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Construction of global solutions for a symmetric system of Keyfitz–Kranzer type with three piecewise constant states

Pengyan Wang[1], Chun Shen[1*] and Xiuli Lin[2]

*Correspondence:
shenchun3641@sina.com
[1]School of Mathematics and Statistics Science, Ludong University, Yantai, P.R. China
Full list of author information is available at the end of the article

Abstract

The exact solutions of the Riemann problem for a symmetric system of Keyfitz–Kranzer type are obtained in fully explicit forms. Furthermore, the global solutions of the double Riemann problems are also constructed explicitly when the initial data are taken to be three piecewise constant states. During the process of constructing the global solutions, all occurring wave interactions have been dealt with in detail by using the method of characteristics. In addition, it is shown that the Riemann solutions are stable with respect to the specific small perturbations of the Riemann initial data.

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1 Introduction

In this paper, we are concerned with the following hyperbolic system of conservation laws in the form

$$\begin{cases} u_t + (u\phi(u^n + v^n))_x = 0, \\ v_t + (v\phi(u^n + v^n))_x = 0, \end{cases} \tag{1.1}$$

which is named a symmetric system of Keyfitz–Kranzer type [1–5]. For convenience, we introduce the notation $r = u^n + v^n$. In the present work, we restrict ourselves only to considering the situation $\phi'(r) > 0$ and $\phi''(r) > 0$ in the quarter (u, v) phase plane, where $u \geq 0$ and $v \geq 0$ are required. It is easily shown that system (1.1) has two real eigenvalues $\lambda_1 = \phi(r)$ and $\lambda_2 = \phi(r) + nr\phi'(r)$. Obviously, we have $\lambda_1 < \lambda_2$ under our assumption $\phi'(r) > 0$ when $r > 0$, which implies that system (1.1) is strictly hyperbolic except for the origin in the quarter (u, v) phase plane. It is worthwhile to notice that the shock curve has the same expression formula as the rarefaction one in the quarter (u, v) phase plane, so that system (1.1) belongs to the so-called Temple class [6, 7].

It is easily seen that system (1.1) is a special situation for the following general Keyfitz–Kranzer system [8]:

$$\begin{cases} u_t + (u\phi(u,v))_x = 0, \\ v_t + (v\phi(u,v))_x = 0. \end{cases} \tag{1.2}$$

The general Keyfitz–Kranzer system (1.2) was used as a tensile elastic model to describe the propagation of longitudinal and transverse waves [8]. It was also used in [9] to illustrate some features of the solar wind in magnetohydrodynamic. On the one hand, if $n = 1$ is taken, then system (1.1) is simplified into

$$\begin{cases} u_t + (u\phi(u+v))_x = 0, \\ v_t + (v\phi(u+v))_x = 0. \end{cases} \tag{1.3}$$

It is remarkable that the system of multi-component chromatography is just a special case of system (1.3) and thus was also studied for example in [10–15]. On the other hand, if $n = 2$ is taken, then system (1.1) turns out to be

$$\begin{cases} u_t + (u\phi(u^2+v^2))_x = 0, \\ v_t + (v\phi(u^2+v^2))_x = 0. \end{cases} \tag{1.4}$$

This symmetric Keyfitz–Kranzer system (1.4) has been widely investigated in [1, 2, 5, 8, 9], in which some peculiar phenomena were shown such as the propagation and cancelation of initial oscillations. In addition, it should be pointed out that the following asymmetric Keyfitz–Kranzer system

$$\begin{cases} \rho_t + (\rho\phi(\rho,u_1,u_2,\dots,u_n))_x = 0, \\ (\rho u_i)_t + (\rho u_i\phi(\rho,u_1,u_2,\dots,u_n))_x = 0, \end{cases} \tag{1.5}$$

was also proposed by Lu [16, 17], in which $i = 1,2,\dots,n$. It should be stressed that some well-known hyperbolic systems, such as the pressureless gas dynamics system [18, 19], the isentropic Chaplygin gas dynamics system [20–24], and the macroscopic production model [25], are recovered by choosing suitable ϕ.

One can see from the above description that system (1.1) has been extensively considered when $n = 1$ or $n = 2$ is taken. However, there are relatively less studies on system (1.1) when n is the other number. It was shown in [5] that the existence of a weak entropy solution to the Cauchy problem for system (1.1) with bounded measurable initial data was obtained by using the vanishing viscosity approach when $n > 1$ was taken. Based on the above result, the present work under consideration is devoted to dealing with the special initial value problem for system (1.1) with $n > 1$ subject to the following three piecewise constant initial data:

$$(u,v)(x,0) = \begin{cases} (u_-,v_-), & -\infty < x < 0, \\ (u_m,v_m), & 0 < x < x_0, \\ (u_+,v_+), & x_0 < x < +\infty, \end{cases} \tag{1.6}$$

where x_0 is a sufficiently small positive number. The initial data in the form (1.6) have been intensively used to investigate the interaction problem of elementary waves [26–32] for various hyperbolic systems of conservation laws. It is worth mentioning that the three piecewise constant initial data (1.6) are also taken to be a special small perturbation of the corresponding Riemann initial data

$$(u,v)(x,0)=\begin{cases}(u_-,v_-), & -\infty<x<0,\\ (u_+,v_+), & 0<x<+\infty.\end{cases} \tag{1.7}$$

Thus, the special initial value problem (1.1) and (1.6) is often named a double Riemann problem (or the perturbed Riemann problem) in the literature.

In the present paper, we construct the global solutions to the double Riemann problem (1.1) and (1.6) for all the possible situations under the assumptions $\phi'(r)>0$ and $\phi''(r)>0$ when the initial data (1.6) lie in the quarter (u,v) phase plane. During the process of constructing the global solutions, the wave interactions have been investigated intensively by using the method of characteristics [26, 33]. First of all, we need to deal with the collision of waves belonging to different families. More precisely, we must consider the situation that the shock (or rarefaction) wave collides with the contact discontinuity. Consequently, we need to deal with the coalescence and cancelation of waves belonging to the same family. More specifically, we are concerned with the situation that the shock wave cancels the rarefaction wave or with the situation that the two shock waves coalesce into a new shock wave. In the end, it is shown that the limits $x_0\to 0$ of the global solutions to the double Riemann problem (1.1) and (1.6) are in agreement with those to the Riemann problem (1.1) and (1.7) for all the possible situations. As a consequence, it can be concluded that the solutions to the Riemann problem (1.1) and (1.7) are stable with respect to the specific small perturbation in the form (1.6) of the Riemann initial data (1.7).

The paper is organized in the following way. In Sect. 2, we study the Riemann problem (1.1) and (1.7) in detail. More precisely, the exact solutions of the Riemann problem (1.1) and (1.7) are obtained in fully explicit forms for all the possible Riemann initial data. In Sect. 3, the global solutions to the special initial value problem (1.1) and (1.6) are also constructed by studying all the occurring wave interactions. Then, the stabilities of Riemann solutions with respect to the small perturbations of the corresponding Riemann initial data can be obtained by taking the limit $x_0\to 0$ in the global solutions to the special initial value problem (1.1) and (1.6).

