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Periodic solutions for p-Laplacian neutral differential equation with multiple delay and variable coefficients

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Abstract

In this paper, we first discuss some properties of the neutral operator with multiple delays and variable coefficients $(Ax)(t) := x(t) - \sum_{i=1}^{n} c_i(t)x(t-\delta_i)$. Afterwards, by using an extension of Mawhin's continuation theorem, a second order p-Laplacian neutral differential equation

$$\left(\phi_p\left(x(t) - \sum_{i=1}^{n} c_i(t)x(t-\delta_i)\right)'\right)' = \tilde{f}(t, x(t), x'(t))$$

is studied. Some new results on the existence of a periodic solution are obtained. Meanwhile, the approaches to estimate a priori bounds of periodic solutions are different from those known in the literature.

MSC: 34C25; 34K14

Keywords: Neutral operator; p-Laplacian; Periodic solution; Extension of Mawhin's continuation theorem; Singularity

1 Introduction

In this paper, we consider a second order p-Laplacian neutral differential equation

$$\left(\phi_p\left(x(t) - \sum_{i=1}^{n} c_i(t)x(t-\delta_i)\right)'\right)' = \tilde{f}\big(t, x(t), x'(t)\big), \tag{1.1}$$

where $\phi_p : \mathbb{R} \to \mathbb{R}$ is given by $\phi_p(s) = |s|^{p-2}s$, here $p > 1$ is a constant, $c_i(t) \in C^1(\mathbb{R}, \mathbb{R})$ and $c_i(t+T) = c_i(t)$ and δ_i are constants in $[0, T)$ for $i = 1, 2, \ldots, n$; $\tilde{f}$: $[0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an L^2-Carathéodory function, i.e., it is measurable in the first variable and continuous in the second variable, and for every $0 < r < s$ there exists $h_{r,s} \in L^2[0, T]$ such that $|\tilde{f}(t, x(t), x'(t))| \le h_{r,s}$ for all $x \in [r, s]$ and a.e. $t \in [0, T]$.

The study of the properties of the neutral operator $(A_1x)(t) := x(t) - cx(t-\delta)$ began with the paper of Zhang [2]. In 2004, Lu and Ge [14] investigated an extension of A_1, namely the neutral operator $(A_2x)(t) := x(t) - \sum_{i=1}^{n} c_i x(t-\delta_i)$. Afterwards, Du [6] discussed the neutral operator $(A_3x)(t) := x(t) - c(t)x(t-\delta)$, here $c(t)$ is a T-periodic function. And by using

Mawhin's continuation theorem and the properties of A_3, they obtained sufficient conditions for the existence of periodic solutions to the following Liénard neutral differential equation:

$$\bigl(x(t)-c(t)x(t-\tau)\bigr)''+f\bigl(x(t)\bigr)x'(t)+g\bigl(x\bigl(t-\gamma(t)\bigr)\bigr)=e(t).$$

In recent years, many works have been published on the existence of periodic solutions of second-order neutral differential equations (see [1, 3–5, 7, 9, 11–13, 16–19]). In 2007, Zhu and Lu [19] discussed the existence of periodic solutions for a p-Laplacian neutral differential equation

$$\bigl(\phi_p\bigl(x(t)-cx(t-\tau)\bigr)'\bigr)'+g\bigl(t,x\bigl(t-\delta(t)\bigr)\bigr)=p(t).$$

Since $(\phi_p(x'(t)))'$ is nonlinear (i.e., quasilinear), Mawhin's continuation theorem [8] cannot be applied directly. In order to get around this difficulty, Zhu and Lu translated the p-Laplacian neutral differential equation into a two-dimensional system

$$\begin{cases}(x_1(t)-cx_1(t-\tau))'(t)=\phi_q(x_2(t))=|x_2(t)|^{q-2}x_2(t),\\ x_2'(t)=-g(t,x_1(t-\delta(t)))+p(t),\end{cases}$$

where $\frac{1}{p}+\frac{1}{q}=1$, for which Mawhin's continuation theorem can be applied. Afterwards, Du [5] discussed the existence of a periodic solution for a p-Laplacian neutral differential equation

$$\bigl(\phi_p\bigl(x(t)-c(t)x(t-\tau)\bigr)'\bigr)'+f\bigl(x(t)\bigr)x'(t)+g\bigl(x\bigl(t-\gamma(t)\bigr)\bigr)=e(t),$$

by applying Mawhin's continuation theorem.

However, the existence of a periodic solution for p-Laplacian neutral differential equation (1.1) has not been studied until now. The obvious difficulty lies in the following two respects. First, although $(Ax)(t)=x(t)-\sum_{i=1}^{n}c_i(t)x(t-\delta_i)$ is a natural generalization of the operators A_1, A_2 and A_3, the class of neutral differential equations with A typically possesses a more complicated nonlinearity than neutral differential equations with A_1, A_2 and A_3. Second, we do not get $(Ax)'(t)=(Ax')(t)$, meanwhile a priori bounds of periodic solutions are not easy to estimate.

The remaining part of the paper is organized as follows. In Sect. 2, we analyze qualitative properties of the generalized neutral operator A. In Sect. 3, by employing an extension of Mawhin's continuation theorem, we state and prove the existence of periodic solutions for Eq. (1.1). In Sect. 4, we investigate the existence of periodic solutions for a p-Laplacian neutral differential equation by applying Theorem 3.2. In comparison to [5] and [19], we avoid translating the equation into a two-dimensional system. In Sect. 5, we discuss the existence of periodic solutions for a p-Laplacian neutral differential equation with singularity by applying Theorem 3.2. In Sect. 6, we give four examples to demonstrate the validity of the methods.

2 Analysis of the generalized neutral operator

Let

$$\|c_i\| := \max_{t\in[0,T]} \left|c_i(t)\right|, \quad i = 1,2,\dots n; \qquad \|c_k\| := \max\left\{\|c_1\|, \|c_2\|, \dots, \|c_n\|\right\}.$$

Set $C_T := \{x \in C(\mathbb{R},\mathbb{R}) : x(t+T) = x(t), t \in \mathbb{R}\}$, then $(C_T, \|\cdot\|)$ is a Banach space. Define operators $A, B : C_T \to C_T$, by

$$(Ax)(t) = x(t) - \sum_{i=1}^{n} c_i(t)x(t-\delta_i), \qquad (Bx)(t) = \sum_{i=1}^{n} c_i(t)x(t-\delta_i).$$

Lemma 2.1 *If $\sum_{i=1}^{n} \|c_i\| \neq 1$, then operator A has a continuous inverse A^{-1} on C_T, satisfying*

(1)

$$\left|\left(A^{-1}x\right)(t)\right| \le \begin{cases} \frac{\|x\|}{1-\sum_{i=1}^{n}\|c_i\|}, & \text{for } \sum_{i=1}^{n}\|c_i\| < 1; \\ \frac{\frac{1}{\|c_k\|}\|x\|}{1-\frac{1}{\|c_k\|}-\sum_{i=1,i\neq k}^{n}\|\frac{c_i}{c_k}\|}, & \text{for } \sum_{i=1}^{n}\|c_i\| > 1; \end{cases}$$

(2)

$$\int_0^T \left|\left(A^{-1}x\right)(t)\right| dt \le \begin{cases} \frac{1}{1-\sum_{i=1}^{n}\|c_i\|}\int_0^T |x(t)|\,dt, & \text{for } \sum_{i=1}^{n}\|c_i\| < 1; \\ \frac{\frac{1}{\|c_k\|}}{1-\frac{1}{\|c_k\|}-\sum_{i=1,i\neq k}^{n}\|\frac{c_i}{c_k}\|}\int_0^T |x(t)|\,dt, & \text{for } \sum_{i=1}^{n}\|c_i\| > 1. \end{cases}$$

Proof Case 1:

$$\sum_{i=1}^{n} \|c_i\| < 1.$$

$$(Bx)(t) = \sum_{i=1}^{n} c_i(t)x(t-\delta_i);$$

$$\left(B^2x\right)(t) = \sum_{l_1=1}^{n} c_{l_1}(t) \sum_{l_2=1}^{n} c_{l_2}(t-\delta_{l_1})x(t-\delta_{l_1}-\delta_{l_2});$$

$$\left(B^3x\right)(t) = \sum_{l_1=1}^{n} c_{l_1}(t) \sum_{l_2=1}^{n} c_{l_2}(t-\delta_{l_1}) \sum_{l_3=1}^{n} c_{l_2}(t-\delta_{l_1}-\delta_{l_2})x(t-\delta_{l_1}-\delta_{l_2}-\delta_{l_3}).$$

Therefore, we have

$$\left(B^jx\right)(t) = \sum_{l_1=1}^{n} c_{l_1}(t) \sum_{l_2=1}^{n} c_{l_2}(t-\delta_{l_1}) \cdots \sum_{l_j=1}^{n} c_{l_j}(t-\delta_{l_1}-\delta_{l_2}\cdots-\delta_{l_{j-1}})x(t-\delta_{l_1}-\delta_{l_2}-\cdots-\delta_{l_j}),$$

and

$$\sum_{j=0}^{\infty}\left(B^j x\right)(t) = x(t) + \sum_{j=1}^{\infty}\sum_{l_1=1}^{n} c_{l_1}(t)\sum_{l_2=1}^{n} c_{l_2}(t-\delta_{l_1})\cdots \\ \times \sum_{l_j=1}^{n} c_{l_j}(t-\delta_{l_1}-\delta_{l_2}-\cdots-\delta_{l_{j-1}})x(t-\delta_{l_1}-\delta_{l_2}-\cdots-\delta_{l_j}).$$

