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Representations of solutions to linear and bilinear difference equations and systems of bilinear difference equations

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Abstract

We represent general solution to a homogeneous linear difference equation of second order in terms of a specially chosen solution to the equation and apply it to get a representation of general solution to the bilinear difference equation in terms of a solution to an associate difference equation of second order, considerably generalizing some recent results in an elegant way. We also present the corresponding representations for some systems of bilinear difference equations. Many historical notes not so known to wide audience are also presented, and we offer an answer to an open question regarding the attribution of the bilinear difference equation.

MSC: Primary 39A05; 39A06

Keywords: Linear difference equation; Bilinear difference equation; General solution; Representation of solutions; System of bilinear difference equations; History of difference equations

1 Introduction

It has now been 300 years since the area of difference equations/recurrent relations started to attract a serious interest of scientist working in various branches of science. During these three centuries scientists have obtained many nice results on the equations, and the area has been growing constantly. However, it has been also noticed that more and more researchers interested in the area work on some difference equations and systems without knowing some basic facts on them, so it frequently happens that some basic results are rediscovered and published. Here, among other things, we will demonstrate it on a concrete example of a nonlinear difference equation. One of the aims of the paper, among other ones, is also to give many useful mathematical and historical information to the experts, as well as to everyone who is interested in the area. The reader can find here many old sources, some of which are most probably original ones, which were forgotten during centuries of investigations of recurrent relations, and can see that some known results and formulas are wrongly attributed, a thing which frequently happens.

1.1 Some history

Since the time of de Moivre, recurrent relations have been attracting interest of mathematicians. The notion recurrent series/sequence was most probably coined in [1], although such sequences had already appeared in [2]. From [1] and [2] we see that de Moivre had all necessary ingredients for solving homogeneous linear difference equations with constant coefficients in closed form, although some explicit formulas for the solutions to the equations of small orders appeared later in [3] (see also [4]). The results by de Moivre can be regarded as a starting point for a serious investigation of recurrence relations.

There had been few results on recurrent relations before de Moivre, such as Cassini's formula for Fibonacci numbers from 1680, but de Moivre seems to be the first who presented some general methods for dealing with the relations. Of course, it is well known that recurrent relations in some descriptive ways had been already known to Fibonacci [5], and certainly to many other ancient researchers, but important analytic results were not known for a long time.

That recurrent relations can be written in the form of difference equations and vice-versa was realized during the eighteenth century by other researchers. Laplace could be the first who noted the connection (see [6]). This is why many researchers have been using both notions interchangeably since that time. This will be also the case in our paper.

Among other ones de Moivre solved the homogeneous second-order linear difference equation with constant coefficients, that is, the following one:

$$a_{n+1} - aa_n - ba_{n-1} = 0, \quad n \in \mathbb{N}, \tag{1}$$

where $b \neq 0$, and obtained the following formula:

$$a_n = \frac{a_1 + (m-a)a_0}{m-p} m^n + \frac{a_1 + (p-a)a_0}{p-m} p^n, \quad n \in \mathbb{N}_0, \tag{2}$$

where m and p are the zeros of the polynomial

$$P_{2,a,b}(t) = t^2 - at - b,$$

when $m \neq p$.

He did it in passing, since he was actually interested in finding closed-form formulas for the sums

$$\sum_n \widehat{a}_n x^n,$$

where the general term $\widehat{a}_n x^n$ satisfies a relative of equation (1).

A more systematic presentation of the de Moivre methods and ideas can be found in Euler's known book [7], where solutions to some other linear recurrent relations can also be found.

By using Viète's formulas, equality (2) can be written in the following form:

$$a_n = a_1 \frac{m^n - p^n}{m-p} + ba_0 \frac{m^{n-1} - p^{n-1}}{m-p} \tag{3}$$

for $n \in \mathbb{N}_0$.

De Moivre also noticed that if a solution $(\widetilde{a}_n)_{n\in\mathbb{N}_0}$ to equation (1) satisfies the following initial conditions:

$$\widetilde{a}_0 = 1 \quad \text{and} \quad \widetilde{a}_1 = a,$$

then

$$\widetilde{a}_n = \frac{m^{n+1} - p^{n+1}}{m - p} \tag{4}$$

for $n \in \mathbb{N}_0$, when $m \neq p$, that is, when $a^2 + 4b \neq 0$.

Combining (3) and (4) it immediately follows that

$$a_n = a_1\widetilde{a}_{n-1} + ba_0\widetilde{a}_{n-2} \tag{5}$$

for $n \geq 2$.

Hence, the representation of solutions to equation (1) given in (5) is a very simple consequence of some results by de Moivre. The simple consequence we could not find in de Moivre's papers, but our observation shows that the representation is/must be a matter of folklore. We have used recently this representation in [8]. The usefulness of solutions (4) has been demonstrated in several recent papers of us on product-type difference equations and systems (see, e.g., [9, 10] and the references therein).

If $b = 1$, from (5) it is obtained

$$a_n = a_1\widetilde{a}_{n-1} + a_0\widetilde{a}_{n-2} \tag{6}$$

for $n \geq 2$.

If additionally it is assumed that $a = 1$, we obtain that representation (6) holds for every solution to the Fibonacci recurrence relation. Moreover, since $\widetilde{a}_0 = \widetilde{a}_1 = 1$, we have

$$\widetilde{a}_n = f_{n+1}, \quad n \in \mathbb{N}_0,$$

where f_n is the Fibonacci sequence [11, 12].

Hence, from (6) we obtain that in this case

$$a_n = a_1 f_n + a_0 f_{n-1}, \quad n \in \mathbb{N}_0, \tag{7}$$

which is a well-known representation of solutions to the Fibonacci difference equation (see, e.g., [11]).

One of the basic nonlinear difference equations solvable in closed form is

$$z_{n+1} = \frac{\alpha z_n + \beta}{\gamma z_n + \delta}, \quad n \in \mathbb{N}_0, \tag{8}$$

where $\alpha\delta \neq \beta\gamma$, $\delta \neq 0$. It is called the *bilinear difference equation* or *linear fractional difference equation* (here we use the first name).

Special cases of equation (8) appeared long time ago since they are connected, for example, to continuous fractions. Note that (8) can be written as follows:

$$\gamma z_{n+1} + \delta = \alpha + \delta + \frac{\beta\gamma - \alpha\delta}{\gamma z_n + \delta}, \quad n \in \mathbb{N}_0.$$

By using the change of variables

$$w_n = \gamma z_n + \delta, \quad n \in \mathbb{N}_0,$$

the equation becomes

$$w_{n+1} = \widetilde{\alpha} + \frac{\widetilde{\beta}}{w_n}, \quad n \in \mathbb{N}_0, \tag{9}$$

where

$$\widetilde{\alpha} = \alpha + \delta \quad \text{and} \quad \widetilde{\beta} = \beta\gamma - \alpha\delta,$$

which defines a continuous fraction.

Since w_n is written as a fraction, we may assume that

$$w_n = \frac{x_{n+1}}{y_{n+1}}, \quad n \in \mathbb{N}_0.$$

If it is naturally assumed that

$$\begin{aligned} x_{n+1} &= \widetilde{\alpha} x_n + \widetilde{\beta} y_n, \\ y_{n+1} &= x_n \end{aligned} \tag{10}$$

for $n \in \mathbb{N}$ (write the right-hand side in (9) as a fraction),

$$x_1 = w_0 \quad \text{and} \quad y_1 = 1, \tag{11}$$

it is easily seen that it must be

$$w_n = \frac{x_{n+1}}{x_n}, \quad n \in \mathbb{N}_0$$

(note that from (10) and (11) with $n = 0$ it is obtained that $x_0 = 1$). These and many other data on various continuous fractions can be found in several papers in [13], see also [14–16].