2 The solutions of the Riemann problem (1.1) and (1.7)

In this section, we are mainly concerned with the Riemann problem for system (1.1) subject to the Riemann initial data (1.7). We can refer to the literature [1, 8, 9] about the Riemann problem for the symmetric Keyfitz–Kranzer system (1.4). In what follows, we first deliver some basic properties of system (1.1), which can also be seen in [5]. A simple calculation shows that the two eigenvalues of system (1.1) are given respectively by

$$\lambda_1=\phi(r),\qquad \lambda_2=\phi(r)+nr\phi'(r). \tag{2.1}$$

It suffices to get $\lambda_1<\lambda_2$ when $r>0$ or $\lambda_1=\lambda_2$ when $r=0$ under the assumption $\phi'(r)>0$. Hence system (1.1) is strictly hyperbolic in the quarter (u,v) phase plane except for the origin $(u,v)=(0,0)$ where it is degenerated to be non-strictly hyperbolic. The corresponding right eigenvalues can be chosen as

$$\overrightarrow{r_1}=\left(-v^{n-1},u^{n-1}\right)^T,\qquad \overrightarrow{r_2}=(u,v)^T. \tag{2.2}$$

Let us introduce the notation $\nabla=(\frac{\partial}{\partial u},\frac{\partial}{\partial v})$, then we can get

$$\nabla\lambda_1\cdot r_1=0,\qquad \nabla\lambda_2\cdot r_2=nr\big((n+1)\phi'(r)+nr\phi''(r)\big)\neq 0\quad \text{when } r>0, \tag{2.3}$$

which implies that the characteristic field associated with λ_1 is always linearly degenerate and the characteristic field associated with λ_2 is genuinely nonlinear except for the origin. As a consequence, the elementary wave corresponding to λ_1 is always the contact discontinuity, and the elementary wave corresponding to λ_2 is either the shock wave or the rarefaction wave depending on the choice of the Riemann initial data (1.7).

Since the Riemann problem (1.1) and (1.7) remains unchanged under the uniform stretching of coordinates: $(x,t)\to(\alpha x,\alpha t)$, we can carry out the corresponding self-similar transformation

$$(u,v)(x,t)=(u,v)(\xi),\quad \text{where } \xi=\frac{x}{t}. \tag{2.4}$$

Under the self-similar transformation (2.4), the Riemann problem (1.1) and (1.7) can be reduced into the boundary value problem [34, 35] of ordinary differential equations as follows:

$$\begin{cases}(\phi(r)-\xi+nu^n\phi'(r))u_\xi+nuv^{n-1}\phi'(r)v_\xi=0,\\ nu^{n-1}v\phi'(r)u_\xi+(\phi(r)-\xi+nv^n\phi'(r))v_\xi=0,\\ (u,v)(\pm\infty)=(u_\pm,v_\pm).\end{cases} \tag{2.5}$$

Furthermore, (2.5) can be rewritten as

$$\begin{pmatrix}\phi(r)-\xi+nu^n\phi'(r) & nuv^{n-1}\phi'(r)\\ nu^{n-1}v\phi'(r) & \phi(r)-\xi+nv^n\phi'(r)\end{pmatrix}\begin{pmatrix}du\\ dv\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}. \tag{2.6}$$

It is easy to know that (2.6) provides either the constant state solution or the singular solution, namely the contact discontinuity corresponds to λ_1 and the rarefaction wave corresponds to λ_2. For a given left state (u_-,v_-) in the interior of the quarter (u,v) phase plane, on the one hand, the contact discontinuity is expressed in the quarter (u,v) phase plane as follows:

$$J:\tau=\phi(r)=\phi(r_-),\quad \text{where } r=u^n+v^n \text{ and } r_-=u_-^n+v_-^n. \tag{2.7}$$

It is easily seen from (2.7) that we have $u^n+v^n=u_-^n+v_-^n$ under the assumption $\phi'(r)>0$. By differentiating u with respect to v in the relation formula $u^n+v^n=u_-^n+v_-^n$, we get

$$\frac{du}{dv}=-\frac{v^{n-1}}{u^{n-1}},\qquad \frac{d^2u}{d^2v}=-\frac{(n-1)v^{n-2}}{u^{n-1}}. \tag{2.8}$$

It can be deduced directly from (2.8) that we have $\frac{du}{dv} < 0$ and $\frac{d^2u}{d^2v} < 0$ when $n > 1$. On the other hand, the rarefaction wave is expressed in the quarter (u, v) phase plane as follows:

$$R : \xi = \phi(r) + n\phi'(r) \quad \text{when } \frac{u}{v} = \frac{u_-}{v_-} \text{ and } r > r_-. \tag{2.9}$$

Let us turn our attention to the discontinuous solutions of system (1.1). For a bounded discontinuity on $\xi = \sigma$, there exist the following Rankine–Hugoniot jump conditions:

$$\begin{cases} \sigma[u] = [u\phi(r)], \\ \sigma[v] = [v\phi(r)], \end{cases} \tag{2.10}$$

where $\sigma = \frac{dx}{dt}$. The jump quantity $[r]$ is calculated by $[r] = r - r_-$, in which r_- (or r) is the value of r on the left-hand (or right-hand) side of the discontinuity. For a given left state (u_-, v_-) in the interior of the quarter (u, v) phase plane, the contact discontinuity is also expressed by (2.7) and the shock wave is expressed in the quarter (u, v) phase plane as follows:

$$S : \sigma = \frac{u\phi(r) - u_-\phi(r_-)}{u - u_-} \quad \text{when } \frac{u}{v} = \frac{u_-}{v_-} \text{ and } r < r_-. \tag{2.11}$$

It can be seen from (2.9) and (2.11) that the rarefaction wave curve has the same expression formula as the shock wave curve. Thus, system (1.1) is attributed to the well-known Temple class [6, 7].

In the end, we use the above elementary waves J, R, and S to construct the solutions of the Riemann problem (1.1) and (1.7) in the quarter (u, v) phase plane by using the method of phase plane analysis. For the given left state (u_-, v_-) in the interior of the quarter (u, v) phase plane, the solutions of the Riemann problem (1.1) and (1.7) are a contact discontinuity when $r_- = r_+$, or a rarefaction wave when $\frac{u_+}{v_+} = \frac{u_-}{v_-}$ and $r_+ > r_-$, or a shock wave when $\frac{u_+}{v_+} = \frac{u_-}{v_-}$ and $r_+ < r_-$ to connect with the two constant states $(u_\pm, v_\pm)$ directly. For other general situations, the solutions of the Riemann problem (1.1) and (1.7) have the following two kinds of combinations of elementary waves according to $0 < r_- < r_+$ or not.

(1) If $0 < r_+ < r_-$, then the solutions of the Riemann problem (1.1) and (1.7) can be expressed by the symbol $J_1 + S_2$ in the following form:

$$(u, v)(x, t) = \begin{cases} (u_-, v_-), & x < \tau_1 t, \\ (u_*, v_*), & \tau_1 t < x < \sigma_2 t, \\ (u_+, v_+), & x > \sigma_2 t, \end{cases} \tag{2.12}$$

in which the intermediate state (u_*, v_*) between J_1 and S_2 is calculated by

$$(u_*, v_*) = \left(\sqrt[n]{\frac{u_-^n + v_-^n}{u_+^n + v_+^n}}\, u_+, \sqrt[n]{\frac{u_-^n + v_-^n}{u_+^n + v_+^n}}\, v_+ \right). \tag{2.13}$$

Here $\tau_1 = \phi(r_-)$ is the propagation speed of the contact discontinuity J_1 and σ_2 is used to denote the propagation speed of the shock wave S_2, which can be calculated by

$$\sigma_2 = \frac{u_+\phi(r_+) - u_*\phi(r_*)}{u_+ - u_*} = \frac{u_+\phi(r_+) - \sqrt[n]{\frac{u_-^n+v_-^n}{u_+^n+v_+^n}}u_+\phi(r_-)}{u_+ - \sqrt[n]{\frac{u_-^n+v_-^n}{u_+^n+v_+^n}}u_+} = \frac{\sqrt[n]{r_+}\phi(r_+) - \sqrt[n]{r_-}\phi(r_-)}{\sqrt[n]{r_+} - \sqrt[n]{r_-}}. \tag{2.14}$$

The symbol $J_1 + S_2$ is used to represent the 1-contact discontinuity J_1 followed by the 2-shock wave S_2 in this paper and the similar symbol $J_1 + R_2$ will also be used later without explanation again.

