\alpha,p}$.

Let $\{u_k\} \subset \mathcal{N}$ be a minimizing sequence of I, then one has

$$I(u_k) = m + o(1), \qquad G(u_k) = 0. \tag{11}$$

Next, we prove that $\{u_k\}$ is bounded in $E_0^{\alpha,p}$. Based on the continuity of $\mu F(t,x) - xf(t,x)$ and (H_3), we see that there exists a constant $c > 0$ such that

$$F(t,x) \leq \frac{1}{\mu} xf(t,x) + c, \quad \forall (t,x) \in [0,T] \times \mathbb{R}.$$

Thus, from (11), we have

$$\begin{aligned} m + o(1) &= I(u_k) \\ &\geq \frac{1}{bp^2} M_{u_k}^p - \frac{1}{\mu}\int_0^T f(t, u_k(t)) u_k(t)\,dt - cT - \frac{a^p}{bp^2} \\ &= \frac{1}{bp^2} M_{u_k}^p - \frac{1}{\mu} M_{u_k}^{p-1} \|u_k\|_{E^{\alpha,p}}^p - cT - \frac{a^p}{bp^2} \\ &= M_{u_k}^{p-1}\left(\left(\frac{1}{p^2} - \frac{1}{\mu}\right)\|u_k\|_{E^{\alpha,p}}^p + \frac{a}{bp^2}\right) - cT - \frac{a^p}{bp^2}. \end{aligned}$$

Hence it follows from $\mu > p^2$ that $\{u_k\}$ is bounded in $E_0^{\alpha,p}$.

Since $E_0^{\alpha,p}$ is a reflexive Banach space (see Lemma 2.6), up to a subsequence, we can assume $u_k \rightharpoonup u$ in $E_0^{\alpha,p}$. Moreover, from Lemma 2.9, one has $u_k \to u$ in $C([0,T],\mathbb{R})$. Next, we prove $u \not\equiv 0$. By (H_2), we get that, for $\forall\varepsilon > 0$, there exists a constant $\delta > 0$ such that

$$f(t,x)x \le \varepsilon|x|^p, \quad \forall(t,x) \in [0,T] \times [-\delta,\delta].$$

Then, assume $\|u_k\|_\infty \le \delta$, we obtain from (3), (4), and $u_k \in \mathcal{N}$ that

$$\begin{aligned} C_\infty^{-p}\bigl(a + bC_\infty^{-p}\|u_k\|_\infty^p\bigr)^{p-1}\|u_k\|_\infty^p &\le \bigl(a + b\|u_k\|_{E^{\alpha,p}}^p\bigr)^{p-1}\|u_k\|_{E^{\alpha,p}}^p \\ &= \int_0^T f\bigl(t,u_k(t)\bigr)u_k(t)\,dt \\ &\le \varepsilon\int_0^T \bigl|u_k(t)\bigr|^p\,dt \\ &\le \varepsilon T\|u_k\|_\infty^p, \end{aligned}$$

which is a contradiction from the arbitrariness of ε. Hence we have

$$\|u\|_\infty = \underline{\lim}_{k\to\infty}\|u_k\|_\infty \ge \delta > 0,$$

and then $u \not\equiv 0$. Thus, by Lemma 3.2, there exists $s \in \mathbb{R}^+$ such that $su \in \mathcal{N}$. Therefore, together with the fact that I is weakly lower semi-continuous, we obtain

$$m \le I(su) \le \underline{\lim}_{k\to\infty} I(su_k) \le \lim_{k\to\infty} I(su_k). \tag{12}$$

Finally, for $\forall u_k \in \mathcal{N}$, we see from (9) and (10) that $s = 1$ is the global maximum point of g_{u_k}. So one has

$$I(su_k) \le I(u_k),$$

which together with (12) implies

$$m \le I(su) \le \lim_{k\to\infty} I(u_k) = m.$$

That is, m is achieved at $su \in \mathcal{N}$. □

Now we give the proof of our main result.

Proof of Theorem 1.1 By Lemma 3.3, we get $u^* \in \mathcal{N}$ such that $I(u^*) = m = \inf_{\mathcal{N}} I > 0$, that is, u^* is a non-zero critical point of $I|_{\mathcal{N}}$. Then, from Lemma 3.1, we know $I'(u^*) = 0$, and so u^* is a nontrivial ground state solution of BVP (1). □

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Authors' contributions
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