Liu et al. *Advances in Difference Equations* (2019) 2019:100
https://doi.org/10.1186/s13662-019-2037-0

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Variational method to a fractional impulsive (p,q)-Laplacian coupled systems with partial sub-(p,q) linear growth

Cuiling Liu[1], Xingyong Zhang[1,3*] and Junping Xie[2]

*Correspondence: zhangxingyong1@163.com
[1]Faculty of Science, Kunming University of Science and Technology, Kunming, P.R. China
[3]School of Mathematics and Statistics, Central South University, Changsha, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, by using the least action principle, an existence result of nontrivial weak solutions for a class of fractional impulsive coupled systems with (p,q)-Laplacian is obtained if the nonlinear term has sub-(p,q) linear growth, and by using an extension of Clark's theorem, infinitely many solutions of the system are obtained if the nonlinear term has partial sub-(p,q) linear growth.

Keywords: Fractional coupled systems; (p,q)-Laplacian; Multiplicity; Clark's theorem; Impulsive effects; The least action principle

1 Introduction and main results

In recent years, because of the important applications of fractional differential equations to engineering, physics, chemistry and biology, the existence and multiplicity of solutions for fractional differential equations have been investigated extensively by different methods such as fixed point theory, degree theory, monotone iterative technique and upper and lower solutions method (for example, see [1–4] and the references therein). It is well known that the variational method is an effective tool to deal with existence and multiplicity of solutions for integer-order ordinary differential equations which have variational structures. For the fractional ordinary differential equation, a pioneering work by a variational method was presented by Jiao and Zhou in [5], where they studied the following fractional differential equations with the left and right Riemann–Liouville fractional integrals:

$$\begin{cases} \frac{d}{dt}\left(\frac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(t))\right) + \nabla F(t,u(t)) = 0, & \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{1}$$

where $T > 0$, ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ denote the left and right Riemann–Liouville fractional integrals of order $0 \le \beta < 1$, respectively, $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ and $\nabla F(t,x)$ is the gradient of F at x. They established the variational structure of system (1), some embedding relations of working spaces, and some existence results of solutions for system (1) under subquadratic and superquadratic conditions, respectively. Subsequently, some authors applied a variational method to different kinds of fractional differential equations and some interesting results were given (for example, see [6–15] and the references therein). Especially, in [9], Zhang and Li considered the following fractional differential equation:

$$\begin{cases} {}_tD_T^{\alpha}({}_0^cD_t^{\alpha}u(t)) = \nabla W(t,u(t)), \quad t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{2}$$

where ${}_0^cD_t^{\alpha}$ is the left Caputo fractional derivative, and they obtained the following theorem.

Theorem A ([9], Theorem 1.1) *Suppose that the following conditions hold*:

(A1) $W(t,0) = 0$ *for all* $t \in [0,T]$, $W(t,u) \geq a(t)|u|^{\theta}$ *and* $|\nabla W(t,u)| \leq b(t)|u|^{\theta-1}$ *for all* $(t,u) \in [0,T] \times \mathbb{R}^N$, *where* $1 < \theta < 2$ *is a constant*, $a : [0,T] \to \mathbb{R}^+$ *is a continuous function and* $b : [0,T] \to \mathbb{R}^+$ *is a continuous function*;

(A2) *there is a constant* $1 < \sigma \leq \theta < 2$ *such that*

$$\bigl(\nabla W(t,u),u\bigr) \leq \sigma W(t,u) \quad \textit{for all } t \in [0,T] \textit{ and } u \in \mathbb{R}^N;$$

(A3) $W(t,u) = W(t,-u)$ *for all* $t \in [0,T]$ *and* $u \in \mathbb{R}^N$.

Then (2) *has infinitely many nontrivial solutions*.