(2) If $0 < r_- < r_+$, then the solution of the Riemann problem (1.1) and (1.7) can be expressed by the symbol $J_1 + R_2$ in the following form:

$$(u,v)(x,t) = \begin{cases} (u_-,v_-), & x < \tau_1 t, \\ (u_*,v_*), & \tau_1 t < x < \lambda_2(r_-)t, \\ (u,v), & \lambda_2(r_-)t \le x \le \lambda_2(r_+)t, \\ (u_+,v_+), & x > \lambda_2(r_+)t, \end{cases} \tag{2.15}$$

in which τ_1 and (u_*,v_*) can be calculated in the same way as before. The state (u,v) in the rarefaction wave fan R_2 varies from (u_*,v_*) to (u_+,v_+), which can be determined uniquely by

$$\frac{u}{v} = \frac{u_+}{v_+}, \qquad \frac{x}{t} = \phi(r) + nr\phi'(r). \tag{2.16}$$

In addition, the propagation speeds of characteristics in the rarefaction wave fan R_2 can be calculated by $\xi = \lambda_2(r) = \phi(r) + nr\phi'(r)$, in which r varies from r_* $(= r_-)$ to r_+.

3 Construction of global solutions to the double Riemann problem (1.1) and (1.6)

In this section, the main purpose is to construct the global solutions to the double Riemann problem (1.1) and (1.6) by using the method of characteristics. In the present work, we restrict ourselves to considering the initial data (1.6) which are located in the quarter (u,v) phase plane. In order to cover all the cases completely, we divide our discussion into the following four cases according to the different combinations of the Riemann solutions starting from the initial points $(0,0)$ and $(x_0,0)$ as follows:

$$\begin{aligned} &(1)\quad J+S \quad\text{and}\quad J+S, \qquad (2)\quad J+R \quad\text{and}\quad J+R, \\ &(3)\quad J+S \quad\text{and}\quad J+R, \qquad (4)\quad J+R \quad\text{and}\quad J+S. \end{aligned}$$

Furthermore, we make a step further to consider the interesting work of determining whether the limits $x_0 \to 0$ of the solutions to the double Riemann problem (1.1) and (1.6) are in accordance with the corresponding ones to the Riemann problem (1.1) and (1.7).

Case 1 $J + S$ and $J + S$.

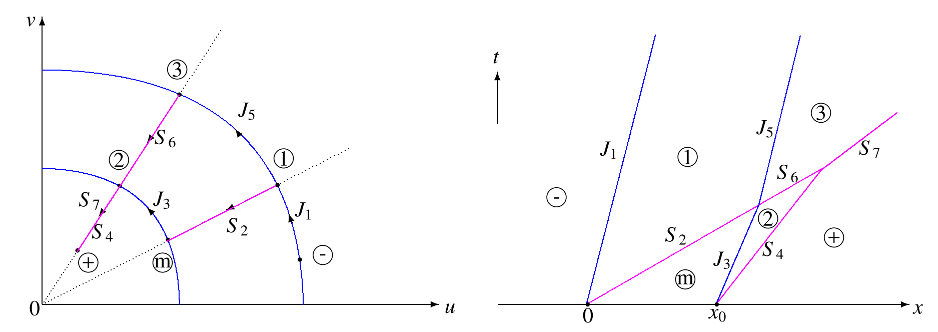

Figure 1 The interaction between $J+S$ and $J+S$ is shown when $0<r_+<r_m<r_-$

We first consider the situation that there is a contact discontinuity followed by a shock wave emitting from the initial points $(0,0)$ and $(x_0,0)$, respectively. Obviously, the occurrence of this case depends on the conditions $0<r_+<r_m<r_-$. For convenience, we use J_1, S_2 and J_3, S_4 to denote the contact discontinuities and the shock waves emitting from the initial points $(0,0)$ and $(x_0,0)$ (see Fig. 1). In this case, when the time t is small enough, the solution to the double Riemann problem (1.1) and (1.6) can be expressed briefly as (see Fig. 1):

$$(u_-,v_-)+J_1+(u_1,v_1)+S_2+(u_m,v_m)+J_3+(u_2,v_2)+S_4+(u_+,v_+), \tag{3.1}$$

where the symbol "+" means "followed by". It follows from (2.13) that the intermediate states (u_1,v_1) and (u_2,v_2) may be given respectively by

$$(u_1,v_1)=\left(\sqrt[n]{\frac{u_-^n+v_-^n}{u_m^n+v_m^n}}u_m,\sqrt[n]{\frac{u_-^n+v_-^n}{u_m^n+v_m^n}}v_m\right), \tag{3.2}$$

$$(u_2,v_2)=\left(\sqrt[n]{\frac{u_m^n+v_m^n}{u_+^n+v_+^n}}u_+,\sqrt[n]{\frac{u_m^n+v_m^n}{u_+^n+v_+^n}}v_+\right). \tag{3.3}$$

It can be seen from Fig. 1 that the relations $r_+<r_2=r_m<r_1=r_-$ can be established directly.

Lemma 3.1 *The shock wave S_2 collides with the contact discontinuity J_3 in finite time. Then the interaction between S_2 and J_3 gives rise to a new contact discontinuity and a new shock wave denoted by J_5 and S_6, respectively. More precisely, the two contact discontinuities J_1 and J_5 are parallel to each other, while the two shock waves S_2 and S_6 share the same propagation speed.*

Proof It is easy to know that the propagation speeds of S_2 and J_3 are given respectively by

$$\sigma_2=\frac{\sqrt[n]{r_m}\phi(r_m)-\sqrt[n]{r_-}\phi(r_-)}{\sqrt[n]{r_m}-\sqrt[n]{r_-}},\qquad \tau_3=\phi(r_m). \tag{3.4}$$

By virtue of $r_m<r_-$ and the assumption $\phi'(r)>0$, we have

$$\sigma_2-\tau_3=\frac{\sqrt[n]{r_-}(\phi(r_m)-\phi(r_-))}{\sqrt[n]{r_m}-\sqrt[n]{r_-}}>0, \tag{3.5}$$

which implies that $\sigma_2 > \tau_3$. In other words, the shock wave S_2 collides with the contact discontinuity J_3 in finite time. The intersection point (x_1, t_1) is determined by

$$\begin{cases} x_1 = \sigma_2 t_1 = \frac{\sqrt[n]{r_m}\phi(r_m) - \sqrt[n]{r_-}\phi(r_-)}{\sqrt[n]{r_m} - \sqrt[n]{r_-}} t_1, \\ x_1 - x_0 = \tau_3 t_1 = \phi(r_m) t_1, \end{cases} \tag{3.6}$$

which yields

$$(x_1, t_1) = \left(\frac{x_0(\sqrt[n]{r_m}\phi(r_m) - \sqrt[n]{r_-}\phi(r_-))}{\sqrt[n]{r_-}(\phi(r_m) - \phi(r_-))}, \frac{x_0(\sqrt[n]{r_m} - \sqrt[n]{r_-})}{\sqrt[n]{r_-}(\phi(r_m) - \phi(r_-))} \right). \tag{3.7}$$