Since $A = I - B$ and $\|B\| < 1$, we get that A has a continuous inverse $A^{-1}\colon C_T \to C_T$ with

$$A^{-1} = (I-B)^{-1} = I + \sum_{j=1}^{\infty} B^j = \sum_{j=0}^{\infty} B^j,$$

where $B^0 = I$. Then

$$\begin{aligned}
\left|\left(A^{-1}x\right)(t)\right| &= \left|\sum_{j=0}^{\infty}\left(B^j x\right)(t)\right| \\
&= \left|x(t) + \sum_{j=1}^{\infty}\left(B^j x\right)(t)\right| \\
&= \Bigg|x(t) \\
&\quad + \sum_{l_1=1}^{n} c_{l_1}(t)\cdots\sum_{l_j=1}^{n} c_{l_j}(t-\delta_{l_1}-\delta_{l_2}-\cdots-\delta_{l_{j-1}})x(t-\delta_{l_1}-\delta_{l_2}-\cdots-\delta_{l_j})\Bigg| \\
&\le \frac{\|x\|}{1-\sum_{i=1}^{n}\|c_i\|}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\int_0^T \left|\left(A^{-1}x\right)(t)\right| dt \\
&\quad = \int_0^T \left|\sum_{j=0}^{\infty}\left(B^j x\right)(t)\right| dt \\
&\quad \le \sum_{j=0}^{\infty}\int_0^T \left|\left(B^j x\right)(t)\right| dt \\
&\quad \le \sum_{j=0}^{\infty}\int_0^T \Bigg|\sum_{l_1=1}^{n} c_{l_1}(t)\sum_{l_2=1}^{n} c_{l_2}(t-\delta_{l_1})\cdots \\
&\qquad \times \sum_{l_j=1}^{n} c_{l_j}(t-\delta_{l_1}-\delta_{l_2}-\cdots-\delta_{l_{j-1}})x(t-\delta_{l_1}-\delta_{l_2}-\cdots-\delta_{l_j})\Bigg| dt \\
&\quad \le \frac{1}{1-\sum_{i=1}^{n}\|c_i\|}\int_0^T \left|x(t)\right| dt.
\end{aligned}$$

Case 2: $\sum_{i=1}^{n}\|c_i\| > 1$.

The operator $(Ax)(t) = x(t) - \sum_{i=1}^{n} c_i(t)x(t-\delta_i)$ can be converted to

$$\begin{aligned}(Ax)(t) &= x(t) - c_k(t)x(t-\delta_k) - \sum_{i=1,i\neq k}^{n} c_i(t)x(t-\delta_i)\\ &= -c_k(t)\left(-\frac{x(t)}{c_k(t)} + x(t-\delta_k) + \sum_{i=1,i\neq k}^{n}\frac{c_i(t)}{c_k(t)}x(t-\delta_i)\right)\\ &= -c_k(t)\left(x(t-\delta_k) - \frac{x(t)}{c_k(t)} + \sum_{i=1,i\neq k}^{n}\frac{c_i(t)}{c_k(t)}x(t-\delta_i)\right).\end{aligned}$$

Let $t_1 = t - \delta_k$, it is clear that

$$(Ax)(t_1+\delta_k) = -c_k(t_1+\delta_k)\left(x(t_1) - \frac{x(t_1+\delta_k)}{c_k(t_1+\delta_k)} + \sum_{i=1,i\neq k}^{n}\frac{c_i(t_1+\delta_k)}{c_k(t_1+\delta_k)}x(t_1+\delta_k-\delta_i)\right).$$

Define

$$(Ex)(t) = -c_k(t_1+\delta_k)\left(x(t_1) - \frac{x(t_1+\delta_k)}{c_k(t_1+\delta_k)} + \sum_{i=1,i\neq k}^{n}\frac{c_i(t_1+\delta_k)}{c_k(t_1+\delta_k)}x(t_1+\delta_k-\delta_i)\right),$$

$$e_i = \begin{cases}\frac{1}{c_k(t_1+\delta_k)}, & \text{for } i = k;\\ -\frac{c_i(t_1+\delta_k)}{c_k(t_1+\delta_k)}, & \text{for } i \neq k.\end{cases} \qquad \varepsilon_i = \begin{cases}-\delta_k, & \text{for } i = k;\\ \delta_i - \delta_k, & \text{for } i \neq k.\end{cases}$$

Therefore, $(Ex)(t_1+\delta_k) = x(t_1+\delta_k) - \sum_{i=1}^{n} e_i(t_1+\delta_k)x(t_1-\varepsilon_i)$ and, from Case 1, we get

$$\left|\left(E^{-1}x\right)(t)\right| \leq \frac{\|x\|}{1-\sum_{i=1}^{n}\|e_i\|}.$$

Moreover, since $(A^{-1}x)(t) = -\frac{1}{c_k(t)}(E^{-1}x)(t)$, we have

$$\begin{aligned}\left|\left(A^{-1}x\right)(t)\right| &\leq \left|-\frac{1}{c_k(t)}\left(E^{-1}x\right)(t)\right|\\ &\leq \frac{\frac{1}{\|c_k\|}\|x\|}{1-\frac{1}{\|c_k\|}-\sum_{i=1,i\neq k}^{n}\|\frac{c_i}{c_k}\|}.\end{aligned}$$

Meanwhile, we obtain

$$\int_0^T \left|\left(A^{-1}x\right)(t)\right| dt \leq \frac{\frac{1}{\|c_k\|}}{1-\frac{1}{\|c_k\|}-\sum_{i=1,i\neq k}^{n}\|\frac{c_i}{c_k}\|}\int_0^T \left|x'(t)\right| dt. \qquad \square$$

3 Periodic solutions for equation (1.1)

In order to use an extension of Mawhin's continuation theorem [10], we recall it firstly.

Let X and Z be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively. A continuous operator $M : X \cap \operatorname{dom} M \to Z$ is said to be quasilinear if

(1) $\operatorname{Im} M := M(X \cap \operatorname{dom} M)$ is a closed subset of Z;

(2) $\ker M := \{x \in X \cap \operatorname{dom} M : Mx = 0\}$ is a subspace of X with $\dim\ker M < +\infty$.

Let $X_1 = \ker M$ and X_2 be the complement space of X_1 in X, then $X = X_1 \oplus X_2$. Furthermore, Z_1 is a subspace of Z and Z_2 is the complement space of Z_1 in Z, so $Z = Z_1 \oplus Z_2$. Suppose that $P : X \to X_1$ and $Q : Z \to Z_1$ are two projections and $\Omega \subset X$ is an open and bounded set with the origin $\theta \in \Omega$.

Let $N_\lambda : \bar{\Omega} \to Z$, $\lambda \in [0,1]$ be a continuous operator. Denote N_1 by N, and let $\sum_\lambda = \{x \in \bar{\Omega} : Mx = N_\lambda x\}$. Then N_λ is said to be M-compact in $\bar{\Omega}$ if

(3) there is a vector subspace Z_1 of Z with $\dim Z_1 = \dim X_1$ and an operator $R : \bar{\Omega} \times X_2$ being continuous and compact such that for $\lambda \in [0,1]$,

$$(I - Q)N_\lambda(\bar{\Omega}) \subset \operatorname{Im} M \subset (I - Q)Z, \tag{3.1}$$

$$QN_\lambda x = 0, \quad \lambda \in (0,1) \Leftrightarrow QNx = 0, \tag{3.2}$$

$$R(\cdot, 0) \text{ is the zero operator and } R(\cdot,\lambda)|_{\sum_\lambda} = (I - P)|_{\sum_\lambda}, \tag{3.3}$$

and

$$M\bigl[P + R(\cdot,\lambda)\bigr] = (I - Q)N_\lambda. \tag{3.4}$$

Let $J : Z_1 \to X_1$ be a homeomorphism with $J(\theta) = \theta$.

Next, we investigate existence of periodic solutions for Eq. (1.1) by applying the extension of Mawhin's continuation theorem.

Lemma 3.1 ([10]) *Let X and Z be Banach spaces with norm $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively, and $\Omega \subset X$ be an open and bounded set with $\theta \in \Omega$. Suppose that $M : X \cap \operatorname{dom} M \to Z$ is a quasilinear operator and*

$$N_\lambda : \bar{\Omega} \to Z, \quad \lambda \in (0,1)$$

is an M-compact mapping. In addition, if

(a) $Mx \neq N_\lambda x$, $\lambda \in (0,1)$, $x \in \partial\Omega$,

(b) $\deg\{JQN, \Omega \cap \ker M, 0\} \neq 0$,

where $N = N_1$, then the abstract equation $Mx = Nx$ has at least one solution in $\bar{\Omega}$.

Theorem 3.2 *Assume $\sum_{i=1}^n \|c_i\| \neq 1$, Ω is an open bounded set in C_T^1. Suppose the following conditions hold*:

(i) *For each $\lambda \in (0,1)$, the equation*

$$\bigl(\phi_p(Ax)'(t)\bigr)' = \lambda \tilde{f}\bigl(t, x(t), x'(t)\bigr) \tag{3.5}$$

has no solution on $\partial\Omega$.

(ii) *The equation*

$$F(a) := \frac{1}{T}\int_0^T \tilde{f}(t,a,0)\,dt = 0$$

has no solution on $\partial\Omega \cap \mathbb{R}$.

(iii) *The Brouwer degree*

$$\deg\{F, \Omega \cap \mathbb{R}, 0\} \neq 0.$$

Then Eq. (1.1) *has at least one* T*-periodic solution on* $\bar{\Omega}$.