Impulse phenomena exist extensively in the real world and impulsive differential equations are often used to describe these phenomena. In the last decade, by using the variational methods, the problems on existence and multiplicity of solutions for integer-order impulsive differential equations with different boundary value conditions have been studied deeply. We refer to the papers in [16–21] and the references therein. In comparison to the integer-order impulsive differential equations, there are less results for the fractional impulsive differential equations by variational methods. In 2014, Rodrìguez-López et al. [22] and Bonanno et al. [23] considered the fractional impulsive differential equation with the right Riemann–Liouville fractional derivative and left Caputo fractional derivative:

$$\begin{cases} {}_tD_T^{\alpha}({}_0^cD_t^{\alpha}u(t)) + a(t)u(t) = \lambda f(t,u(t)), \quad t \neq t_i, \text{a.e. } t \in [0,T], \\ \Delta({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u))(t_i) = \mu Q_i(u(t_i)), \quad i = 1,2,\ldots,l, \\ u(0) = u(T) = 0, \end{cases} \tag{3}$$

where $\alpha \in (\frac{1}{2},1]$, both λ and μ are positive parameters, $f \in C([0,T] \times \mathbb{R},\mathbb{R})$, $Q_i \in C(\mathbb{R},\mathbb{R})$ and $a \in C([0,T],\mathbb{R})$. By using variational methods, they found that Eq. (3) has at least one or three solutions. Subsequently, in [24–27], several results are given along this direction by variational methods. For example, recently, Heidarkhani etc. [12] considered the following perturbed impulsive fractional differential system:

$$\begin{cases} {}_tD_T^{\alpha_i}(a_i(t){}_0D_t^{\alpha_i}u_i(t)) = \lambda F_{u_i}(t,u) + \mu G_{u_i}(t,u) + h_i(u_i), \quad t \neq t_j, t \in (0,T), \\ \Delta({}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u_i))(t_j) = I_{ij}(u_i(t_j)), \quad j = 1,2,\ldots,m, \\ u_i(0) = u_i(T) = 0, \end{cases} \tag{4}$$

for $1 \leq i \leq N$, where $u = (u_1, \ldots, u_N)$, $0 < \alpha_i \leq 1$, $\lambda > 0$, $\mu \geq 0$, $T > 0$, $a_i \in L^\infty([0,T])$, $F, G : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ are measurable with respect to t for all $u \in \mathbb{R}^N$ and continuously differentiable in u for almost every $t \in [0,T]$, and $h_i : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function for $1 \leq i \leq N$. By using a three critical point theorem due Bonanno and Candito [28], they found that system (4) has at least three distinct weak solutions.

As a natural extension of Eq. (3), Zhao and Tang [29] considered the p-Laplacian fractional impulsive differential equations:

$$\begin{cases} {}_tD_T^\alpha(\Phi_p({}_0^cD_t^\alpha u(t))) + |u(t)|^{p-2}u(t) = f(t,u(t)), \quad t \in (0,T), \\ \Delta({}_tD_T^{\alpha-1}(\Phi_p({}_0^cD_t^\alpha u)))(t_i) = Q_i(u(t_i)), \quad i = 1,2,\ldots,l, \\ u(0) = u(T) = 0, \end{cases} \tag{5}$$

where $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$, $Q_i \in C(\mathbb{R},\mathbb{R})$ and $\Phi_p(x) = |x|^{p-2}x$ ($p > 1$) for all $x \in \mathbb{R}^N$. By using critical point theory, they obtained two multiplicity results of solutions for Eq. (5) when f satisfies the superquadratic conditions.

Motivated by the work in [5, 6, 11] and [29], Xie-Zhang [30] investigated the existence of infinitely many solutions for the following (p,q)-Laplacian fractional impulsive differential system:

$$\begin{cases} {}_tD_T^\alpha(\rho(t)\Phi_p({}_0^cD_t^\alpha u(t))) = \nabla_u W(t,u(t),v(t)), \quad \text{a.e. } t \in [0,T], \\ {}_tD_T^\beta(\gamma(t)\Phi_q({}_0^cD_t^\beta v(t))) = \nabla_v W(t,u(t),v(t)), \quad \text{a.e. } t \in [0,T], \\ \Delta({}_tD_T^{\alpha-1}(\rho\Phi_p({}_0^cD_t^\alpha u)))(t_i) = \nabla I_i(u(t_i)), \quad i = 1,2,\ldots,l, \\ \Delta({}_tD_T^{\beta-1}(\gamma\Phi_q({}_0^cD_t^\beta v)))(s_j) = \nabla H_j(v(s_j)), \quad j = 1,2,\ldots,m, \\ u(0) = u(T) = 0, \qquad v(0) = v(T) = 0, \end{cases} \tag{6}$$