The collision between the shock wave S_2 and the contact discontinuity J_3 occurs at the point (x_1, t_1), where we again have a local Riemann problem for system (1.1) subject to the Riemann-type initial data (u_1, v_1) and (u_2, v_2). Due to the relations $r_2 = r_m < r_- = r_1$, the solution to the local Riemann problem at the point (x_1, t_1) is still a contact discontinuity followed by a shock wave, which are denoted by J_5 and S_6, respectively. In addition, the intermediate state (u_3, v_3) between J_5 and S_6 can be obtained by

$$(u_3, v_3) = \left(\sqrt[n]{\frac{u_-^n + v_-^n}{u_+^n + v_+^n}} u_+, \sqrt[n]{\frac{u_-^n + v_-^n}{u_+^n + v_+^n}} v_+ \right). \tag{3.8}$$

The propagation speeds of J_5 and S_6 can be computed by $\tau_5 = \phi(r_1) = \phi(r_-)$ and $\sigma_6 = \frac{\sqrt[n]{r_2}\phi(r_2) - \sqrt[n]{r_3}\phi(r_3)}{\sqrt[n]{r_2} - \sqrt[n]{r_3}} = \frac{\sqrt[n]{r_m}\phi(r_m) - \sqrt[n]{r_-}\phi(r_-)}{\sqrt[n]{r_m} - \sqrt[n]{r_-}}$, respectively. On the one hand, it is worth mentioning that the propagation speed of J_1 is also calculated by $\tau_1 = \phi(r_-)$, which implies that J_1 is parallel to J_5. On the other hand, we can also obtain $\sigma_2 = \sigma_6$, which means that the shock wave keeps its propagation direction unchanged when it passes through the contact discontinuity. □

In the following, we consider the confluence of the two shock waves S_6 and S_4.

Lemma 3.2 *The two shock waves S_4 and S_6 coalesce into a new shock wave denoted by S_7.*

Proof The propagation speed of S_4 is given by $\sigma_4 = \frac{\sqrt[n]{r_+}\phi(r_+) - \sqrt[n]{r_2}\phi(r_2)}{\sqrt[n]{r_+} - \sqrt[n]{r_2}} = \frac{\sqrt[n]{r_+}\phi(r_+) - \sqrt[n]{r_m}\phi(r_m)}{\sqrt[n]{r_+} - \sqrt[n]{r_m}}$. Comparing with σ_6, we have

$$\begin{aligned} &\sigma_6 - \sigma_4 \\ &= \frac{\sqrt[n]{r_m}\phi(r_m) - \sqrt[n]{r_-}\phi(r_-)}{\sqrt[n]{r_m} - \sqrt[n]{r_-}} - \frac{\sqrt[n]{r_+}\phi(r_+) - \sqrt[n]{r_m}\phi(r_m)}{\sqrt[n]{r_+} - \sqrt[n]{r_m}} \\ &= \frac{(\sqrt[n]{r_+} - \sqrt[n]{r_m})(\sqrt[n]{r_m}\phi(r_m) - \sqrt[n]{r_-}\phi(r_-)) - (\sqrt[n]{r_m} - \sqrt[n]{r_-})(\sqrt[n]{r_+}\phi(r_+) - \sqrt[n]{r_m}\phi(r_m))}{(\sqrt[n]{r_m} - \sqrt[n]{r_-})(\sqrt[n]{r_+} - \sqrt[n]{r_m})} \\ &= \frac{\sqrt[n]{r_m r_+}(\phi(r_m) - \phi(r_+)) + \sqrt[n]{r_+ r_-}(\phi(r_+) - \phi(r_-)) + \sqrt[n]{r_- r_m}(\phi(r_-) - \phi(r_m))}{(\sqrt[n]{r_m} - \sqrt[n]{r_-})(\sqrt[n]{r_+} - \sqrt[n]{r_m})} \\ &= \frac{(\sqrt[n]{r_m r_+} - \sqrt[n]{r_+ r_-})(\phi(r_m) - \phi(r_+)) + (\sqrt[n]{r_m r_-} - \sqrt[n]{r_+ r_-})(\phi(r_-) - \phi(r_m))}{(\sqrt[n]{r_m} - \sqrt[n]{r_-})(\sqrt[n]{r_+} - \sqrt[n]{r_m})} \\ &= \sqrt[n]{r_-} \frac{\phi(r_-) - \phi(r_m)}{\sqrt[n]{r_-} - \sqrt[n]{r_m}} - \sqrt[n]{r_+} \frac{\phi(r_m) - \phi(r_+)}{\sqrt[n]{r_m} - \sqrt[n]{r_+}}. \end{aligned} \tag{3.9}$$

By using the mean value theorem, there exist $\bar{r_1}$ and $\bar{r_2}$ satisfying $r_+ < \bar{r_2} < r_m < \bar{r_1} < r_-$ such that we have

$$\sigma_6 - \sigma_4 = n\bar{r_1}^{\frac{n-1}{n}}\phi'(\bar{r_1})\sqrt[n]{r_-} - n\bar{r_2}^{\frac{n-1}{n}}\phi'(\bar{r_2})\sqrt[n]{r_+}, \tag{3.10}$$

in which we have used the formula $\frac{\partial\phi}{\partial\sqrt[n]{r}} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial\sqrt[n]{r}} = nr^{\frac{n-1}{n}}\phi'(r)$. Under the assumption $\phi''(r) > 0$, we can take a step further to get $\phi'(\bar{r_1}) > \phi'(\bar{r_2})$ when $\bar{r_1} > \bar{r_2}$. Noticing that $n > 1$, we also have $\bar{r_1}^{\frac{n-1}{n}} > \bar{r_2}^{\frac{n-1}{n}}$ when $\bar{r_1} > \bar{r_2}$. Hence, we have $\sigma_6 > \sigma_4$. In other words, the shock wave S_6 catches up with S_4 in finite time.

The intersection point (x_2, t_2) is determined by

$$\begin{cases} x_2 = \sigma_6 t_2 = \frac{\sqrt[n]{r_m}\phi(r_m) - \sqrt[n]{r_-}\phi(r_-)}{\sqrt[n]{r_m} - \sqrt[n]{r_-}}t_2, \\ x_2 - x_0 = \sigma_4 t_2 = \frac{\sqrt[n]{r_+}\phi(r_+) - \sqrt[n]{r_m}\phi(r_m)}{\sqrt[n]{r_+} - \sqrt[n]{r_m}}t_2, \end{cases} \tag{3.11}$$

which yields

$$\begin{cases} x_2 = \frac{x_0(\sqrt[n]{r_m} - \sqrt[n]{r_+})(\sqrt[n]{r_-}\phi(r_-) - \sqrt[n]{r_m}\phi(r_m))}{\sqrt[n]{r_+}(\sqrt[n]{r_m} - \sqrt[n]{r_-})(\phi(r_m) - \phi(r_+)) + \sqrt[n]{r_-}(\sqrt[n]{r_m} - \sqrt[n]{r_+})(\phi(r_-) - \phi(r_m))}, \\ t_2 = \frac{x_0(\sqrt[n]{r_m} - \sqrt[n]{r_+})(\sqrt[n]{r_-} - \sqrt[n]{r_m})}{\sqrt[n]{r_+}(\sqrt[n]{r_m} - \sqrt[n]{r_-})(\phi(r_m) - \phi(r_+)) + \sqrt[n]{r_-}(\sqrt[n]{r_m} - \sqrt[n]{r_+})(\phi(r_-) - \phi(r_m))}. \end{cases} \tag{3.12}$$

