Perveen et al. *Advances in Difference Equations* (2019) 2019:88
https://doi.org/10.1186/s13662-019-2027-2

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Generalized contraction principle under relatively weaker contraction in partial metric spaces

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Abstract

In this paper, we introduce the concept of a generalized weak $(\phi, \mathcal{R})$-contraction and employ this to prove some fixed point results for self-mappings in partial metric spaces endowed with a binary relation $\mathcal{R}$. We also establish some consequences in ordered partial metric spaces and metric spaces with a binary relation and exemplify that our results are a sharpened version of results of Zhiqun Xue (Nonlinear Funct. Anal. Appl. 21(3):497–500, 2016) and Alam and Imdad (J. Fixed Point Theory Appl. 17(4):693–702, 2015). Finally, we provide the existence of a solution for integral and fuzzy partial differential equations.

MSC: Primary 54H25; secondary 47H10

Keywords: Partial metric space; Binary relation; Preserving mappings; $\mathcal{R}^{\neq}$-precompleteness; $\mathcal{R}^{\neq}$-continuous; Partial order

1 Introduction

The very first contribution to fixed point theory was due to Banach [3] in 1922. He conferred the celebrated result in his thesis, namely the Banach contraction principle. Later on, many researchers, fascinated by his idea, extended this result in various directions. One of these is by generalizing the metric. In this direction, Matthews [4] in 1994 presented the idea of a partial metric by extending the concept of metric and proved a supplementary result of the Banach contraction principle in partial metric spaces. Thereafter, many results on the fixed points in partial metric spaces were established (see [5–14] and the references therein).

In 1986, Turinici [15] initiated the idea of order theoretic metric fixed point theory, which was later modified and generalized by Ran and Reurings [16], Nieto and Rodríguez-López [13, 17] and others. In the recent past, Alam and Imdad [2] extended the Banach contraction principle to complete metric space endowed with a binary relation, which generalizes several existing results. Then [18–23] did a variety of work in this field and proved common and relation theoretic fixed point fixed point results in various distance spaces under different conditions.

The growing applications of fixed point theory in various domains, such as mathematics, economics, engineering and game theory, are very encouraging. In mathematics, the applications of suitable fixed point results to establish the existence and uniqueness to the solutions of differential and integral equations to find their solutions are proving very fruitful these days. Very recently, Long et al. [24] have given novel and innovative results for the existence of a solution of some uncertain differential equations. Some more results in this direction were obtained in [25, 26] and the references therein, which are expected to attract the attention of various researchers in the near future.

In this paper, we introduce the notions, e.g. $\mathcal{R}$-precompleteness, ρ-self-closedness and $\mathcal{R}$-continuity in the setting of partial metric spaces endowed with a binary relation $\mathcal{R}$ and establish fixed point results for generalized weak $(\phi, \mathcal{R})$-contraction mappings. We also present the variants of our results in a metric space. Moreover, some examples are furnished to validate the utility of our results and we deduce some consequences via our results. In the end, we furnish sufficient conditions for the existence of solutions for integral equations and fuzzy partial differential equations by utilizing our results.

2 Preliminaries

Matthews [4] defined the partial metric spaces as follows.

Definition 2.1 ([4]) Let M be a non-empty set and $\rho : M \times M \to [0,\infty)$ a mapping satisfying the following conditions:

(ρ_1) $z_1 = z_2 \iff \rho(z_1,z_1) = \rho(z_1,z_2) = \rho(z_2,z_2)$;

(ρ_2) $\rho(z_1,z_1) \leq \rho(z_1,z_2)$;

(ρ_3) $\rho(z_1,z_2) = \rho(z_2,z_1)$;

(ρ_4) $\rho(z_1,z_2) \leq \rho(z_1,z_3) + \rho(z_3,z_2) - \rho(z_3,z_3)$,

$\forall z_1,z_2,z_3 \in M$. Then the mapping ρ is known as a partial metric and the pair (M,ρ) is called partial metric space.

It is observed that the self-distance of a point in partial metric need not be zero. If it zero for all points $z \in M$, i.e., $\rho(z,z) = 0$, $\forall z \in M$, then that partial metric is a metric.

The topology, say τ_ρ, generated by a partial metric ρ on M is a T_0-topology and the base is the family of open balls $\mathcal{B}_\rho(z,\epsilon)$ ($z \in M$ and $\epsilon > 0$) defined by

$$\mathcal{B}_\rho(z,\epsilon) = \bigl\{w \in M : \rho(z,w) \leq \rho(z,z) + \epsilon\bigr\}.$$

Let ρ be a partial metric on M. Then the mapping $d_\rho : M \times M \to [0,\infty)$ defined by

$$d_\rho(z_1,z_2) = 2\rho(z_1,z_2) - \rho(z_1,z_1) - \rho(z_2,z_2), \quad z_1,z_2 \in M,$$

is a metric on M and hence, (M,d_ρ) is a metric space.

Definition 2.2 ([4]) Let (M,ρ) be a partial metric space.

(a) A sequence $\{z_n\}$ is convergent to a point $z \in M$, if $\lim_{n\to\infty} \rho(z_n,z) = \rho(z,z)$.

(b) A sequence $\{z_n\}$ is Cauchy if $\lim_{m,n\to\infty} \rho(z_n,z_m)$ exists and is finite.

(c) (M,ρ) is said to be complete if every Cauchy sequence $\{z_n\}$ in M converges (with respect to τ_ρ) to a point $z \in M$ and $\rho(z,z) = \lim_{n,m\to\infty} \rho(z_n,z_m)$.

Lemma 2.1 ([4]) *Let (M,ρ) be a partial metric space.*

(a) *A sequence $\{z_n\}$ is Cauchy in (M,ρ) if and only if it is Cauchy in (M,d_ρ).*

(b) (M,ρ) *is complete if and only if* (M,d_ρ) *is complete. Besides,*

$$\lim_{n\to\infty} d_\rho(z_n,z) = 0 \quad\Longleftrightarrow\quad \rho(z,z) = \lim_{n\to\infty}\rho(z_n,z) = \lim_{m,n\to\infty}\rho(z_n,z_m).$$

Lemma 2.2 ([12]) *Let* (M,ρ) *be a partial metric space and* $\{z_n\}\subseteq M$ *such that* $\{z_n\}\to z$, *for some* $z\in M$ *with* $\rho(z,z)=0$. *Then, for any* $z^*\in M$, *we have* $\lim_{n\to\infty}\rho(z_n,z^*)=\rho(z,z^*)$.

3 Relation theoretic notions and auxiliary results

For a non-empty subset M, a binary relation $\mathcal{R}$ on M is a subset of $M\times M$. Now, we write some relation theoretic notions as follows:

$(z_1,z_2)\in\mathcal{R}$ (also denoted $z_1\mathcal{R}z_2$) if z_1 is related to z_2 under $\mathcal{R}$;

$(z_1,z_2)\in\mathcal{R}^{\neq}$ (also denoted $(z_1,z_2)\in\mathcal{R}^{\neq}$) if $(z_1,z_2)\in\mathcal{R}$ such that z_1 and z_2 are distinct;

$\mathcal{R}^{-1}$ is the inverse, transpose or dual relation of $\mathcal{R}$, which is defined by $\mathcal{R}^{-1}=\{(z_1,z_2)\in M\times M:(z_2,z_1)\in\mathcal{R}\}$;

$\mathcal{R}^s$ is the the symmetric closure of $\mathcal{R}$, which is defined by $\mathcal{R}^s=\mathcal{R}\cup\mathcal{R}^{-1}$.

It is observed that $\mathcal{R}^{\neq}\subseteq\mathcal{R}$ is also a binary relation. $M\times M$ and $\emptyset$ are trivial binary relations on M, specifically called a universal relation and an empty relation.

Throughout the manuscript, M is a non-empty set, N a non-empty subset of M. $\mathcal{R}$ and S stand for a binary relation and a self-mapping on M, respectively.

Definition 3.1 ([2]) For a binary relation $\mathcal{R}$:

(a) Two elements $z_1,z_2\in M$ are said to be $\mathcal{R}$-comparative if $(z_1,z_2)\in\mathcal{R}$ or $(z_2,z_1)\in\mathcal{R}$. We denote it by $[z_1,z_2]\in\mathcal{R}$.

(b) $\mathcal{R}$ is said to be complete if $[z_1,z_2]\in\mathcal{R}$, $\forall z_1,z_2\in M$.

Proposition 3.1 ([2]) *For a binary relation* $\mathcal{R}$ *on* M,

$$(z_1,z_2)\in\mathcal{R}^s \quad\Longleftrightarrow\quad [z_1,z_2]\in\mathcal{R},\quad \forall z_1,z_2\in M.$$

Definition 3.2 ([2]) A sequence $\{z_n\}\subseteq M$ is said to be $\mathcal{R}$-preserving if $(z_n,z_{n+1})\in\mathcal{R}$, $\forall n\in\mathbb{N}_0$ and $\mathcal{R}^{\neq}$-preserving if $(z_n,z_{n+1})\in\mathcal{R}^{\neq}$, $\forall n\in\mathbb{N}_0$.

Here, we follow the notion (of $\mathcal{R}$-preserving) as used by Alam and Imdad [2]. Notice that Roldán and Shahzad [27] and Shahzad et al. [28] used the term "$\mathcal{R}$-nondecreasing" instead of "$\mathcal{R}$-preserving".

Definition 3.3 ([29]) $N\subseteq M$ is said to be $\mathcal{R}$-directed if for each $z_1,z_2\in N$, there exists a point $z_3\in M$ such that $(z_1,z_3)\in\mathcal{R}$ and $(z_2,z_3)\in\mathcal{R}$.