If in (9) we take the change of variables

$$\widetilde{w}_n = \frac{1}{w_n}, \quad n \in \mathbb{N}_0,$$

then it is obtained

$$\widetilde{w}_{n+1} = \frac{1}{\widetilde{\alpha} + \widetilde{\beta}\widetilde{w}_n}, \quad n \in \mathbb{N}_0, \tag{12}$$

and from the above consideration we have

$$\widetilde{w}_n = \frac{x_n}{x_{n+1}}, \quad n \in \mathbb{N}_0. \tag{13}$$

This is an explanation why equation (8) is usually solved by using the change of variables

$$z_n = \frac{1}{\gamma}\left(\frac{x_{n+1}}{x_n} - \delta\right), \quad n \in \mathbb{N}_0, \tag{14}$$

and why the generalized bilinear difference equation

$$z_{n+1} = \frac{\alpha_n z_n + \beta_n}{\gamma_n z_n + \delta_n}, \quad n \in \mathbb{N}_0,$$

is transformed to a linear second-order difference equation by a similar change of variables.

The change of variables can already be found in papers by Laplace, see, e.g., [6]. Hence, the use of the change of variables is a common method which has been applied since (see, e.g., [17–27]). For a different approach in solving equation (8), which uses a linear system of difference equations, see [15, 28]. A related method has been recently used in [29] for solving another system of difference equations. See also [30]. Methods for solving some linear systems of difference equations appeared also in [6]. We have to mention that both methods transform equation (8) to an equation of the form (1). For some applications of equation (8), see also [31–33].

If $\gamma = 0$, then equation (8) is reduced to the following linear first-order difference equation:

$$z_{n+1} = \widetilde{a} z_n + \widetilde{b}, \quad n \in \mathbb{N}_0, \tag{15}$$

where

$$\widetilde{a} = \frac{\alpha}{\delta} \quad \text{and} \quad \widetilde{b} = \frac{\beta}{\delta}.$$

General solution to equation (15) is

$$z_n = \widetilde{a}^n\left(z_0 + \frac{\widetilde{b}}{\widetilde{a} - 1}\right) + \frac{\widetilde{b}}{1 - \widetilde{a}}, \quad \text{when } \widetilde{a} \neq 1; \tag{16}$$

$$z_n = z_0 + n\widetilde{b}, \quad \text{when } \widetilde{a} = 1, \tag{17}$$

and can be found in [34].

Equation (15) is a special case of the *nonhomogeneous linear first-order difference equation with variable coefficients*

$$z_{n+1} = a_n z_n + b_n, \quad n \in \mathbb{N}_0, \tag{18}$$

which was also solved by Lagrange in [34]. Another method for solving equation (15) can be found in Laplace's paper [6].

There are three standard methods for solving equation (18) which correspond to the three methods for solving the linear differential equation of first order. A presentation of the methods can be found in [26]. Many books on difference equations which deal with equation (18) solve it by using one of the methods [15, 20, 22–24, 35]. Usefulness of the equation in solving many classes of nonlinear difference equations and systems has been recently demonstrated in many papers [32, 36–42] (see also the references therein; see also [43, 44] and compare the methods therein with the solvability ones). For some applications and qualitative analysis of the solutions to equation (18), see [23, 45–47].

Regarding our investigations, we will only say here that our renewed considerable interest, which essentially always existed to some extent, in solvability of difference equations started about fifteen years ago when we noted that the following nonlinear difference equation

$$x_{n+1} = \frac{x_{n-1}}{a + bx_n x_{n-1}}, \quad n \in \mathbb{N}_0, \tag{19}$$

can be solved in closed form using a change of variables, which transforms equation (19) to a special case of equation (18), showing a relative simplicity of the equation. Because of the simplicity we did not expect a special interest in it. However, a bit surprisingly, the method has attracted a considerable interest of researchers and many of them started employing our ideas, which consequently motivated us to study solvability of extensions of equation (19) such as those in [37, 38], and [41], as well as of some related systems of difference equation [40, 42]. Some of our recent methods, ideas, and new classes of solvable equations and systems can be found, for example, in [10] and [39]; see also the references therein.

For some other solvable difference equations and systems or the ones which have invariant integrals and their applications, see, e.g., [48–58] and the references therein.

1.2 On a representation of an equation of form (8)

It is easy to check that solution (4) to equation (1) holds for every $n \in \mathbb{Z}$, and that $a_{-1} = 0$. Hence, the solution is the one with initial conditions $a_{-1} = 0$ and $a_0 = 1$. As we have already mentioned, the Fibonacci sequence is of this type, but with shifted indices (recall that $f_0 = 0$ and $f_1 = 1$). This was one of the motivations for us to introduce in [8] the *generalized Fibonacci sequences* as those solutions to equation (1) satisfying the initial conditions

$$a_0 = 0 \quad \text{and} \quad a_1 = 1. \tag{20}$$

Our idea in [8] was to generalize some representations to solutions to equations (1) and (8) which are given in terms of the Fibonacci sequence.

One of our motivations was the following representation:

$$z_n = \frac{z_0 f_{n-1} + f_n}{z_0 f_n + f_{n+1}}, \tag{21}$$

of well-defined solutions to the following difference equation:

$$z_{n+1} = \frac{1}{1 + z_n}, \quad n \in \mathbb{N}_0, \tag{22}$$

which can be found in [59] and was not theoretically explained there.

Equation (22) is one of the most known difference equations and can be found in many books dealing with sequences or problem books (see, e.g., [60]). Some other special cases of equation (8) can be also found in [61]. Moreover, formula (21) immediately follows from very old results and is essentially well known. Namely, as explained above, equation (22) is solved by using the change of variables (13), known to Laplace already [6], and the following is obtained:

$$x_{n+2} = x_{n+1} + x_n, \quad n \in \mathbb{N}_0. \tag{23}$$

By the known representation (7) [11], essentially known to de Moivre [3], we have

$$x_n = x_0 f_{n-1} + x_1 f_n, \quad n \in \mathbb{N}_0, \tag{24}$$

and consequently

$$z_n = \frac{x_n}{x_{n+1}} = \frac{x_0 f_{n-1} + x_1 f_n}{x_0 f_n + x_1 f_{n+1}} = \frac{z_0 f_{n-1} + f_n}{z_0 f_n + f_{n+1}}$$

for $n \in \mathbb{N}_0$.

1.3 On a recent extension of representation (21)

In [8] we generalized representation formula (21) for the case of solutions to general equation (8) by proving the following result.

Theorem 1 *Consider equation* (8), *with* $\gamma \neq 0$ *and*

$$\alpha\delta \neq \beta\gamma. \tag{25}$$

Let $(s_n)_{n\in\mathbb{N}_0}$ *be the solution to equation*

$$c_{n+1} - (\alpha + \delta)c_n + (\alpha\delta - \beta\gamma)c_{n-1} = 0, \quad n \in \mathbb{N}, \tag{26}$$

satisfying the initial conditions

$$c_0 = 0 \quad \text{and} \quad c_1 = 1. \tag{27}$$

Then every well-defined solution to equation (8) *has the following representation*:

$$z_n = \frac{(\beta\gamma - \alpha\delta)z_0 s_{n-1} + (\alpha z_0 + \beta)s_n}{(\gamma z_0 - \alpha)s_n + s_{n+1}} \tag{28}$$

for $n \in \mathbb{N}_0$.

Using formulas (16) and (17) and some calculations it is easily proved that Theorem 1 also holds in the case when $\gamma = 0$ [62]. Bearing in mind this fact we see that the following somewhat more general result holds.

Theorem 2 *Consider equation (8). Let condition (25) hold and let $(s_n)_{n\in\mathbb{N}_0}$ be the solution to equation (26) satisfying the initial conditions in (27). Then every well-defined solution to equation (8) has representation (28).*

1.4 Problems treated in the paper

Although the solution to equation (1) satisfying initial conditions (20) enables a representation of all the solutions to the equation in a nice form, it is a natural question if some of the other solutions to the equation can also represent all the solutions in a similar form, and if so, which ones of them can do this. More precisely, the problem is to find all $\vec{v} = (v_0, v_1) \in \mathbb{R}^2$ (or $\in \mathbb{C}^2$) such that for every solution $(x_n)_{n\in\mathbb{N}_0}$ to the equation the following representation holds:

$$x_n = c_1 s_n(\vec{v}) + c_2 s_{n-1}(\vec{v}), \quad n \in \mathbb{N}_0, \tag{29}$$

for some constants c_j, $j = 1, 2$, depending on $\vec{v}$, x_0, and x_1.