Proof In order to use Lemma 3.1 we study the existence of periodic solutions to Eq. (1.1). We set $X := \{x \in C[0,T] : x(0) = x(T)\}$ and $Z := C[0,T]$,

$$M : X \cap \operatorname{dom} M \to Z, \qquad (Mx)(t) = \bigl(\phi_p(Ax)'(t)\bigr)', \tag{3.6}$$

where $\operatorname{dom} M := \{u \in X : \phi_p(Au)' \in C^1(\mathbb{R},\mathbb{R})\}$. Then $\ker M = \mathbb{R}$. In fact,

$$\begin{aligned}
\ker M &= \bigl\{x \in X : \bigl(\phi_p(Ax)'(t)\bigr)' = 0\bigr\} \\
&= \bigl\{x \in X : \phi_p(Ax)' \equiv c\bigr\} \\
&= \bigl\{x \in X : (Ax)' \equiv \phi_q(c) := c_1\bigr\} \\
&= \bigl\{x \in X : (Ax)(t) \equiv c_1 t + c_2\bigr\},
\end{aligned}$$

where $q > 1$ is a constant with $\frac{1}{p} + \frac{1}{q} = 1$ and c, c_1, c_2 are constants in $\mathbb{R}$. Since $(Ax)(0) = (Ax)(T)$, then we get $\ker M = \{x \in X : (Ax)(t) \equiv c_2\}$. In addition,

$$\begin{aligned}
\operatorname{Im} M = \biggl\{ & y \in Z, \text{ for } x(t) \in X \cap \operatorname{dom} M, \bigl(\phi_p(Ax)'\bigr)' = y(t), \\
& \int_0^T y(t)\,dt = \int_0^T \bigl(\phi_p\bigl((Ax)'\bigr)'\,dt = 0 \biggr\}.
\end{aligned}$$

So M is quasilinear. Let

$$\begin{aligned}
&X_1 = \ker M, \qquad X_2 = \bigl\{x \in X : x(0) = x(T) = 0\bigr\}, \\
&Z_1 = \mathbb{R}, \qquad Z_2 = \operatorname{Im} M.
\end{aligned}$$

Clearly, $\dim X_1 = \dim Z_1 = 1$, and $X = X_1 \oplus X_2$, $P : X \to X_1$, $Q : Z \to Z_1$, are defined by

$$Px = x(0), \qquad Qy = \frac{1}{T}\int_0^T y(s)\,ds.$$

For $\forall \bar{\Omega} \subset X$, define $N_\lambda : \bar{\Omega} \to Z$ by

$$(N_\lambda x)(t) = \lambda \tilde{f}\bigl(t, x(t), x'(t)\bigr).$$

We claim that $(I - Q)N_\lambda(\bar{\Omega}) \subset \operatorname{Im} M = (I - Q)Z$ holds. In fact, for $x \in \bar{\Omega}$, we observe that

$$\begin{aligned}
&\int_0^T (I - Q)N_\lambda x(t)\,dt \\
&\quad = \int_0^T (I - Q)\lambda \tilde{f}\bigl(t, x(t), x'(t)\bigr)\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \lambda \tilde{f}\big(t, x(t), x'(t)\big)\, dt - \int_0^T \frac{\lambda}{T} \int_0^T \tilde{f}\big(s, x(s), x'(s)\big)\, ds\, dt \\
&= 0.
\end{aligned}
$$

Therefore, we have $(I - Q)N_\lambda(\bar{\Omega}) \subset \operatorname{Im} M$.

Moreover, for any $x \in Z$, it is obvious that

$$
\int_0^T (I - Q)x(t)\, dt = \int_0^T \left(x(t) - \int_0^T \frac{1}{T} \int_0^T x(t)\, dt \right) dt = 0.
$$

So, we have $(I - Q)Z \subset \operatorname{Im} M$. On the other hand, $x \in \operatorname{Im} M$ and $\int_0^T x(t)\, dt = 0$, so we have $x(t) = x(t) - \int_0^T x(t)\, dt$. Hence, we get $x(t) \in (I - Q)Z$. Therefore, $\operatorname{Im} M = (I - Q)Z$.

From $QN_\lambda x = 0$, we get $\frac{\lambda}{T} \int_0^T \tilde{f}(t, x(t), x'(t))\, dt = 0$. Since $\lambda \in (0, 1)$, we have $\frac{1}{T} \int_0^T \tilde{f}(t, x(t), x'(t))\, dt = 0$. Therefore, $QNx = 0$, and so Eq. (3.4) also holds.

Let $J : Z_1 \to X_1$, $J(x) = x$, then $J(0) = 0$. Define $R : \bar{\Omega} \times [0, 1] \to X_2$,

$$
\begin{aligned}
R(x, \lambda)(t) = A^{-1} \int_0^t \phi_p^{-1} \Bigg(& a + \int_0^s \lambda \tilde{f}\big(u, x(u), x'(u)\big)\, du \\
& - \frac{\lambda s}{T} \int_0^T \tilde{f}\big(u, x(u), x'(u)\big)\, du \Bigg)\, ds,
\end{aligned} \tag{3.7}
$$

where $a \in R$ is a constant such that

$$
\begin{aligned}
R(x, \lambda)(T) &= A^{-1} \int_0^T \phi_p^{-1} \left(a + \int_0^s \lambda \tilde{f}\big(u, x(u), x'(u)\big)\, du - \frac{\lambda s}{T} \int_0^T \tilde{f}\big(u, x(t), x'(u)\big)\, du \right) ds \\
&= 0.
\end{aligned} \tag{3.8}
$$

From Lemma 2.3 of [15], we know that a is uniquely defined by

$$
a = \tilde{a}(x, \lambda),
$$

where $\tilde{a}(x, \lambda)$ is continuous on $\bar{\Omega} \times [0, 1]$ and maps bounded sets of $\bar{\Omega} \times [0, 1]$ into bounded sets of $\mathbb{R}$.

From Eq. (3.4), one can find that

$$
\mathrm{R} : \bar{\Omega} \times [0, 1] \to X_2.
$$

Now, for any $x \in \sum_\lambda = \{x \in \bar{\Omega} : Mx = N_\lambda x\} = \{x \in \bar{\Omega} : (\phi_p(Ax)'(t))' = \lambda \tilde{f}(t, x(t), x'(t))\}$, we have $\int_0^T \tilde{f}(t, x(t), x'(t))\, dt = 0$, which, together with Eq. (3.7), gives

$$
\begin{aligned}
R(x, \lambda)(t) &= A^{-1} \int_0^t \phi_p^{-1} (a + \int_0^s \lambda \tilde{f}\big(u, x(u), x'(u)\, du\big)\, ds \\
&= A^{-1} \int_0^t \phi_p^{-1} \left(a + \int_0^s \big(\phi_p(Ax)'(u)\big)'\, du \right) ds \\
&= A^{-1} \int_0^t \phi_p^{-1} \big(a + \phi_p(Ax)'(s) - \phi_p(Ax)'(0) \big)\, ds.
\end{aligned}
$$

Taking $a = \phi_p(Ax)'(0)$, we then have

$$\begin{aligned} R(x,\lambda)(T) &= A^{-1}\int_0^T \left(\phi_p^{-1}\left(\phi_p(Ax)'(s)\right)\right) ds \\ &= A^{-1}\int_0^T (Ax)'(t)\, ds \\ &= A^{-1}\left((Ax)(T) - (Ax)(0)\right) \\ &= x(T) - x(0) \\ &= 0, \end{aligned}$$

where a is unique, and we see that

$$a = \tilde{a}(x,\lambda) = \phi_p(Ax)'(0), \quad \forall \lambda \in [0,1].$$

Thus, we derive

$$\begin{aligned} R(x,\lambda)(t)|_{x\in\sum_\lambda} &= A^{-1}\int_0^t \left(\phi_p^{-1}\left(\phi_p(Ax)'(0) + \int_0^s \lambda\tilde{f}\left(t,u,x(u),x'(u)\right) du\right)\right) ds \\ &= A^{-1}\int_0^t \left(\phi_p^{-1}\left(\phi_p(Ax)'(s)\right)\right) ds \\ &= A^{-1}\int_0^t (Ax)'(s)\, ds \\ &= x(t) - x(0) \\ &= (I-P)x(t), \end{aligned}$$

which yields the second part of Eq. (3.3). Meanwhile, if $\lambda = 0$, then

$$\sum_\lambda = \{x \in \bar{\Omega} : Mx = N_\lambda x\} = \{x \in \bar{\Omega} : (\phi_p(Ax)'(t))' = \lambda\tilde{f}(t,x(t),x'(t))\} = c_3,$$

where $c_3 \in \mathbb{R}$ is a constant, so by the continuity of $\tilde{a}(x,\lambda)$ with respect to (x,λ), $a = \tilde{a}(x,0) = \phi_p(Ac)'(0) = 0$. Hence,

$$R(x,0)(t) = A^{-1}\int_0^t \phi_p^{-1}(0)\, ds = 0, \quad \forall x \in \bar{\Omega},$$

which yields the first part of Eq. (3.3). Furthermore, we consider

$$M(P+R) = (I-Q)N_\lambda,$$

and, in fact,

$$\frac{d}{dt}\phi_p\left(A(P+R)\right)' = (I-Q)N_\lambda. \tag{3.9}$$

Integrating both sides of (3.9) over $[0,s]$, we have

$$\int_0^s \frac{d}{dt}\phi_p\big(A(P+R)\big)'\,ds = \int_0^s (I-Q)N_\lambda\,ds.$$

Therefore, we arrive at

$$\begin{aligned}\phi_p\big(A(P+R)\big)'(s) - a &= \lambda\int_0^s \tilde{f}\big(u,x(u),x'(u)\big)\,du - \int_0^s \frac{\lambda}{T}\int_0^T \tilde{f}\big(u,x(u),x'(u)\big)\,du\,dt \\ &= \lambda\int_0^s \tilde{f}\big(u,x(u),x'(u)\big)\,du - \frac{\lambda s}{T}\int_0^T \tilde{f}\big(u,x(u),x'(u)\big)\,du,\end{aligned}$$

where $a := \phi_p(A(P+R))'(0)$. Then, we get

$$\begin{aligned}&\big(A(P+R)\big)'(s) \\ &\quad = \phi_p^{-1}\biggl(a + \lambda\int_0^s \tilde{f}\big(u,x(u),x'(u)\big)\,du - \frac{\lambda s}{T}\int_0^T \tilde{f}\big(u,x(u),x'(u)\big)\,du\biggr).\end{aligned} \tag{3.10}$$