Definition 3.4 ([30]) For $z_1,z_2\in M$, a path of length $l(\in\mathbb{N})$ in $\mathcal{R}$ from z_1 to z_2 is a finite sequence $\{p_0,p_1,\dots,p_l\}\subseteq M$ such that $p_0=z_1$, $p_l=z_2$ and $(p_i,p_{i+1})\in\mathcal{R}$, for each $0\le i\le l-1$.

Definition 3.5 ([18]) $N\subseteq M$ is said to be $\mathcal{R}$-connected if, for each $z_1,z_2\in N$, there exists a path in $\mathcal{R}$ from z_1 to z_2.

Definition 3.6 ([2]) $\mathcal{R}$ is said to be S-closed if $\forall z_1,z_2\in M$ such that $(z_1,z_2)\in\mathcal{R}$, we have $(Sz_1,Sz_2)\in\mathcal{R}$.

Proposition 3.2 ([2]) *If $\mathcal{R}$ is S-closed, then $\mathcal{R}^s$ is also S-closed.*

Definition 3.7 ([19]) $\mathcal{R}$ is said to be locally S-transitive if for each $\mathcal{R}$-preserving sequence $S(M)$ with range $E = \{z_n : n \in \mathbb{N}_0\}$, the binary relation $\mathcal{R}|_E$ is transitive.

Motivated by Alam and Imdad [18], we present the notion of $\mathcal{R}$-continuity in the setting of partial metric spaces as follows.

Definition 3.8 Let $(M, \rho, \mathcal{R})$ be a partial metric space endowed with a binary relation $\mathcal{R}$. Then a self-mapping S is said to be $\mathcal{R}$-continuous at a point $z \in M$ if for any $\mathcal{R}$-preserving sequence $\{z_n\}$ such that $\{z_n\} \to z$, we have $\{Sz_n\} \to Sz$. S is $\mathcal{R}$-continuous, if it is $\mathcal{R}$-continuous at each point of M.

Following Imdad et al. [22], we introduce the following in the setting of partial metric spaces.

Definition 3.9 Let $(M, \rho, \mathcal{R})$ be a partial metric space endowed with binary relation $\mathcal{R}$. A subset $N \subseteq M$ is said to be $\mathcal{R}$-precomplete if each $\mathcal{R}$-preserving Cauchy sequence $\{z_n\} \subseteq N$ converges to some $z \in M$.

Definition 3.10 Let $(M, \rho, \mathcal{R})$ be a partial metric space endowed with binary relation $\mathcal{R}$. Then $\mathcal{R}$ is said to be ρ-self closed if for each $\mathcal{R}$-preserving sequence $\{z_n\} \subseteq M$ with $\{z_n\} \to z$, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $[z_{n_k}, z] \in \mathcal{R}$, $\forall k \in \mathbb{N}_0$.

In the sequel, for convenience, we use $\mathbb{N}_0$ for $\mathbb{N} \cup \{0\}$ and other notations are used in their natural meaning.

4 Main result

Before presenting our main result, we define the following.

Let Φ denote the set of all mappings $\phi : [0, \infty) \to [0, \infty)$ satisfying the following:

($\Phi 1$) ϕ is nondecreasing;

($\Phi 2$) $\phi(\delta) = 0$ iff $\delta = 0$ and $\liminf_{n\to\infty} \phi(\delta_n) > 0$ if $\lim_{n\to\infty} \delta_n > 0$.

Notice that [1] used the condition that ϕ is continuous. Inspired by [31], we replace this condition by a weaker condition ($\Phi 2$). In fact this condition is also weaker than that ϕ is lower semicontinuous. Indeed, if ϕ is lower semicontinuous function, then, for a sequence $\{\delta_n\}$ with $\lim_{n\to\infty} \delta_n = \delta > 0$, we have $\liminf_{n\to\infty} \phi(\delta_n) \geq \phi(\delta) > 0$.

Now, we embark on the first result in this section.

Theorem 4.1 *Let $(M, \rho, \mathcal{R})$ be a partial metric space equipped with a binary relation $\mathcal{R}$ and $S : M \to M$. Assume that the following conditions are satisfied:*

(a) *$\exists\, z_0 \in M$ such that $(z_0, Sz_0) \in \mathcal{R}$;*

(b) *$\mathcal{R}$ is S-closed and locally S-transitive;*

(c) *$\exists\, N \subseteq M$ such that N is $\mathcal{R}^{\neq}$-precomplete and $S(M) \subseteq N$;*

(d) *S satisfies generalized weak $(\phi, \mathcal{R})$-contraction, i.e.,*

$$\rho(Sz, Sw) \leq \rho(z, w) - \phi\bigl(\rho(Sz, Sw)\bigr), \tag{4.1}$$

$\forall z, w \in M$ with $(z, w) \in \mathcal{R}^{\neq}$ and $\phi \in \Phi$;

(e) *S is $\mathcal{R}^{\neq}$-continuous or $\mathcal{R}^{\neq}|_N$ is ρ-self closed.*
Then S has a fixed point $z^* \in M$ with $\rho(z^*, z^*) = 0$.

Proof Choose $z_0 \in M$ as in (a) and construct a sequence $\{z_n\}$ in M defined by $z_n = Sz_{n-1} = S^n z_0$ based on z_0. If there is some $m_0 \in \mathbb{N}_0$ such that $z_{m_0} = z_{m_0+1}$, then z_{m_0} is the fixed point of S and we are done. Assume that $z_n \neq z_{n+1}$, for every $n \in \mathbb{N}_0$ which along with (b) ensures that $(z_n, z_{n+1}) \in \mathcal{R}^{\neq}$, $\forall n \in \mathbb{N}_0$. Then, by employing condition (d), we obtain

$$\rho(Sz_{n-1}, Sz_n) \leq \rho(z_{n-1}, z_n) - \phi\bigl(\rho(Sz_{n-1}, Sz_n)\bigr), \tag{4.2}$$

which implies

$$\rho(z_n, z_{n+1}) = \rho(Sz_{n-1}, Sz_n) \leq \rho(z_{n-1}, z_n), \tag{4.3}$$

i.e., $\{\rho(z_n, z_{n+1})\}$ is a nondecreasing sequence of positive real numbers (also bounded below by 0). So, there exists $r \geq 0$ such that $\lim_{n\to\infty} \rho(z_n, z_{n+1}) = r$. Next, we have to show that $r = 0$. Suppose to the contrary that it is not so, i.e., $r > 0$. Passing the limit $n \to \infty$ in (4.2), we get

$$r \leq r - \liminf_{n\to\infty} \phi\bigl(\rho(z_n, z_{n+1})\bigr);$$

a contradiction (due to ($\Phi 2$)). Hence,

$$\lim_{n\to\infty} \rho(z_n, z_{n+1}) = 0. \tag{4.4}$$

We also have

$$\begin{aligned} d_\rho(z_n, z_{n+1}) &= 2\rho(z_n, z_{n+1}) - \rho(z_n, z_n) - \rho(z_{n+1}, z_{n+1}) \\ &\leq 2\rho(z_n, z_{n+1}), \end{aligned}$$

which on letting $n \to \infty$ and applying (4.4) yields

$$\lim_{n\to\infty} d_\rho(z_n, z_{n+1}) = 0.$$

Next, our claim is that $\{z_n\}$ is a Cauchy sequence in (N, d_ρ). Otherwise, there exist subsequences $\{z_{m_k}\}$ and $\{z_{n_k}\}$ of $\{z_n\}$ such that n_k is the smallest integer for which

$$n_k > m_k > k \quad \text{and} \quad d_\rho(z_{m_k}, z_{n_k}) \geq \epsilon. \tag{4.5}$$

Since $d_\rho(z, w) \leq 2\rho(z, w)$, $\forall z, w \in M$, so (4.5) gives

$$n_k > m_k > k, \qquad \rho(z_{m_k}, z_{n_k}) \geq \frac{\epsilon}{2} \quad \text{and} \quad \rho(z_{m_k}, z_{n_k-1}) < \frac{\epsilon}{2}. \tag{4.6}$$

Now, using the triangular inequality, we have

$$\frac{\epsilon}{2} \leq \rho(z_{m_k}, z_{n_k}) \leq \rho(z_{m_k}, z_{n_k-1}) + \rho(z_{n_k-1}, z_{n_k}) - \rho(z_{n_k-1}, z_{n_k-1})$$

$$< \frac{\epsilon}{2} + \rho(z_{n_k-1}, z_{n_k}).$$

Taking $k \to \infty$ in the above inequality and using (4.4), we get

$$\lim_{k\to\infty} \rho(z_{m_k}, z_{n_k}) = \frac{\epsilon}{2}. \tag{4.7}$$

Again, (4.3), (4.6) and (ρ_4) give rise to

$$\begin{aligned}\frac{\epsilon}{2} &\le \rho(z_{m_k}, z_{n_k}) \\ &\le \rho(z_{m_k-1}, z_{n_k-1}) \\ &\le \rho(z_{m_k-1}, z_{m_k}) + \rho(z_{m_k}, z_{n_k-1}) - \rho(z_{m_k}, z_{m_k}) \\ &< \rho(z_{m_k-1}, z_{m_k}) + \frac{\epsilon}{2}.\end{aligned}$$