where $T > 0$, $\alpha \in (\frac{1}{p}, 1]$ with $p > 1$, $\beta \in (\frac{1}{q}, 1]$ with $q > 1$, $\Phi_s(x) = |x|^{s-2}x$ ($s > 1$ and $s = p, q$), ${}_tD_T^\alpha$ (or ${}_tD_T^\beta$) denotes the right Riemann–Liouville fractional derivative of order α (or β), ${}_0^cD_t^\alpha$ (or ${}_0^cD_t^\beta$) is the left Caputo fractional derivative of order α (or β), $\rho, \gamma \in L^\infty([0,T], \mathbb{R}^+)$, $I_i, H_j : \mathbb{R}^N (N \geq 1) \to \mathbb{R}$ are continuously differentiable, $i = 1,2,\ldots,l$, $j = 1,2,\ldots,m$, $0 = t_0 < t_1 < \cdots < t_{l+1} = T$, $0 = s_0 < s_1 < \cdots < s_{m+1} = T$, and

$$\begin{aligned} \Delta\bigl({}_tD_T^{\alpha-1}\bigl(\rho\Phi_p\bigl({}_0^cD_t^\alpha u\bigr)\bigr)\bigr)(t_i) &= {}_tD_T^{\alpha-1}\bigl(\rho\Phi_p\bigl({}_0^cD_t^\alpha u\bigr)\bigr)\bigl(t_i^+\bigr) - {}_tD_T^{\alpha-1}\bigl(\rho\Phi_p\bigl({}_0^cD_t^\alpha u\bigr)\bigr)\bigl(t_i^-\bigr), \\ \Delta\bigl({}_tD_T^{\beta-1}\bigl(\gamma\Phi_q\bigl({}_0^cD_t^\beta v\bigr)\bigr)\bigr)(s_j) &= {}_tD_T^{\beta-1}\bigl(\gamma\Phi_q\bigl({}_0^cD_t^\beta v\bigr)\bigr)\bigl(s_j^+\bigr) - {}_tD_T^{\beta-1}\bigl(\gamma\Phi_q\bigl({}_0^cD_t^\beta v\bigr)\bigr)\bigl(s_j^-\bigr), \end{aligned}$$

where

$$\begin{aligned} {}_tD_T^{\alpha-1}\bigl(\rho\Phi_p\bigl({}_0^cD_t^\alpha u\bigr)\bigr)\bigl(t_i^+\bigr) &= \lim_{t \to t_i^+} {}_tD_T^{\alpha-1}\bigl(\rho\Phi_p\bigl({}_0^cD_t^\alpha u\bigr)\bigr)(t), \\ {}_tD_T^{\alpha-1}\bigl(\rho\Phi_p\bigl({}_0^cD_t^\alpha u\bigr)\bigr)\bigl(t_i^-\bigr) &= \lim_{t \to t_i^-} {}_tD_T^{\alpha-1}\bigl(\rho\Phi_p\bigl({}_0^cD_t^\alpha u\bigr)\bigr)(t), \\ {}_tD_T^{\beta-1}\bigl(\gamma\Phi_q\bigl({}_0^cD_t^\beta v\bigr)\bigr)\bigl(s_j^+\bigr) &= \lim_{s \to s_j^+} {}_tD_T^{\beta-1}\bigl(\gamma\Phi_q\bigl({}_0^cD_t^\beta v\bigr)\bigr)(s), \\ {}_tD_T^{\alpha-1}\bigl(\gamma\Phi_q\bigl({}_0^cD_t^\beta v\bigr)\bigr)\bigl(s_j^-\bigr) &= \lim_{s \to s_j^-} {}_tD_T^{\beta-1}\bigl(\gamma\Phi_q\bigl({}_0^cD_t^\beta v\bigr)\bigr)(s), \end{aligned}$$

$W : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

(W0) *$W(t,x,y)$ is continuously differentiable in $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ for a.e. $t \in [0,T]$, measurable in t for every $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, and there are $a_1, a_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in L^\infty([0,T];\mathbb{R}^+)$ such that*

$$\begin{aligned} &\bigl|W(t,x,y)\bigr| + \bigl|\nabla W(t,x,y)\bigr| \le \bigl[a_1\bigl(|x|\bigr) + a_2\bigl(|y|\bigr)\bigr]b(t), \\ &\bigl|I_i(x)\bigr| + \bigl|\nabla I_i(x)\bigr| \le a_1\bigl(|x|\bigr), \quad i = 1,2,\dots,l, \\ &\bigl|H_j(y)\bigr| + \bigl|\nabla H_j(y)\bigr| \le a_2\bigl(|y|\bigr), \quad j = 1,2,\dots,m \end{aligned}$$

for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $t \in [0,T]$. They assumed $W(t,x,y) = -K(t,x,y) + F(t,x,y)$ for a.e. $t \in [0,T]$ and all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, where K has sub-(p,q) linear growth and F has super-(p,q) liear growth. By using the symmetric mountain pass theorem, they obtained system (6) has infinitely many weak solutions.