Integrating both sides of (3.10) over $[0,t]$, we derive

$$\begin{aligned}&\int_0^t \big(A(P+R)\big)'(s)\,ds \\ &\quad = \int_0^t \phi_p^{-1}\biggl(a + \lambda\int_0^s \tilde{f}\big(u,x(u),x'(u)\big)\,du - \frac{\lambda s}{T}\int_0^T \tilde{f}\big(u,x(u),x'(u)\big)\,du\biggr)\,ds,\end{aligned}$$

i.e.,

$$\begin{aligned}&(P+R)(t) - (P+R)(0) \\ &\quad = A^{-1}\biggl(\int_0^t \biggl(\phi_p^{-1}\biggl(\biggl(a + \lambda\int_0^s \tilde{f}\big(u,x(u),x'(u)\big)\,du \\ &\qquad - \frac{\lambda s}{T}\int_0^T \tilde{f}\big(u,x(u),x'(u)\big)\,du\biggr)\biggr)\biggr)\,ds\biggr).\end{aligned}$$

Since $R(x,\lambda)(0) = 0$, $P(t) = P(0)$, we obtain

$$\begin{aligned}&R(x,\lambda)(t) \\ &\quad = A^{-1}\biggl(\int_0^t \phi_p^{-1}\biggl(a + \lambda\int_0^s \tilde{f}\big(u,x(u),x'(u)\big)\,du - \frac{\lambda s}{T}\int_0^T \tilde{f}\big(u,x(u),x'(u)\big)\,du\biggr)\,dt\biggr).\end{aligned}$$

Hence, we have that N_λ is M-compact on $\bar{\Omega}$. Obviously, the equation

$$\big(\phi_p(Ax)'(t)\big)' = \lambda\tilde{f}\big(t,x(t),x'(t)\big)$$

can be converted to

$$Mx = N_\lambda x, \quad \lambda \in (0,1),$$

where M and N_λ are defined by Eqs. (3.6) and (3.7), respectively. As proved above,

$$N_\lambda : \bar{\Omega} \to Z, \quad \lambda \in (0,1),$$

is an M-compact mapping. From assumption (i), one finds

$$Mx \neq N_\lambda x, \quad \lambda \in (0,1), x \in \partial\Omega,$$

and assumptions (ii) and (iii) imply that $\deg\{JQN, \Omega \cap \ker M, \theta\}$ is valid and

$$\deg\{JQN, \Omega \cap \ker M, \theta\} \neq 0.$$

So by applications of Lemma 3.1, we see that Eq. (1.1) has a T-periodic solution. □

4 Application of Theorem 3.2: p-Laplacian equation

As an application, we consider the following p-Laplacian neutral Liénard equation:

$$\left(\phi_p\left(x(t) - \sum_{i=1}^{n} c_i(t)x(t-\delta_i)\right)'\right)' + f\big(x(t)\big)x'(t) + g\big(t, x(t)\big) = e(t), \tag{4.1}$$

where $\phi_p : \mathbb{R} \to \mathbb{R}$ is given by $\phi_p(s) = |s|^{p-2}s$, here $p > 1$ is a constant, g is a continuous function defined on $\mathbb{R}^2$ and periodic in t with $g(t,\cdot) = g(t+T,\cdot)$, $f \in C(\mathbb{R},\mathbb{R})$, e is a continuous periodic function defined on $\mathbb{R}$ with period T and $\int_0^T e(t)\,dt = 0$. Next, by applications of Theorem 3.2, we will investigate the existence of periodic solution for Eq. (4.1) in the case that $\sum_{i=1}^n \|c_i\| \neq 1$.

Define

$$\sigma := \begin{cases} \frac{1}{1-\sum_{i=1}^n \|c_i\|}, & \text{for } \sum_{i=1}^n \|c_i\| < 1; \\ \frac{\frac{1}{\|c_k\|}}{1-\frac{1}{\|c_k\|}-\|\frac{c_i}{c_k}\|}, & \text{for } \sum_{i=1}^n \|c_i\| > 1. \end{cases}$$

Theorem 4.1 *Suppose $\sum_{i=1}^n \|c_i\| \neq 1$ holds. Assume the following conditions hold*:

(H_1) *There exists a constant $D > 0$ such that*

$$xg(t,x) > 0, \quad \forall (t,x) \in [0,T] \times \mathbb{R}, \text{ with } |x| > D.$$

(H_2) *There exist positive constants m, $\tilde{n}$ such that*

$$\big|f(x)\big| \leq m|x|^{p-2} + \tilde{n}, \quad x \in \mathbb{R}.$$

(H_3) *There exist positive constants a, b, B such that*

$$\big|g(t,x)\big| \leq a|x|^{p-1} + b, \quad \text{for } |x| > B \text{ and } t \in [0,T].$$

Then Eq. (4.1) has at least one T-periodic solution, if

$$\sigma T^{\frac{1}{q}}\left(\frac{m\sum_{i=1}^n \|c_i\|}{2^{p-1}} + \frac{aT(1+\sum_{i=1}^n \|c_i\|)}{2^p}\right)^{\frac{1}{p}} + \frac{\sigma T \sum_{i=1}^n \|c_i'\|}{2} < 1.$$

Proof Consider the homotopic equation

$$\left(\phi_p\left(x(t)-\sum_{i=1}^{n}c_i(t)x(t-\delta_i)\right)'\right)'+\lambda f\big(x(t)\big)x'(t)+\lambda g\big(t,x(t)\big)=\lambda e(t). \tag{4.2}$$

Firstly, we claim that the set of all T-periodic solutions of Eq. (4.2) is bounded. Let $x(t)\in C_T$ be an arbitrary T-periodic solution of Eq. (4.2). Integrating both sides of (4.2) over $[0,T]$, we have

$$\int_0^T g\big(t,x(t)\big)\,dt=0. \tag{4.3}$$

From the mean-value theorem for integrals, there is a constant $\xi\in[0,T]$ such that

$$g\big(\xi,x(\xi)\big)=0.$$

In view of condition (H_1), we obtain

$$\big|x(\xi)\big|\le D.$$

Then, we have

$$\begin{aligned}
\|x\| &= \max_{t\in[0,T]}\big|x(t)\big| = \max_{t\in[\xi,\xi+T]}\big|x(t)\big| \\
&= \frac{1}{2}\max_{t\in[\xi,\xi+T]}\big(\big|x(t)\big|+\big|x(t-T)\big|\big) \\
&= \frac{1}{2}\max_{t\in[\xi,\xi+T]}\left(\left|x(\xi)+\int_\xi^T x'(s)\,ds\right|+\left|x(\xi)-\int_{t-T}^{\xi}x'(s)\,ds\right|\right) \\
&\le D+\frac{1}{2}\left(\int_\xi^t\big|x'(s)\big|\,ds+\int_{t-T}^{\xi}\big|x'(s)\big|\,ds\right) \\
&\le D+\frac{1}{2}\int_0^T\big|x'(s)\big|\,ds.
\end{aligned} \tag{4.4}$$

Multiplying both sides of Eq. (4.2) by $(Ax)(t)$ and integrating over the interval $[0,T]$, we get

$$\begin{aligned}
&\int_0^T\big(\phi_p(Ax)'(t)\big)'(Ax)(t)\,dt+\lambda\int_0^T f\big(x(t)\big)x'(t)(Ax)(t)\,dt+\lambda\int_0^T g\big(t,x(t)\big)(Ax)(t)\,dt \\
&\quad=\lambda\int_0^T e(t)(Ax)(t)\,dt.
\end{aligned} \tag{4.5}$$

Substituting $\int_0^T(\phi_p(Ax)'(t))'(Ax)(t)\,dt=-\int_0^T|(Ax)'(t)|^p\,dt$, $\int_0^T f(x(t))x'(t)x(t)\,dt=0$ into Eq. (4.5), we see that

$$-\int_0^T\big|(Ax)'(t)\big|^p\,dt=\lambda\int_0^T f\big(x(t)\big)x'(t)\left(\sum_{i=1}^{n}c_i(t)x(t-\delta_i)\right)dt$$

$$
\begin{aligned}
&-\lambda\int_0^T g\bigl(t,x(t)\bigr)(Ax)(t)\,dt\\
&+\lambda\int_0^T e(t)(Ax)(t)\,dt.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\int_0^T \bigl|(Ax)'(t)\bigr|^p\,dt\\
&\quad\le \int_0^T \bigl|f\bigl(x(t)\bigr)\bigr|\bigl|x'(t)\bigr|\left|\sum_{i=1}^n c_i(t)x(t-\delta_i)\right|dt\\
&\qquad+\int_0^T \bigl|g\bigl(t,x(t)\bigr)\bigr|\left|x(t)-\sum_{i=1}^n c_i(t)x(t-\delta_i)\right|dt\\
&\qquad+\int_0^T \bigl|e(t)\bigr|\left|x(t)-\sum_{i=1}^n c_i(t)x(t-\delta_i)\right|dt\\
&\quad\le \sum_{i=1}^n \|c_i\|\|x\|\int_0^T \bigl|f\bigl(x(t)\bigr)\bigr|\bigl|x'(t)\bigr|\,dt+\left(1+\sum_{i=1}^n\|c_i\|\right)\|x\|\int_0^T \bigl|g\bigl(t,x(t)\bigr)\bigr|\,dt\\
&\qquad+\left(1+\sum_{i=1}^n\|c_i\|\right)\|x\|\int_0^T \bigl|e(t)\bigr|\,dt.
\end{aligned}
$$

Define

$$
E_1:=\bigl\{t\in[0,T]\bigm|\bigl|x(t)\bigr|\le B\bigr\},\qquad E_2:=\bigl\{t\in[0,T]\bigm|\bigl|x(t)\bigr|>B\bigr\}.
$$

