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Oscillatory behavior of solutions of certain fractional difference equations

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Abstract

In this paper, we consider the oscillation behavior of solutions of the following fractional difference equation:

$$\Delta(c(t)\Delta(a(t)\Delta(r(t)\Delta^{\alpha}x(t)))) + q(t)G(t) = 0,$$

where $t \in \mathbf{N}_{t_0+1-\alpha}$, $G(t) = \sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{-\alpha}x(s)$, and Δ^{α} denotes a Riemann–Liouville fractional difference operator of order $0 < \alpha \le 1$. By using the generalized Riccati transformation technique, we obtain some oscillation criteria. Finally we give an example.

Keywords: Oscillation; Oscillation criteria; Fractional difference operator; Riemann–Liouville; Fractional difference equations; Riccati technique; Hardy inequalities

1 Introduction and preliminaries

Fractional differential (or difference) equations are a more general form of differential equations with integer order. And there is an increasing interest in the study of them due to some important contributions [1, 2].

Many authors have been focused on various equations like ordinary and partial differential equations [3–6], difference equations [7–9], dynamic equations on time scales [10–14], and fractional differential (difference) equations [15–31] obtaining some oscillation criteria. Recently, oscillation studies have become a very hot topic. That is why, we consider the following fractional difference equation:

$$\Delta\big(c(t)\Delta\big(a(t)\Delta\big(r(t)\Delta^{\alpha}x(t)\big)\big)\big) + q(t)G(t) = 0, \tag{1}$$

where $t \in \mathbf{N}_{t_0+1-\alpha}$, $G(t) = \sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)$, $c(t)$, $a(t)$, $r(t)$, and $q(t)$ are positive sequences, and Δ^{α} denotes the Riemann–Liouville fractional difference operator of order $0 < \alpha \le 1$.

By a solution of Eq. (1), we mean a real-valued sequence $x(t)$ satisfying Eq. (1) for $t \in \mathbf{N}_{t_0}$. A solution $x(t)$ of Eq. (1) is called oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called non-oscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

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Definition 1 ([32]) Let $\nu > 0$. The νth fractional sum f is defined by

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\nu-1} f(s), \tag{2}$$

where f is defined for $s \equiv a\,\mathbf{mod}(1)$, $\Delta^{-\nu} f$ is defined for $t \equiv (a+\nu)\mathbf{mod}(1)$, and $t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t-\nu+1)}$. The fractional sum $\Delta^{-\nu} f$ maps functions defined on $\mathbf{N}_a$ to functions defined on $\mathbf{N}_{a+\nu}$, where $\mathbf{N}_t = \{t, t+1, t+2, \ldots\}$.

Definition 2 ([32]) Let $\nu > 0$ and $m-1 < \mu < m$, where m denotes a positive integer, $m = \lceil \mu \rceil$. Set $\nu = m - \mu$. The μth fractional difference is defined as

$$\Delta^{\mu} f(t) = \Delta^{m-\nu} f(t) = \Delta^{m} \Delta^{-\nu} f(t), \tag{3}$$

where $\lceil \mu \rceil$ is the ceiling function of μ.

Lemma 1 ([33]) *Assume that A and B are nonnegative real numbers. Then*

$$\lambda A B^{\lambda-1} - A^{\lambda} \leq (\lambda - 1) B^{\lambda} \tag{4}$$

for all $\lambda > 1$.

2 Main results

Throughout this paper, we denote

$$\phi(t) = \sum_{s=t_1}^{t-1} \frac{1}{c(s)}; \qquad \vartheta(t) = \sum_{s=t_2}^{t-1} \frac{\phi(s)}{a(s)}; \qquad \delta(t) = \sum_{s=t_3}^{t-1} \frac{\vartheta(s)}{r(s)}.$$

For simplification, we consider

$$\Delta\gamma_+(s) = \max\bigl\{0, \Delta\gamma(s)\bigr\}$$

and

$$\Delta\beta_+(s) = \max\bigl\{0, \Delta\beta(s)\bigr\}.$$

Lemma 2 ([28]) *Let $x(t)$ be a solution of Eq.* (1), *and let*

$$G(t) = \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s), \tag{5}$$

then

$$\Delta\bigl(G(t)\bigr) = \Gamma(1-\alpha)\Delta^{\alpha} x(t). \tag{6}$$

Lemma 3 *Assume that $x(t)$ is an eventually positive solution of Eq.* (1). *If*

$$\sum_{s=t_0}^{\infty} \frac{1}{c(s)} = \sum_{s=t_0}^{\infty} \frac{1}{a(s)} = \sum_{s=t_0}^{\infty} \frac{1}{r(s)} = \infty, \tag{7}$$

then we have two possible cases for $t \in [t_1, \infty)$, $t_1 > t_0$ is sufficiently large:

Case 1 $\Delta^\alpha x(t) > 0$, $\Delta(r(t)\Delta^\alpha x(t)) > 0$, $\Delta(a(t)\Delta(r(t)\Delta^\alpha x(t))) > 0$ *or*

Case 2 $\Delta^\alpha x(t) > 0$, $\Delta(r(t)\Delta^\alpha x(t)) < 0$, $\Delta(a(t)\Delta(r(t)\Delta^\alpha x(t))) > 0$.