Note here that since $b \neq 0$, for every solution $(x_n)_{n\in\mathbb{N}_0}$ to equation (1) the value x_{-1} can be naturally defined by using the recurrent relation (1) for $n = 0$, as

$$x_{-1} = \frac{x_1 - a x_0}{b}.$$

Another aim of ours here is to give a positive answer to the question and to apply the obtained result in getting representations of solutions to equation (8) as well as to the corresponding representations for some bilinear systems of difference equations. Besides, we offer here an answer to an open problem circulating among some experts regarding the attribution of equation (8).

2 General representation of solutions to equation (8)

In this section we prove our main results. The first one is a generalization of Theorem 2.

Theorem 3 *Let $\vec{v} = (v_0, v_1) \in \mathbb{R}^2$ (or $\in \mathbb{C}^2$) and $(s_n(\vec{v}))_{n\in\mathbb{N}_0}$ be the solution to equation (1) such that*

$$s_0 = v_0 \quad \text{and} \quad s_1 = v_1. \tag{30}$$

Then all the solutions to equation (1) have representation (29) if and only if

$$b v_0^2 + a v_0 v_1 \neq v_1^2. \tag{31}$$

Further, if (31) holds, then for every solution $(x_n)_{n\in\mathbb{N}_0}$ to equation (1) the following representation holds:

$$x_n = \frac{(b v_0 x_0 + (a v_0 - v_1) x_1) s_n(\vec{v}) + b(v_0 x_1 - v_1 x_0) s_{n-1}(\vec{v})}{b v_0^2 + a v_0 v_1 - v_1^2} \tag{32}$$

for $n \in \mathbb{N}_0$.

Proof Since $s_n(\vec{v})$ is a solution to linear equation (1) so is every sequence of the following form:

$$c_1 s_n(\vec{v}) + c_2 s_{n-1}(\vec{v}), \quad n \in \mathbb{N}_0,$$

where c_1 and c_2 are arbitrary constants.

This means that the set

$$S := \{c_1 s_n(\vec{v}) + c_2 s_{n-1}(\vec{v}) : c_j \in \mathbb{R}\,(\text{or} \in \mathbb{C}), j = 1, 2\}$$

is a subset of the set of all solutions to equation (1). Now we prove that these two sets are equal if and only if condition (31) is satisfied.

Since each solution $(x_n)_{n\in\mathbb{N}_0}$ to equation (1) is completely determined by its initial values x_0 and x_1, we see that (29) holds for every solution $(x_n)_{n\in\mathbb{N}_0}$ to the equation if and only if, for every $(x_0, x_1) \in \mathbb{R}^2$ (or $\in \mathbb{C}^2$),

$$\begin{aligned} x_0 &= c_1 s_0(\vec{v}) + c_2 s_{-1}(\vec{v}), \\ x_1 &= c_1 s_1(\vec{v}) + c_2 s_0(\vec{v}) \end{aligned}$$

for some constants c_1 and c_2.

Due to (30), this means that (29) holds for every solution to the equation if and only if the linear system

$$\begin{aligned} v_0 c_1 + v_{-1} c_2 &= x_0, \\ v_1 c_1 + v_0 c_2 &= x_1 \end{aligned} \tag{33}$$

in variables c_1 and c_2 has a solution for every $(x_0, x_1) \in \mathbb{R}^2$ (or $\in \mathbb{C}^2$).

This will happen if and only if the determinant of system (33) is different from zero, that is, if and only if

$$\Delta := \begin{vmatrix} v_0 & v_{-1} \\ v_1 & v_0 \end{vmatrix} = v_0^2 - v_1 v_{-1} = \frac{b v_0^2 + a v_0 v_1 - v_1^2}{b} \neq 0, \tag{34}$$

from which the first part of the theorem follows.

Now assume that (31) holds. Then from (33) it follows that

$$c_1 = \frac{1}{\Delta} \begin{vmatrix} x_0 & (v_1 - a v_0)/b \\ x_1 & v_0 \end{vmatrix} = \frac{b v_0 x_0 + (a v_0 - v_1) x_1}{b v_0^2 + a v_0 v_1 - v_1^2} \tag{35}$$

and

$$c_2 = \frac{1}{\Delta} \begin{vmatrix} v_0 & x_0 \\ v_1 & x_1 \end{vmatrix} = \frac{b(v_0 x_1 - v_1 x_0)}{b v_0^2 + a v_0 v_1 - v_1^2}. \tag{36}$$

Using (35) and (36) in (29), we get (32), as claimed. □

Remark 1 The set $\widetilde{S}$ consisting of all $\vec{v} \in \mathbb{C}^2$ such that (31) does not hold consists of the zero vector $\vec{0} \in \mathbb{C}^2$, and of those $\vec{v}$ such that

$$\left(\frac{v_1}{v_0}\right)^2 - a\frac{v_1}{v_0} - b = 0,$$

that is,

$$v_1 = \frac{a + \sqrt{a^2 + 4b}}{2} v_0$$

or

$$v_1 = \frac{a - \sqrt{a^2 + 4b}}{2} v_0,$$

so

$$\widetilde{S} = \left\{\left(v_0, \frac{a \pm \sqrt{a^2 + 4b}}{2} v_0\right) : v_0 \in \mathbb{C}\right\}.$$

By representation (32) a generalization of the representation of solutions to difference equation (8) given in (28) can be obtained. Namely, the following theorem holds.

Theorem 4 *Consider equation* (8) *with* $\alpha\delta \neq \beta\gamma$. *Assume that* $\vec{v} = (v_0, v_1) \in \mathbb{R}^2$ (*or* $\in \mathbb{C}^2$) *satisfies the following condition*:

$$(\beta\gamma - \alpha\delta)v_0^2 + (\alpha + \delta)v_0 v_1 \neq v_1^2. \tag{37}$$

Let $(s_n)_{n\in\mathbb{N}_0}$ *be the solution to equation* (26) *such that* $s_0 = v_0$ *and* $s_1 = v_1$. *Then every well-defined solution* $(z_n)_{n\in\mathbb{N}_0}$ *to equation* (8) *has the following representation*:

$$z_n = \frac{(\beta\gamma - \alpha\delta)(\beta v_0 + (\alpha v_0 - v_1)z_0)s_{n-1} + (\beta((\alpha + \delta)v_0 - v_1) + z_0(v_0(\beta\gamma + \alpha^2) - \alpha v_1))s_n}{((\beta\gamma - \alpha\delta)v_0 + \alpha v_1 - \gamma v_1 z_0)s_n + (\delta v_0 - v_1 + \gamma v_0 z_0)s_{n+1}} \tag{38}$$

for $n \in \mathbb{N}_0$.

Proof First assume that $\gamma \neq 0$. Then from (14) we see that for every well-defined solution $(z_n)_{n\in\mathbb{N}_0}$ to equation (8) the following holds:

$$z_n = \frac{1}{\gamma}\left(\frac{x_{n+1}}{x_n} - \delta\right), \quad n \in \mathbb{N}_0, \tag{39}$$

where x_n is a solution to equation (26).

On the other hand, since (37) holds, from Theorem 3 we have

$$x_n = \frac{((\beta\gamma - \alpha\delta)v_0 x_0 + ((\alpha + \delta)v_0 - v_1)x_1)s_n + (\beta\gamma - \alpha\delta)(v_0 x_1 - v_1 x_0)s_{n-1}}{(\beta\gamma - \alpha\delta)v_0^2 + (\alpha + \delta)v_0 v_1 - v_1^2} \tag{40}$$

for every solution to equation (26).