Using conditions (H_2) and (H_3), we arrive at

$$
\begin{aligned}
&\int_0^T \bigl|(Ax)'(t)\bigr|^p\,dt\\
&\quad\le \sum_{i=1}^n \|c_i\|\|x\|\int_0^T \bigl|f\bigl(x(t)\bigr)\bigr|\bigl|x'(t)\bigr|\,dt+\left(1+\sum_{i=1}^n\|c_i\|\right)\|x\|\int_{E_1+E_2} \bigl|g\bigl(t,x(t)\bigr)\bigr|\,dt\\
&\qquad+\left(1+\sum_{i=1}^n\|c_i\|\right)\|x\|\int_0^T \bigl|e(t)\bigr|\,dt\\
&\quad\le \sum_{i=1}^n \|c_i\|\|x\|\bigl(m\|x\|^{p-2}+\tilde{n}\bigr)\int_0^T \bigl|x'(t)\bigr|\,dt+\left(1+\sum_{i=1}^n\|c_i\|\right)T\|g_B\|\|x\|\\
&\qquad+\left(1+\sum_{i=1}^n\|c_i\|\right)bT\|x\|+aT\left(1+\sum_{i=1}^n\|c_i\|\right)\|x\|^p+\|e\|T\left(1+\sum_{i=1}^n\|c_i\|\right)\|x\|\\
&\quad\le m\sum_{i=1}^n \|c_i\|\|x\|^{p-1}\int_0^T \bigl|x'(t)\bigr|\,dt+\tilde{n}\sum_{i=1}^n \|c_i\|\|x\|\int_0^T \bigl|x'(t)\bigr|\,dt\\
&\qquad+aT\left(1+\sum_{i=1}^n\|c_i\|\right)\|x\|^p+N_1\|x\|,
\end{aligned}
\tag{4.6}
$$

where $\|e\| := \max_{t\in[0,T]} |e(t)|$, $\|g_B\| := \max_{|x(t)|\le B} |g(t,x(t))|$ and $N_1 := (1+\sum_{i=1}^n \|c_i\|)T(\|g_B\| + b + \|e\|)$. Substituting Eq. (4.4) into Eq. (4.6), we get

$$\begin{aligned}\int_0^T \left|(Ax)'(t)\right|^p dt \le{}& m\sum_{i=1}^n \|c_i\|\left(D+\frac{1}{2}\int_0^T \left|x'(t)\right| dt\right)^{p-1}\int_0^T \left|x'(t)\right| dt \\ &+ aT\left(1+\sum_{i=1}^n \|c_i\|\right)\left(D+\frac{1}{2}\int_0^T \left|x'(t)\right| dt\right)^p \\ &+ \tilde{n}\sum_{i=1}^n \|c_i\|\left(D+\frac{1}{2}\int_0^T \left|x'(t)\right| dt\right)\int_0^T \left|x'(t)\right| dt \\ &+ N_1\left(D+\frac{1}{2}\int_0^T \left|x'(t)\right| dt\right). \end{aligned} \tag{4.7}$$

Next, we introduce a classical inequality: there exists a $\kappa(p) > 0$, which is depends on p only, such that

$$(1+x)^p \le 1+(1+p)x, \quad \text{for } x \in \left[0,\kappa(p)\right]. \tag{4.8}$$

Then, we consider the following two cases:

Case 1: If $\frac{D}{\frac{1}{2}\int_0^T |x'(t)|\,dt} > \kappa(p)$, we deduce

$$\int_0^T \left|x'(t)\right| dt < \frac{2D}{\kappa(p)}.$$

From Eq. (4.4), it is clear that

$$\begin{aligned}\|x\| &\le D+\frac{1}{2}\int_0^T \left|x'(t)\right| dt \\ &\le D+\frac{1}{2}\frac{2D}{\kappa(p)} \\ &= D+\frac{D}{\kappa(p)} := M_{11}. \end{aligned} \tag{4.9}$$

Case 2: If $\frac{D}{\frac{1}{2}\int_0^T |x'(t)|\,dt} < \kappa(p)$, then

$$\begin{aligned}&\int_0^T \left|(Ax)'(t)\right|^p dt \\ &\quad\le aT\left(1+\sum_{i=1}^n \|c_i\|\right)\left(\frac{1}{2}\int_0^T \left|x'(t)\right| dt\right)^p \\ &\qquad + m\sum_{i=1}^n \|c_i\|\left(\frac{1}{2}\int_0^T \left|x'(t)\right| dt\right)^{p-1}\int_0^T \left|x'(t)\right| dt \\ &\qquad + aT\left(1+\sum_{i=1}^n \|c_i\|\right)(1+p)D\left(\frac{1}{2}\int_0^T \left|x'(t)\right| dt\right)^{p-1} \\ &\qquad + mpD\sum_{i=1}^n \|c_i\|\left(\frac{1}{2}\int_0^T \left|x'(t)\right| dt\right)^{p-2}\int_0^T \left|x'(t)\right| dt \end{aligned}$$

$$
\begin{aligned}
&+\frac{\tilde{n}\sum_{i=1}^{n}\|c_i\|}{2}\left(\int_0^T\left|x'(t)\right|dt\right)^2+\left(\tilde{n}\sum_{i=1}^{n}\|c_i\|+\frac{N_1}{2}\right)\int_0^T\left|x'(t)\right|dt+N_1D\\
&=\left(\frac{aT(1+\sum_{i=1}^{n}\|c_i\|)}{2^p}+\frac{m\sum_{i=1}^{n}\|c_i\|}{2^{p-1}}\right)\left(\int_0^T\left|x'(t)\right|dt\right)^p\\
&\quad+\left(\frac{aT(1+p)D(1+\sum_{i=1}^{n}\|c_i\|)}{2^{p-1}}+\frac{mpD\sum_{i=1}^{n}\|c_i\|}{2^{p-2}}\right)\left(\int_0^T\left|x'(t)\right|dt\right)^{p-1}\\
&\quad+\frac{\tilde{n}\sum_{i=1}^{n}\|c_i\|}{2}\left(\int_0^T\left|x'(t)\right|dt\right)^2+\left(\tilde{n}\sum_{i=1}^{n}\|c_i\|+\frac{N_1}{2}\right)\int_0^T\left|x'(t)\right|dt+N_1D. \qquad (4.10)
\end{aligned}
$$

Since $(Ax)(t)=x(t)-\sum_{i=1}^{n}x(t-\delta_i)$, we have

$$
\begin{aligned}
(Ax)'(t)&=\left(x(t)-\sum_{i=1}^{n}c_i(t)x(t-\delta_i)\right)'\\
&=x'(t)-\sum_{i=1}^{n}c_i'(t)x(t-\delta_i)-\sum_{i=1}^{n}c_i(t)x'(t-\delta_i)\\
&=\left(Ax'\right)(t)-\sum_{i=1}^{n}c_i'(t)x(t-\delta_i)
\end{aligned}
$$

and

$$
\left(Ax'\right)(t)=(Ax)'(t)+\sum_{i=1}^{n}c_i'(t)x(t-\delta_i).
$$

By applying Lemma 2.1 and Hölder inequality, we get

$$
\begin{aligned}
\int_0^T\left|x'(t)\right|dt&=\int_0^T\left|\left(A^{-1}Ax'\right)(t)\right|dt\\
&\le\sigma\int_0^T\left|\left(Ax'\right)(t)\right|dt\\
&=\sigma\int_0^T\left|(Ax)'(t)+\sum_{i=1}^{n}c_i'(t)x(t-\delta_i)\right|dt\\
&\le\sigma\int_0^T\left|(Ax)'(t)\right|dt+\sigma\int_0^T\left|\sum_{i=1}^{n}c_i'(t)x(t-\delta_i)\right|dt\\
&\le\sigma T^{\frac{1}{q}}\left(\int_0^T\left|(Ax)'(t)\right|^p dt\right)^{\frac{1}{p}}+\sigma T\sum_{i=1}^{n}\left\|c_i'\right\|\|x\|, \qquad (4.11)
\end{aligned}
$$

where $\|c_i'\|:=\max_{t\in[0,T]}|c_i'(t)|$, for $i=1,2,\ldots,n$. Substituting Eq. (4.10) into Eq. (4.11), since $(\tilde{a}+\tilde{b})^k\le\tilde{a}^k+\tilde{b}^k$, $0<k\le 1$, we have

$$
\begin{aligned}
&\int_0^T\left|x'(t)\right|dt\\
&\quad\le\sigma T^{\frac{1}{q}}\left(\frac{aT(1+\sum_{i=1}^{n}\|c_i\|)}{2^p}+\frac{m\sum_{i=1}^{n}\|c_i\|}{2^{p-1}}\right)^{\frac{1}{p}}\int_0^T\left|x'(t)\right|dt
\end{aligned}
$$

$$
\begin{aligned}
&+\sigma T^{\frac{1}{q}}\left(\frac{aT(1+\sum_{i=1}^{n}\|c_i\|)(1+p)D}{2^{p-1}}+\frac{m\sum_{i=1}^{n}\|c_i\|pD}{2^{p-2}}\right)^{\frac{1}{p}}\left(\int_0^T\left|x'(t)\right|dt\right)^{\frac{p}{p-1}}\\
&+\sigma T^{\frac{1}{q}}\left(\frac{\tilde{n}\sum_{i=1}^{n}\|c_i\|}{2}\right)^{\frac{1}{p}}\left(\int_0^T\left|x'(t)\right|dt\right)^{\frac{2}{p}}\\
&+\sigma T^{\frac{1}{q}}\left(\tilde{n}\sum_{i=1}^{n}\|c_i\|D+\frac{N_1}{2}\right)^{\frac{1}{p}}\left(\int_0^T\left|x'(t)\right|dt\right)^{\frac{1}{p}}\\
&+\sigma T^{\frac{1}{q}}(N_1D)^{\frac{1}{p}}+\frac{\sigma T\sum_{i=1}^{n}\|c_i'\|}{2}\int_0^T\left|x'(t)\right|dt+\sigma T\sum_{i=1}^{n}\|c_i'\|D.
\end{aligned}
\tag{4.12}
$$