Proof From the hypothesis, there exists t_1 such that $x(t) > 0$ on $[t_1, \infty)$, so that $G(t) > 0$ on $[t_1, \infty)$, and from Eq. (1), we have

$$\Delta\bigl(c(t)\Delta\bigl(a(t)\Delta\bigl(r(t)\Delta^\alpha x(t)\bigr)\bigr)\bigr) = -q(t)G(t) < 0. \tag{8}$$

Then $c(t)\Delta(a(t)\Delta(r(t)\Delta^\alpha x(t)))$ is an eventually non-increasing sequence on $[t_1, \infty)$. We know that $\Delta^\alpha x(t)$, $\Delta(r(t)\Delta^\alpha x(t))$, and $\Delta(a(t)\Delta(r(t)\Delta^\alpha x(t)))$ are eventually of one sign. For $t_2 > t_1$ is sufficiently large, we claim that $\Delta(a(t)\Delta(r(t)\Delta^\alpha x(t))) > 0$ on $[t_2, \infty)$. Otherwise, assume that there exists sufficiently large $t_3 > t_2$ such that $\Delta(a(t)\Delta(r(t)\Delta^\alpha x(t))) < 0$ on $[t_3, \infty)$. For $[t_3, \infty)$ and there exists a constant $l_1 > 0$, we have

$$\Delta\bigl(a(t)\Delta\bigl(r(t)\Delta^\alpha x(t)\bigr)\bigr) \le -\frac{l_1}{c(t)} < 0.$$

Hence, there exist a constant $l_2 > 0$ and sufficiently large $t_4 > t_3$ such that

$$\Delta\bigl(r(t)\Delta^\alpha x(t)\bigr) \le -\frac{l_2}{a(t)} < 0. \tag{9}$$

Then there exist a constant $l_3 > 0$ and sufficiently large $t_5 > t_4$ such that

$$\Delta^\alpha x(t) \le -\frac{l_3}{r(t)},$$

that is,

$$\Delta G(t) \le -\frac{\Gamma(1-\alpha) l_3}{r(t)} < 0.$$

By (7), we obtain $\lim_{t\to\infty} G(t) = -\infty$. This is a contradiction. If $\Delta(r(t)\Delta^\alpha x(t)) < 0$, then $\Delta^\alpha x(t) > 0$ due to $\sum_{s=t_0}^{\infty} \frac{1}{r(s)} = \infty$. If $\Delta(r(t)\Delta^\alpha x(t)) > 0$, then $\Delta^\alpha x(t) > 0$ due to $\Delta(a(t)\Delta(r(t)\Delta^\alpha x(t))) > 0$. So, the proof is complete. □

Lemma 4 *Assume that $x(t)$ is an eventually positive solution of Eq.* (1), *which satisfies Case* 1 *of Lemma* 3. *Then*

$$a(t)\Delta\bigl(r(t)\Delta^\alpha x(t)\bigr) \ge c(t)\Delta\bigl(a(t)\Delta\bigl(r(t)\Delta^\alpha x(t)\bigr)\bigr) \sum_{s=t_0}^{t-1} \frac{1}{c(s)}.$$

If there exists a positive sequence ϕ *such that, for* $t \in [t_1, \infty)$,

$$\frac{\phi(t)}{c(t)\sum_{s=t_0}^{t-1}\frac{1}{c(s)}} - \Delta\phi(t) \le 0,$$

where t_1 *is sufficiently large, then* $a(t)\Delta(r(t)\Delta^\alpha x(t))/\phi(t)$ *is a non-increasing sequence on* $[t_1, \infty)$ *and*

$$r(t)\Delta^\alpha x(t) \ge \Delta\bigl(r(t)\Delta^\alpha x(t)\bigr)\frac{a(t)}{\phi(t)}\sum_{s=t_1}^{t-1}\frac{\phi(s)}{a(s)}.$$

Furthermore, if there exists a positive sequence ϑ *and* $t_2 > t_1$ *is sufficiently large such that, for* $t \in [t_2, \infty)$,

$$\frac{\vartheta(t)}{\frac{a(t)}{\phi(t)}\sum_{s=t_2}^{t-1}\frac{\phi(s)}{a(s)}} - \Delta\vartheta(t) \le 0,$$

then $r(t)\Delta^\alpha x(t)/\vartheta(t)$ *is a non-increasing sequence on* $[t_2, \infty)$ *and*

$$G(t) \ge \Delta G(t)\frac{r(t)}{\vartheta(t)}\sum_{s=t_2}^{t-1}\frac{\vartheta(s)}{r(s)}.$$

Suppose also that there exists a positive sequence δ *and* $t_3 > t_2$ *is sufficiently large such that, for* $t \in [t_3, \infty)$,

$$\frac{\delta(t)}{\frac{r(t)}{\vartheta(t)}\sum_{s=t_2}^{t-1}\frac{\vartheta(s)}{r(s)}} - \Delta\delta(t) \le 0.$$

Then $G(t)/\delta(t)$ *is a non-increasing sequence on* $[t_3, \infty)$.