From (39), (40), after some standard but time-consuming calculations and use of (26), we have

$$\begin{aligned} z_n &= \frac{1}{\gamma}\left(\frac{((\beta\gamma-\alpha\delta)v_0x_0+((\alpha+\delta)v_0-v_1)x_1)s_{n+1}+(\beta\gamma-\alpha\delta)(v_0x_1-v_1x_0)s_n}{((\beta\gamma-\alpha\delta)v_0x_0+((\alpha+\delta)v_0-v_1)x_1)s_n+(\beta\gamma-\alpha\delta)(v_0x_1-v_1x_0)s_{n-1}}-\delta\right) \\ &= \frac{1}{\gamma}\frac{((\beta\gamma-\alpha\delta)v_0+((\alpha+\delta)v_0-v_1)(\gamma z_0+\delta))(\alpha s_n+(\beta\gamma-\alpha\delta)s_{n-1})}{((\beta\gamma-\alpha\delta)v_0+((\alpha+\delta)v_0-v_1)(\gamma z_0+\delta))s_n+(v_0(\gamma z_0+\delta)-v_1)(s_{n+1}-(\alpha+\delta)s_n)} \\ &\quad+\frac{1}{\gamma}\frac{(\beta\gamma-\alpha\delta)(v_0(\gamma z_0+\delta)-v_1)(s_n-\delta s_{n-1})}{((\beta\gamma-\alpha\delta)v_0+((\alpha+\delta)v_0-v_1)(\gamma z_0+\delta))s_n+(v_0(\gamma z_0+\delta)-v_1)(s_{n+1}-(\alpha+\delta)s_n)} \\ &= \frac{(\beta\gamma-\alpha\delta)(\beta v_0+(\alpha v_0-v_1)z_0)s_{n-1}+(\beta((\alpha+\delta)v_0-v_1)+z_0(v_0(\beta\gamma+\alpha^2)-\alpha v_1))s_n}{((\beta\gamma-\alpha\delta)v_0+\alpha v_1-\gamma v_1z_0)s_n+(\delta v_0-v_1+\gamma v_0z_0)s_{n+1}}, \end{aligned}$$

finishing the proof, in this case.

Now assume that $\gamma=0$. Then equation (8) becomes

$$z_{n+1}=\frac{\alpha}{\delta}z_n+\frac{\beta}{\delta},\quad n\in\mathbb{N}_0 \tag{41}$$

(note that since $\alpha\delta\neq\beta\gamma=0$, δ cannot be equal to zero so that equation (41) is defined).

From (16) and (17), it follows that

$$z_n=\left(\frac{\alpha}{\delta}\right)^n\left(z_0+\frac{\beta}{\alpha-\delta}\right)-\frac{\beta}{\alpha-\delta},\quad \text{when } \alpha\neq\delta, \tag{42}$$

$$z_n=z_0+\frac{\beta}{\delta}n,\quad \text{when } \alpha=\delta, \tag{43}$$

for $n\in\mathbb{N}_0$.

Let the right-hand side in (38) with $\gamma=0$ be denoted by $\tilde{z}_n$. Then we have

$$\tilde{z}_n:=\frac{\alpha\delta(\beta v_0+(\alpha v_0-v_1)z_0)s_{n-1}-(\beta((\alpha+\delta)v_0-v_1)+\alpha(\alpha v_0-v_1)z_0)s_n}{\alpha(\delta v_0-v_1)s_n+(v_1-\delta v_0)s_{n+1}} \tag{44}$$

for $n\in\mathbb{N}_0$.

By de Moivre's formula (2) we know that

$$s_n=\frac{(\delta v_0-v_1)\alpha^n+(v_1-\alpha v_0)\delta^n}{\delta-\alpha},\quad n\in\mathbb{N}_0, \tag{45}$$

if $\alpha\neq\delta$, while

$$s_n=\big(v_1n+\alpha v_0(1-n)\big)\alpha^{n-1},\quad n\in\mathbb{N}_0, \tag{46}$$

if $\alpha=\delta$, since α and δ are the zeros of the characteristic polynomial

$$P_2(\lambda)=\lambda^2-(\alpha+\delta)\lambda+\alpha\delta$$

associated to equation (26) in the case $\gamma=0$.

Assume that $\alpha \neq \delta$. Then, by using (45) in (44), after some calculation we obtain

$$\begin{aligned}
\tilde{z}_n &= \frac{\alpha\delta(\beta\nu_0 + (\alpha\nu_0 - \nu_1)z_0)((\delta\nu_0 - \nu_1)\alpha^{n-1} + (\nu_1 - \alpha\nu_0)\delta^{n-1})}{\alpha(\delta\nu_0 - \nu_1)((\delta\nu_0 - \nu_1)\alpha^n + (\nu_1 - \alpha\nu_0)\delta^n) + (\nu_1 - \delta\nu_0)((\delta\nu_0 - \nu_1)\alpha^{n+1} + (\nu_1 - \alpha\nu_0)\delta^{n+1})} \\
&\quad - \frac{(\beta((\alpha+\delta)\nu_0 - \nu_1) + \alpha(\alpha\nu_0 - \nu_1)z_0)((\delta\nu_0 - \nu_1)\alpha^n + (\nu_1 - \alpha\nu_0)\delta^n)}{\alpha(\delta\nu_0 - \nu_1)((\delta\nu_0 - \nu_1)\alpha^n + (\nu_1 - \alpha\nu_0)\delta^n) + (\nu_1 - \delta\nu_0)((\delta\nu_0 - \nu_1)\alpha^{n+1} + (\nu_1 - \alpha\nu_0)\delta^{n+1})} \\
&= \frac{(\nu_1 - \alpha\nu_0)(\nu_1 - \delta\nu_0)(\beta\delta^n + (-\beta + (\delta - \alpha)z_0)\alpha^n)}{(\nu_1 - \alpha\nu_0)(\nu_1 - \delta\nu_0)(\delta - \alpha)\delta^n}.
\end{aligned} \tag{47}$$

Because of (37) it must be

$$\nu_1 \neq \alpha\nu_0 \quad \text{and} \quad \nu_1 \neq \delta\nu_0. \tag{48}$$

From (48) and since $\alpha \neq \delta$, from (47) we obtain

$$\tilde{z}_n = \frac{\beta}{\delta - \alpha} + \left(-\frac{\beta}{\delta - \alpha} + z_0\right)\left(\frac{\alpha}{\delta}\right)^n. \tag{49}$$

From (42) and (49) we see that

$$\tilde{z}_n = z_n, \quad n \in \mathbb{N}_0, \tag{50}$$

from which the result follows in this case.

If $\alpha = \delta$, then by using the assumption and (46) in (44), after some calculation we obtain

$$\begin{aligned}
\tilde{z}_n &= \frac{\alpha\delta(\beta\nu_0 + (\alpha\nu_0 - \nu_1)z_0)((\nu_1 - \alpha\nu_0)(n-1) + \alpha\nu_0)\alpha^{n-2}}{\alpha(\delta\nu_0 - \nu_1)((\nu_1 - \alpha\nu_0)n + \alpha\nu_0)\alpha^{n-1} + (\nu_1 - \delta\nu_0)((\nu_1 - \alpha\nu_0)(n+1) + \alpha\nu_0)\alpha^n} \\
&\quad - \frac{(\beta((\alpha+\delta)\nu_0 - \nu_1) + \alpha(\alpha\nu_0 - \nu_1)z_0)((\nu_1 - \alpha\nu_0)n + \alpha\nu_0)\alpha^{n-1}}{\alpha(\delta\nu_0 - \nu_1)((\nu_1 - \alpha\nu_0)n + \alpha\nu_0)\alpha^{n-1} + (\nu_1 - \delta\nu_0)((\nu_1 - \alpha\nu_0)(n+1) + \alpha\nu_0)\alpha^n} \\
&= \frac{(\beta\nu_0 + (\alpha\nu_0 - \nu_1)z_0)((\nu_1 - \alpha\nu_0)n + 2\alpha\nu_0 - \nu_1)\alpha^n}{((\alpha\nu_0 - \nu_1)((\nu_1 - \alpha\nu_0)n + \alpha\nu_0) + (\nu_1 - \alpha\nu_0)((\nu_1 - \alpha\nu_0)n + \nu_1))\alpha^n} \\
&\quad - \frac{(\beta(2\alpha\nu_0 - \nu_1) + \alpha(\alpha\nu_0 - \nu_1)z_0)((\nu_1 - \alpha\nu_0)n + \alpha\nu_0)\alpha^{n-1}}{((\alpha\nu_0 - \nu_1)((\nu_1 - \alpha\nu_0)n + \alpha\nu_0) + (\nu_1 - \alpha\nu_0)((\nu_1 - \alpha\nu_0)n + \nu_1))\alpha^n} \\
&= \frac{(\nu_1 - \alpha\nu_0)^2(\beta n + \alpha z_0)\alpha^{n-1}}{(\nu_1 - \alpha\nu_0)^2\alpha^n}
\end{aligned} \tag{51}$$

for $n \in \mathbb{N}_0$.