Since $\sigma T^{\frac{1}{q}}(\frac{m\sum_{i=1}^{n}\|c_i\|}{2^{p-1}}+\frac{aT(1+\sum_{i=1}^{n}\|c_i\|)}{2^p})^{\frac{1}{p}}+\frac{\sigma T\sum_{i=1}^{n}\|c_i'\|}{2}<1$, it is easily see that there exists a constant $M_1'>0$ (independent of λ) such that

$$
\int_0^T\left|x'(t)\right|dt\le M_1'. \tag{4.13}
$$

From Eq. (4.4), we obtain

$$
\|x\|\le D+\frac{1}{2}\int_0^T\left|x'(s)\right|ds\le D+\frac{1}{2}M_1':=M_{12}. \tag{4.14}
$$

Let $M_1=\sqrt{M_{11}^2+M_{12}^2}+1$. As $(Ax)(0)=(Ax)(T)$, there exists a point $t_0\in(0,T)$ such that $(Ax)'(t_0)=0$. Moreover, since $\phi_p(0)=0$, due to Eq. (4.14), it is obvious that

$$
\begin{aligned}
\left|\phi_p(Ax)'(t)\right|&=\left|\int_{t_0}^{t}\left(\phi_p(Ax)'(s)\right)'ds\right|\\
&\le\lambda\int_0^T\left|f\left(x(t)\right)\right|\left|x'(t)\right|dt+\lambda\int_0^T\left|g\left(t,x(t)\right)\right|dt+\lambda\int_0^T\left|e(t)\right|dt\\
&\le\|f_{M_1}\|\int_0^T\left|x'(t)\right|dt+T\|g_{M_1}\|+T\|e\|\\
&\le\|f_{M_1}\|M_1'+T\|g_{M_1}\|+T\|e\|:=M_2',
\end{aligned}
$$

where $\|f_{M_1}\|:=\max_{|x(t)|\le M_1}|f(x(t))|$ and $\|g_{M_1}\|:=\max_{|x(t)|\le M_1}|g(t,x(t))|$. Next we claim that there exists a positive constant $M_2^*>M_2'+1$, such that, for all $t\in\mathbb{R}$,

$$
\left\|(Ax)'\right\|\le M_2^*. \tag{4.15}
$$

In fact, if $(Ax)'$ is not bounded, there exists a positive constant M_2'' such that $\|(Ax)'\|>M_2''$ for some $(Ax)'\in\mathbb{R}$. Therefore, we have $\|\phi_p(Ax)'\|=\|(Ax)'^{p-1}\|\ge M_2''$, which is a contra-

diction. Hence, Eq. (4.15) holds. From Lemma 2.1 and Eq. (4.15), we have

$$\begin{aligned}
\|x'\| &= \|A^{-1}Ax'\| \\
&= \|A^{-1}(Ax')(t)\| \\
&\le \sigma \left\| (Ax)'(t) + \sum_{i=1}^{n} c_i'(t)x(t-\delta_i) \right\| \\
&\le \sigma \|(Ax)'\| + \sigma \left(\sum_{i=1}^{n} \|c_i'\| \|x\| \right) \\
&\le \sigma M_2^* + \sigma \sum_{i=1}^{n} \|c_i'\| M_1 := M_2.
\end{aligned} \tag{4.16}$$

Setting $M = \sqrt{M_1^2 + M_2^2} + 1$, we get

$$\Omega = \{x \in C_T^1(\mathbb{R},\mathbb{R}) \mid \|x\| \le M+1, \|x'\| \le M+1\},$$

and we know that Eq. (4.1) has no solution on $\partial\Omega$ as $\lambda \in (0,1)$. When $x(t) \in \partial\Omega \cap \mathbb{R}$, $x(t) = M+1$ or $x(t) = -M-1$, and from Eq. (4.4) we know that $M+1 > D$. Thus, from condition (H_1), we see that

$$\begin{aligned}
&\frac{1}{T}\int_0^T g(t, M+1)\,dt > 0, \\
&\frac{1}{T}\int_0^T g(t, -M-1)\,dt < 0,
\end{aligned}$$

since $\int_0^T e(t)\,dt = 0$. So condition (ii) of Theorem 3.2 is also satisfied. Set

$$H(x,\mu) = \mu x + (1-\mu)\frac{1}{T}\int_0^T g(t,x)\,dt, \quad x \in \partial\Omega \cap \mathbb{R}, \mu \in [0,1].$$

Obviously, from condition (H_1), we can get $xH(x,\mu) > 0$ and thus $H(x,\mu)$ is a homotopic transformation, as well as

$$\begin{aligned}
\deg\{F, \Omega \cap \mathbb{R}, 0\} &= \deg\left\{ \frac{1}{T}\int_0^T g(t,x)\,dt, \Omega \cap \mathbb{R}, 0 \right\} \\
&= \deg\{x, \Omega \cap \mathbb{R}, 0\} \neq 0.
\end{aligned}$$

So condition (iii) of Theorem 3.2 is satisfied. In view of Theorem 3.2, there exists at least one T-periodic solution. □

5 Application of Theorem 3.2: *p*-Laplacian equation with singularity

In this section, we consider Eq. (4.1) with a singularity. Here $g(t,x(t)) = g_0(x) + g_1(t,x(t))$, $g_0 \in C((0,\infty);R)$ and g_1 is an L^2-Carathéodory function, and g_0 has a singularity at $x = 0$, i.e.,

$$\int_0^1 g_0(x)\,dx = -\infty. \tag{5.1}$$

Next, we consider the existence of periodic solutions for Eq. (4.1) with singularity by applying Theorem 3.2.

Theorem 5.1 *Suppose* $\sum_{i=1}^{n}\|c_i\| \neq 1$ *and condition* (H_2) *hold. Assume that the following conditions hold*:

(H_4) *There exist positive constants* $0 < D_1 < D_2$ *such that* x *is a positive continuous* T*-periodic function satisfying* $\int_0^T g(t,x(t))\,dt < 0$, *for some* $x \in (0,D_1)$ *and* $\int_0^T g(t,x(t))\,dt > 0$, *for some* $x \in (D_2,\infty)$.

(H_5) *There exist positive constants* α *and* β *such that*

$$g(t,x) \leq \alpha x^{p-1} + \beta, \quad \text{for } t \in [0,T], \text{ and } x > 0. \tag{5.2}$$

Then Eq. (4.1) *has at least one* T*-periodic solution if*

$$\sigma T^{\frac{1}{q}}\left(\frac{m\sum_{i=1}^{n}\|c_i\|}{2^{p-1}} + \frac{\alpha T(1+\sum_{i=1}^{n}\|c_i\|)}{2^{p-1}}\right)^{\frac{1}{p}} + \frac{\sigma T\sum_{i=1}^{n}\|c_i'\|}{2} < 1.$$

Proof Consider the homotopic equation

$$\bigl(\phi_p(Ax)'(t)\bigr)' + \lambda f\bigl(x(t)\bigr)x'(t) + \lambda g\bigl(t,x(t)\bigr) = \lambda e(t). \tag{5.3}$$

We follow the same strategy and notation as in the proof of Theorem 4.1. From condition (H_4), we know that there exists a constant $D_2 > 0$ such that

$$\bigl|x(t)\bigr| \leq D_2 + \frac{1}{2}\int_0^T \bigl|x'(t)\bigr|\,dt. \tag{5.4}$$

From Eq. (4.5), we have

$$\begin{aligned}\int_0^T \bigl|(Ax)'(t)\bigr|^p\,dt \leq{}& \sum_{i=1}^{n}\|c_i\|\|x\|\int_0^T \bigl|f\bigl(x(t)\bigr)\bigr|\bigl|x'(t)\bigr|\,dt \\ &+ \left(1+\sum_{i=1}^{n}\|c_i\|\right)\|x\|\int_0^T \bigl|g\bigl(t,x(t)\bigr)\bigr|\,dt \\ &+ \left(1+\sum_{i=1}^{n}\|c_i\|\right)\|x\|\int_0^T \bigl|e(t)\bigr|\,dt.\end{aligned}$$

From Eq. (4.3) and condition (H_5), we get

$$\begin{aligned}\int_0^T \bigl|g\bigl(t,x(t)\bigr)\bigr|\,dt &= \int_{g(t,x(t))\geq 0} g^+\bigl(t,x(t)\bigr)\,dt - \int_{g(t,x(t))<0} g^-\bigl(t,x(t)\bigr)\,dt \\ &= 2\int_{g(t,x(t))\geq 0} g^+\bigl(t,x(t)\bigr)\,dt \\ &\leq 2\int_0^T \bigl(\alpha x^{p-1}+\beta\bigr)\,dt \\ &\leq 2\alpha T\|x\|^{p-1} + 2\beta T,\end{aligned} \tag{5.5}$$

where $g^+ := \max\{g(t,x),0\}$. Using condition (H_2) and Eq. (5.5), we derive

$$\begin{aligned}\int_0^T \left|(Ax)'(t)\right|^p dt \le{}& m\sum_{i=1}^n \|c_i\|\|x\|^{p-1}\int_0^T \left|x'(t)\right| dt + \tilde{n}\sum_{i=1}^n \|c_i\|\|x\|\int_0^T \left|x'(t)\right| dt \\ &+ 2\alpha T\left(1+\sum_{i=1}^n \|c_i\|\right)\|x\|^p + 2\beta\left(1+\sum_{i=1}^n \|c_i\|\right)\|x\|T \\ &+ \left(1+\sum_{i=1}^n \|c_i\|\right)\|x\|\|e\|T. \end{aligned} \tag{5.6}$$

Following the same strategy and notation as in the proof of Theorem 4.1, we can obtain, since $\sigma T^{\frac{1}{q}}(\frac{m\sum_{i=1}^n \|c_i\|}{2^{p-1}} + \frac{\alpha T(1+\sum_{i=1}^n \|c_i\|)}{2^{p-1}})^{\frac{1}{p}} + \frac{\sigma T\sum_{i=1}^n \|c_i'\|}{2} < 1$, that there exists a constant $M_3' > 0$ (independent of λ) such that

$$\int_0^T \left|x'(t)\right| dt \le M_3'. \tag{5.7}$$

From Eq. (5.7), we get

$$\|x\| \le D_2 + \frac{1}{2}\int_0^T \left|x'(s)\right| ds \le D_2 + \frac{1}{2}M_3' := M_3. \tag{5.8}$$