Proof Assume that x is an eventually positive solution of Eq. (1). Then we have that $\Delta(r(t)\Delta^\alpha x(t)) > 0$ and $\Delta(c(t)\Delta(a(t)\Delta(r(t)\Delta^\alpha x(t)))) < 0$ on $[t_0, \infty)$. So,

$$\begin{aligned} a(t)\Delta\bigl(r(t)\Delta^\alpha x(t)\bigr) &= a(t_0)\Delta\bigl(r(t_0)\Delta^\alpha x(t_0)\bigr) \\ &\quad + \sum_{s=t_0}^{t-1}\frac{c(s)\Delta(a(s)\Delta(r(s)\Delta^\alpha x(s)))}{c(s)} \\ &\ge c(t)\Delta\bigl(a(t)\Delta\bigl(r(t)\Delta^\alpha x(t)\bigr)\bigr)\sum_{s=t_0}^{t-1}\frac{1}{c(s)}, \end{aligned}$$

and then

$$\begin{aligned} &\Delta\left(\frac{a(t)\Delta(r(t)\Delta^\alpha x(t))}{\phi(t)}\right) \\ &\quad = \frac{\Delta(a(t)\Delta(r(t)\Delta^\alpha x(t)))\phi(t) - a(t)\Delta(r(t)\Delta^\alpha x(t))\Delta\phi(t)}{\phi(t)\phi(t+1)} \\ &\quad \le \frac{\Delta(a(t)\Delta(r(t)\Delta^\alpha x(t)))}{\phi(t)\phi(t+1)}\left(\frac{\phi(t)}{c(t)\sum_{s=t_1}^{t-1}\frac{1}{c(s)}} - \Delta\phi(t)\right) \le 0. \end{aligned}$$

Hence, $a(t)\Delta(r(t)\Delta^\alpha x(t))/\phi(t)$ is a non-increasing sequence on $[t_1,\infty)$ where $t_1 > t_0$ is sufficiently large. Then we have

$$\begin{aligned} r(t)\Delta^\alpha x(t) &= r(t_1)\Delta^\alpha x(t_1) + \sum_{s=t_1}^{t-1} \frac{a(s)\Delta(r(s)\Delta^\alpha x(s))}{\phi(s)} \frac{\phi(s)}{a(s)} \\ &\geq \frac{a(t)\Delta(r(t)\Delta^\alpha x(t))}{\phi(t)} \sum_{s=t_1}^{t-1} \frac{\phi(s)}{a(s)} \end{aligned}$$

and

$$\begin{aligned} \Delta\left(\frac{r(t)\Delta^\alpha x(t)}{\vartheta(t)}\right) &= \frac{\Delta(r(t)\Delta^\alpha x(t))\vartheta(t) - r(t)\Delta^\alpha x(t)\Delta\vartheta(t)}{\vartheta(t)\vartheta(t+1)} \\ &\leq \frac{r(t)\Delta^\alpha x(t)}{\vartheta(t)\vartheta(t+1)}\left(\frac{\vartheta(t)}{\frac{a(t)}{\phi(t)}\sum_{s=t_1}^{t-1}\frac{\phi(s)}{a(s)}} - \Delta\vartheta(t)\right) \leq 0. \end{aligned}$$

So $r(t)\Delta^\alpha x(t)/\vartheta(t)$ is a non-increasing sequence on $[t_2,\infty)$ where $t_2 > t_1$ is sufficiently large. Then we have

$$\begin{aligned} G(t) &= G(t_2) + \Gamma(1-\alpha)\sum_{s=t_2}^{t-1} \frac{r(s)\Delta^\alpha x(s)}{\vartheta(s)} \frac{\vartheta(s)}{r(s)} \\ &\geq \frac{r(t)\Gamma(1-\alpha)\Delta^\alpha x(t)}{\vartheta(t)} \sum_{s=t_2}^{t-1} \frac{\vartheta(s)}{r(s)} \\ &= \Delta G(t)\frac{r(t)}{\vartheta(t)} \sum_{s=t_2}^{t-1} \frac{\vartheta(s)}{r(s)}, \end{aligned}$$

and then

$$\begin{aligned} \Delta\left(\frac{G(t)}{\delta(t)}\right) &= \frac{(\Delta G(t))\delta(t) - G(t)\Delta\delta(t)}{\delta(t)\delta(t+1)} \\ &\leq \frac{G(t)}{\delta(t)\delta(t+1)}\left(\frac{\delta(t)}{\frac{r(t)}{\vartheta(t)}\sum_{s=t_2}^{t-1}\frac{\vartheta(s)}{r(s)}} - \Delta\delta(t)\right) \leq 0. \end{aligned}$$

Then $G(t)/\delta(t)$ is a non-increasing sequence on $[t_3,\infty)$ where $t_3 > t_2$ is sufficiently large. So the proof is complete. □

Theorem 1 *Assume that (7) holds and there exists a positive sequence γ such that, for all sufficiently large t,*

$$\limsup_{t\to\infty} \sum_{s=t_3}^{t-1}\left(\frac{\Gamma(1-\alpha)\gamma(s)q(s)}{\vartheta(s)\phi(s+1)} \sum_{u=t_2}^{s-1} \frac{\vartheta(u)}{r(u)} \sum_{u=t_1}^{s-1} \frac{\phi(u)}{a(u)} - \frac{c(s)(\Delta\gamma_+(s))^2}{4\gamma(s)}\right) = \infty. \tag{10}$$

If there exist positive sequences β, λ such that, for all sufficiently large t,

$$\frac{\lambda(t)}{r(t)\sum_{s=t_1}^{t-1}\frac{1}{r(s)}} - \Delta\lambda(t) \leq 0 \tag{11}$$

and

$$\lim_{t\to\infty}\sup\sum_{\zeta=t_2}^{t-1}\left(\frac{\beta(\zeta)\lambda(\zeta)}{\lambda(\zeta+1)a(\zeta)}\sum_{s=\zeta}^{\infty}\left(\frac{1}{c(s)}\sum_{\nu=s}^{\infty}q(\nu)\right)-\frac{r(\zeta)(\Delta\beta_+(\zeta))^2}{4\Gamma(1-\alpha)\beta(\zeta)}\right)=\infty. \tag{12}$$