Now note that due to (37) it must be

$$\nu_1 \neq \alpha\nu_0. \tag{52}$$

From (52) and since $\alpha \neq 0$, from (51) we obtain

$$\tilde{z}_n = \frac{\beta n + \alpha z_0}{\alpha} \tag{53}$$

for $\in \mathbb{N}_0$.

From (43) and (53) we see that (50) holds also in this case, finishing the proof of the theorem. □

2.1 On a bilinear system of difference equations

It is a natural problem to see if there is a representation similar to the one in Theorem 4 for the case of bilinear systems of difference equations.

The corresponding close-to-symmetric system (a recently introduced terminology, see, e.g, [58]) to equation (8) is

$$z_{n+1}=\frac{\alpha w_n+\beta}{\gamma w_n+\delta},\qquad w_{n+1}=\frac{az_n+b}{cz_n+d},\quad n\in\mathbb{N}_0, \tag{54}$$

where $\alpha,\beta,\gamma,\delta,a,b,c,d,z_0,w_0\in\mathbb{R}$ (or $\in\mathbb{C}$).

From (54) we easily obtain

$$z_{n+1}=\frac{(a\alpha+\beta c)z_{n-1}+\alpha b+\beta d}{(a\gamma+c\delta)z_{n-1}+b\gamma+d\delta},$$
$$w_{n+1}=\frac{(a\alpha+b\gamma)w_{n-1}+a\beta+b\delta}{(\alpha c+\gamma d)w_{n-1}+\beta c+d\delta}$$

for $n\in\mathbb{N}$.

Hence, the sequences

$$z_n^{(j)}:=z_{2n+j},\quad n\in\mathbb{N}_0, j=0,1,$$

are two solutions to the equation

$$\widetilde{z}_{n+1}=\frac{\widetilde{\alpha}\widetilde{z}_n+\widetilde{\beta}}{\widetilde{\gamma}\widetilde{z}_n+\widetilde{\delta}},\quad n\in\mathbb{N}_0, \tag{55}$$

where

$$\widetilde{\alpha}=a\alpha+\beta c,\qquad \widetilde{\beta}=\alpha b+\beta d,\qquad \widetilde{\gamma}=a\gamma+c\delta,\qquad \widetilde{\delta}=b\gamma+d\delta, \tag{56}$$

while the sequences

$$w_n^{(j)}:=w_{2n+j},\quad n\in\mathbb{N}_0, j=0,1,$$

are two solutions to the equation

$$\widetilde{w}_{n+1}=\frac{\widehat{\alpha}\widetilde{w}_n+\widehat{\beta}}{\widehat{\gamma}\widetilde{w}_n+\widehat{\delta}},\quad n\in\mathbb{N}_0, \tag{57}$$

where

$$\widehat{\alpha}=a\alpha+b\gamma,\qquad \widehat{\beta}=a\beta+b\delta,\qquad \widehat{\gamma}=\alpha c+\gamma d,\qquad \widehat{\delta}=\beta c+d\delta. \tag{58}$$

For the case of bilinear equations (55) and (57), the corresponding equations in (26) are the same and are given by

$$s_{n+1}-(a\alpha+b\gamma+c\beta+d\delta)s_n+(ad-bc)(\alpha\delta-\beta\gamma)s_{n-1}=0 \tag{59}$$

for $n\in\mathbb{N}_0$.

It is not difficult to see that

$$a\alpha + b\gamma + c\beta + d\delta = \widetilde{\alpha} + \widetilde{\delta} = \widehat{\alpha} + \widehat{\delta} \tag{60}$$

and

$$(ad - bc)(\alpha\delta - \beta\gamma) = \widetilde{\alpha}\widetilde{\delta} - \widetilde{\beta}\widetilde{\gamma} = \widehat{\alpha}\widehat{\delta} - \widehat{\beta}\widehat{\gamma}. \tag{61}$$

From (60) and (61), we see that (59) can be written in the following two forms:

$$s_{n+1} - (\widetilde{\alpha} + \widetilde{\delta})s_n + (\widetilde{\alpha}\widetilde{\delta} - \widetilde{\beta}\widetilde{\gamma})s_{n-1} = 0, \quad n \in \mathbb{N}_0, \tag{62}$$

and

$$s_{n+1} - (\widehat{\alpha} + \widehat{\delta})s_n + (\widehat{\alpha}\widehat{\delta} - \widehat{\beta}\widehat{\gamma})s_{n-1} = 0, \quad n \in \mathbb{N}_0, \tag{63}$$

which are the corresponding equations in (26) associated to equations (55) and (57), respectively.

By using Theorem 4, equations (62) and (63), as well as the following two equalities:

$$z_1 = \frac{\alpha w_0 + \beta}{\gamma w_0 + \delta} \quad \text{and} \quad w_1 = \frac{az_0 + b}{cz_0 + d},$$

we obtain the following theorem.

Theorem 5 *Consider system (54) with $(\alpha\delta - \beta\gamma)(ad - bc) \neq 0$. Assume that $\vec{v} = (v_0, v_1) \in \mathbb{R}^2$ (or $\in \mathbb{C}^2$) satisfies the condition*

$$(\beta\gamma - \alpha\delta)(ad - bc)v_0^2 + (a\alpha + b\gamma + c\beta + d\delta)v_0 v_1 \neq v_1^2. \tag{64}$$

Let $(s_n)_{n\in\mathbb{N}_0}$ be the solution to equation (59) such that $s_0 = v_0$ and $s_1 = v_1$. Then every well-defined solution $(z_n, w_n)_{n\in\mathbb{N}_0}$ to system (54) has the following representation:

$$z_{2n} = \frac{(\widetilde{\beta}\widetilde{\gamma} - \widetilde{\alpha}\widetilde{\delta})(\widetilde{\beta}v_0 + (\widetilde{\alpha}v_0 - v_1)z_0)s_{n-1} + (\widetilde{\beta}((\widetilde{\alpha} + \widetilde{\delta})v_0 - v_1) + z_0(v_0(\widetilde{\beta}\widetilde{\gamma} + \widetilde{\alpha}^2) - \widetilde{\alpha}v_1))s_n}{((\widetilde{\beta}\widetilde{\gamma} - \widetilde{\alpha}\widetilde{\delta})v_0 + \widetilde{\alpha}v_1 - \widetilde{\gamma}v_1 z_0)s_n + (\widetilde{\delta}v_0 - v_1 + \widetilde{\gamma}v_0 z_0)s_{n+1}},$$