From Eqs. (4.15), (4.16) and (5.8), we get that there exists a constant M_3^*, such that, for all $t \in \mathbb{R}$,

$$\left\|x'\right\| \le M_3^*. \tag{5.9}$$

On the other hand, multiplying both sides of (5.3) by $x'(t)$, we get

$$\left(\phi_p(Ax)'(t)\right)x'(t) + \lambda f(x(t)x'(t)x'(t) + \lambda\left(g_1\left(t,x(t)\right) + g_0\left(x(t)\right)\right)x'(t) = \lambda e(t)x'(t), \tag{5.10}$$

since $g(t,x(t)) = g_0(x) + g_1(t,x(t))$. Letting $\tau \in [0,T]$, for any $\tau \le t \le T$, we integrate Eq. (5.10) on $[\tau,t]$ and get

$$\begin{aligned}\lambda\int_\tau^t g_0(x)\,dt ={}& \int_\tau^t \left(\phi_p(Ax)'(t)\right)'x'(t)\,dt - \lambda\int_\tau^t f\left(x(t)\right)x'(t)x'(t)\,dt \\ &- \lambda\int_\tau^t g_1\left(t,x(t)\right)x'(t)\,dt + \lambda\int_\tau^t e(t)x'(t)\,dt.\end{aligned}$$

Furthermore,

$$\begin{aligned}\lambda\left|\int_{x(\tau)}^{x(t)} g_0(u)\,du\right| \le{}& \int_\tau^t \left|\left(\phi_p(Ax)'(t)\right)'\right|\left|x'(t)\right| dt + \lambda\int_\tau^t \left|f(x(t)\right|\left|x'(t)\right|\left|x'(t)\right| dt \\ &+ \lambda\int_\tau^t \left|g_1\left(t,x(t)\right)\right|\left|x'(t)\right| dt + \lambda\int_\tau^t \left|e(t)\right|\left|x'(t)\right| dt.\end{aligned}$$

From Eq. (5.3), we have

$$\begin{aligned}
&\int_\tau^t \left|\left(\phi_p(Ax)'(t)\right)'\right|\left|x'(t)\right| dt \\
&\quad = \|x'\| \int_\tau^t \left|\left(\phi_p(Ax)'(t)\right)'\right| dt \\
&\quad \le \|x'\| \left(\lambda \int_0^T \left|f\left(x(t)\right)\right|\left|x'(t)\right| dt + \lambda \int_0^T \left|g\left(t,x(t)\right)\right| dt + \lambda \int_0^T \left|e(t)\right| dt\right) \\
&\quad \le \lambda M_3^* \left(\|f_{M_3}\| \int_0^T \left|x'(t)\right| dt + 2\alpha T\|x\|^{p-1} + 2\beta T + T\|e\|\right) \\
&\quad \le \lambda M_3^* \left(\|f_{M_3}\| M_3' + 2\alpha T(M_3)^{p-1} + 2\beta T + T\|e\|\right),
\end{aligned}$$

where $\|f_{M_3}\| := \max_{|x(t)|\le M_3} |f(x(t))|$. From Eqs. (5.7) and (5.8), we obtain

$$\begin{aligned}
\lambda \int_\tau^t \left|f\left(x(t)\right)\right|\left|x'(t)\right|\left|x'(t)\right| dt &\le \lambda \int_0^T \left|f\left(x(t)\right)\right|\left|x'(t)\right|\left|x'(t)\right| dt \\
&\le \lambda \|f_{M_3}\| \left(\int_0^T \left|x'(t)\right|^2 dt\right) \\
&\le \lambda \|f_{M_3}\| \left(M_3^*\right)^2 T, \\
\lambda \int_\tau^t \left|g_1(t,x)\right|\left|x'(t)\right| dt &\le \lambda \int_0^T \left|g_1(t,x)\right|\left|x'(t)\right| dt \\
&\le \lambda \|g_{1M_3}\| M_3',
\end{aligned}$$

where $\|g_{1M_3}\| := \max_{|x(t)|\le M_3} |g_1(t,x)|$,

$$\begin{aligned}
\lambda \int_\tau^t \left|e(t)\right|\left|x'(t)\right| dt &\le \lambda \int_0^T \left|e(t)\right|\left|x'(t)\right| dt \\
&\le \lambda \|e\| M_3'.
\end{aligned}$$

From these inequalities, we get

$$\begin{aligned}
\left|\int_{x(\tau)}^{x(t)} g_0(u)\, du\right| &\le M_3^* \left(\|f_{M_3}\| M_3' + 2\alpha T(M_3)^{p-1} + 2\beta T + T\|e\|\right) \\
&\quad + \|f_{M_3}\| \left(M_3^*\right)^2 T + \|g_{1M_3}\| M_3' + \|e\| M_3'.
\end{aligned}$$

In view of Eq. (5.1), we know that there exists a constant $M_4 > 0$ such that

$$x(t) \ge M_4, \quad \forall t \in [\tau, T]. \tag{5.11}$$

The case $t \in [0,\tau]$ can be treated similarly.

From Eqs. (5.8), (5.9) and (5.11), we have

$$\Omega = \left\{x \in C_T^1(\mathbb{R},\mathbb{R}) | E_1 \le x \le E_2, \left\|x'\right\| \le M_3^*, \forall t \in [0,T]\right\},$$

where $0 < E_1 < \min(M_4, D_1)$, $E_2 > \max(M_3, D_2)$. This proves the claim, and the rest of the proof is identical to that of Theorem 4.1. □

6 Examples

Example 6.1 Consider the p-Laplacian Liénard equation in the case $\sum_{i=1}^{n}\|c_i\|<1$:

$$\begin{aligned}&\left(\phi_p\left(x(t)-\left(\frac{1}{40}\sin(4t)x(t-\delta_1)+\frac{1}{60}\cos\left(4t-\frac{\pi}{3}\right)x(t-\delta_2)\right)\right)'\right)'\\&\quad+\left(\frac{1}{20}x\right)x'(t)+\frac{1}{40}\big(2+\sin(4t)\big)x^2=\sin(4t),\end{aligned}\tag{6.1}$$

where $p=3$, δ_1, δ_2 are constants and $0<\delta_1,\delta_2<T$.

Comparing Eq. (6.1) with Eq. (4.1), it is easy to see that $c_1(t)=\frac{1}{40}\sin(4t)$, $c_2(t)=\frac{1}{60}\cos(4t-\frac{\pi}{3})$, $f(x)=\frac{1}{20}x$, $g(t,x)=\frac{1}{40}(2+\sin 4t)x^2$, $e(t)=\sin(4t)$, $T=\frac{\pi}{2}$. It is easy to see that there exists a constant $D=1$ such that condition (H_1) holds. Obviously, we get $|f(x)|=|\frac{1}{20}x|\le\frac{1}{20}|x|+3$, here $m=\frac{1}{20}$, $\tilde{n}=3$, and condition (H_2) holds. Consider $|g(t,x)|=|\frac{1}{40}(2+\sin(4t))x^2|\le\frac{3}{40}|x|^2+1$, here $a=\frac{3}{40}$, $b=1$. So, condition (H_3) is satisfied. Moreover, $\|c_i\|=\frac{1}{40}$, $\|c_2\|=\frac{1}{60}$. So, we have $\sum_{i=1}^{2}\|c_i\|=\|c_1\|+\|c_2\|=\frac{1}{24}<1$. Also $\sigma=\frac{1}{1-\|c_1\|-\|c_2\|}=\frac{24}{23}$, $\|c_1'\|=\frac{1}{10}$ and $\|c_2'\|=\frac{1}{15}$. Next, we consider the condition

$$\begin{aligned}&\sigma T^{\frac{1}{q}}\left(\frac{m\sum_{i=1}^{n}\|c_i\|}{2^{p-1}}+\frac{aT(1+\sum_{i=1}^{n}\|c_i\|)}{2^p}\right)^{\frac{1}{p}}+\frac{\sigma T\sum_{i=1}^{n}\|c_i'\|}{2}\\&=\frac{24}{23}\times\left(\frac{\pi}{2}\right)^{\frac{2}{3}}\times\left(\frac{\frac{1}{20}\times\frac{1}{24}}{2^2}+\frac{\frac{3}{40}\times\frac{\pi}{2}\times\frac{25}{24}}{2^3}\right)^{\frac{1}{3}}+\frac{\frac{24}{23}\times\frac{\pi}{2}\times\frac{1}{6}}{2}\\&\approx 0.4909<1.\end{aligned}$$

Therefore, by Theorem 4.1, we know that Eq. (6.1) has at least one positive $\frac{\pi}{2}$-periodic solution.

Example 6.2 Consider the p-Laplacian Liénard equation in the case $\sum_{i=1}^{n}\|c_i\|>1$:

$$\begin{aligned}&\left(\phi_p\left(x(t)-\left(\left(\frac{1}{8}\cos(8t)+\frac{15}{8}\right)x(t-\delta_3)+\frac{1}{64}\sin\left(8t-\frac{\pi}{6}\right)x(t-\delta_4)\right)\right)'\right)'\\&\quad+\left(\frac{1}{24}x^2+1\right)x'(t)+\frac{1}{16}(2+\sin 8t)x^3=\cos\left(8t-\frac{\pi}{4}\right),\end{aligned}\tag{6.2}$$

where $p=5$, δ_3, δ_4 are constants and $0<\delta_3,\delta_4<T$.