Then every solution of Eq. (1) *is oscillatory.*

Proof Suppose to the contrary that $x(t)$ is a non-oscillatory solution of Eq. (1). Then, without loss of generality, we may assume that there is a solution $x(t)$ of Eq. (1) such that $x(t)>0$ on $[t_0,\infty)$, where t_0 is sufficiently large. From Lemma 3, $x(t)$ satisfies Case 1 or Case 2. Firstly, let Case 1 hold. Then we define the following function:

$$\omega(t)=\gamma(t)\frac{c(t)\Delta(a(t)\Delta(r(t)\Delta^\alpha x(t)))}{a(t)\Delta(r(t)\Delta^\alpha x(t))}.$$

For $t\in[t_0,\infty)$, we have

$$\begin{aligned}\Delta\omega(t)&=\Delta\gamma(t)\frac{\omega(t+1)}{\gamma(t+1)}+\gamma(t)\Delta\left(\frac{c(t)\Delta(a(t)\Delta(r(t)\Delta^\alpha x(t)))}{a(t)\Delta(r(t)\Delta^\alpha x(t))}\right)\\&=\Delta\gamma(t)\frac{\omega(t+1)}{\gamma(t+1)}-\gamma(t)\frac{q(t)G(t)}{a(t+1)\Delta(r(t+1)\Delta^\alpha x(t+1))}\\&\quad-\gamma(t)\frac{c(t)\Delta(a(t)\Delta(r(t)\Delta^\alpha x(t)))\Delta(a(t)\Delta(r(t)\Delta^\alpha x(t)))}{a(t)\Delta(r(t)\Delta^\alpha x(t))a(t+1)\Delta(r(t+1)\Delta^\alpha x(t+1))}.\end{aligned}$$

Since $a(t)\Delta(r(t)\Delta^\alpha x(t))/\phi(t)$ is a non-increasing sequence on $[t_1,\infty)$, we have

$$\frac{a(t+1)\Delta(r(t+1)\Delta^\alpha x(t+1))}{\phi(t+1)}\le\frac{a(t)\Delta(r(t)\Delta^\alpha x(t))}{\phi(t)}.$$

From Lemma 4, we obtain

$$\begin{aligned}&\frac{G(t)}{a(t+1)\Delta(r(t+1)\Delta^\alpha x(t+1))}\\&\quad=\frac{1}{a(t+1)}\frac{G(t)}{\Delta G(t)}\frac{\Delta G(t)}{\Delta(r(t)\Delta^\alpha x(t))}\frac{\Delta(r(t)\Delta^\alpha x(t))}{\Delta(r(t+1)\Delta^\alpha x(t+1))}\\&\quad\ge\frac{1}{a(t+1)}\left(\frac{r(t)}{\vartheta(t)}\sum_{s=t_2}^{t-1}\frac{\vartheta(s)}{r(s)}\right)\left(\frac{\Gamma(1-\alpha)}{r(t)}\frac{a(t)}{\phi(t)}\sum_{s=t_1}^{t-1}\frac{\phi(s)}{a(s)}\right)\frac{\phi(t)a(t+1)}{\phi(t+1)a(t)}\\&\quad=\frac{\Gamma(1-\alpha)}{\vartheta(t)\phi(t+1)}\sum_{s=t_2}^{t-1}\frac{\vartheta(s)}{r(s)}\left(\sum_{s=t_1}^{t-1}\frac{\phi(s)}{a(s)}\right)\end{aligned}$$

and

$$\begin{aligned}\Delta\omega(t)&\le\Delta\gamma_+(t)\frac{\omega(t+1)}{\gamma(t+1)}-\gamma(t)q(t)\frac{\Gamma(1-\alpha)}{\vartheta(t)\phi(t+1)}\sum_{s=t_2}^{t-1}\frac{\vartheta(s)}{r(s)}\left(\sum_{s=t_1}^{t-1}\frac{\phi(s)}{a(s)}\right)\\&\quad-\frac{\gamma(t)}{c(t)}\frac{\omega^2(t+1)}{\gamma^2(t+1)}.\end{aligned}$$

Setting $\lambda = 2$, $A = (\frac{\gamma(t)}{c(t)})^{1/2}\frac{\omega(t+1)}{\phi(t+1)}$, and $B = \frac{1}{2}(\frac{c(t)}{\gamma(t)})^{1/2}\Delta\gamma_+(t)$ using Lemma 1, we obtain

$$\Delta\omega(t) \le -\gamma(t)q(t)\frac{\Gamma(1-\alpha)}{\vartheta(t)\phi(t+1)}\sum_{s=t_2}^{t-1}\frac{\vartheta(s)}{r(s)}\left(\sum_{s=t_1}^{t-1}\frac{\phi(s)}{a(s)}\right) + \frac{c(t)}{4\gamma(t)}\left(\Delta\gamma_+(t)\right)^2.$$

Summing both sides of the above inequality from t_3 to $t-1$, we get

$$\begin{aligned}&\sum_{s=t_3}^{t-1}\left(\frac{\Gamma(1-\alpha)\gamma(s)q(s)}{\vartheta(s)\phi(s+1)}\sum_{u=t_2}^{s-1}\frac{\vartheta(u)}{r(u)}\left(\sum_{u=t_1}^{s-1}\frac{\phi(u)}{a(u)}\right) - \frac{c(s)(\Delta\gamma_+(s))^2}{4\gamma(s)}\right)\\ &\quad\le \omega(t_3) - \omega(t) \le \omega(t_3).\end{aligned}$$