Now, on taking $k \to \infty$, the above inequality yields

$$\lim_{k\to\infty} \rho(z_{m_k-1}, z_{n_k-1}) = \frac{\epsilon}{2}. \tag{4.8}$$

Using local S-transitivity, we have $(z_{m_k-1}, z_{n_k-1}) \in \mathcal{R}^{\neq}$ and hence, (4.1) implies

$$\rho(z_{m_k}, z_{n_k}) \le \rho(z_{m_k-1}, z_{n_k-1})) - \phi\big(\rho(z_{m_k}, z_{n_k})\big).$$

Using (4.7), (4.8) and opting for $k \to \infty$ in the above inequality, we get

$$\frac{\epsilon}{2} \le \frac{\epsilon}{2} - \liminf_{k\to\infty} \phi\big(\rho(z_{m_k}, z_{n_k})\big);$$

a contradiction. Hence, $\{z_n\}$ is Cauchy in (N, d_ρ) and also $\mathcal{R}^{\neq}$-preserving. By Lemma 2.1, $\{z_n\}$ is also Cauchy in (N, ρ). The $\mathcal{R}^{\neq}$-precompleteness of N in M ensures the existence of a point $z^* \in M$ such that

$$\lim_{n\to\infty} z_n = z^*, \quad \text{i.e., } \lim_{n\to\infty} \rho\big(z_n, z^*\big) = \rho\big(z^*, z^*\big) \implies \lim_{n\to\infty} d_\rho\big(z_n, z^*\big) = 0. \tag{4.9}$$

Again, Lemma 2.1 gives

$$\rho\big(z^*, z^*\big) = \lim_{n\to\infty} \rho\big(z_n, z^*\big) = \lim_{m,n\to\infty} \rho(z_m, z_n) = 0. \tag{4.10}$$

Now, $\mathcal{R}^{\neq}$-continuity of S implies that (as $\{z_n\} \to z^*$ and $(z_n, z_{n+1}) \in \mathcal{R}^{\neq}$, $\forall n \in \mathbb{N}_0$)

$$\lim_{n\to\infty} Sz_n = \lim_{n\to\infty} z_{n+1} = Sz^*.$$

Thus, by virtue of uniqueness of the limit, we obtain $Sz^* = z^*$.

Alternatively, if $\mathcal{R}^{\neq}|_N$ is ρ-self closed, then, for any $\mathcal{R}^{\neq}$-preserving sequence $\{z_n\}$ in N with $\{z_n\} \to z^*$, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $[z_{n_k}, z^*] \in \mathcal{R}$, $\forall k \in \mathbb{N}_0$. Now, applying condition (d) with $z = z_{n_k}$ and $w = z^*$, we obtain

$$\rho\big(Sz_{n_k}, Sz^*\big) \le \rho\big(z_{n_k}, z^*\big) - \phi\big(\rho\big(Sz_{n_k}, Sz^*\big)\big) \le \rho\big(z_{n_k}, z^*\big),$$

which, on letting $n \to \infty$ and using Lemma 2.2, gives $\rho(z^*, Sz^*) \leq 0$, yielding thereby $z^* = Sz^*$. This completes the proof. □

Theorem 4.2 *If we add the following assumption, in addition to the assumptions of Theorem 4.1:*
(f) *$S(M)$ is $\mathcal{R}^s|_N$-connected,*
then S has a unique fixed point.

Proof Theorem 4.1 ensures the existence of at least one fixed point of S. Assume that it has two fixed points, say $z, z^* \in M$. Then we have $z = Sz$ and $z^* = Sz^*$. Our claim is that $z = z^*$. As $z, z^* \in S(M) \subseteq N$, so condition (f) ensures the existence of a path, say $\{w_0, w_1, w_2, \dots, w_l\} \subseteq M$ of some finite length l in $\mathcal{R}^s|_N$ from z to z^*, where $w_0 = z$ and $w_l = z^*$. Henceforth,

$$[w_i, w_{i+1}] \in \mathcal{R}, \quad \text{for each } 0 \leq i \leq l-1. \tag{4.11}$$

Define two constant sequences $\{w_n^0 = z\}$ and $\{w_n^l = z^*\}$, then we have $Sw_n^0 = Sz = z$ and $Sw_n^l = Sz^* = z^*$, $\forall n \in \mathbb{N}_0$. Also, put

$$w_0^i = w_i, \quad \text{for each } 0 \leq i \leq l \tag{4.12}$$

and define sequences $\{w_n^1\}, \{w_n^2\}, \dots, \{w_n^{k-1}\}$ by

$$w_{n+1}^i = Sw_n^i, \quad \forall n \in \mathbb{N}_0 \text{ and for each } 1 \leq i \leq l-1.$$

Hence

$$w_{n+1}^i = Sw_n^i, \quad \forall n \in \mathbb{N}_0 \text{ and for each } 0 \leq i \leq l.$$

Next, we prove that

$$[w_n^i, w_n^{i+1}] \in \mathcal{R}, \quad \forall n \in \mathbb{N}_0 \text{ and for each } 0 \leq i \leq l-1.$$

Indeed, owing to (4.11) and (4.12), we obtain $[w_0^i, w_0^{i+1}] \in \mathcal{R}$ and further S-closedness of $\mathcal{R}$ implies

$$\left[w_n^i, w_n^{i+1}\right] \in \mathcal{R}, \quad \text{for each } 0 \leq i \leq l-1. \tag{4.13}$$

Now, $\forall n \in \mathbb{N}_0$ and for each $0 \leq i \leq l-1$, define $\alpha_n^i = \rho(w_n^i, w_n^{i+1})$. Our claim is that

$$\lim_{n\to\infty} \alpha_n^i = 0.$$

Suppose to the contrary that $\lim_{n\to\infty} \alpha_n^i = \alpha > 0$. Since $[w_n^i, w_n^{i+1}] \in \mathcal{R}$, either $(w_n^i, w_n^{i+1}) \in \mathcal{R}$ or $(w_n^{i+1}, w_n^i) \in \mathcal{R}$ (and are distinct) $\forall n \in \mathbb{N}_0$ and for each $0 \leq i \leq k-1$, so (4.1) gives

$$\rho\left(Sw_n^i, Sw_n^{i+1}\right) \leq \rho\left(w_n^i, w_n^{i+1}\right) - \phi\left(\rho\left(Sw_n^i, Sw_n^{i+1}\right)\right)$$

or

$$\rho\left(w_{n+1}^{i}, w_{n+1}^{i+1}\right) \leq \rho\left(w_{n}^{i}, w_{n}^{i+1}\right) - \phi\left(\rho\left(w_{n+1}^{i}, w_{n+1}^{i+1}\right)\right), \tag{4.14}$$

which on taking the limit gives rise to

$$\alpha \leq \alpha - \liminf_{n\to\infty} \phi\left(\rho\left(w_{n+1}^{i}, w_{n+1}^{i+1}\right)\right);$$

a contradiction. Hence, $\lim_{n\to\infty} \alpha_n^i = 0$.

Next, we have

$$\begin{aligned}
\rho\left(z, z^*\right) = \rho\left(w_n^0, w_n^l\right) &\leq \sum_{i=0}^{k-1} \rho\left(w_n^i, w_n^{i+1}\right) - \sum_{i=1}^{k-1} \rho\left(w_n^i, w_n^{i+1}\right) \\
&\leq \sum_{i=0}^{k-1} \rho\left(w_n^i, w_n^{i+1}\right) \\
&= \sum_{i=0}^{k-1} \alpha_n^i \to 0 \quad (\text{as } n \to \infty).
\end{aligned}$$

Hence, (by ρ_1 and ρ_2) $z = z^*$ and the proof is completed. □

Corollary 4.1 *The conclusion of Theorem 4.2 remains valid if condition (f) is replaced by any one of the following*:

(f*) $\mathcal{R}|_{S(M)}$ *is complete*;

(f**) $S(M)$ *is* $\mathcal{R}^s|_N$*-directed.*

Proof If (f*) holds true, then, for any $z_1, z_2 \in S(M)$, we have $[z_1, z_2] \in \mathcal{R}|_{S(M)} \subseteq \mathcal{R}|_N$ (by condition (c)), i.e., $\{z_1, z_2\}$ is a path of length 1 in $\mathcal{R}^s|_N$ from z_1 to z_2. Hence, condition (f) of Theorem 4.2 is fulfilled and the result is concluded to by Theorem 4.2.

On the other hand, if condition (f**) holds, then, for each $z_1, z_2 \in S(M)$, there exists $z_3 \in N$ such that $[z_1, z_3]$ and $[z_2, z_3] \in \mathcal{R}|_N$. This amounts to saying that there exists a path of length 2 (say $\{z_1, z_3, z_2\}$) in $\mathcal{R}^s|_N$ from z_1 to z_2. Hence, again by Theorem 4.2, the conclusion follows. □

The following example exhibits the utility of our results.

Example 4.1 Let $M = [0, \infty)$ with a partial metric $\rho : M \times M \to [0, \infty)$ defined by

$$\rho(z_1, z_2) = \max\{z_1, z_2\}, \quad \forall z_1, z_2 \in M.$$

We define the binary relation $\mathcal{R}$ by

$$(z_1, z_2) \in \mathcal{R} \quad \Longleftrightarrow \quad \rho(z_1, z_1) = \rho(z_1, z_2) \quad \Longleftrightarrow \quad z_1 = \max\{z_1, z_2\}.$$

It is clear that (M, ρ) is complete as (M, d_ρ) is complete. Define $S : M \to M$ as

$$Sz = \frac{z}{2}, \quad \forall z \in M.$$

Then S is continuous and hence $\mathcal{R}^{\neq}$-continuous. Also, conditions (a) and (b) are trivially satisfied. Further, let us define $\phi : [0,\infty) \to [0,\infty)$ as

$$\phi(t) = \frac{3t}{4}, \quad \forall t \in [0,\infty).$$

For any $z_1, z_2 \in M$ such that $(z_1, z_2) \in \mathcal{R}$, we have

$$\begin{aligned} \rho(Sz_1, Sz_2) = \frac{z_1}{2} &\leq \rho(z_1, z_2) - \phi\bigl(\rho(Sz_1, Sz_2)\bigr) \\ &= z_1 - \frac{3}{4}\left(\frac{z_1}{2}\right) \\ &= \frac{5z_1}{8}. \end{aligned}$$

Thus all the conditions of Theorem 4.1 are satisfied. Hence, S has a fixed point, namely $z = 0$. Moreover, $\mathcal{R}|_{S(M)}$ is complete, so, from Corollary 4.1 one deduces that 0 is the unique fixed point.