It can be deduced from (3.8) that $\frac{u_3}{v_3} = \frac{u_+}{v_+}$. In addition, one also has $r_+ < r_- = r_3$. Thus, the conclusion can be drawn that the two states (u_3, v_3) and (u_+, v_+) are connected directly by a new shock wave S_7, which can also be seen from the (u, v) phase plane in Fig. 1. It means that the two shock waves S_4 and S_6 coalesce into a new shock wave which is denoted by S_7, whose propagation speed is given by $\sigma_7 = \frac{\sqrt[n]{r_+}\phi(r_+) - \sqrt[n]{r_-}\phi(r_-)}{\sqrt[n]{r_+} - \sqrt[n]{r_-}}$. On the one hand, we have

$$\begin{aligned} \sigma_6 - \sigma_7 &= \frac{\sqrt[n]{r_m}\phi(r_m) - \sqrt[n]{r_-}\phi(r_-)}{\sqrt[n]{r_m} - \sqrt[n]{r_-}} - \frac{\sqrt[n]{r_+}\phi(r_+) - \sqrt[n]{r_-}\phi(r_-)}{\sqrt[n]{r_+} - \sqrt[n]{r_-}} \\ &= \sqrt[n]{r_m}\frac{\phi(r_-) - \phi(r_m)}{\sqrt[n]{r_-} - \sqrt[n]{r_m}} - \sqrt[n]{r_+}\frac{\phi(r_-) - \phi(r_+)}{\sqrt[n]{r_-} - \sqrt[n]{r_+}}. \end{aligned} \tag{3.13}$$

On the other hand, we also have

$$\begin{aligned} \sigma_4 - \sigma_7 &= \frac{\sqrt[n]{r_+}\phi(r_+) - \sqrt[n]{r_m}\phi(r_m)}{\sqrt[n]{r_+} - \sqrt[n]{r_m}} - \frac{\sqrt[n]{r_+}\phi(r_+) - \sqrt[n]{r_-}\phi(r_-)}{\sqrt[n]{r_+} - \sqrt[n]{r_-}} \\ &= \sqrt[n]{r_m}\frac{\phi(r_m) - \phi(r_+)}{\sqrt[n]{r_m} - \sqrt[n]{r_+}} - \sqrt[n]{r_-}\frac{\phi(r_-) - \phi(r_+)}{\sqrt[n]{r_-} - \sqrt[n]{r_+}}. \end{aligned} \tag{3.14}$$

With the similar calculation as that in Lemma 3.2, the following inequalities $\sigma_6 - \sigma_7 > 0$ and $\sigma_4 - \sigma_7 < 0$ can also be established, which means that the propagation speed of the shock wave S_7 is between those of S_4 and S_6. □

Case 2 $J + R$ and $J + R$.

We consider the situation that there is a contact discontinuity followed by a rarefaction wave emitting from the initial points $(0, 0)$ and $(x_0, 0)$, respectively. The occurrence of this case depends on the condition $0 < r_- < r_m < r_+$. We use J_1, R_2 and J_3, R_4 to denote

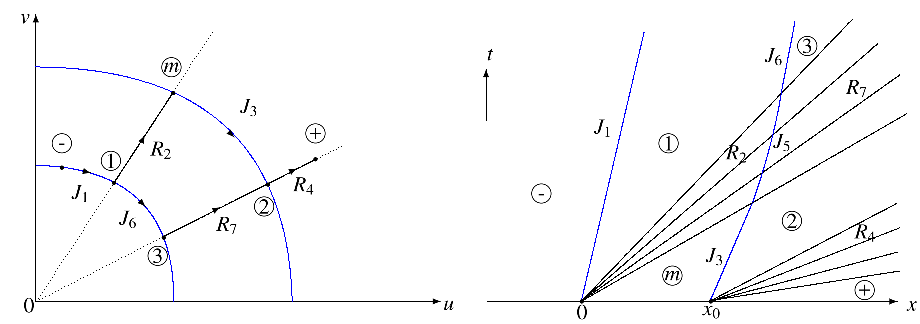

Figure 2 The interaction between $J+R$ and $J+R$ is shown when $0<r_-<r_m<r_+$

the contact discontinuities and the shock waves emitting from the initial points $(0,0)$ and $(x_0,0)$ (see Fig. 2). In this case, when the time t is small enough, the solution to the double Riemann problem (1.1) and (1.6) may be represented succinctly as follows (see Fig. 2):

$$(u_-,v_-)+J_1+(u_1,v_1)+R_2+(u_m,v_m)+J_3+(u_2,v_2)+R_4+(u_+,v_+). \tag{3.15}$$

The intermediate states (u_1,v_1), (u_2,v_2), and (u_3,v_3) are the same as in Case 1. Furthermore, it can be seen from Fig. 2 that the relations $r_1=r_-<r_2=r_m<r_+$ can be established directly.

Lemma 3.3 *The wave front of the rarefaction wave R_2 collides with the contact discontinuity J_3 in finite time. Then the interaction between R_2 and J_3 gives rise to a new contact discontinuity and a new rarefaction wave denoted by J_6 and R_7, respectively. More precisely, the two contact discontinuities J_1 and J_6 are parallel to each other, while the wave front of the rarefaction wave R_7 and the wave back of the rarefaction wave R_4 share the same propagation speed.*

Proof A simple calculation shows that the propagation speeds of the wave front of the rarefaction wave R_2 and the contact discontinuity J_3 are given respectively by

$$\xi_2(u_m,v_m)=\phi(r_m)+nr_m\phi'(r_m),\qquad \tau_3=\phi(r_m). \tag{3.16}$$

We have $\xi_2(u_m,v_m)-\tau_3=nr_m\phi'(r_m)>0$ under the assumption $\phi'(r)>0$, which implies that $\xi_2(u_m,v_m)>\tau_3$. In other words, the rarefaction wave R_2 collides with the contact discontinuity J_3 in finite time. The intersection point (x_1,t_1) is determined by

$$\begin{cases} x_1=\xi_2(u_m,v_m)t_1=(\phi(r_m)+nr_m\phi'(r_m))t_1, \\ x_1-x_0=\tau_3t_1=\phi(r_m)t_1, \end{cases} \tag{3.17}$$

from which it is easy to get

$$(x_1,t_1)=\left(\frac{x_0(\phi(r_m)+nr_m\phi'(r_m))}{nr_m\phi'(r_m)},\frac{x_0}{nr_m\phi'(r_m)}\right). \tag{3.18}$$

The collision between the wave front of the rarefaction wave R_2 and the contact discontinuity J_3 occurs at the point (x_1,t_1). The contact discontinuity J_3 begins to penetrate the rarefaction wave R_2 after t_1, which is denoted by J_5 during the process of penetration. The expression for J_5 during the process of penetration is determined by

$$\begin{cases} \frac{dx}{dt} = \phi(r), \\ \frac{x}{t} = \phi(r) + nr\phi'(r), \\ x(t_1) = x_1, \end{cases} \tag{3.19}$$

in which r changes from r_m to r_1 $(= r_-)$. It can be deduced from the second equation in (3.19) that

$$\frac{dx}{dt} = \phi(r) + nr\phi'(r) + \bigl((n+1)\phi'(r) + nr\phi''(r)\bigr)t \cdot \frac{dr}{dt}, \tag{3.20}$$

which, together with the first equation in (3.19), yields

$$\frac{dt}{t} = -\frac{((n+1)\phi'(r) + nr\phi''(r))\,dr}{nr\phi'(r)}, \tag{3.21}$$

which enables us to have

$$d\ln t = -\frac{d\ln r}{n} - d\ln\bigl(nr\phi'(r)\bigr) = -d\ln\bigl(nr^{\frac{1+n}{n}}\phi'(r)\bigr). \tag{3.22}$$

With the initial condition $r(t_1) = r_m$ in mind, one can only obtain the following implicit expression:

$$r^{\frac{1+n}{n}}\phi'(r)t = (r_m)^{\frac{1+n}{n}}\phi'(r_m)t_1 = \frac{(r_m)^{\frac{1}{n}}x_0}{n}. \tag{3.23}$$

In addition, it follows from (3.21) that

$$\frac{dr}{dt} = \frac{-nr\phi'(r)}{((n+1)\phi'(r) + nr\phi''(r))t}. \tag{3.24}$$

In the same way as before, by differentiating the first equation in (3.19) with respect to t, we have

$$\frac{d^2x}{dt^2} = \phi'(r)\frac{dr}{dt} = \frac{-nr(\phi'(r))^2}{((n+1)\phi'(r) + nr\phi''(r))t} < 0, \tag{3.25}$$

in which (3.24) and the assumptions $\phi' > 0$ and $\phi'' > 0$ have been used, which implies that the contact discontinuity slows down during the process of penetration.