This contradicts (10). Now we consider Case 2. Then we define the following function:

$$\omega_2(t) = \beta(t)\frac{r(t)\Delta^\alpha x(t)}{G(t)}.$$

Then

$$\begin{aligned}\Delta\omega_2(t) &= \Delta\beta(t)\frac{\omega(t+1)}{\beta(t+1)} + \beta(t)\Delta\left(\frac{r(t)\Delta^\alpha x(t)}{G(t)}\right)\\ &= \Delta\beta(t)\frac{\omega(t+1)}{\beta(t+1)} + \beta(t)\left(\frac{\Delta(r(t)\Delta^\alpha x(t))G(t) - r(t)\Delta^\alpha x(t)\Delta G(t)}{G(t)G(t+1)}\right)\\ &= \Delta\beta(t)\frac{\omega(t+1)}{\beta(t+1)} + \beta(t)\frac{\Delta(r(t)\Delta^\alpha x(t))}{G(t+1)} - \beta(t)\frac{r(t)\Delta^\alpha x(t)\Delta G(t)}{G(t)G(t+1)}.\end{aligned}$$

Hence we have

$$\begin{aligned}G(t) &= G(t_1) + \Gamma(1-\alpha)\sum_{s=t_1}^{t-1}\frac{r(s)\Delta^\alpha x(s)}{r(s)}\\ &\ge \Gamma(1-\alpha)r(t)\Delta^\alpha x(t)\sum_{s=t_1}^{t-1}\frac{1}{r(s)}.\end{aligned}$$

That is,

$$\frac{G(t)}{r(t)\sum_{s=t_1}^{t-1}\frac{1}{r(s)}} \ge \Gamma(1-\alpha)\Delta^\alpha x(t) = \Delta G(t)$$

and

$$\begin{aligned}\Delta\left(\frac{G(t)}{\lambda(t)}\right) &= \frac{\Delta G(t)\lambda(t) - G(t)\Delta\lambda(t)}{\lambda(t)\lambda(t+1)}\\ &\le \frac{G(t)}{\lambda(t)\lambda(t+1)}\left(\frac{\lambda(t)}{r(t)\sum_{s=t_1}^{t-1}\frac{1}{r(s)}} - \Delta\lambda(t)\right) \le 0.\end{aligned}$$

Thus we have $G(t)/\lambda(t)$ is eventually non-increasing and

$$\frac{G(t)}{G(t+1)} \ge \frac{\lambda(t)}{\lambda(t+1)}. \tag{13}$$

Using the fact that $r(t)\Delta^\alpha x(t)$ is strictly decreasing, we have

$$r(t)\Delta^\alpha x(t) \geq r(t+1)\Delta^\alpha x(t+1)$$

and $\Delta G(t) > 0$, then $G(t+1) > G(t)$, it follows that

$$\begin{aligned}\Delta\omega_2(t) &\leq \Delta\beta_+(t)\frac{\omega(t+1)}{\beta(t+1)} + \beta(t)\frac{\Delta(r(t)\Delta^\alpha x(t))}{G(t+1)} \\ &\quad - \frac{\Gamma(1-\alpha)\beta(t)}{r(t)}\frac{\omega_2^2(t+1)}{\beta^2(t+1)}.\end{aligned}$$

From 8, we have

$$\begin{aligned}&c(u)\Delta\bigl(a(u)\Delta\bigl(r(u)\Delta^\alpha x(u)\bigr)\bigr) - c(t)\Delta\bigl(a(t)\Delta\bigl(r(t)\Delta^\alpha x(t)\bigr)\bigr) \\ &\quad = -\sum_{s=t}^{u-1} q(s)G(s)\end{aligned}$$

for $\Delta G(t) > 0$, and letting $u \to \infty$, we get

$$-c(t)\Delta\bigl(a(t)\Delta\bigl(r(t)\Delta^\alpha x(t)\bigr)\bigr) \leq -G(t)\sum_{s=t}^{\infty} q(s)$$

or

$$\Delta\bigl(a(t)\Delta\bigl(r(t)\Delta^\alpha x(t)\bigr)\bigr) \geq \frac{G(t)}{c(t)}\sum_{s=t}^{\infty} q(s).$$

And so

$$a(u)\Delta\bigl(r(u)\Delta^\alpha x(u)\bigr) - a(t)\Delta\bigl(r(t)\Delta^\alpha x(t)\bigr) \geq G(t)\sum_{s=t}^{u-1}\left(\frac{1}{c(s)}\sum_{\nu=s}^{\infty} q(\nu)\right).$$

Letting $u \to \infty$, we have

$$\Delta\bigl(r(t)\Delta^\alpha x(t)\bigr) \leq -G(t)\frac{1}{a(t)}\sum_{s=t}^{\infty}\left(\frac{1}{c(s)}\sum_{\nu=s}^{\infty} q(\nu)\right)$$

due to $\lim_{u\to\infty} a(u)\Delta(r(u)\Delta^\alpha x(u)) = k < 0$. Then, by (13), we obtain

$$\begin{aligned}\frac{\Delta(r(t)\Delta^\alpha x(t))}{G(t+1)} &\leq -\frac{G(t)}{G(t+1)}\frac{1}{a(t)}\sum_{s=t}^{\infty}\left(\frac{1}{c(s)}\sum_{\nu=s}^{\infty} q(\nu)\right) \\ &\leq -\frac{\lambda(t)}{\lambda(t+1)}\frac{1}{a(t)}\sum_{s=t}^{\infty}\left(\frac{1}{c(s)}\sum_{\nu=s}^{\infty} q(\nu)\right).\end{aligned}$$