$$\begin{aligned}
z_{2n+1} = {} & \frac{(\widetilde{\beta}\widetilde{\gamma} - \widetilde{\alpha}\widetilde{\delta})(\widetilde{\beta}v_0(\gamma w_0 + \delta) + (\widetilde{\alpha}v_0 - v_1)(\alpha w_0 + \beta))s_{n-1}}{(((\widetilde{\beta}\widetilde{\gamma} - \widetilde{\alpha}\widetilde{\delta})v_0 + \widetilde{\alpha}v_1)(\gamma w_0 + \delta) - \widetilde{\gamma}v_1(\alpha w_0 + \beta))s_n + ((\widetilde{\delta}v_0 - v_1)(\gamma w_0 + \delta) + \widetilde{\gamma}v_0(\alpha w_0 + \beta))s_{n+1}} \\
& + \frac{(\widetilde{\beta}((\widetilde{\alpha} + \widetilde{\delta})v_0 - v_1)(\gamma w_0 + \delta) + (v_0(\widetilde{\beta}\widetilde{\gamma} + \widetilde{\alpha}^2) - \widetilde{\alpha}v_1)(\alpha w_0 + \beta))s_n}{(((\widetilde{\beta}\widetilde{\gamma} - \widetilde{\alpha}\widetilde{\delta})v_0 + \widetilde{\alpha}v_1)(\gamma w_0 + \delta) - \widetilde{\gamma}v_1(\alpha w_0 + \beta))s_n + ((\widetilde{\delta}v_0 - v_1)(\gamma w_0 + \delta) + \widetilde{\gamma}v_0(\alpha w_0 + \beta))s_{n+1}},
\end{aligned}$$

$$w_{2n} = \frac{(\widehat{\beta}\widehat{\gamma} - \widehat{\alpha}\widehat{\delta})(\widehat{\beta}v_0 + (\widehat{\alpha}v_0 - v_1)w_0)s_{n-1} + (\widehat{\beta}((\widehat{\alpha} + \widehat{\delta})v_0 - v_1) + w_0(v_0(\widehat{\beta}\widehat{\gamma} + \widehat{\alpha}^2) - \widehat{\alpha}v_1))s_n}{((\widehat{\beta}\widehat{\gamma} - \widehat{\alpha}\widehat{\delta})v_0 + \widehat{\alpha}v_1 - \widehat{\gamma}v_1 w_0)s_n + (\widehat{\delta}v_0 - v_1 + \widehat{\gamma}v_0 w_0)s_{n+1}},$$

$$\begin{aligned}
w_{2n+1} = {} & \frac{(\widehat{\beta}\widehat{\gamma} - \widehat{\alpha}\widehat{\delta})(\widehat{\beta}v_0(cz_0 + d) + (\widehat{\alpha}v_0 - v_1)(az_0 + b))s_{n-1}}{(((\widehat{\beta}\widehat{\gamma} - \widehat{\alpha}\widehat{\delta})v_0 + \widehat{\alpha}v_1)(cz_0 + d) - \widehat{\gamma}v_1(az_0 + b))s_n + ((\widehat{\delta}v_0 - v_1)(cz_0 + d) + \widehat{\gamma}v_0(az_0 + b))s_{n+1}} \\
& + \frac{(\widehat{\beta}((\widehat{\alpha} + \widehat{\delta})v_0 - v_1)(cz_0 + d) + (v_0(\widehat{\beta}\widehat{\gamma} + \widehat{\alpha}^2) - \widehat{\alpha}v_1)(az_0 + b))s_n}{(((\widehat{\beta}\widehat{\gamma} - \widehat{\alpha}\widehat{\delta})v_0 + \widehat{\alpha}v_1)(cz_0 + d) - \widehat{\gamma}v_1(az_0 + b))s_n + ((\widehat{\delta}v_0 - v_1)(cz_0 + d) + \widehat{\gamma}v_0(az_0 + b))s_{n+1}},
\end{aligned}$$

$n \in \mathbb{N}_0$, where $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\gamma}$, $\widetilde{\delta}$, $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}$, $\widehat{\delta}$ are defined in (56) and (58).

2.2 On a three-dimensional generalization of equation (8)

Here we show that it is also possible to generalize Theorem 4 for the case of a three-dimensional close-to-cyclic system of bilinear difference equations which extends difference equation (8).

The following three-dimensional close-to-cyclic system of bilinear difference equations is a natural generalization of difference equation (8), as well as of system of equations (54)

$$z_{n+1}=\frac{aw_n+b}{cw_n+d},\qquad w_{n+1}=\frac{eu_n+f}{gu_n+h},\qquad u_{n+1}=\frac{pz_n+q}{rz_n+s},\quad n\in\mathbb{N}_0,\tag{65}$$

where $a,b,c,d,e,f,g,h,p,q,r,s,z_0,w_0,u_0\in\mathbb{R}$ (or $\in\mathbb{C}$).

By using the method applied to system (54), it is easy to see that from the equations in (65) it follows that

$$\begin{aligned}
z_{n+1}&=\frac{(aep+bgp+afr+bhr)z_{n-2}+aeq+bgq+afs+bhs}{(cep+dgp+cfr+dhr)z_{n-2}+ceq+dgq+cfs+dhs},\\
w_{n+1}&=\frac{(aep+afr+ceq+cfs)w_{n-2}+bep+bfr+deq+dfs}{(agp+ahr+cgq+chs)w_{n-2}+bgp+bhr+dgq+dhs},\\
u_{n+1}&=\frac{(pae+qce+pbg+qdg)w_{n-2}+paf+qcf+pbh+qdh}{(rae+sce+rbg+sdg)w_{n-2}+raf+scf+rbh+sdh}
\end{aligned}$$

for $n\ge 2$.

Hence, the sequences

$$z_n^{(j)}:=z_{3n+j},\quad n\in\mathbb{N}_0, j=0,1,2,$$

are three solutions to the equation

$$\widetilde{z}_{n+1}=\frac{\widetilde{a}\widetilde{z}_n+\widetilde{b}}{\widetilde{c}\widetilde{z}_n+\widetilde{d}},\quad n\in\mathbb{N}_0,\tag{66}$$

where

$$\widetilde{a}=aep+bgp+afr+bhr,\tag{67}$$

$$\widetilde{b}=aeq+bgq+afs+bhs,\tag{68}$$

$$\widetilde{c}=cep+dgp+cfr+dhr,\tag{69}$$

$$\widetilde{d}=ceq+dgq+cfs+dhs,\tag{70}$$

the sequences

$$w_n^{(j)}:=w_{3n+j},\quad n\in\mathbb{N}_0, j=0,1,2,$$

are three solutions to the equation

$$\widetilde{w}_{n+1}=\frac{\widetilde{e}\widetilde{w}_n+\widetilde{f}}{\widetilde{g}\widetilde{w}_n+\widetilde{h}},\quad n\in\mathbb{N}_0,\tag{71}$$

where

$$\widetilde{e} = aep + afr + ceq + cfs, \tag{72}$$

$$\widetilde{f} = bep + bfr + deq + dfs, \tag{73}$$

$$\widetilde{g} = agp + ahr + cgq + chs, \tag{74}$$

$$\widetilde{h} = bgp + bhr + dgq + dhs, \tag{75}$$

while the sequences

$$u_n^{(j)} := u_{3n+j}, \quad n \in \mathbb{N}_0, j = 0, 1, 2,$$

are three solutions to the equation

$$\widetilde{u}_{n+1} = \frac{\widetilde{p}\widetilde{u}_n + \widetilde{q}}{\widetilde{r}\widetilde{u}_n + \widetilde{s}}, \quad n \in \mathbb{N}_0, \tag{76}$$

where

$$\widetilde{p} = pae + qce + pbg + qdg, \tag{77}$$

$$\widetilde{q} = paf + qcf + pbh + qdh, \tag{78}$$

$$\widetilde{r} = rae + sce + rbg + sdg, \tag{79}$$

$$\widetilde{s} = raf + scf + rbh + sdh. \tag{80}$$

For the case of bilinear equations (66), (71), and (76), the corresponding equations in (26) are the same and are given by

$$\begin{aligned} &s_{n+1} - (aep + bgp + afr + bhr + ceq + dgq + cfs + dhs)s_n \\ &\quad + (ad - bc)(eh - fg)(ps - qr)s_{n-1} = 0 \end{aligned} \tag{81}$$

for $n \in \mathbb{N}_0$.