Due to the facts $r_2 = r_m$ and $r_1 = r_3 = r_-$, it is worthwhile to notice that

$$\begin{aligned} &\phi(r_m) + nr_m\phi'(r_m) = \phi(r_2) + nr_2\phi'(r_2) \quad \text{and} \\ &\phi(r_1) + nr_1\phi'(r_1) = \phi(r_3) + nr_3\phi'(r_3). \end{aligned} \tag{3.26}$$

Thus, it can be seen clearly that the rarefaction wave does not change the propagation direction during the process of penetration. But here it should be stressed that the state in the rarefaction wave fan R_2 varying from (u_m, v_m) to (u_1, v_1) is determined by $\frac{u}{v} = \frac{u_m}{v_m}$ as well as the state in the rarefaction wave fan R_7 varying from (u_2, v_2) to (u_3, v_3) is governed by $\frac{u}{v} = \frac{u_+}{v_+}$.

It is clear to see that (u_3, v_3) can be directly connected to (u_1, v_1) through a contact discontinuity J_6 for the reason that $r_1 = r_3 = r_-$. Thus, J_5 is able to penetrate R_2 completely and ends at the point (x_2, t_2), which is determined by

$$\begin{cases} x_2 = \xi_2(u_-, v_-)t_2 = (\phi(r_-) + nr_-\phi'(r_-))t_2, \\ x_2 = (\phi((\frac{nr_m^{-\frac{1}{n}}\phi'(r)t}{x_0})^{-\frac{n}{n+1}}) + n(\frac{nr_m^{-\frac{1}{n}}\phi'(r)t}{x_0})^{-\frac{n}{n+1}}\phi'((\frac{nr_m^{-\frac{1}{n}}\phi'(r)t}{x_0})^{-\frac{n}{n+1}}))t, \end{cases} \tag{3.27}$$

which enables us to get

$$\begin{cases} x_2 = (\phi((\frac{nr_m^{-\frac{1}{n}}\phi'(r)t}{x_0})^{-\frac{n}{n+1}}) + n(\frac{nr_m^{-\frac{1}{n}}\phi'(r)t}{x_0})^{-\frac{n}{n+1}}\phi'((\frac{nr_m^{-\frac{1}{n}}\phi'(r)t}{x_0})^{-\frac{n}{n+1}}))t, \\ t_2 = \frac{(\phi((\frac{nr_m^{-\frac{1}{n}}\phi'(r)t}{x_0})^{-\frac{n}{n+1}}) + n(\frac{nr_m^{-\frac{1}{n}}\phi'(r)t}{x_0})^{-\frac{n}{n+1}}\phi'((\frac{nr_m^{-\frac{1}{n}}\phi'(r)t}{x_0})^{-\frac{n}{n+1}}))t}{\phi(r_-)+nr_-\phi'(r_-)}. \end{cases} \tag{3.28}$$

The propagation speeds of J_6 and J_1 are $\tau_6 = \tau_1 = \phi(r_-)$. In other words, the two contact discontinuities J_1 and J_6 are parallel to each other. In the same way, we can also see that $\xi_7(u_2, v_2) = \xi_4(u_2, v_2) = \phi(r_2) + nr_2\phi'(r_2) = \phi(r_m) + nr_m\phi'(r_m)$ due to the fact $r_2 = r_m$, which means that the wave front of the rarefaction wave R_7 and the wave back of the rarefaction wave R_4 share the same propagation speed. □

Case 3 $J + S$ and $J + R$.

Let us turn to consider the situation that there are a contact discontinuity followed by a shock wave emitting from the initial point $(0, 0)$ and a contact discontinuity followed by a rarefaction wave emitting from the initial point $(x_0, 0)$. The occurrence of this case depends on the conditions $0 < r_m < r_-$ and $0 < r_m < r_+$. We use J_1, S_2 and J_3, R_4 to denote them respectively. In this case, when the time t is small enough, the solution to the double Riemann problem (1.1) and (1.6) can be expressed clearly as follows (see Figs. 3 and 4):

$$(u_-, v_-) + J_1 + (u_1, v_1) + S_2 + (u_m, v_m) + J_3 + (u_2, v_2) + R_4 + (u_+, v_+). \tag{3.29}$$

The intermediate states (u_1, v_1), (u_2, v_2), and (u_3, v_3) are also the same as those in Case 1. Similarly, the intersection point (x_1, t_1) is also the same as that in Case 1. As before, we

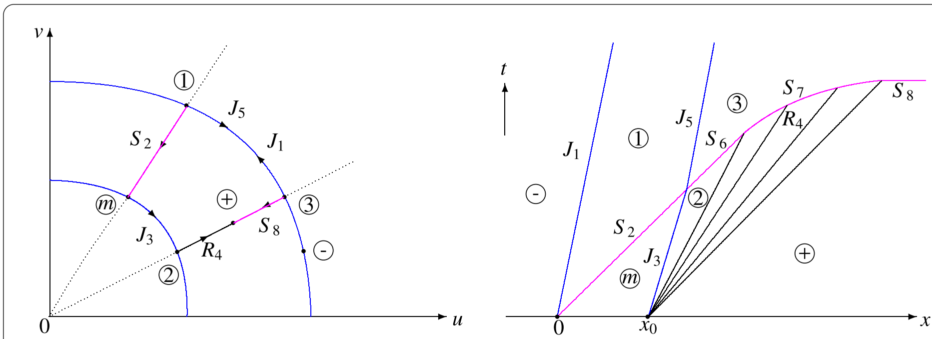

Figure 3 The interaction between $J + S$ and $J + R$ is displayed when $0 < r_m < r_+ < r_-$, in which the shock wave S_6 is able to penetrate the whole rarefaction wave R_4 completely

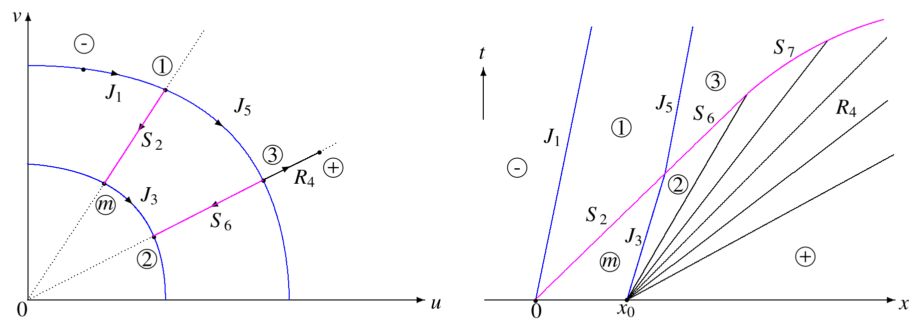

Figure 4 The interaction between $J+S$ and $J+R$ is displayed when $0<r_m<r_-<r_+$, in which the shock wave S_6 cannot penetrate R_4 completely and ultimately has the line $x=(\phi(r_-)+nr_-\phi'(r_-))t+x_0$ as its asymptote

again have a local Riemann problem at the point (x_1,t_1) for system (1.1) subject to the left state (u_1,v_1) and the right state (u_2,v_2). Due to the relations $r_2=r_m<r_1=r_-$, the solution to the local Riemann problem at the point (x_1,t_1) is still a contact discontinuity followed by a shock wave, which are denoted by J_5 and S_6, respectively. As in those in Case 1, it can be obtained that $\tau_1=\tau_5$ and $\sigma_2=\sigma_6$. That is to say, J_1 is parallel to J_5 as well as the propagation speed of shock wave is invariant when across the contact discontinuity. In the following, we shall draw our attention to how the shock wave S_6 penetrates the rarefaction wave R_4.