As in [1], it can easily be seen that in a partial metric space (M, ρ), $\forall (z_1, z_2) \in \mathcal{R}^{\neq}$ (also $\forall z_1, z_2 \in M$), the condition

$$\rho(Sz_1, Sz_2) \leq \rho(z_1, z_2) - \phi\bigl(\rho(Sz_1, Sz_2)\bigr) \tag{4.15}$$

is weaker than

$$\rho(Sz_1, Sz_2) \leq \rho(z_1, z_2) - \phi\bigl(\rho(z_1, z_2)\bigr). \tag{4.16}$$

But, the converse need not be true in general.

Indeed, consider $M = [1,\infty)$ with partial metric $\rho(z_1, z_2) = \max\{z_1, z_2\}$ and binary relation $\mathcal{R}$ such that $(z_1, z_2) \in \mathcal{R}$ if and only if $z_1 \leq z_2$. Let $S : M \to M$ be defined by

$$Sz = \frac{z}{2}$$

and $\phi : [0,\infty) \to [0,\infty)$ be defined by

$$\phi(t) = \frac{t+1}{3}.$$

Then (4.15) is satisfied for all $(z_1, z_2) \in \mathcal{R}^{\neq}$. But, for $z_1 = 1$ and $z_2 = \frac{5}{4}$ in (4.16), we have

$$\frac{5}{8} = \rho\left(S1, S\frac{5}{4}\right) \leq \rho\left(1, \frac{5}{4}\right) - \phi\left(\rho\left(1, \frac{5}{4}\right)\right) = \frac{5}{4} - \frac{9}{12} = \frac{1}{2};$$

a contradiction.

Hence, in view of the above observation and Theorems 4.1 and 4.2, the following result is obvious.

Corollary 4.2 *Let $(M, \rho, \mathcal{R})$ be a partial metric space equipped with binary relation $\mathcal{R}$ and $S : M \to M$. Assume that the following conditions are satisfied*:

(a) $\exists\, z_0 \in M$ *such that* $(z_0, Sz_0) \in \mathcal{R}$;
(b) $\mathcal{R}$ *is S-closed and locally S-transitive*;
(c) $\exists\, N \subseteq M$ *such that* N *is* $\mathcal{R}^{\neq}$*-precomplete and* $S(M) \subseteq N$;
(d) S *satisfies a weak* $(\phi, \mathcal{R})$*-contraction, i.e.*,

$$\rho(Sz, Sw) \leq \rho(z, w) - \phi\bigl(\rho(z, w)\bigr),$$

$\forall z, w \in M$ *with* $(z, w) \in \mathcal{R}^{\neq}$ *and* $\phi \in \Phi$;
(e) S *is* $\mathcal{R}^{\neq}$*-continuous or* $\mathcal{R}^{\neq}|_N$ *is* ρ*-self closed.*

Then S *has a fixed point. Moreover, the fixed point is unique if* $S(M)$ *is* $\mathcal{R}^s|_N$*-connected.*

5 Order theoretic results in partial metric spaces

In this section, from now on, $\preceq$ denotes the partial order on a non-empty set M. In what follows, $(M, \preceq)$ denotes a partially ordered set, $(M, \rho, \preceq)$ stands for a partial metric space with partial order $\preceq$, we call it an ordered partial metric space.

Definition 5.1 ([32]) A self-mapping S on $(M, \preceq)$ is said to be increasing (or isotone or order preserving) if $Sz_1 \preceq Sz_2$, for any $z_1, z_2 \in M$ with $z_1 \preceq z_2$.

Remark 5.1 Notice that S to be increasing coincides with the notion of $\preceq$ to be S-closed in our sense.

Definition 5.2 ([32]) Let $\{z_n\}$ be a sequence in an ordered set $(M, \preceq)$. Then

(a) $\{z_n\}$ is increasing if $\forall m, n \in \mathbb{N}_0$,

$$m \leq n \quad \Longrightarrow \quad z_m \preceq z_n.$$

(b) $\{z_n\}$ is decreasing if $\forall m, n \in \mathbb{N}_0$,

$$m \leq n \quad \Longrightarrow \quad z_n \preceq z_m.$$

(c) $\{z_n\}$ is monotone if it is either increasing on decreasing.

Definition 5.3 Let $(M, \rho, \preceq)$ be an ordered partial metric space. We say that $(M, \rho, \preceq)$ has the ICU (increasing-convergence-upper bound) property if every increasing sequence $\{z_n\} \subseteq M$ such that $z_n \to z$ is bounded above by its limit, i.e., $z_n \preceq z$.

Remark 5.2 If $(M, \rho, \preceq)$ has the ICU property, then $\preceq$ is ρ-self closed.

Notice that Alam et al. [33] defined the ICU property in the setting of ordered metric spaces.

Definition 5.4 In an ordered partial metric space $(M, \rho, \preceq)$, we define the following:

(a) $(M, \rho, \preceq)$ is said to be $\overline{O}$-complete (respectively, $\underline{O}$-complete, O-complete) if every increasing (respectively, decreasing, monotone) Cauchy sequence in M converges.

(b) A self-mapping S on M is said to be $\overline{O}$-continuous (respectively, $\underline{O}$-continuous, O-continuous) at $z \in M$, if for any increasing (respectively, decreasing, monotone) sequence $\{z_n\} \subseteq M$ such that $\{z_n\} \to z$, we have $\{Sz_n\} \to Sz$.

S is $\overline{O}$-continuous (respectively, $\underline{O}$-continuous, O-continuous) on M if it $\overline{O}$-continuous (respectively, $\underline{O}$-continuous, O-continuous) at every $z \in M$.

The above notions were defined by Kutbi et al. [34] in the setting of ordered metric spaces.

Next, we introduce the following notion.

Definition 5.5 A subset N of an ordered partial metric space $(M, \rho, \preceq)$ is said to be $\overline{O}$-precomplete (respectively, $\underline{O}$-precomplete, O-complete) if every increasing (respectively, decreasing, monotone) Cauchy sequence in N converges to a point of M.

Now, we are equipped to state the following result, which is a more refined and generalized version of Theorem 2.1 of [7].

Theorem 5.1 *Let $(M, \rho, \preceq)$ be an ordered partial metric space and $S : M \to M$. Assume that the following conditions are satisfied*:

(a) *$\exists\, z_0 \in M$ such that $z_0 \preceq Sz_0$*;

(b) *S is increasing*;

(c) *$\exists\, N \subseteq M$ such that N is $\bar{O}$-precomplete and $S(M) \subseteq N$*;

(d) *S satisfies*

$$\rho(Sz, Sw) \leq \rho(z, w) - \phi\big(\rho(Sz, Sw)\big),$$

$\forall z, w \in M$ with $z \preceq w$ and $\phi \in \Phi$;

(f) *S is $\bar{O}$-continuous or $(N, \rho, \preceq)$ has the ICU property.*

Then S has a fixed point $z^* \in M$. Moreover, $\rho(z^*, z^*) = 0$.

Proof The result holds by taking $\mathcal{R} = \preceq$ in Theorem 4.1. □

6 Results in metric space

By virtue of the fact that each metric is a partial metric, the following result is apparent via Theorems 4.1 and 4.2.

Theorem 6.1 *Let $(M, d, \mathcal{R})$ be a metric space endowed with relation $\mathcal{R}$ and $S : M \to M$. Assume that the following conditions are satisfied*:

(a) *$\exists\, z_0 \in M$ such that $(z_0, Sz_0) \in \mathcal{R}$*;

(b) *$\mathcal{R}$ is S-closed and locally S-transitive*;

(c) *$\exists\, N \subseteq M$ such that N is $\mathcal{R}^{\neq}$-precomplete and $S(M) \subseteq N$*;

(d) *S satisfies generalized weak $(\phi, \mathcal{R})$-contraction, i.e.,*

$$d(Sz, Sw) \leq d(z, w) - \phi\big(d(Sz, Sw)\big),$$

$\forall z, w \in M$ with $(z, w) \in \mathcal{R}^{\neq}$ and $\phi \in \Phi$;

(e) *S is $\mathcal{R}^{\neq}$-continuous or $\mathcal{R}^{\neq}|_N$ is d-self closed.*

Then S has a fixed point. Moreover, the fixed point is unique if $S(M)$ is $\mathcal{R}^s|_N$-connected.

Remark 6.1 Theorem 6.1 is a version of Theorem 2.1 of [1] improved in the following way:

- In place of the usual notion of completeness of the whole set M, we have used $\mathcal{R}^{\neq}$-precompleteness of a subset $N \subseteq M$.
- We have dropped the continuity of the function ϕ and use a weaker condition ($\Phi 1$), which is weaker than that ϕ is lower semicontinuous.