In this paper, we investigate the existence and multiplicity of solutions for system (6). Different from [30], we use the least action principle and an extension of Clark's theorem to prove our results. We assume W has sub-(p,q) linear growth on $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ so that system (6) has at least one weak solution, and we assume W has partially sub-(p,q) linear growth on $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ and is even for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ so that system (6) has infinitely many weak solutions. To be precise, we obtain the results below.

Theorem 1.1 *Suppose that* (W0) *and the following conditions hold*:

($\mathcal{A}$) $\rho^- := \operatorname{essinf}_{[0,T]} \rho(t) > 0$, $\gamma^- := \operatorname{essinf}_{[0,T]} \gamma(t) > 0$;

(W1) *there exist constants $d_1 \in [0, \frac{\rho^-}{pC_\alpha^p})$, $d_2 \in [0, \frac{\gamma^-}{qC_\beta^q})$, $\gamma_1 \in (0,p]$, $\gamma_2 \in (0,q]$ and $g_1, g_2, g_3 \in L^1([0,T],\mathbb{R}^+)$ such that*

$$W(t,x,y) \le d_1|x|^p + d_2|y|^q + g_1(t)|x|^{p-\gamma_1} + g_2(t)|y|^{q-\gamma_2} + g_3(t)$$

for a.e. $t \in [0,T]$ and all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, where $C_\alpha = \frac{T^\alpha}{\Gamma(\alpha+1)}$ and $C_\beta = \frac{T^\beta}{\Gamma(\beta+1)}$;

(I1) *there exist positive constants k_1, k_2 and $\nu_1 \in [0,p)$ such that*

$$I_i(x) \ge -k_1|x|^{\nu_1} - k_2$$

for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}^N$, $i = 1,\dots,l$;

(H1) *there exist positive constants k_3, k_4 and $\nu_2 \in [0,q)$ such that*

$$H_j(y) \ge -k_3|y|^{\nu_2} - k_4$$

for a.e. $t \in [0,T]$ and all $y \in \mathbb{R}^N$, $j = 1,\dots,m$.

Then system (6) *has at least one weak solution.*

Remark 1.1 There exist examples of functions satisfying the assumptions in Theorem 1.1. For example, let $p > 1$, $q > 1$, $\rho(t) = \gamma(t) = e^t + 1$, $I_i(x) = -\ln(1 + |x|^p)$, $i = 1,\dots,l$, $H_j(y) = -\ln(1 + |y|^q)$, $j = 1,\dots,m$, and

$$W(t,x,y) = \bigl(e^t + 1\bigr)\bigl[\ln\bigl(1 + |x|^p\bigr) + \ln\bigl(1 + |y|^q\bigr)\bigr]$$

for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $t \in [0,T]$.

Theorem 1.2 *Assume that* (W0), ($\mathcal{A}$), (I1), (H1) *and the following conditions hold*:

(W1)′ *there exist constants* $\delta_1 > 0$, $\delta_2 > 0$, $\mu_1 \in [0,p)$, $\mu_2 \in [0,q)$ *and* $f_1, f_2 \in L^1([0,T];\mathbb{R}^+)$ *such that*

$$W(t,x,y) \le f_1(t)|x|^{\mu_1} + f_2(t)|y|^{\mu_2}$$

for a.e. $t \in [0,T]$ *and all* $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ *with* $|x| \le \delta_1$ *and* $|y| \le \delta_2$;

(W2) $W(t,0,0) = 0$ *and* $W(t,x,y) = W(t,-x,-y)$ *for a.e.* $t \in [0,T]$ *and all* $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$;

(W3)

$$\lim_{|x|+|y|\to 0} \frac{W(t,x,y)}{|x|^p + |y|^q} = +\infty \quad \textit{uniformly for a.e. } t \in [0,T];$$

(I1)′ *there exist positive constants* l_1, l_2, δ_3 *and* $\nu_3 \in [0,p)$ *such that*

$$I_i(x) \ge -l_1|x|^{\nu_3} - l_2$$

for a.e. $t \in [0,T]$ *and all* $x \in \mathbb{R}^N$ *with* $|x| \le \delta_3$, $i = 1,\dots,l$;