Lemma 3.4 *The shock wave S_6 keeps up with the wave back of the rarefaction wave R_4 in finite time. More specifically, if $r_+<r_-$, then S_6 is able to penetrate R_4 completely. Otherwise, if $r_+>r_-$, then S_6 cannot penetrate R_4 completely and finally takes the line $x=(\phi(r_-)+nr_-\phi'(r_-))t+x_0$ as its asymptote.*

Proof It suffices to get the propagation speeds of the wave back of the rarefaction wave R_4 and the shock wave S_6, which are given respectively by

$$\xi_4(u_2,v_2)=\xi_4(u_m,v_m)=\phi(r_m)+nr_m\phi'(r_m),\qquad \sigma_6=\frac{\sqrt[n]{r_m}\phi(r_m)-\sqrt[n]{r_-}\phi(r_-)}{\sqrt[n]{r_m}-\sqrt[n]{r_-}}. \tag{3.30}$$

Hence, we get

$$\begin{aligned}\sigma_6-\xi_4(u_m,v_m)&=\frac{\sqrt[n]{r_m}\phi(r_m)-\sqrt[n]{r_-}\phi(r_-)}{\sqrt[n]{r_m}-\sqrt[n]{r_-}}-\bigl(\phi(r_m)+nr_m\phi'(r_m)\bigr)\\&=\sqrt[n]{r_-}\frac{\phi(r_-)-\phi(r_m)}{\sqrt[n]{r_-}-\sqrt[n]{r_m}}-nr_m\phi'(r_m).\end{aligned} \tag{3.31}$$

By using the mean value theorem again, there exists $\bar{r_4}\in(r_m,r_-)$ such that we have

$$\sigma_6-\xi_4(u_m,v_m)=nr_-^{\frac{1}{n}}\bar{r_4}^{\frac{n-1}{n}}\phi'(\bar{r_4})-nr_m\phi'(r_m)>n\bar{r_4}\phi'(\bar{r_4})-nr_m\phi'(r_m). \tag{3.32}$$

It is worthwhile to notice that $r_4>r_m>0$, which implies that $\phi'(\bar{r_4})>\phi'(r_m)>0$ under the assumptions $\phi''>0$ and $\phi'>0$. Thus, we have $\sigma_6>\xi_4(u_m,v_m)$. In other words, the shock wave S_6 catches up with the wave back of the rarefaction wave R_4 in finite time.

The intersection point (x_2, t_2) is determined by

$$\begin{cases} x_2 = \sigma_6 t_2 = \frac{\sqrt[n]{r_m}\phi(r_m) - \sqrt[n]{r_-}\phi(r_-)}{\sqrt[n]{r_m} - \sqrt[n]{r_-}} t_2, \\ x_2 - x_0 = \xi_4(u_m, v_m) t_2 = (\phi(r_m) + n r_m \phi'(r_m)) t_2, \end{cases} \tag{3.33}$$

which yields

$$\begin{aligned} (x_2, t_2) = \biggl(& \frac{x_0(\sqrt[n]{r_m}\phi(r_m) - \sqrt[n]{r_-}\phi(r_-))}{\sqrt[n]{r_-}(\phi(r_m) - \phi(r_-)) - n r_m \phi'(r_m)(\sqrt[n]{r_m} - \sqrt[n]{r_-})}, \\ & \frac{x_0(\sqrt[n]{r_m} - \sqrt[n]{r_-})}{\sqrt[n]{r_-}(\phi(r_m) - \phi(r_-)) - n r_m \phi'(r_m)(\sqrt[n]{r_m} - \sqrt[n]{r_-})} \biggr). \end{aligned} \tag{3.34}$$

The interaction between the wave back of the rarefaction wave R_4 and the shock wave S_6 occurs at the point (x_2, t_2), where we again have a local Riemann problem for system (1.1). The shock wave S_6 begins to penetrate the rarefaction wave R_4 after t_2, which is recorded as S_7 during the process of penetration.

The expression for S_7 during the process of penetration is calculated by

$$\begin{cases} \frac{dx}{dt} = \frac{\sqrt[n]{r}\phi(r) - \sqrt[n]{r_-}\phi(r_-)}{\sqrt[n]{r} - \sqrt[n]{r_-}}, \\ \frac{x - x_0}{t} = \phi(r) + n r \phi'(r), \\ x(t_2) = x_2, \end{cases} \tag{3.35}$$

in which r varies from r_2 $(= r_m)$ to r_+. Depending on the ordering relation between r_+ and r_-, there are two possible situations to occur during the process of penetration as follows:

(1) If $r_+ < r_-$, then the shock wave S_6 is able to penetrate the whole rarefaction wave R_4 completely in finite time and subsequently forms a new shock wave S_8 (see Fig. 3), whose propagation speed is given by $\sigma_8 = \frac{\sqrt[n]{r_+}\phi(r_+) - \sqrt[n]{r_-}\phi(r_-)}{\sqrt[n]{r_+} - \sqrt[n]{r_-}}$.

(2) If $r_+ > r_-$, then the shock wave S_6 cannot penetrate R_4 completely in finite time and ultimately has the line $x = (\phi(r_-) + n r_- \phi'(r_-))t + x_0$ as its asymptote (see Fig. 4). □

Case 4 $J + R$ and $J + S$.

In the end, we mainly consider the situation that there are a contact discontinuity followed by a rarefaction wave emitting from the initial point $(0, 0)$ and a contact discontinuity followed by a shock wave emitting from the initial point $(x_0, 0)$. The occurrence of this case depends on the conditions $0 < r_- < r_m$ and $0 < r_+ < r_m$. Moreover, we use J_1, R_2 and J_3, S_4 to denote them respectively (see Fig. 5). In this case, when the time t is small enough, the solution to the double Riemann problem (1.1) and (1.6) may be expressed simply as follows (see Figs. 5 and 6):

$$(u_-, v_-) + J_1 + (u_1, v_1) + R_2 + (u_m, v_m) + J_3 + (u_2, v_2) + S_4 + (u_+, v_+). \tag{3.36}$$

Here and below the intermediate states (u_1, v_1), (u_2, v_2), and (u_3, v_3) and all the situations before the time t_1 are the same as those in Case 2. Thus, we shall draw our attention to the process of the shock wave S_4 penetrating the rarefaction wave R_7.