So,

$$\begin{aligned}\Delta\omega_2(t) &\leq \Delta\beta_+(t)\frac{\omega_2(t+1)}{\beta(t+1)} - \beta(t)\frac{\lambda(t)}{\lambda(t+1)}\frac{1}{a(t)}\sum_{s=t}^{\infty}\left(\frac{1}{c(s)}\sum_{\nu=s}^{\infty} q(\nu)\right) \\ &\quad - \frac{\Gamma(1-\alpha)\beta(t)}{r(t)}\frac{\omega_2^2(t+1)}{\beta^2(t+1)}.\end{aligned}$$

Setting $\lambda = 2$, $A = (\frac{\Gamma(1-\alpha)\beta(t)}{r(t)})^{1/2}\frac{\omega_2(t+1)}{\beta(t+1)}$, and $B = \frac{1}{2}(\frac{r(t)}{\Gamma(1-\alpha)\beta(t)})^{1/2}\Delta\beta_+(t)$ using Lemma 1, we obtain

$$\Delta\omega_2(t) \le -\beta(t)\frac{\lambda(t)}{\lambda(t+1)}\frac{1}{a(t)}\sum_{s=t}^{\infty}\left(\frac{1}{c(s)}\sum_{\nu=s}^{\infty}q(\nu)\right) + \frac{r(t)(\Delta\beta_+(t))^2}{4\Gamma(1-\alpha)\beta(t)}.$$

Summing both sides of the above inequality from t_2 to $t-1$, we have

$$\begin{aligned}&\sum_{\zeta=t_2}^{t-1}\left(\beta(\zeta)\frac{\lambda(\zeta)}{\lambda(\zeta+1)}\frac{1}{a(\zeta)}\sum_{s=\zeta}^{\infty}\left(\frac{1}{c(s)}\sum_{\nu=s}^{\infty}q(\nu)\right) - \frac{r(\zeta)(\Delta\beta_+(\zeta))^2}{4\Gamma(1-\alpha)\beta(\zeta)}\right)\\ &\quad\le \omega_2(t_2) - \omega_2(t) \le \omega_2(t_2) < \infty,\end{aligned}$$

which contradicts (12). So, the proof is complete. □

Theorem 2 *Let* (7) *hold. Assume that there exists a positive sequence* γ *such that, for all sufficiently large* t,

$$\lim_{t\to\infty}\sup\sum_{s=t_3}^{t-1}\left(\gamma(s)q(s)\frac{\Gamma(1-\alpha)}{\vartheta(s+1)}\sum_{u=t_2}^{s-1}\frac{\vartheta(u)}{r(u)} - \frac{a(s)\vartheta(s+1)(\Delta\gamma_+(s))^2}{4\gamma(s)\vartheta(s)\sum_{u=t_0}^{s-1}\frac{1}{c(u)}}\right) = \infty. \tag{14}$$

If there exist positive sequences β, λ *such that* (11) *and* (12) *hold, then Eq.* (1) *is oscillatory.*

Proof Suppose to the contrary that $x(t)$ is a non-oscillatory solution of (1). Then, without loss of generality, we may assume that there is a solution $x(t)$ of Eq. (1) such that $x(t) > 0$ on $[t_0,\infty)$ where t_0 is sufficiently large. From Lemma 3, $x(t)$ satisfies Case 1 or Case 2. Firstly, let Case 1 hold. Then we define the following function:

$$\pi(t) = \gamma(t)\frac{c(t)\Delta(a(t)\Delta(r(t)\Delta^\alpha x(t)))}{r(t)\Delta^\alpha x(t)}.$$

For $t \in [t_0,\infty)$, we have

$$\begin{aligned}\Delta\pi(t) &= \Delta\gamma(t)\frac{\pi(t+1)}{\gamma(t+1)} + \gamma(t)\Delta\left(\frac{c(t)\Delta(a(t)\Delta(r(t)\Delta^\alpha x(t)))}{r(t)\Delta^\alpha x(t)}\right)\\ &= \Delta\gamma(t)\frac{\pi(t+1)}{\gamma(t+1)} - \gamma(t)\frac{q(t)G(t)}{r(t+1)\Delta^\alpha x(t+1)}\\ &\quad - \gamma(t)\frac{c(t)\Delta(a(t)\Delta(r(t)\Delta^\alpha x(t)))\Delta(r(t)\Delta^\alpha x(t))}{r(t)\Delta^\alpha x(t)r(t+1)\Delta^\alpha x(t+1)}.\end{aligned}$$

From Lemma 4, we obtain

$$\begin{aligned}&\Delta\bigl(r(t)\Delta^\alpha x(t)\bigr) \ge \frac{\sum_{s=t_0}^{t-1}\frac{1}{c(s)}}{a(t)}c(t)\Delta\bigl(a(t)\Delta\bigl(r(t)\Delta^\alpha x(t)\bigr)\bigr),\\ &1 \le \frac{r(t+1)\Delta^\alpha x(t+1)}{r(t)\Delta^\alpha x(t)} \le \frac{\vartheta(t+1)}{\vartheta(t)},\\ &\frac{\vartheta(t)}{\vartheta(t+1)} \le \frac{r(t+1)\Delta^\alpha x(t+1)}{r(t)\Delta^\alpha x(t)}\end{aligned}$$