It is not difficult to see that

$$aep + bgp + afr + bhr + ceq + dgq + cfs + dhs = \widetilde{a} + \widetilde{d} = \widetilde{e} + \widetilde{h} = \widetilde{p} + \widetilde{s} \tag{82}$$

and

$$(ad - bc)(eh - fg)(ps - qr) = \widetilde{a}\widetilde{d} - \widetilde{b}\widetilde{c} = \widetilde{e}\widetilde{h} - \widetilde{f}\widetilde{g} = \widetilde{p}\widetilde{s} - \widetilde{q}\widetilde{r}. \tag{83}$$

From (82) and (83), we see that (81) can be written in the following three forms:

$$s_{n+1} - (\widetilde{a} + \widetilde{d})s_n + (\widetilde{a}\widetilde{d} - \widetilde{b}\widetilde{c})s_{n-1} = 0, \tag{84}$$

$$s_{n+1} - (\widetilde{e} + \widetilde{h})s_n + (\widetilde{e}\widetilde{h} - \widetilde{f}\widetilde{g})s_{n-1} = 0, \tag{85}$$

and

$$s_{n+1}-(\tilde{p}+\tilde{s})s_n+(\tilde{p}\tilde{s}-\tilde{q}\tilde{r})s_{n-1}=0 \tag{86}$$

for $n\in\mathbb{N}_0$, which are the corresponding equations in (26) associated to equations (66), (71), and (76), respectively.

Hence, by using Theorem 4, (84)–(86), and the following equalities:

$$z_1=\frac{aw_0+b}{cw_0+d},\qquad w_1=\frac{eu_0+f}{gu_0+h},\qquad u_1=\frac{pz_0+q}{rz_0+s},$$

we get the following result.

Theorem 6 *Consider system (65). Assume that*

$$(ad-bc)(eh-fg)(ps-qr)\neq 0,$$

$\vec{v}=(v_0,v_1)\in\mathbb{R}^2$ *(or* $\in\mathbb{C}^2$*) satisfies the condition*

$$(bc-ad)(eh-fg)(ps-qr)v_0^2+(aep+bgp+afr+bhr+ceq+dgq+cfs+dhs)v_0v_1\neq v_1^2. \tag{87}$$

Let $(s_n)_{n\in\mathbb{N}_0}$ *be the solution to equation (81) such that* $s_0=v_0$ *and* $s_1=v_1$. *Then every well-defined solution* $(z_n,w_n,u_n)_{n\in\mathbb{N}_0}$ *to system (65) has the following representation*:

$$\begin{aligned}
z_{3n}&=\frac{(\tilde{b}\tilde{c}-\tilde{a}\tilde{d})(\tilde{b}v_0+(\tilde{a}v_0-v_1)z_0)s_{n-1}+(\tilde{b}((\tilde{a}+\tilde{d})v_0-v_1)+z_0(v_0(\tilde{b}\tilde{c}+\tilde{a}^2)-\tilde{a}v_1))s_n}{((\tilde{b}\tilde{c}-\tilde{a}\tilde{d})v_0+\tilde{a}v_1-\tilde{c}v_1z_0)s_n+(\tilde{d}v_0-v_1+\tilde{c}v_0z_0)s_{n+1}},\\
w_{3n}&=\frac{(\tilde{f}\tilde{g}-\tilde{e}\tilde{h})(\tilde{f}v_0+(\tilde{e}v_0-v_1)w_0)s_{n-1}+(\tilde{f}((\tilde{e}+\tilde{h})v_0-v_1)+w_0(v_0(\tilde{f}\tilde{g}+\tilde{e}^2)-\tilde{e}v_1))s_n}{((\tilde{f}\tilde{g}-\tilde{e}\tilde{h})v_0+\tilde{e}v_1-\tilde{g}v_1w_0)s_n+(\tilde{h}v_0-v_1+\tilde{g}v_0w_0)s_{n+1}},\\
u_{3n}&=\frac{(\tilde{q}\tilde{r}-\tilde{p}\tilde{s})(\tilde{q}v_0+(\tilde{p}v_0-v_1)u_0)s_{n-1}+(\tilde{q}((\tilde{p}+\tilde{s})v_0-v_1)+u_0(v_0(\tilde{q}\tilde{r}+\tilde{p}^2)-\tilde{p}v_1))s_n}{((\tilde{q}\tilde{r}-\tilde{p}\tilde{s})v_0+\tilde{p}v_1-\tilde{r}v_1u_0)s_n+(\tilde{s}v_0-v_1+\tilde{r}v_0u_0)s_{n+1}},\\
z_{3n+1}&=\frac{(\tilde{b}\tilde{c}-\tilde{a}\tilde{d})(\tilde{b}v_0(cw_0+d)+(\tilde{a}v_0-v_1)(aw_0+b))s_{n-1}}{(((\tilde{b}\tilde{c}-\tilde{a}\tilde{d})v_0+\tilde{a}v_1)(cw_0+d)-\tilde{c}v_1(aw_0+b))s_n+((\tilde{d}v_0-v_1)(cw_0+d)+\tilde{c}v_0(aw_0+b))s_{n+1}}\\
&\quad+\frac{(\tilde{b}((\tilde{a}+\tilde{d})v_0-v_1)(cw_0+d)+(v_0(\tilde{b}\tilde{c}+\tilde{a}^2)-\tilde{a}v_1)(aw_0+b))s_n}{(((\tilde{b}\tilde{c}-\tilde{a}\tilde{d})v_0+\tilde{a}v_1)(cw_0+d)-\tilde{c}v_1(aw_0+b))s_n+((\tilde{d}v_0-v_1)(cw_0+d)+\tilde{c}v_0(aw_0+b))s_{n+1}},\\
w_{3n+1}&=\frac{(\tilde{f}\tilde{g}-\tilde{e}\tilde{h})(\tilde{f}v_0(gu_0+h)+(\tilde{e}v_0-v_1)(eu_0+f))s_{n-1}}{(((\tilde{f}\tilde{g}-\tilde{e}\tilde{h})v_0+\tilde{e}v_1)(gu_0+h)-\tilde{g}v_1(eu_0+f))s_n+((\tilde{h}v_0-v_1)(gu_0+h)+\tilde{g}v_0(eu_0+f))s_{n+1}}\\
&\quad+\frac{(\tilde{f}((\tilde{e}+\tilde{h})v_0-v_1)(gu_0+h)+(v_0(\tilde{f}\tilde{g}+\tilde{e}^2)-\tilde{e}v_1)(eu_0+f))s_n}{(((\tilde{f}\tilde{g}-\tilde{e}\tilde{h})v_0+\tilde{e}v_1)(gu_0+h)-\tilde{g}v_1(eu_0+f))s_n+((\tilde{h}v_0-v_1)(gu_0+h)+\tilde{g}v_0(eu_0+f))s_{n+1}},\\
u_{3n+1}&=\frac{(\tilde{q}\tilde{r}-\tilde{p}\tilde{s})(\tilde{q}v_0(rz_0+s)+(\tilde{p}v_0-v_1)(pz_0+q))s_{n-1}}{(((\tilde{q}\tilde{r}-\tilde{p}\tilde{s})v_0+\tilde{p}v_1)(rz_0+s)-\tilde{r}v_1(pz_0+q))s_n+((\tilde{s}v_0-v_1)(rz_0+s)+\tilde{r}v_0(pz_0+q))s_{n+1}}\\
&\quad+\frac{(\tilde{q}((\tilde{p}+\tilde{s})v_0-v_1)(rz_0+s)+(v_0(\tilde{q}\tilde{r}+\tilde{p}^2)-\tilde{p}v_1)(pz_0+q))s_n}{(((\tilde{q}\tilde{r}-\tilde{p}\tilde{s})v_0+\tilde{p}v_1)(rz_0+s)-\tilde{r}v_1(pz_0+q))s_n+((\tilde{s}v_0-v_1)(rz_0+s)+\tilde{r}v_0(pz_0+q))s_{n+1}},\\
z_{3n+2}&=\frac{(\tilde{b}\tilde{c}-\tilde{a}\tilde{d})(\tilde{b}v_0+(\tilde{a}v_0-v_1)z_2)s_{n-1}+(\tilde{b}((\tilde{a}+\tilde{d})v_0-v_1)+z_2(v_0(\tilde{b}\tilde{c}+\tilde{a}^2)-\tilde{a}v_1))s_n}{((\tilde{b}\tilde{c}-\tilde{a}\tilde{d})v_0+\tilde{a}v_1-\tilde{c}v_1z_2)s_n+(\tilde{d}v_0-v_1+\tilde{c}v_0z_2)s_{n+1}},\\
w_{3n+2}&=\frac{(\tilde{f}\tilde{g}-\tilde{e}\tilde{h})(\tilde{f}v_0+(\tilde{e}v_0-v_1)w_2)s_{n-1}+(\tilde{f}((\tilde{e}+\tilde{h})v_0-v_1)+w_2(v_0(\tilde{f}\tilde{g}+\tilde{e}^2)-\tilde{e}v_1))s_n}{((\tilde{f}\tilde{g}-\tilde{e}\tilde{h})v_0+\tilde{e}v_1-\tilde{g}v_1w_2)s_n+(\tilde{h}v_0-v_1+\tilde{g}v_0w_2)s_{n+1}},\\
u_{3n+2}&=\frac{(\tilde{q}\tilde{r}-\tilde{p}\tilde{s})(\tilde{q}v_0+(\tilde{p}v_0-v_1)u_2)s_{n-1}+(\tilde{q}((\tilde{p}+\tilde{s})v_0-v_1)+u_2(v_0(\tilde{q}\tilde{r}+\tilde{p}^2)-\tilde{p}v_1))s_n}{((\tilde{q}\tilde{r}-\tilde{p}\tilde{s})v_0+\tilde{p}v_1-\tilde{r}v_1u_2)s_n+(\tilde{s}v_0-v_1+\tilde{r}v_0u_2)s_{n+1}}
\end{aligned} \tag{88}$$