  Indeed, if ϕ is lower semicontinuous, then, for a sequence $\{\delta_n\}$ such that $\lim_{n\to\infty} \delta_n = \delta > 0$, we have $\liminf_{n\to\infty} \phi(\delta_n) \geq \phi(\delta) > 0$.

Example 6.1 Let $M = [0,4]$ equipped with the usual metric. Then M is a complete metric space. Define a binary relation $\mathcal{R}$ on M as

$$\mathcal{R} = \bigl\{(0,0),(0,2),(2,0),(2,2),(0,4)\bigr\}.$$

Define a mapping $S : M \times M$ by

$$Sz = \begin{cases} 0 & \text{if } 0 \leq z \leq 2, \\ 2 & \text{if } 2 < z \leq 4, \end{cases}$$

and $\phi : [0,\infty) \to [0,\infty)$ by

$$\phi(t) = \frac{t}{2}, \quad \forall t \in [0,\infty).$$

Then all the hypotheses of Theorem 6.1 are satisfied and we observe that 0 is the unique fixed point of S. Observe that Theorem 2.1 of [1] cannot be applied. For $x = 2$ and $y = 4$ in (2.1) of Theorem 2.1 [1], we have

$$\begin{aligned} 2 = d(0,2) = d(S2,S4) &\leq d(2,4) - \phi\bigl(d(0,2)\bigr) \\ &= 2 - 1 = 1; \end{aligned}$$

a contradiction.

Corollary 6.1 *The conclusion of Theorem 6.1 holds true even if we replace the condition* (*d*) *by the following*:

(d*) *S satisfies a weak* $(\phi,\mathcal{R})$*-contraction, i.e.*,

$$d(Sz,Sw) \leq d(z,w) - \phi\bigl(d(z,w)\bigr),$$

$$\forall z,w \in M \textit{ with } (z,w) \in \mathcal{R}^{\neq} \textit{ and } \phi \in \Phi.$$

Next, putting $\phi(t) = (1-h)t$, $h \in [0,1)$ in Corollary 6.1, we obtain a relation theoretic variant of the Banach contraction principle (a more sharpened version of Theorem 3.1 of [2]), since

- instead of using completeness of M, we have used $\mathcal{R}^{\neq}$-precompleteness of $N \subseteq M$;
- in place of continuity of S, we have used $\mathcal{R}^{\neq}$-continuity and d-self-closedness of $\mathcal{R}$ is replaced by d-self-closedness of $\mathcal{R}^{\neq}|_N$;
- we have used $\mathcal{R}^s|_N$-connectedness of $S(M)$ instead of only $\mathcal{R}^s$-connectedness of M.

Corollary 6.2 *Let $(M,d,\mathcal{R})$ be a metric space endowed with binary relation $\mathcal{R}$ and $S: M \to M$. Assume that the following conditions are satisfied*:

(a) $\exists\, z_0 \in M$ *such that* $(z_0, Sz_0) \in \mathcal{R}$;

(b) $\mathcal{R}$ *is S-closed and locally S-transitive*;

(c) $\exists\, N \subseteq M$ *such that N is $\mathcal{R}^{\neq}$-precomplete and* $S(M) \subseteq N$;

(d) *there exists* $h \in [0,1)$ *such that*

$$d(Sz,Sw) \le hd(z,w),$$

$\forall z,w \in M$ *with* $(z,w) \in \mathcal{R}^{\neq}$;

(e) *S is $\mathcal{R}^{\neq}$-continuous or $\mathcal{R}^{\neq}|_N$ is increasingly d-self closed.*

Then S has a fixed point. Moreover, the fixed point is unique if $S(M)$ is $\mathcal{R}^s|_N$-connected.

Remark 6.2 The assumption that $\mathcal{R}$ is locally S-transitive is not necessary in Corollary 6.2.

In the following example, it is observed that the results of both [2] and [1] cannot be applied, while our result is applicable.

Example 6.2 Let $M = \{p,q,r,s\}$ have the binary relation

$$\mathcal{R} = \bigl\{(p,p),(q,q),(r,r),(p,q),(p,r),(r,s)\bigr\}.$$

Let us define the metric $d: M \times M \to [0,\infty)$ as follows:

$$\begin{cases} d(z,z) = 0, \quad \forall z \in M \quad \text{and} \quad d(z,w) = d(w,z), \quad \forall z,w \in M; \\ d(p,q) = 2; \qquad d(p,r) = d(p,s) = d(r,s) = 3; \\ d(q,r) = d(q,s) = \frac{3}{2}. \end{cases}$$

Also define $S: M \to M$ by

$$Sp = Sq = Sr = p; \qquad Ss = q,$$

and $\phi: [0,\infty) \to [0,\infty)$ by

$$\phi(t) = \frac{t}{3}, \quad \forall t \in [0,\infty).$$

Then all the conditions of Theorem 6.1 are satisfied and p is the unique fixed point of S.

But Theorem 3.1 of [2] does not hold true (as M is not $\mathcal{R}^s$-connected). Indeed, for $x = q$ and $y = s$, there exists no path in $\mathcal{R}$. Also, Theorem 2.1 of [1] cannot be applied. For $q,s \in M$, we have

$$\begin{aligned} d(Sq,Ss) = d(p,q) = 2 &\le d(q,s) - \phi\bigl(d(Sq,Ss)\bigr) \\ &= \frac{3}{2} - \frac{2}{3}; \end{aligned}$$

a contradiction.

7 Applications

7.1 Application to integral equation

In this subsection, we study the sufficient condition for the existence of solution of the following integral equation in the framework of a partial metric space under some binary relation:

$$z(t)=\int_0^t K\bigl(t,\tau,z(\tau)\bigr)\,d\tau,\quad t\in\Omega=[0,T],T>0,\tag{7.1}$$

where $K:\Omega\times\Omega\times\mathbb{R}\to\mathbb{R}$ is continuous.

Here, we consider the complete partial metric space $M=C(\Omega,\mathbb{R})$, the space of all continuous functions from Ω to $\mathbb{R}$, with partial metric ρ on M defined by

$$\rho(f,g)=\max\Bigl\{\sup_{t\in\Omega}f(t),\sup_{t\in\Omega}g(t)\Bigr\}.$$

Also, suppose $\beta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a function with $\beta(t,s)=0$ if and only if $t=s$ and

$$\beta(t,s)\le 0\quad\text{and}\quad\beta(s,w)\le 0\quad\Longrightarrow\quad\beta(t,w)\le 0.$$

Theorem 7.1 *Suppose the following conditions are satisfied*:

(H_1) $\exists\, z_0\in M$ *such that*

$$\beta\biggl(z_0(t),\int_0^t K\bigl(t,\tau,z_0(\tau)\bigr)\,d\tau\biggr)\le 0;$$

(H_2) $\forall z_1,z_2\in M$ *and* $t\in\Omega$,

$$\beta\bigl(z_1(t),z_2(t)\bigr)\le 0\quad\Longrightarrow\quad\beta\biggl(\int_0^t K\bigl(t,\tau,z_1(\tau)\bigr)\,d\tau,\int_0^t K\bigl(t,\tau,z_2(\tau)\bigr)\,d\tau\biggr)\le 0;$$

(H_3) *for each* $t,\tau\in\Omega$ *and* $z\in M$, *there exists a number* $h\in[0,1)$ *such that*

$$\int_0^t K\bigl(t,\tau,z(\tau)\bigr)\,d\tau\le hz(t).$$

Then 7.1 *has a solution, say* $\bar{z}\in M$.

Proof Define a binary relation $\mathcal{R}$ on M by

$$(z_1,z_2)\in\mathcal{R}\quad\Longleftrightarrow\quad\beta\bigl(z_1(t),z_2(t)\bigr)\le 0,\quad\forall t\in\Omega.$$

Also, define $S:M\to M$ by

$$Sz(t)=\int_0^t K\bigl(t,\tau,z(\tau)\bigr)\,d\tau.$$

Then, by condition (H_1), there exists z_0 such that $(z_0,Sz_0)\in\mathcal{R}$. Now, suppose $(z_1,z_2)\in\mathcal{R}$, for some $z_1,z_2\in M$, i.e., $\beta(z_1(t),z_2(t))\le 0$, $\forall t\in\Omega$. Then, by condition (H_2), we obtain

$$\beta\bigl(z_1(t),z_2(t)\bigr)\le 0\quad\Longrightarrow\quad\beta\biggl(\int_0^t K\bigl(t,\tau,z_1(\tau)\bigr)\,d\tau,\int_0^t K\bigl(t,\tau,z_2(\tau)\bigr)\,d\tau\biggr)\le 0$$

$$\begin{aligned}
&\Longrightarrow \quad \beta\bigl(Sz_1(t), Sz_2(t)\bigr) \leq 0 \\
&\Longrightarrow \quad (Sz_1, Sz_2) \in \mathcal{R},
\end{aligned}$$

i.e., $\mathcal{R}$ is S-closed. Also, for $(z_1, z_2) \in \mathcal{R}^{\neq}$, i.e., $\beta(z_1(t), z_2(t)) < 0$ ($\forall\, t \in \Omega$), we have

$$\begin{aligned}
\rho(Sz_1, Sz_2) &= \max\Bigl\{\sup_{t\in\Omega}(Sz_1)(t), \sup_{t\in\Omega}(Sz_2)(t)\Bigr\} \\
&= \max\biggl\{\sup_{t\in\Omega}\int_0^t K\bigl(t,\tau,z_1(\tau)\bigr)\,d\tau, \sup_{t\in\Omega}\int_0^t K\bigl(t,\tau,z_2(\tau)\bigr)\,d\tau\biggr\} \\
&\leq \max\Bigl\{\sup_{t\in\Omega} hz_1(t), \sup_{t\in\Omega} hz_2(t)\Bigr\}\bigl(\text{by condition } (H_3)\bigr) \\
&= h\max\Bigl\{\sup_{t\in\Omega} z_1(t), \sup_{t\in\Omega} z_2(t)\Bigr\} \\
&= h\rho(z_1, z_2).
\end{aligned}$$

Define $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = (1-h)t$, $h \in [0,1)$. It is easily seen that $\phi \in \Phi$. Now, applying it in the above inequality, we obtain

$$\begin{aligned}
\rho(Sz_1, Sz_2) &\leq \rho(z_1, z_2) - \phi\bigl(\rho(z_1, z_2)\bigr) \\
&\leq \rho(z_1, z_2) - \phi\bigl(\rho(Sz_1, Sz_2)\bigr).
\end{aligned}$$

Thus, all the hypotheses of Theorem 4.1 are satisfied, so by Theorem 4.1 one concludes that (7.1) has a solution, $\bar{z} \in M$. □

7.2 Application to fuzzy partial differential equations

Before presenting the main results of this subsection, we discuss the following relevant notions and results.