or

$$\frac{r(t+1)\vartheta(t)}{r(t)\vartheta(t+1)} \le \frac{\Delta G(t)}{\Delta G(t+1)}$$

and

$$\begin{aligned}\frac{G(t)}{r(t+1)\Delta^{\alpha}x(t+1)} &= \frac{\Gamma(1-\alpha)}{r(t+1)}\frac{G(t)}{\Delta G(t)}\frac{\Delta G(t)}{\Delta G(t+1)} \\ &\ge \frac{\Gamma(1-\alpha)}{r(t+1)}\left(\frac{r(t)}{\vartheta(t)}\sum_{s=t_2}^{t-1}\frac{\vartheta(s)}{r(s)}\right)\frac{r(t+1)\vartheta(t)}{r(t)\vartheta(t+1)} \\ &= \frac{\Gamma(1-\alpha)}{\vartheta(t+1)}\sum_{s=t_2}^{t-1}\frac{\vartheta(s)}{r(s)}.\end{aligned}$$

Hence,

$$\begin{aligned}\Delta\pi(t) \le{}& \Delta\gamma_+(t)\frac{\pi(t+1)}{\gamma(t+1)} - \gamma(t)q(t)\frac{\Gamma(1-\alpha)}{\vartheta(t+1)}\sum_{s=t_2}^{t-1}\frac{\vartheta(s)}{r(s)} \\ &- \frac{\gamma(t)\vartheta(t)}{\vartheta(t+1)}\frac{\sum_{s=t_0}^{t-1}\frac{1}{c(s)}}{a(t)}\frac{\pi^2(t+1)}{\gamma^2(t+1)}.\end{aligned}$$

In Lemma 1, choosing $\lambda = 2$, $A = (\frac{\gamma(t)\vartheta(t)}{\vartheta(t+1)}\frac{\sum_{s=t_1}^{t-1}\frac{1}{c(s)}}{a(t)})^{1/2}\frac{\pi(t+1)}{\gamma(t+1)}$, and $B = \frac{1}{2}(\frac{a(t)\vartheta(t+1)}{\gamma(t)\vartheta(t)\sum_{s=t_0}^{t-1}\frac{1}{c(s)}})^{1/2} \times \Delta\gamma_+(t)$, we obtain

$$\Delta\pi(t) \le -\gamma(t)q(t)\frac{\Gamma(1-\alpha)}{\vartheta(t+1)}\sum_{s=t_2}^{t-1}\frac{\vartheta(s)}{r(s)} + \frac{a(t)\vartheta(t+1)(\Delta\gamma_+(t))^2}{4\gamma(t)\vartheta(t)\sum_{s=t_0}^{t-1}\frac{1}{c(s)}}.$$

Summing both sides of the above inequality from t_3 to $t-1$, we have

$$\begin{aligned}&\sum_{s=t_3}^{t-1}\left(\gamma(s)q(s)\frac{\Gamma(1-\alpha)}{\vartheta(s+1)}\sum_{u=t_2}^{s-1}\frac{\vartheta(u)}{r(u)} - \frac{a(s)\vartheta(s+1)(\Delta\gamma_+(s))^2}{4\gamma(s)\vartheta(s)\sum_{u=t_0}^{s-1}\frac{1}{c(u)}}\right) \\ &\quad\le \pi(t_1) - \pi(t) \\ &\quad\le \pi(t_2) < \infty,\end{aligned}$$

which contradicts (14). And the proof of Case 2 is the same as that of Theorem 1 and hence is omitted. This completes the proof. □

Theorem 3 *Let (7) hold. Assume that there exists a positive sequence γ such that, for all sufficiently large t,*

$$\lim_{t\to\infty}\sup\sum_{s=t_2}^{t-1}\left(\gamma(s)q(s)\frac{\delta(s)}{\delta(s+1)} - \frac{r(s)\phi(s)(\Delta\gamma_+(s))^2}{4\gamma(s)\sum_{s=t_1}^{u-1}\frac{\phi(u)}{a(u)}\sum_{u=t_0}^{s-1}\frac{1}{c(u)}}\right) = \infty. \tag{15}$$

If there exist positive sequences β, λ such that (11) and (12) hold, then Eq. (1) is oscillatory.

Proof Suppose to the contrary that $x(t)$ is a non-oscillatory solution of (1). Then, without loss of generality, we may assume that there is a solution $x(t)$ of Eq. (1) such that $x(t) > 0$ on $[t_0, \infty)$, where t_0 is sufficiently large. From Lemma 3, $x(t)$ satisfies Case 1 or Case 2. Firstly, let Case 1 hold. Then we define the following function:

$$\nu(t) = \gamma(t)\frac{c(t)\Delta(a(t)\Delta(r(t)\Delta^\alpha x(t)))}{G(t)}.$$

For $t \in [t_0, \infty)$, we get

$$\begin{aligned}\Delta\nu(t) &= \Delta\gamma(t)\frac{\nu(t+1)}{\gamma(t+1)} + \gamma(t)\Delta\left(\frac{c(t)\Delta(a(t)\Delta(r(t)\Delta^\alpha x(t)))}{G(t)}\right)\\ &= \Delta\gamma(t)\frac{\nu(t+1)}{\alpha(t+1)} - \gamma(t)\frac{q(t)G(t)}{G(t+1)}\\ &\quad - \gamma(t)\frac{c(t)\Delta(a(t)\Delta(r(t)\Delta^\alpha x(t)))\Delta G(t)}{G(t)G(t+1)}.\end{aligned}$$