Comparing Eq. (6.2) with Eq. (4.1), it is easy to see that $c_1(t)=\frac{1}{8}\cos(8t)+\frac{15}{8}$, $c_2(t)=\frac{1}{64}\sin(8t-\frac{\pi}{6})$, $f(x)=\frac{1}{24}x^2+1$, $g(t,x)=\frac{1}{16}(2+\sin 8t)x^3$, $e(t)=\cos(8t-\frac{\pi}{4})$. $T=\frac{\pi}{4}$. It is easy to see that there exists a constant $D=1$ such that (H_1) holds. Obviously, we get $|f(x)|=|\frac{1}{24}x^2+1|\le\frac{1}{24}|x|^2+2$, here $m=\frac{1}{24}$, $\tilde{n}=2$, and condition (H_2) holds. Consider $|g(t,x)|=|\frac{1}{16}(2+\sin 8t)x^3|\le\frac{3}{16}|x|^3+1$, here $a=\frac{3}{16}$, $b=1$. So, condition (H_3) is satisfied. Furthermore, $\|c_1\|=\frac{1}{8}+\frac{15}{8}=2$, $\|c_2\|=\frac{1}{64}$, so we have $\sum_{i=1}^{2}\|c_i\|=\|c_1\|+\|c_2\|=\frac{129}{64}>1$, $\sigma=\frac{\frac{1}{\|c_k\|}}{1-\frac{1}{\|c_k\|}-\sum_{i=1,i\neq k}^{n}\|\frac{c_i}{c_k}\|}=$

$\frac{\frac{1}{2}}{1-\frac{1}{2}-\frac{\frac{1}{2}}{\frac{1}{64}}} = \frac{64}{63}$, $\|c_1'\| = 1$ and $\|c_2'\| = \frac{1}{8}$. Next, we consider the condition

$$\begin{aligned}
&\sigma T^{\frac{1}{q}}\left(\frac{m\sum_{i=1}^{n}\|c_i\|}{2^{p-1}} + \frac{aT(1+\sum_{i=1}^{n}\|c_i\|)}{2^p}\right)^{\frac{1}{p}} + \frac{\sigma T\sum_{i=1}^{n}\|c_i'\|}{2} \\
&\quad = \frac{64}{63}\times\left(\frac{\pi}{4}\right)^{\frac{4}{5}}\times\left(\frac{\frac{1}{24}\times\frac{129}{64}}{2^4} + \frac{\frac{3}{16}\times\frac{\pi}{4}\times\frac{193}{64}}{2^5}\right)^{\frac{1}{5}} + \frac{\frac{64}{63}\times\frac{\pi}{4}\times(1+\frac{1}{8})}{2} \\
&\quad \approx 0.8283 < 1.
\end{aligned}$$

Therefore, by Theorem 4.1, we know that Eq. (6.2) has at least one positive $\frac{\pi}{4}$-periodic solution.

Example 6.3 Consider the following p-Laplacian Liénard equation with singularity

$$\begin{aligned}
&\left(\phi_p\left(x(t) - \left(\left(\frac{1}{16}\cos\left(8t-\frac{\pi}{16}\right)\right)x(t-\delta_1) + \frac{1}{24}\sin(8t)x(t-\delta_2)\right.\right.\right. \\
&\quad \left.\left.\left. + \frac{1}{48}\cos\left(8t+\frac{\pi}{6}\right)x(t-\delta_3)\right)\right)'\right)' + \left(\frac{1}{10}x^3\right)x'(t) + \frac{1}{32}\left(\frac{1}{2}+2\sin 8t\right)x^4 - \frac{1}{x^\mu} \\
&= \cos\left(8t+\frac{\pi}{4}\right),
\end{aligned} \tag{6.3}$$

where $p = 5$, $\mu \geq 1$, δ_1, δ_2 and δ_3 are constants and $0 < \delta_1, \delta_2, \delta_3 < T$.

Comparing Eq. (6.3) with Eq. (4.1), it is easy to see that $c_1(t) = \frac{1}{16}\cos(8t - \frac{\pi}{16}))$, $c_2(t) = \frac{1}{24}\sin(8t)$, $c_3(t) = \frac{1}{48}\cos(8t + \frac{\pi}{6})$, $f(x) = \frac{1}{10}x^3$, $g(t,x) = \frac{1}{32}(\frac{1}{2} + 2\sin(8t))x^4 - \frac{1}{x^\mu}$, $e(t) = \cos(8t + \frac{\pi}{4})$. $T = \frac{\pi}{4}$. It is obvious that (H_2), (H_4) and (H_5) hold. $\|c_1\| = \frac{1}{16}$, $\|c_2\| = \frac{1}{24}$ and $\|c_3\| = \frac{1}{48}$, so we have $\sum_{i=1}^{3}\|c_i\| = \frac{1}{8} < 1$, $\sigma = \frac{1}{1-\sum_{i=1}^{3}\|c_i\|} = \frac{1}{1-\frac{1}{16}-\frac{1}{24}-\frac{1}{48}} = \frac{8}{7}$. Furthermore, $\|c_1'\| = \frac{1}{2}$, $\|c_2'\| = \frac{1}{4}$ and $\|c_3'\| = \frac{1}{6}$. Next, we consider the condition

$$\begin{aligned}
&\sigma T^{\frac{1}{q}}\left(\frac{m\sum_{i=1}^{n}\|c_i\|}{2^{p-1}} + \frac{\alpha T(1+\sum_{i=1}^{n}\|c_i\|)}{2^{p-1}}\right)^{\frac{1}{p}} + \frac{\sigma T\sum_{i=1}^{n}\|c_i'\|}{2} \\
&\quad = \frac{8}{7}\times\left(\frac{\pi}{4}\right)^{\frac{4}{5}}\times\left(\frac{\frac{1}{10}\times\frac{1}{8}}{2^4} + \frac{\frac{5}{64}\times\frac{\pi}{4}\times\frac{9}{8}}{2^4}\right)^{\frac{1}{4}} + \frac{\frac{8}{7}\times\frac{\pi}{4}\times\frac{11}{12}}{2} \\
&\quad \approx 0.7391 < 1.
\end{aligned}$$

Therefore, by Theorem 5.1, we know that Eq. (6.3) has at least one positive $\frac{\pi}{4}$-periodic solution.

Example 6.4 Consider the following p-Laplacian Liénard equation with singularity

$$\begin{aligned}
&\left(\phi_p\left(x(t) - \left(\left(\frac{1}{6}\cos\left(6t+\frac{\pi}{5}\right) + \frac{11}{6}\right)x(t-\delta_1) + \frac{1}{36}\cos\left(6t+\frac{\pi}{3}\right)x(t-\delta_2)\right.\right.\right. \\
&\quad \left.\left.\left. - \frac{1}{24}\cos\left(6t+\frac{\pi}{6}\right)x(t-\delta_3)\right)\right)'\right)' \\
&\quad + \left(\frac{1}{149}x^4 + 2\right)x'(t) + \frac{1}{256}\left(1+\sin(6t)\right)x^3 - \frac{1}{x^\mu} = \cos\left(6t+\frac{\pi}{4}\right),
\end{aligned} \tag{6.4}$$

where $p = 3$, $\mu \geq 1$, δ_1, δ_2, and δ_3 are constants and $0 < \delta_1, \delta_2, \delta_3 < T$.

Comparing Eq. (6.4) with Eq. (4.1), it is easy to see that $c_1(t) = \frac{1}{6}\cos(6t + \frac{\pi}{5}) + \frac{11}{6}$, $c_2(t) = \frac{1}{36}\cos(6t + \frac{\pi}{3})$, $c_3(t) = -\frac{1}{24}\cos(6t + \frac{\pi}{6})$. $f(x) = \frac{1}{149}x^4 + 2$, $g(t,x) = \frac{1}{256}(1 + \sin(6t))x^5 - \frac{1}{x^\mu}$, $e(t) = \sin(6t - \frac{\pi}{4})$, $T = \frac{\pi}{3}$. It is obvious that (H_2), (H_4) and (H_5) hold. Furthermore, $\|c_1\| = \frac{1}{6} + \frac{11}{6} = 2$, $\|c_2\| = \frac{1}{36}$ and $\|c_3\| = \frac{1}{24}$, so we have $\sum_{i=1}^{3}\|c_i\| = \frac{149}{72} > 1$, $\sigma = \frac{\frac{1}{\|c_k\|}}{1-\frac{1}{\|c_k\|}-\sum_{i=1,i\neq k}^{n}\|\frac{c_i}{c_k}\|} = \frac{\frac{1}{2}}{1-\frac{1}{2}-\frac{\frac{1}{2}}{\frac{1}{36}}-\frac{\frac{1}{2}}{\frac{1}{24}}} = \frac{576}{571}$, $\|c_1'\| = 1$, $\|c_2'\| = \frac{1}{6}$ and $\|c_3'\| = \frac{1}{4}$. Next, we consider the condition

$$\begin{aligned}
&\sigma T^{\frac{1}{q}}\left(\frac{m\sum_{i=1}^{n}\|c_i\|}{2^{p-1}} + \frac{\alpha T(1+\sum_{i=1}^{n}\|c_i\|)}{2^{p-1}}\right)^{\frac{1}{p}} + \frac{\sigma T\sum_{i=1}^{n}\|c_i'\|}{2} \\
&\quad = \frac{576}{571}\times\left(\frac{\pi}{3}\right)^{\frac{2}{3}}\times\left(\frac{\frac{1}{149}\times\frac{149}{72}}{2^2} + \frac{\frac{1}{128}\times\frac{\pi}{3}\times\frac{221}{72}}{2^2}\right)^{\frac{1}{3}} + \frac{\frac{576}{571}\times\frac{\pi}{3}\times(1+\frac{7}{6}+\frac{1}{4})}{2} \\
&\quad \approx 0.9705 < 1.
\end{aligned}$$

Therefore, by Theorem 5.1, we know that Eq. (6.4) has at least one positive $\frac{\pi}{3}$-periodic solution.

7 Conclusions

In this paper, we first investigated some properties of the neutral operator with multiple delays and variable coefficients $(Ax)(t) := x(t) - \sum_{i=1}^{n} c_i(t)x(t-\delta_i)$. Afterwards, by using an extension of Mawhin's continuation theorem due to Ge and Ren, properties of the neutral operator A, we studied the existence of a periodic solution for equation (1.1). At last, by applying Theorem 3.2, we discussed the existence of a periodic solution for two p-Laplacian neutral differential equations. In comparison to [5] and [19], we avoided translating the equation into a two-dimensional system.

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