(H1)′ *there exist positive constants* l_3, l_4, δ_4 *and* $\nu_4 \in [0,q)$ *such that*

$$H_j(y) \ge -l_3|y|^{\nu_4} - l_4$$

for a.e. $t \in [0,T]$ *and all* $y \in \mathbb{R}^N$ *with* $|y| \le \delta_4$, $j = 1,\dots,m$;

(I2) *there exists a constant* $b_1 > 0$ *such that*

$$\lim_{|x|\to 0} \frac{I_i(x)}{|x|^p} < b_1, \quad i = 1,\dots,l;$$

(H2) *there exists a constant* $b_2 > 0$ *such that*

$$\lim_{|y|\to 0} \frac{H_j(y)}{|y|^q} < b_2, \quad j = 1,\dots,m;$$

(I3) $I_i(0) = 0$ *and* $I_i(x)$ *is even in* $x \in \mathbb{R}^N$, $i = 1,\dots,l$;

(H3) $H_j(0) = 0$ *and* $H_j(y)$ *is even in* $y \in \mathbb{R}^N$, $j = 1,\dots,m$.

Then system (6) *has infinitely many weak solutions*.

Remark 1.2 There exist examples of functions satisfying the assumptions in Theorem 1.2. For example, let $p = 4$, $q = 5$, $\rho(t) = \gamma(t) = e^t + 1$, $I_i(x) = -e^{|x|^4} + 1$, $i = 1,\dots,l$, $H_j(y) = -e^{|y|^5} + 1$, $j = 1,\dots,m$, and $W(t,x,y) = (t^2+1)(|x|^3 + |y|^3)$ for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $t \in [0,T]$.

It is easy to obtain similar theorems to Theorem 1.1 and Theorem 1.2 for the p-Laplacian fractional impulsive differential system:

$$\begin{cases} {}_tD_T^{\alpha}(\rho(t)\Phi_p({}_0^cD_t^{\alpha}u(t))) = \nabla_u W(t,u(t)), & \text{a.e. } t \in [0,T], \\ \Delta({}_tD_T^{\alpha-1}(\rho\Phi_p({}_0^cD_t^{\alpha}u)))(t_i) = \nabla I_i(u(t_i)), & i = 1,2,\dots,l, \\ u(0) = u(T) = 0. \end{cases} \tag{7}$$

Theorem 1.3 *Suppose that the following conditions hold*:

(W0)′ $W(t,x)$ *is continuously differential in* $x \in \mathbb{R}^N$ *for a.e.* $t \in [0,T]$, *measurable in* t *for each* $x \in \mathbb{R}^N$, *and there are functions* $a \in C(\mathbb{R}^+,\mathbb{R}^+)$ *and* $b \in L^1([0,T];\mathbb{R}^+)$ *such that*

$$\begin{aligned}&\bigl|W(t,x)\bigr| + \bigl|\nabla W(t,x)\bigr| \le a\bigl(|x|\bigr)b(t),\\ &\bigl|I_i(x)\bigr| + \bigl|\nabla I_i(x)\bigr| \le a\bigl(|x|\bigr), \quad i = 1,2,\ldots,l,\end{aligned}$$

for all $x \in \mathbb{R}^N$ *and a.e.* $t \in [0,T]$;

$(\mathcal{A})'$ $\rho^- := \operatorname{essinf}_{[0,T]} \rho(t) > 0$;

(W1)″ *there exist constants* $d \in [0, \frac{\rho^-}{pC_\alpha^p})$, $\gamma \in (0,p]$, *and* $g_1, g_2 \in L^1([0,T],\mathbb{R}^+)$ *such that*

$$W(t,x,y) \le d|x|^p + g_1(t)|x|^{p-\gamma} + g_2(t)$$

for a.e. $t \in [0,T]$ *and all* $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$;

(I1)″ *there exist positive constants* k_1, k_2 *and* $\nu_1 \in [0,p)$ *such that*

$$I_i(x) \ge -k_1|x|^{\nu_1} - k_2$$

for a.e. $t \in [0,T]$ *and all* $x \in \mathbb{R}^N$, $i = 1,\ldots,l$.

Then system (7) *has at least one weak solution*.