From Lemma 4, we have

$$\Delta G(t) \geq \frac{1}{r(t)}\left(\frac{a(t)}{\phi(t)}\sum_{s=t_1}^{t-1}\frac{\phi(s)}{a(s)}\right)\frac{\sum_{s=t_0}^{t-1}\frac{1}{c(s)}}{a(t)}c(t)\Delta\big(a(t)\Delta\big(r(t)\Delta^\alpha x(t)\big)\big)$$

and

$$\frac{G(t)}{G(t+1)} \geq \frac{\delta(t)}{\delta(t+1)}.$$

Thus we obtain

$$\begin{aligned}\Delta\nu(t) &\leq \Delta\gamma_+(t)\frac{\nu(t+1)}{\gamma(t+1)} - \gamma(t)p(t)\frac{\delta(t)}{\delta(t+1)}\\ &\quad - \frac{\gamma(t)}{r(t)\phi(t)}\sum_{s=t_1}^{t-1}\frac{\phi(s)}{a(s)}\sum_{s=t_0}^{t-1}\frac{1}{c(s)}\frac{\nu^2(t+1)}{\gamma^2(t+1)}.\end{aligned}$$

Then, setting $\lambda = 2$,

$$\begin{aligned}A &= \left(\frac{\gamma(t)}{r(t)\phi(t)}\sum_{s=t_1}^{t-1}\frac{\phi(s)}{a(s)}\sum_{s=t_0}^{t-1}\frac{1}{c(s)}\right)^{1/2}\frac{\nu(t+1)}{\gamma(t+1)}, \quad \text{and}\\ B &= \frac{1}{2}\left(\frac{r(t)\phi(t)}{\gamma(t)\sum_{s=t_1}^{t-1}\frac{\phi(s)}{a(s)}\sum_{s=t_0}^{t-1}\frac{1}{c(s)}}\right)^{1/2}\Delta\gamma_+(t)\end{aligned}$$

using Lemma 1, we obtain

$$\Delta\nu(t) \leq -\gamma(t)q(t)\frac{\delta(t)}{\delta(t+1)} + \frac{r(t)\phi(t)(\Delta\gamma_+(t))^2}{4\gamma(t)\sum_{s=t_1}^{t-1}\frac{\phi(s)}{a(s)}\sum_{s=t_0}^{t-1}\frac{1}{c(s)}}.$$

Summing both sides of the above inequality from t_2 to $t-1$, we have

$$\sum_{s=t_2}^{t-1}\left(\gamma(s)q(s)\frac{\delta(s)}{\delta(s+1)}-\frac{r(s)\phi(s)(\Delta\gamma_+(s))^2}{4\gamma(s)\sum_{s=t_1}^{u-1}\frac{\phi(u)}{a(u)}\sum_{u=t_0}^{s-1}\frac{1}{c(u)}}\right)\le v(t_2)-v(t) \le v(t_2)<\infty,$$

which contradicts (15). The proof of Case 2 is the same as that of Theorem 1 and hence is omitted. This completes the proof. □

3 Applications

Example 1 Consider the following fractional difference equation for $t\ge 2$:

$$\Delta^{3+\alpha}x(t)+t^{-2}\left(\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right)=0. \tag{16}$$

This corresponds to Eq. (1) with $\alpha\in(0,1]$, $t_0=2$, $c(t)=a(t)=r(t)=1$, and $q(t)=t^{-2}$. Then $\phi(t)=\lambda(t)=t-t_1$, $\vartheta(t)=\sum_{s=t_2}^{t-1}(s-t_1)$, $\gamma(t)=\beta(t)=t$. For $k\in(0,1)$, it can be written $kt\le\phi(t)\le t$, $k^2t^2/2\le\vartheta(t)\le t^2/2$, $k^3t^3/3\le\sum_{s=t_3}^{t-1}k^2s^2\le t^3/3$. So,

$$\begin{aligned}&\limsup_{t\to\infty}\sum_{s=t_3}^{t-1}\left(\frac{\Gamma(1-\alpha)\gamma(s)q(s)}{\vartheta(s)\phi(s+1)}\sum_{u=t_2}^{s-1}\frac{\vartheta(u)}{r(u)}\sum_{u=t_1}^{s-1}\frac{\phi(u)}{a(u)}-\frac{c(s)(\Delta\gamma_+(s))^2}{4\gamma(s)}\right)\\&\quad\ge\limsup_{t\to\infty}\sum_{s=t_3}^{t-1}\left(\frac{\Gamma(1-\alpha)k^5s^2}{6(s+1)}-\frac{1}{4s}\right)=\infty\end{aligned}$$

and

$$\begin{aligned}&\limsup_{t\to\infty}\sum_{\zeta=t_2}^{t-1}\left(\frac{\beta(\zeta)\lambda(\zeta)}{\lambda(\zeta+1)a(\zeta)}\sum_{s=\zeta}^{\infty}\left(\frac{1}{c(s)}\sum_{v=s}^{\infty}q(v)\right)-\frac{r(\zeta)(\Delta\beta_+(\zeta))^2}{4\Gamma(1-\alpha)\beta(\zeta)}\right)\\&\quad\ge\limsup_{t\to\infty}\sum_{\zeta=t_2}^{t-1}\left(\frac{\zeta^2}{(\zeta+1)}\sum_{s=\zeta}^{\infty}\left(\sum_{v=s}^{\infty}v^{-2}\right)-\frac{1}{4\Gamma(1-\alpha)\zeta}\right)\\&\quad=\infty.\end{aligned}$$

Thus, (16) is oscillatory from Theorem 1.

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