Let $\mathbb{R}_{\mathcal{F}}$ be the space of fuzzy sets on $\mathbb{R}$ which are non-empty, normal, fuzzy convex, upper semicontinuous and compact-supported fuzzy sets defined over $\mathbb{R}$. For $\alpha \in [0,1]$, the α-cut (also known as α-level set) of $\mu \in \mathbb{R}_{\mathcal{F}}$ is defined by

$$[\mu]^\alpha = \bigl\{z \in \mathbb{R} : \mu(z) \geq \alpha\bigr\},$$

which is a non-empty, convex and compact subset of $\mathbb{R}$. These properties are also enjoyed by $[\mu]^0 = \overline{\{z \in \mathbb{R} : \mu(z) > 0\}}$, known as support of μ. Particularly, it is often written as $[\mu]^\alpha = [\mu_{l\alpha}, \mu_{r\alpha}]$, $\forall\alpha \in [0,1]$.

The supremum metric d_∞ in $\mathbb{R}_{\mathcal{F}}$ is defined by

$$d_\infty(\mu,\nu) = \sup_{0\leq\alpha\leq 1}\bigl\{d_H\bigl([\mu]^\alpha, [\nu]^\alpha\bigr)\bigr\}, \quad \forall \mu,\nu \in \mathbb{R}_{\mathcal{F}},$$

where d_H is the Hausdorff metric in the space of all non-empty, convex and compact subsets of $\mathbb{R}$. $(\mathbb{R}_{\mathcal{F}}, d_\infty)$ is a complete metric space (see [35]). For $\mu,\nu \in \mathbb{R}_{\mathcal{F}}$, $\alpha \in [0,1]$ and $k \in \mathbb{R}$, the following hold:

$$[\mu+\nu]^\alpha = [\mu]^\alpha + [\nu]^\alpha \quad \text{(Addition);}$$

$$[k\mu]^\alpha = k[\mu]^\alpha \quad \text{(Scalar Multiplication)}.$$

Suppose that there exists $\omega \in \mathbb{R}_{\mathcal{F}}$ with $\mu = \nu + \omega$, then the Hukuhara (H-) difference of μ and ν is defined by $\omega = \mu \ominus \nu$. If it exists, then ($\forall\alpha \in [0,1]$)

$$[\mu \ominus \nu]^\alpha = [\mu_{l\alpha} - \nu_{l\alpha}, \mu_{r\alpha} - \mu_{r\alpha}].$$

A partial ordering in $\mathbb{R}_{\mathcal{F}}$ is defined by

$$\mu \le \nu \quad \text{if } \mu_{l\alpha} \le \nu_{l\alpha} \quad \text{and} \quad \mu_{r\alpha} \le \nu_{r\alpha}, \quad \forall\alpha \in [0,1],$$

where $\mu, \nu \in \mathbb{R}_{\mathcal{F}}$.

Lemma 7.1 ([36]) *For $\mu, \nu, \omega, \delta \in \mathbb{R}_{\mathcal{F}}$, if $\mu \ominus \nu$ and $\omega \ominus \delta$ exist, then we have*

$$d_\infty(\mu \ominus \nu, \omega \ominus \delta) \le d_\infty(\mu, \omega) + d_\infty(\nu, \delta).$$

Lemma 7.2 ([24]) *For $\mu, \nu, \omega \in \mathbb{R}_{\mathcal{F}}$, if $\omega \le \nu$ and $\mu \ominus \nu$ and $\mu \ominus \omega$ exist, then we have $\mu \ominus \nu \le \mu \ominus \omega$.*

For $J \subseteq \mathbb{R}^2$, define the mapping $H_\lambda : C(J, \mathbb{R}_{\mathcal{F}}) \times C(J, \mathbb{R}_{\mathcal{F}}) \to [0,\infty)$ by

$$H_\lambda(\mu, \nu) = \sup_{(z,w)\in\Omega} \bigl\{ d_\infty\bigl(\mu(z,w), \nu(z,w)\bigr) e^{\lambda(z+w)} \bigr\},$$

for $(\mu, \nu) \in C(J, \mathbb{R}_{\mathcal{F}})$, where $\lambda > 0$. $(C(J, \mathbb{R}_{\mathcal{F}}), H_\lambda)$ is a complete metric spaces (see [35]) and hence a complete partial metric space. We define (for $\mu, \nu \in C(J, \mathbb{R}_{\mathcal{F}})$)

$$(\mu, \nu) \in \mathcal{R} \quad \Longleftrightarrow \quad \mu \le \nu \quad \Longleftrightarrow \quad \mu(z,w) \le \nu(z,w), \quad \forall(z,w) \in J.$$

$\mu(z,w) \le \nu(z,w)$ means $\mu_{l\alpha}(z,w) \le \nu_{l\alpha}(z,w)$ and $\mu_{r\alpha}(z,w) \le \nu_{r\alpha}(z,w)$, $\forall\alpha \in [0,1]$ and $(z,w) \in J$.

Lemma 7.3 ([24]) *Let $(\mathbb{R}_{\mathcal{F}}, \le)$ be the space of fuzzy numbers with partial ordering $\le$. Then every pair of elements in $C(J, \mathbb{R}_{\mathcal{F}})$ has an upper bound and a lower bound in $C(J, \mathbb{R}_{\mathcal{F}})$.*

The generalized Hukuhara (gH-) difference of $\mu, \nu \in \mathbb{R}_F$, denoted by $\mu \ominus_{gH} \nu$, is $\omega \in \mathbb{R}_{\mathcal{F}}$, is defined by

$$\mu \ominus_{gH} \nu \quad \Longleftrightarrow \quad \mu = \nu + \omega \quad \text{or} \quad \nu = \mu + (-1)\omega.$$

If $\mu \ominus \nu$ exists, then $\mu \ominus_{gH} \nu = \mu \ominus \nu$. The generalized Kukuhara (gH-p-) derivatives of a fuzzy-valued mapping $s : J \subseteq \mathbb{R}^2 \to \mathbb{R}_{\mathcal{F}}$ are defined in Definitions 2.9 and 3.4 of [37] and for more details, we refer the reader to [37].

Here, $C^2(J, \mathbb{R}_{\mathcal{F}})$ denotes the set of all mappings $\mu \in C(J, \mathbb{R}_{\mathcal{F}})$ having continuous gH-p-derivatives on J up to order 2 on J.

Definition 7.1 ([24]) For $\mu \in C^2(J,\mathbb{R}_\mathcal{F})$ and $(z,w) \in J$, let μ_w be the gH-p-derivative at (z_0,w_0) with respect to w and we do not have any switching points on J. Then μ_{zw} is in type-1 (resp. type-2) of gH-p-derivatives if the type of gH-p-derivatives of both μ and μ_w are the same (resp. different). Then for $\alpha \in [0,1]$

$$\begin{aligned}&\big[{}_1D_{zw}\mu(z_0,w_0)\big]^\alpha = \big[\delta_{zw}\mu_{l\alpha}(z_0,w_0), \delta_{zw}\mu_{r\alpha}(z_0,w_0)\big],\\ &\quad\big(\text{resp. } \big[{}_2D_{zw}\mu(z_0,w_0)\big]^\alpha = \big[\delta_{zw}\mu_{r\alpha}(z_0,w_0), \delta_{zw}\mu_{l\alpha}(z_0,w_0)\big]\big).\end{aligned}$$

Lemma 7.4 ([24]) *Consider $\mu,\nu \in C(J,\mathbb{R}_\mathcal{F})$ such that $\mu \le \nu$, where J is a compact subset of $\mathbb{R}^2$. Then*

$$\int_J \mu(t,\tau)\,dt\,d\tau \le \int_J \nu(t,\tau)\,dt\,d\tau.$$

Now, we come to the main problem to be discussed here. We consider the following problem:

$${}_kD_{zw}\mu(z,w) = s\big(z,w,\nu(z,w)\big), \quad k = 1,2 \quad \text{and} \quad (z,w) \in \Omega = \Omega_1 \times \Omega_2, \tag{7.2}$$

where $\Omega_1 = [0,a]$ and $\Omega_2 = [0,b]$ with the condition

$$\mu(z,0) = \xi_1(z); \quad z \in \Omega_1 \quad \text{and} \quad \mu(0,w) = \xi_2(w); \quad w \in \Omega_2, \tag{7.3}$$

where $\xi_1 \in C(\Omega_1,\mathbb{R}_\mathcal{F})$, $\xi_2 \in C(\Omega_2,\mathbb{R}_\mathcal{F})$ such that $\xi_1(0) = \xi_2(0)$ and $\mu : \Omega \to \mathbb{R}_\mathcal{F}$, is a fuzzy-valued mapping and ${}_kD_{zw}$ represents gH-partial derivative operators. In the following, $I_{zw}s(z,w,\mu)$ denotes the integral $\int_0^w\int_0^z s(t,\tau,\mu(t,\tau))\,dt\,d\tau$, for $(z,w) \in \Omega$.