Theorem 1.4 *Assume that* (W0)′, $(\mathcal{A})'$, (I1)″ *and the following conditions hold*:

(W1)‴ *there exist constants* $\delta_1 > 0$, $\mu \in [0,p)$ *and* $f \in L^1([0,T];\mathbb{R}^+)$ *such that*

$$W(t,x) \le f(t)|x|^\mu$$

for a.e. $t \in [0,T]$ *and all* $x \in \mathbb{R}^N$ *with* $|x| \le \delta_1$;

(W2)′ $W(t,0) = 0$ *and* $W(t,x) = W(t,-x)$ *for a.e.* $t \in [0,T]$ *and all* $x \in \mathbb{R}^N$;

(W3)′

$$\lim_{|x|\to 0} \frac{W(t,x)}{|x|^p} = +\infty;$$

(I1)‴ *there exist positive constants* l_1, l_2, δ_2 *and* $\nu_2 \in [0,p)$ *such that*

$$I_i(x) \ge -l_1|x|^{\nu_2} - l_2$$

for a.e. $t \in [0,T]$ *and all* $x \in \mathbb{R}^N$ *with* $|x| \le \delta_2$, $i = 1,\ldots,l$;

(I2)′ *there exists a constant* $b > 0$ *such that*

$$\lim_{|x|\to 0} \frac{I_i(x)}{|x|^p} < b, \quad i = 1,\ldots,l;$$

(I3)′ $I_i(0) = 0$, $I_i(x)$ *is even in* $x \in \mathbb{R}^N$, $i = 1,\ldots,l$.

Then system (7) *has infinitely many weak solutions*.

Remark 1.3 If $p = 2$, $\rho(t) \equiv 1$ and $I_i(x) \equiv 0$ for a.e. $t \in [0, T]$ and all $x \in \mathbb{R}^N$, $i = 1, \dots, l$, then Theorem 1.3 reduces to Theorem 5.46 in [31]. Hence, Theorem 1.3 generalizes Theorem 5.46 in [31]. In Theorem A (Theorem 1.1 in [9]), W is required to satisfy the subquadratic condition for all $x \in \mathbb{R}^N$ while (W1)$'''$ is a partial sub-p linear growth condition, which is only for all $x \in \mathbb{R}^N$ with $|x| \le \delta$. Hence, Theorem 1.4 is still different from Theorem A even if $p = 2$ and $I_i(x) \equiv 0$ for all $x \in \mathbb{R}^N$, $i = 1, \dots, l$. Moreover, in Theorem 1.4, a condition like (A2) in Theorem A ((W2) in Theorem 1.1 in [9]) is not required. There exist examples of functions satisfying the assumptions in Theorem 1.4 with $p = 2$ but not satisfying those in Theorem A. For example, let $N = 1$, $p = 2$ and

$$W(t,x) = \begin{cases} (t^2+1)|x|^{\frac{3}{2}}, & \text{if } |x| \le 1, \\ (t^2+1)|x|^3, & \text{if } |x| > 1. \end{cases}$$

2 Preliminaries

Let E be a real Banach space and $\Phi \in C^1(E, \mathbb{R})$. If any sequence $\{u_k\}$ possesses a convergent subsequence, where $\{u_k\}$ satisfies $\Phi(u_k)$ is bounded and $\Phi'(u_k) \to 0$ as $k \to \infty$, then one say that Φ satisfies the Palais–Smale (PS) condition (see [32]). We use the following two lemmas to prove our results.

Lemma 2.1 ([32]) *Let E be a real Banach space and $\Phi \in C^1(E, \mathbb{R})$ satisfy the (PS) condition. If Φ is bounded from below, then $c = \inf_E \Phi$ is a critical value of Φ.*

Lemma 2.2 ([33]) *Assume that $(E, \|\cdot\|)$ is a Banach space and $\Phi \in C^1(X, \mathbb{R})$. Suppose that Φ satisfies the (PS) condition, is even and bounded from below, and $\Phi(0) = 0$. If for any $k \in \mathbb{N}$, there exists a k-dimensional subspace X^k of E and $\rho_k > 0$ such that $\sup_{X^k \cap S_{\rho_k}} \Phi < 0$, where $S_\rho = \{u \in E \mid \|u\| = \rho\}$, then at least one of the following conclusions holds:*

(i) *there exists a sequence of critical points $\{u_k\}$ satisfying $\Phi(u_k) < 0$ for all k and $\|u_k\| \to 0$ as $k \to \infty$;*