for every $n \in \mathbb{N}_0$ *and each* $j = 0, 1, 2$, *where* $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, $\widetilde{d}$, $\widetilde{e}$, $\widetilde{f}$, $\widetilde{g}$, $\widetilde{h}$, $\widetilde{p}$, $\widetilde{q}$, $\widetilde{r}$, $\widetilde{s}$ *are defined by* (67)–(70), (72)–(75), (77)–(80), *respectively.*

Remark 2 In order to get all the formulas in Theorem 6 in terms of initial values z_0, w_0, and u_0, in the formulas for z_{3n+2}, w_{3n+2}, and u_{3n+2}, the following equalities should be used:

$$
\begin{aligned}
z_2 &= \frac{(ae+bg)u_0 + af + bh}{(ce+dg)u_0 + cf + dh},\\
w_2 &= \frac{(ep+fr)z_0 + eq + fs}{(gp+hr)z_0 + gq + hs},\\
u_2 &= \frac{(pa+qc)w_0 + pb + qd}{(ra+sc)w_0 + rb + sd},
\end{aligned}
$$

which follows from the equations in (65). We have not done this since such obtained formulas are so long that cannot be written in a line without a drastic reduction of letters. Hence, we leave the simple task to the interested reader.

Remark 3 It is time consuming but not difficult to check that the above procedure employed to the two-dimensional close-to-symmetric and three-dimensional close-to-cyclic bilinear systems of difference equations can be also applied to close-to-cyclic bilinear systems of difference equations of fourth order. However, since the procedure leads to some long calculations and quite long formulas, we also leave getting the corresponding formulas for solutions to the system to the interested reader.

3 On the name of equation (8)

As we have already mentioned, difference equations appeared in the form of recurrent relations/series in 1718 at the latest (before that time they usually had appeared, in a way, indirectly or descriptively). The methods by de Moivre were systematized and studied further by Euler and can be found in [7]. Later in 1759 Lagrange in [34] studied the "integrability" (i.e., solvability) of linear difference equations by modifying the methods which had been used in studying differential equations and essentially laid a cornerstone for further investigations.

There has been a recent custom that the bilinear difference equation is called Riccati difference equation. During the last two decades several colleagues, some of them who used the terminology, asked me if the attribution is correct, bearing in mind that it is known that there were not so many investigations on difference equations during the life of Riccati and that nobody has seen a paper written by him on the topic. The frequent question and recent studies of solvability of difference equations motivated me to conduct a considerable, but, of course, not thorough, literature investigation, to try to give a possible answer for the "open problem" in history of difference equations.

As far as we could see, the bilinear difference equation had not been investigated by Riccati at all, which is not so strange bearing in mind that he lived until 1754 and that a serious investigation of the solvability of difference equations practically started in Lagrange's 1759 paper [34].

Reputable sources from the end of eighteenth century, which are almost of an encyclopedic character, such us Cousin's book [63] (1796) and famous Lacroix book [64] (1800) did not mention Riccati in chapters devoted to difference equations at all. Moreover, Lacroix, besides already mentioned mathematicians and other French ones, mentioned in [64] several Italians, such as Brunacci, Paoli, and Malfatti, who had worked on difference equations. If we follow further the nineteenth century literature on difference equations all over Europe (see, for example, Lacroix [65] (1816), Schlömilch [66] (1848), Boole [20] (1880, 1st ed. 1860), Markoff [67, 68] (1896, 2nd ed. 1910)), we see that nobody mention Riccati at all. Although, for example, Boole devoted considerable space in his book to studying bilinear difference equations. It should be also mentioned that the elementary course on differential and integral calculus [65] by Lacroix (1816) considers the bilinear difference equation and solves it by using the change of variables (14), which means that already at the time the solvability of the equation was regarded as a matter of general mathematical culture, as something which is easily understandable. Nörlund in his known book [69] did not consider equation (8), but the book contains a huge number of references not mentioning Riccati at all.

The first encounter of any connection to the bilinear difference equation with Riccati's name, that we managed to find so far, is in book [24] by Milne-Thomson (1933). Namely, he called the following difference equation

$$u(x)u(x+1)+p(x)u(x+1)+q(x)u(x)+r(x)=0 \tag{89}$$

with variable coefficients as "of Riccati's form" and only said that "it is a difference equation corresponding to Riccati's differential one". This means that he did not attribute the equation to Riccati but simply made an analogy, which is nowadays a well-known fact to any expert on difference equations. It should be also noticed that this part of his book seems to be essentially taken from Boole's book [20], which, as we have already mentioned, does not mention Riccati, at all. Whether or not Milne-Thomson is the first who used the terminology is not known to us, but bearing in mind that his book was one of a few books on difference equations at that time, it certainly had some influence in spreading it. It should be also noticed that other books on difference equations up to the 70s did not mention Riccati either (see, e.g., Fort [23] (1948), Richardson [27] (1954), Levy and Lessman [28] (1961)).

The cherry on the cake could be the classroom note [21] by Brand (1955) published in a widely read popular journal, which solves the bilinear difference equation in the way as it was suggested by old French masters [6, 65] and describes the long-term behavior of its solutions. The note starts with: "The sequences defined by a difference equation of Riccati type", but the note is devoted only to the bilinear difference equation, that is, to equation (8) (with constant coefficients). Interestingly [21] did not cite any paper except [16] because of the Kronecker lemma. Bearing in mind that the terminology is similar to the one in [24], we can suppose that Brand borrowed it from there. As we can see, Brand also did not attribute the bilinear difference equation to Riccati, but it seems this, combined with book [24], caused the chain reaction among some experts who started calling equation (8) by a wrong name, that is, a Riccati equation.

Remark 4 Recall that there has been a related problem with solution (2) to equation (1), which has been frequently called Binet's solution (we personally met the attribution for the first time in a Russian issue of book [12], which could have been simply another case of a wrong chain reaction). Nowadays it is well known that, in fact, formula (2) belongs to de Moivre. The formula and the method that leads to getting it can be also regarded as a cornerstone in the study of solvability of difference equations (one of several cornerstones by de Moivre, besides the formula $(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi$, and de Moivre–Laplace theorem).

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