Definition 7.2 ([24]) A function $\mu \in C(\Omega,\mathbb{R}_\mathcal{F})$ is said to be an integral solution of type-1 (resp. type-2) of the Problem corresponding to (7.2)–(7.3), if $(\forall (z,w) \in \Omega)$

$$\mu(z,w) = p(z,w) + I_{zw}s(z,w,\mu) \quad \big(\text{resp. } \mu(z,w) = p(z,w) \ominus (-1)I_{zw}s(z,w,\mu)\big),$$

where $p(z,w) = \xi_1(z) + \xi_2(w) \ominus \xi_1(0)$.

Definition 7.3 ([24]) A function $\mu \in C^2(\Omega,\mathbb{R}_\mathcal{F})$ is said to be (k)-lower (resp. (k)-upper) solution of Problem (7.2)–(7.3) if for $(z,w) \in \Omega = \Omega_1 \times \Omega_2$, we have ${}_kD_{zw}\mu(z,w) \le s(z,w,\mu(z,w)$; $\mu(z,0) \le \xi_1(z), \mu(0,w) \le \xi_2(w)$ and $\mu(0,0) = \xi_1(0)$ (resp. ${}_kD_{zw}\mu(z,w) \le s(z,w,\mu(z,w)$; $\mu(z,0) \le \xi_1(z), \mu(0,w) \le \xi_2(w)$ and $\mu(0,0) = \xi_1(0)$).

Now, we state and prove the result below to establish the sufficient conditions for the existence and uniqueness of the solution of Problem (7.2)–(7.3) (with $k = 1$).

Theorem 7.2 *Let $s : \Omega \times \mathbb{R}_\mathcal{F} \to \mathbb{R}_\mathcal{F}$ be a continuous mapping. Assume that the following assumptions are satisfied*:

(F_1) *s is nondecreasing in the third variable, i.e., for $\mu,\nu \in \mathbb{R}_\mathcal{F}$ such that*

$$\mu \le \nu \quad \Longrightarrow \quad s(z,w,\mu) \le s(z,w,\nu), \quad \forall (z,w) \in \Omega.$$

(F_2) *There exists* $\phi \in \Phi$ *such that* $\forall (z,w) \in \Omega$ *and* $\mu \preceq \nu$ $(\mu, \nu \in \mathbb{R}_{\mathcal{F}})$, *we have*

$$d_\infty\bigl(s(z,w,\mu), s(z,w,\nu)\bigr) \le d_\infty(\mu,\nu) - \phi\bigl(d_\infty(\mu,\nu)\bigr).$$

(F_3) *Problem* (7.2)–(7.3) *has a* (1)*-lower solution.*
Then Problem (7.2)–(7.3) *has a unique integral solution of type*-1 *on* Ω.

Proof Define the mapping $S_1 : C(\Omega, \mathbb{R}_{\mathcal{F}}) \to C(\Omega, \mathbb{R}_{\mathcal{F}})$ as follows:

$$(S_1\mu)(z,w) = p(z,w) + I_{zw}s(z,w,\mu), \quad (z,w) \in \Omega.$$

Also, define a relation $\mathcal{R}$ on $C(\Omega, \mathbb{R}_{\mathcal{F}})$ by $(\mu,\nu) \in \mathcal{R} \iff \mu \preceq \nu$. Notice that Problem (7.2)–(7.3) has a unique solution if S_1 has a unique fixed point. Now, we establish the hypotheses of Theorem 4.1. Firstly, we claim that $\mathcal{R}$ is S_1-closed. To substantiate our claim, assume that, for arbitrary $\mu, \nu \in C(\Omega, \mathbb{R}_{\mathcal{F}})$, $(\mu,\nu) \in \mathcal{R}$, i.e., $\mu \preceq \nu$. By condition (F_1), we have

$$s(t,\tau,\mu) \preceq s(t,\tau,\nu), \quad \forall (t,\tau) \in \Omega.$$

Then, by Lemma 7.4, we have

$$I_{zw}s(z,w,\mu) \preceq I_{zw}s(z,w,\nu), \quad \forall (z,w) \in \Omega,$$

i.e.,

$$(S_1\mu)(z,w) \preceq (S_1\nu)(z,w), \quad \forall (z,w) \in \Omega.$$

Hence, $S_1\mu \preceq S_1\nu$, i.e., $(S_1\mu, S_1\nu) \in \mathcal{R}$ and the claim is established. It can easily be seen that $\mathcal{R}$ is transitive and hence, locally S_1-transitive.

Secondly, we prove that S_1 satisfies generalized weak $(\phi, \mathcal{R})$-contraction. For all $\mu, \nu \in C(\Omega, \mathbb{R}_{\mathcal{F}})$ such that $\mu \prec \nu$, we have

$$\begin{aligned} d_\infty\bigl(S_1\mu(z,w), S_1\nu(z,w)\bigr) &= d_\infty\bigl(p(z,w) + I_{zw}s(z,w,\mu), p(z,w) + I_{zw}s(z,w,\nu)\bigr) \\ &= d_\infty\bigl(I_{zw}s(z,w,\mu), I_{zw}s(z,w,\nu)\bigr) \\ &\le \int_0^w \int_0^z d_\infty\bigl(s\bigl(t,\tau,\mu(t,\tau)\bigr), s\bigl(t,\tau,\nu(t,\tau)\bigr)\bigr)\,dt\,d\tau \\ &\le \int_0^w \int_0^z d_\infty\bigl(\mu(t,\tau), \nu(t,\tau)\bigr)\,dt\,d\tau \\ &\le \int_0^w \int_0^z H_\lambda(\mu,\nu)e^{\lambda(t+\tau)}\,dt\,d\tau \\ &= \frac{1}{\lambda^2}H_\lambda(\mu,\nu)\bigl(e^{\lambda z} - 1\bigr)\bigl(e^{\lambda w} - 1\bigr). \end{aligned}$$

Then $\forall (z,w) \in \Omega$, we have

$$d_\infty\bigl(S_1\mu(z,w), S_1\nu(z,w)\bigr)e^{-\lambda(z+w)} \le \frac{1}{\lambda^2}H_\lambda(\mu,\nu)\bigl(1 - e^{-\lambda z}\bigr)\bigl(1 - e^{-\lambda w}\bigr).$$

Therefore, we have

$$H_\lambda(S_1\mu, S_1\nu) \leq \frac{1}{\lambda^2} H_\lambda(\mu,\nu)\big(1-e^{-\lambda a}\big)\big(1-e^{-\lambda b}\big)$$

or

$$H_\lambda(S_1\mu, S_1\nu) \leq H_\lambda(\mu,\nu) - \left[H_\lambda(\mu,\nu) - \frac{1}{\lambda^2} H_\lambda(\mu,\nu)\big(1-e^{-\lambda a}\big)\big(1-e^{-\lambda b}\big)\right]. \tag{7.4}$$

Taking $\phi(t) = t - \frac{1}{\lambda^2}(1-e^{-\lambda a})(1-e^{-\lambda b})t$. Then on choosing $\frac{1}{\lambda^2}(1-e^{-\lambda a})(1-e^{-\lambda b}) < 1$, with a routine calculation, one can show that $\phi \in \Phi$. So, (7.4) reduces to

$$\begin{aligned} H_\lambda(S_1\mu, S_1\nu) &\leq H_\lambda(\mu,\nu) - \phi\big(H_\lambda(\mu,\nu)\big) \\ &\leq H_\lambda(\mu,\nu) - \phi\big(H_\lambda(S_1\mu, S_1\nu)\big). \end{aligned}$$

Hence, S_1 is a generalized weak $(\phi,\mathcal{R})$-contraction. Now, we show that condition (a) of Theorem 4.1 is satisfied. By assumption (F_3), there exists a (1)-lower solution, say $\bar{\mu} \in C^2(\Omega,\mathbb{R}_{\mathcal{F}})$, such that (for $\alpha \in [0,1]$ and $(z,w) \in \Omega$)

$$\begin{aligned} \bar{\mu}_{l\alpha}(z,w) &\leq \bar{\mu}_{l\alpha}(z,0) + \bar{\mu}_{l\alpha}(0,w) - \bar{\mu}_{l\alpha}(0,0) + \int_0^w \int_0^z s_{l\alpha}\big(t,\tau,\bar{\mu}(t,\tau)\big)\,dt\,d\tau \\ &\leq (\xi_1)_{l\alpha}(z) + (\xi_2)_{l\alpha}(w) - (\xi_1)_{l\alpha}(0) + \int_0^w \int_0^z s_{l\alpha}\big(t,\tau,\bar{\mu}(t,\tau)\big)\,dt\,d\tau \end{aligned}$$

and

$$\begin{aligned} \bar{\mu}_{r\alpha}(z,w) &\leq \bar{\mu}_{r\alpha}(z,0) + \bar{\mu}_{r\alpha}(0,w) - \bar{\mu}_{r\alpha}(0,0) + \int_0^w \int_0^z s_{r\alpha}\big(t,\tau,\bar{\mu}(t,\tau)\big)\,dt\,d\tau \\ &\leq (\xi_1)_{r\alpha}(z) + (\xi_2)_{r\alpha}(w) - (\xi_1)_{r\alpha}(0) + \int_0^w \int_0^z s_{r\alpha}\big(t,\tau,\bar{\mu}(t,\tau)\big)\,dt\,d\tau. \end{aligned}$$