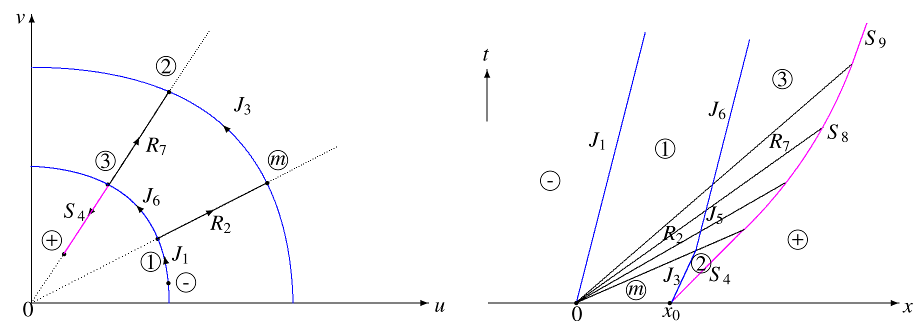

Figure 5 The interaction between $J + R$ and $J + S$ is shown when $0 < r_+ < r_- < r_m$, in which the shock wave S_4 is able to penetrate the rarefaction wave R_7 completely in finite time

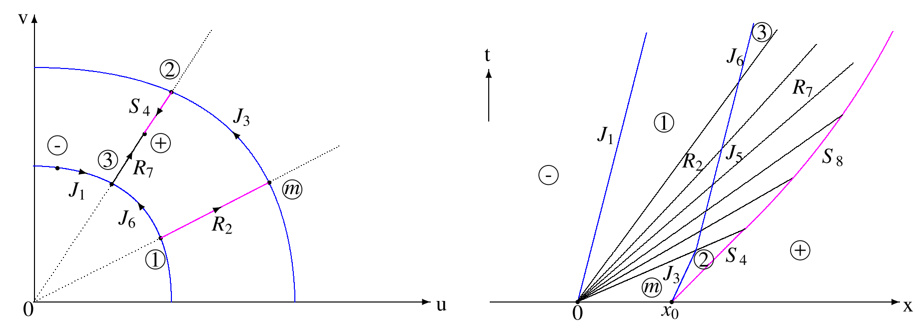

Figure 6 The interaction between $J + R$ and $J + S$ is shown when $0 < r_- < r_+ < r_m$, in which the shock wave S_4 cannot penetrate the rarefaction wave R_7 completely and ultimately has the line $x = (\phi(r_+) + nr_+\phi'(r_+))t$ as its asymptote

Lemma 3.5 *The wave front of the rarefaction wave R_7 catches up with the shock wave S_4 in finite time. More specifically, if $r_+ < r_-$, then S_4 is able to penetrate R_7 completely. Otherwise, if $r_+ > r_-$, then S_4 cannot penetrate R_7 completely and ultimately has the line $x = (\phi(r_+) + nr_+\phi'(r_+))t$ as its asymptote.*

Proof It is easily shown that the propagation speeds of the wave fronts of R_7 and S_4 are given respectively by

$$\xi_7(u_2, v_2) = \xi_7(u_m, v_m) = \phi(r_m) + nr_m\phi'(r_m), \qquad \sigma_4 = \frac{\sqrt[n]{r_+}\phi(r_+) - \sqrt[n]{r_m}\phi(r_m)}{\sqrt[n]{r_+} - \sqrt[n]{r_m}}. \tag{3.37}$$

Thus, we get

$$\begin{aligned} \sigma_4 - \xi_7(u_m, v_m) &= \frac{\sqrt[n]{r_+}\phi(r_+) - \sqrt[n]{r_m}\phi(r_m)}{\sqrt[n]{r_+} - \sqrt[n]{r_m}} - \big(\phi(r_m) + nr_m\phi'(r_m)\big) \\ &= \sqrt[n]{r_+}\frac{\phi(r_+) - \phi(r_m)}{\sqrt[n]{r_+} - \sqrt[n]{r_m}} - nr_m\phi'(r_m). \end{aligned} \tag{3.38}$$

With the similar method as before, there exists $\bar{r}_5 \in (r_+, r_m)$ such that one also has

$$\sigma_4 - \xi_7(u_m, v_m) = nr_+^{\frac{1}{n}}\bar{r}_5^{\frac{n-1}{n}}\phi'(\bar{r}_5) - nr_m\phi'(r_m) < n\bar{r}_5\phi'(\bar{r}_5) - nr_m\phi'(r_m). \tag{3.39}$$

It is obvious to see that $0 < \bar{r_5} < r_m$, which implies that $0 < \phi'(\bar{r_5}) < \phi'(r_m)$ under the assumptions $\phi'' > 0$ and $\phi' > 0$. Thus, it can be concluded that $\sigma_4 < \xi_7(u_m, v_m)$. In other words, the wave front of the rarefaction wave R_7 catches up with the shock wave S_4 in finite time. The intersection point (x_3, t_3) can be calculated by

$$\begin{cases} x_3 = \xi_7(u_m, v_m)t_3 = \phi(r_m) + nr_m\phi'(r_m)t_3, \\ x_3 - x_0 = \sigma_4 t_3 = \frac{\sqrt[n]{r_+}\phi(r_+) - \sqrt[n]{r_m}\phi(r_m)}{\sqrt[n]{r_+} - \sqrt[n]{r_m}} t_3. \end{cases} \tag{3.40}$$

Thus, we have

$$\begin{aligned} (x_3, t_3) = \biggl(& \frac{x_0(\sqrt[n]{r_+} - \sqrt[n]{r_m})(\phi(r_m) + nr_m\phi'(r_m))}{nr_m\phi'(r_m)(\sqrt[n]{r_+} - \sqrt[n]{r_m}) - \sqrt[n]{r_+}(\phi(r_+) - \phi(r_m))}, \\ & \frac{x_0(\sqrt[n]{r_+} - \sqrt[n]{r_m})}{nr_m\phi'(r_m)(\sqrt[n]{r_+} - \sqrt[n]{r_m}) - \sqrt[n]{r_+}(\phi(r_+) - \phi(r_m))} \biggr). \end{aligned} \tag{3.41}$$

It is clear to see that the shock wave S_4 penetrates the rarefaction wave R_7 with the varying propagation speed after the time t_3. During the process of penetration, the curve of S_8 is determined by

$$\begin{cases} \frac{dx}{dt} = \frac{\sqrt[n]{r_+}\phi(r_+) - \sqrt[n]{r}\phi(r)}{\sqrt[n]{r_+} - \sqrt[n]{r}}, \\ \frac{x}{t} = \phi(r) + nr\phi'(r), \\ x(t_3) = x_3, \end{cases} \tag{3.42}$$

in which r varies from r_2 $(= r_m)$ to r_3 $(= r_1 = r_-)$. Similar to those in Case 3, there are also two possible situations in the process of penetration as follows:

(1) If $r_+ < r_-$, then the shock wave S_4 is able to penetrate the rarefaction wave R_7 completely in finite time and subsequently forms a new shock wave S_9 (see Fig. 5), whose propagation speed is given by $\sigma_9 = \frac{\sqrt[n]{r_+}\phi(r_+) - \sqrt[n]{r_-}\phi(r_-)}{\sqrt[n]{r_+} - \sqrt[n]{r_-}}$.

(2) If $r_+ > r_-$, then the shock wave S_4 cannot penetrate the rarefaction wave R_7 completely in finite time and ultimately has the line $x = (\phi(r_+) + nr_+\phi'(r_+))t$ as its asymptote (see Fig. 6). □

Up to now, we have finished the discussions on all kinds of wave interactions for system (1.1) under our assumptions $\phi''(r) > 0$ and $\phi'(r) > 0$. The global solutions to the double Riemann problem (1.1) and (1.6) are constructed completely in explicit forms for all kinds of situations. It can be seen clearly from the above cases that the main conclusion of this paper can be drawn as follows:

Theorem 3.6 *If the perturbed parameter x_0 tends to zero, then the global solutions to the double Riemann problem* (1.1) *and* (1.6) *are exactly identical with those to the Riemann problem* (1.1) *and* (1.7) *for the same Riemann initial data. Thus, it turns out that the solutions to the Riemann problem* (1.1) *and* (1.7) *are stable with respect to the specific small perturbation in the form* (1.6) *of the Riemann initial data* (1.7).

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Competing interests
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Authors' contributions
The authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Statistics Science, Ludong University, Yantai, P.R. China. [2]School of Mathematical Sciences, Qufu Normal University, Qufu, P.R. China.

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