(ii) *there exists $r > 0$ such that for any $0 < a < r$ there exists a critical point u such that $\|u\| = a$ and $\Phi(u) = 0$.*

Next we recall the definitions of Riemann–Liouville fractional derivatives and Caputo fractional derivatives and some related lemmas. Assume $a, b \in \mathbb{R}$. Suppose that AC$([a, b])$ denote the space which consists of all absolutely continuous functions $u : [a, b] \to \mathbb{R}^N$. Set

$$C_0^\infty([0,T],\mathbb{R}^N) := \{u \mid u \in C^\infty([0,T],\mathbb{R}^N), u(0) = u(T) = 0\}$$

and the norm $\|u\|_\infty = \max_{[0,T]} |u(t)|$ and for $s > 1$,

$$L^s([0,T],\mathbb{R}^N) := \left\{ u \mid u : [0,T] \to \mathbb{R}^N, \int_0^T |u(t)|^s \, dt < \infty \right\}$$

and the norm $\|u\|_{L^s} = (\int_0^T |u(t)|^s \, dt)^{1/s}$.

Definition 2.1 ([31, 34]) Assume that $f \in \mathrm{AC}[a,b]$ and $\theta \in (0,1)$. Define

$$
\begin{aligned}
{}_aD_t^\theta f(t) &= \frac{1}{\Gamma(1-\theta)}\frac{d}{dt}\int_a^t (t-s)^{-\theta} f(s)\,ds, \quad t > a,\\
{}_tD_b^\theta f(t) &= -\frac{1}{\Gamma(1-\theta)}\frac{d}{dt}\int_a^t (s-t)^{-\theta} f(s)\,ds, \quad t < b.
\end{aligned}
$$

Then ${}_aD_t^\theta$ and ${}_tD_b^\theta$ are called the left and right Riemann–Liouville fractional derivatives of order θ of the function f, respectively.

Definition 2.2 ([31, 34]) Assume that $f \in \mathrm{AC}[a,b]$ and $\theta \in (0,1)$. Define

$$
\begin{aligned}
{}_a^cD_t^\theta f(t) &= {}_aD_t^{\theta-1} f'(t) = \frac{1}{\Gamma(1-\theta)}\int_a^t (t-s)^{-\theta} f'(s)\,ds,\\
{}_t^cD_b^\theta f(t) &= {}_tD_b^{\theta-1} f'(t) = \frac{1}{\Gamma(1-\theta)}\int_t^b (s-t)^{-\theta} f'(s)\,ds.
\end{aligned}
$$

Then ${}_a^cD_t^\theta$ and ${}_t^cD_b^\theta$ are called the left and right Caputo fractional derivatives of order θ of the function f, respectively.

Let $E_0^{\theta,s}(0,T)$ be the closure of $C_0^\infty([0,T],\mathbb{R}^N)$ with the norm:

$$
\|u\|_s = \left(\int_0^T \left|{}_0^cD_t^\theta u(t)\right|^s dt + \int_0^T \left|u(t)\right|^s dt\right)^{1/s}, \quad \forall u \in E_0^{\theta,s}(0,T),
$$

where $\theta \in (0,1]$ and $s > 1$. Then $E_0^{\theta,s}$ is reflexive and separable Banach space and $u, {}_0^cD_t^\theta u \in L^s([0,T],\mathbb{R})$ if $u \in E_0^{\theta,s}(0,T)$ (see [5]).

Proposition 2.1 ([5]) *Assume that $\theta \in (0,1]$ and $s > 1$. For any $u \in E_0^{\theta,s}(0,T)$,*

$$
\|u\|_{L^s} \le C_\theta \|{}_0^cD_t^\theta u\|_{L^s},
$$

where $C_\theta = \frac{T^\theta}{\Gamma(\theta+1)}$. If $\theta > \frac{1}{s}$, then

$$
\|u\|_\infty \le C_{\theta,s,\infty} \|{}_0^cD_t^\theta u\|_{L^s},
$$

where $C_{\theta,s,\infty} := \frac{T^{\theta-\frac{1}{s}}}{\Gamma(\theta)(\theta s - s' + 1)^{\frac{1}{s'}}}$ and $s' = \frac{s}{s-1}$.