Therefore, we have $(\forall (z,w) \in \Omega)$

$$\bar{\mu}(z,w) \leq \xi_1(z) + \xi_2(w) \ominus \xi_1(0) + I_{zw}s(z,w,\bar{\mu}) = (S_1\bar{\mu})(z,w).$$

Thus, we get $(\bar{\mu}, S_1\bar{\mu}) \in \mathcal{R}$ and with this we observe that all the hypotheses of Theorem 4.1 are satisfied and thus, S_1 has a fixed point. Moreover, the condition of Corollary 4.1 is satisfied in view of Lemma 7.3 and hence we conclude that the fixed point of S_1 is unique. This completes the proof. □

In the next result, we investigate the existence of integral solution of type-2. Before presenting our desired result, we first give the following notions:

$$C^*(\Omega,\mathbb{R}_{\mathcal{F}}) = \big\{\mu \in C(\Omega,\mathbb{R}_{\mathcal{F}}) : p(z,w) \ominus (-1)I_{zw}s(z,w,\mu) \text{ exists}, \ \forall (z,w) \in \Omega\big\}. \tag{7.5}$$

Lemma 7.5 ([24]) *Let $C^*(\Omega,\mathbb{R}_{\mathcal{F}}) \neq \emptyset$ and s in (7.5) is continuous. If $(C(\Omega,\mathbb{R}_{\mathcal{F}}),d)$ is a complete metric space, then $C^*(\Omega,\mathbb{R}_{\mathcal{F}})$ is also a complete metric space.*

Now, we are ready to embark on our result.

Theorem 7.3 *Let $s : \Omega \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ be a continuous mapping. In addition to hypotheses (F_1) and (F_2) of Theorem 7.2, assume that the following conditions are satisfied*:

- (F_4) *$C^*(\Omega, \mathbb{R}_{\mathcal{F}}) \neq \emptyset$ and if $\mu \in C^*(\Omega, \mathbb{R}_{\mathcal{F}})$, then $\mu(z,w) = p(z,w) \ominus (-1)I_{zw}s(z,w,\mu)$, $\forall (z,w) \in \Omega$*;
- (F_5) *Problem (7.2)–(7.3) has a (2)-lower solution*;
- (F_6) *for each fixed pair $\mu, \nu \in C^*(\Omega, \mathbb{R}_{\mathcal{F}})$, there exists an upper or lower bound of μ, ν, say $\omega \in C^*(\Omega, \mathbb{R}_{\mathcal{F}})$, such that the H-difference $p(z,w) \ominus (-1)I_{zw}s(z,w,\omega)$ exists, $\forall (z,w) \in \Omega$.*

Then Problem (7.2)–(7.3) has a unique integral solution of type-2.

Proof By assumption (F_4), we have $C^*(\Omega, \mathbb{R}_{\mathcal{F}}) \neq \emptyset$ and hence, we define $S_2 : C^*(\Omega, \mathbb{R}_{\mathcal{F}}) \to C^*(\Omega, \mathbb{R}_{\mathcal{F}})$ by

$$(S_2\mu) = p(z,w) \ominus (-1)I_{zw}s(z,w,\mu), \quad \forall (z,w) \in \Omega.$$

Notice that Problem (7.2)–(7.3) has a unique solution if S_2 has a unique fixed point.

First, we prove that $\mathcal{R}$ is S_2-closed. For this purpose, assume that $(\mu,\nu) \in \mathcal{R}$, i.e., $\mu(t,\tau) \leq \nu(t,\tau)$, $\forall (t,\tau) \in \Omega$. Then following the steps of Theorem 7.2, we have $(\forall (z,w) \in \Omega)$

$$I_{zw}s(z,w,\mu) \leq I_{zw}s(z,w,\nu)$$

or

$$(-1)I_{zw}s(z,w,\mu) \geq (-1)I_{zw}s(z,w,\nu).$$

Applying Lemma 7.2, we get

$$\begin{aligned}(S_2\mu)(z,w) &= p(z,w) \ominus (-1)I_{zw}s(z,w,\mu) \\ &\leq p(z,w) \ominus (-1)I_{zw}s(z,w,\nu) = (S_2\nu)(z,w),\end{aligned}$$

$\forall (z,w) \in \Omega$. Hence, $(S_2\mu, S_2\nu) \in \mathcal{R}$. Next, we prove that S_2 satisfies a generalized weak $(\phi,\mathcal{R})$-contraction. Following the steps of Theorem 7.2 and utilizing Lemma 7.1, $\forall \mu,\nu \in C^*(\Omega, \mathbb{R}_{\mathcal{F}})$ such that $(\mu,\nu) \in \mathcal{R}$, i.e., $\mu \leq \nu$, we have

$$\begin{aligned}d_\infty\big((S_2\mu)(z,w),(S_2\nu)(z,w)\big) &= d_\infty\big(p(z,w) \ominus (-1)I_{zw}s(z,w,\mu), p(z,w) \\ &\quad \ominus (-1)I_{zw}s(z,w,\nu)\big) \\ &\leq d_\infty\big(I_{zw}s(z,w,\mu), I_{zw}s(z,w,\nu)\big) \\ &\leq \frac{1}{\lambda^2} H_\lambda(\mu,\nu)\big(e^{\lambda z}-1\big)\big(\big(e^{\lambda w}-1\big)\big).\end{aligned}$$

Again, following the same steps as in Theorem 7.2, we find that S_2 satisfies a generalized weak $(\phi,\mathcal{R})$-contraction. Finally, to satisfy condition (a) of Theorem 4.1, we use (F_5) and find that there exists a (2)-lower solution, say $\bar{\mu} \in C^2(\Omega, \mathbb{R}_{\mathcal{F}}) \cap C^*(\Omega, \mathbb{R}_{\mathcal{F}})$. As

$\bar{\mu} \in C^*(\Omega, \mathbb{R}_{\mathcal{F}})$, so $(S_2\bar{\mu})(z,w)$ exists for all $(z,w) \in \Omega$. Due to the existence of a (2)-lower solution $\bar{\mu}$, we get (for $\alpha \in [0,1]$ and $(z,w) \in \Omega$)

$$\begin{aligned}\bar{\mu}_{l\alpha}(z,w) &\leq \bar{\mu}_{l\alpha}(z,0) + \bar{\mu}_{l\alpha}(0,w) - \bar{\mu}_{l\alpha}(0,0) + \int_0^w \int_0^z s_{l\alpha}\big(t,\tau,\bar{\mu}(t,\tau)\big)\,dt\,d\tau \\ &\leq (\xi_1)_{l\alpha}(z) + (\xi_2)_{l\alpha}(w) - (\xi_1)_{l\alpha}(0) + \int_0^w \int_0^z s_{l\alpha}\big(t,\tau,\bar{\mu}(t,\tau)\big)\,dt\,d\tau\end{aligned}$$

and

$$\begin{aligned}\bar{\mu}_{r\alpha}(z,w) &\leq \bar{\mu}_{r\alpha}(z,0) + \bar{\mu}_{r\alpha}(0,w) - \bar{\mu}_{r\alpha}(0,0) + \int_0^w \int_0^z s_{r\alpha}\big(t,\tau,\bar{\mu}(t,\tau)\big)\,dt\,d\tau \\ &\leq (\xi_1)_{r\alpha}(z) + (\xi_2)_{r\alpha}(w) - (\xi_1)_{r\alpha}(0) + \int_0^w \int_0^z s_{r\alpha}\big(t,\tau,\bar{\mu}(t,\tau)\big)\,dt\,d\tau.\end{aligned}$$

Thus, $\forall (z,w) \in \Omega$, we get

$$\begin{aligned}\bar{\mu}(z,w) &\leq \xi_1(z) + \xi_2(w) \ominus \xi_1(0) \ominus (-1) \int_0^w \int_0^z s\big(t,\tau,\bar{\mu}(t,\tau)\big)\,dt\,d\tau \\ &= p(z,w) \ominus (-1) I_{zw} s(z,w,\bar{\mu}) \\ &= (S_2\bar{\mu})(z,w).\end{aligned}$$

Thus, taking all this in account, we have $\bar{\mu} \leq S_2\bar{\mu}$, i.e., $(\bar{\mu}, S_2\bar{\mu}) \in \mathcal{R}$ so that the claim is established.

Lemma 7.5 ensures that $(C^*(\Omega, \mathbb{R}_{\mathcal{F}}), H_{\lambda)}$ is a complete metric space. By a routine calculation, one can easily show that all other conditions of Theorem 4.1 are satisfied and we conclude from the theorem that S_2 has a fixed point in $C^*(\Omega, \mathbb{R}_{\mathcal{F}})$. Moreover, due to assumption (F_6), the hypothesis of Corollary 4.1 is satisfied. Hence, the fixed point of S_2 is unique. This completes the proof. □

Remark 7.1 The conclusions of Theorems 7.2 and 7.3 are still valid if we consider that a (k)-upper solution $(k = 1, 2)$ of Problem (7.2)–(7.3) exists.

Acknowledgements
All the authors are thankful to the referees for their fruitful comments and suggestions in the final preparation of this article.

Availability of data and materials
No data were used to support this study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 8 November 2018 Accepted: 15 February 2019 Published online: 04 March 2019

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