Proposition 2.2 ([5]) *Assume that $\frac{1}{p} < \theta \le 1$ and $1 < p < \infty$, and the sequence $\{u_k\}$ converges weakly to u in $E_0^{\theta,p}$. Then $u_k \to u$ in $C([0,T],\mathbb{R}^N)$.*

Define $E = E_0^{\alpha,p}(0,T) \times E_0^{\beta,q}(0,T)$ with the norm $\|(u,v)\|_E = \|u\|_p + \|v\|_q$ for all $(u,v) \in E$, where $\alpha, \beta \in (0,1]$. Then E is a reflexive and separable Banach space.

Definition 2.3 ([30]) For any $(h,w)\in E$, if the following conditions hold:

$$\begin{aligned}&\int_0^T\bigl(\rho(t)\bigl|{}_0^cD_t^\alpha u(t)\bigr|^{p-2}{}_0^cD_t^\alpha u(t),{}_0^cD_t^\alpha h(t)\bigr)\,dt+\sum_{i=1}^{l}\bigl(\nabla I_i\bigl(u(t_i)\bigr),h(t_i)\bigr)\\&\quad-\int_0^T\bigl(\nabla_u W\bigl(t,u(t),v(t)\bigr),h(t)\bigr)\,dt=0,\\&\int_0^T\bigl(\gamma(t)\bigl|{}_0^cD_t^\beta v(t)\bigr|^{q-2}{}_0^cD_t^\beta v(t),{}_0^cD_t^\beta h(t)\bigr)\,dt+\sum_{j=1}^{m}\bigl(\nabla H_j\bigl(v(s_j)\bigr),w(s_j)\bigr)\\&\quad-\int_0^T\bigl(\nabla_v W\bigl(t,u(t),v(t)\bigr),h(t)\bigr)\,dt=0,\end{aligned}$$

then $(u,v)\in E$ is defined as a weak solution of (6).

Define the functional $\Phi:E\to\mathbb{R}$ by

$$\begin{aligned}\Phi(u,v)&=\frac{1}{p}\int_0^T\rho(t)\bigl|{}_0^cD_t^\alpha u(t)\bigr|^p\,dt+\frac{1}{q}\int_0^T\gamma(t)\bigl|{}_0^cD_t^\beta v(t)\bigr|^q\,dt\\&\quad+\sum_{i=1}^{l}I_i\bigl(u(t_i)\bigr)+\sum_{j=1}^{m}H_j\bigl(v(s_j)\bigr)-\int_0^T W\bigl(t,u(t),v(t)\bigr)\,dt.\end{aligned}\tag{8}$$

Then $\Phi\in C^1(E,\mathbb{R})$ and

$$\begin{aligned}&\bigl\langle\Phi'(u,v),(h,w)\bigr\rangle\\&\quad=\int_0^T\bigl(\rho(t)\bigl|{}_0^cD_t^\alpha u(t)\bigr|^{p-2}{}_0^cD_t^\alpha u(t),{}_0^cD_t^\alpha h(t)\bigr)\,dt\\&\qquad+\int_0^T\bigl(\gamma(t)\bigl|{}_0^cD_t^\alpha v(t)\bigr|^{q-2}{}_0^cD_t^\beta v(t),{}_0^cD_t^\alpha w(t)\bigr)\,dt\\&\qquad-\int_0^T\bigl(\nabla_u W\bigl(t,u(t),v(t)\bigr),h(t)\bigr)\,dt-\int_0^T\bigl(\nabla_v W\bigl(t,u(t),v(t)\bigr),w(t)\bigr)\,dt\\&\qquad+\sum_{i=1}^{l}\bigl(\nabla I_i\bigl(u(t_i)\bigr),h(t_i)\bigr)+\sum_{j=1}^{m}\bigl(\nabla H_j\bigl(v(s_j)\bigr),w(s_j)\bigr).\end{aligned}$$

A critical point of Φ is a weak solution of system (6) (see [29–31]).

3 Proofs of theorems

Lemma 3.1 *Assume that* (W0), ($\mathcal{A}$), (W1), (I1) *and* (H1) *hold. Then Φ is bounded from below on E.*

Proof It follows from (W0), ($\mathcal{A}$), (W1), (I1) and (H1) that

$$\begin{aligned}\Phi(u,v)&=\frac{1}{p}\int_0^T\rho(t)\bigl|{}_0^cD_t^\alpha u(t)\bigr|^p\,dt+\frac{1}{q}\int_0^T\gamma(t)\bigl|{}_0^cD_t^\beta v(t)\bigr|^q\,dt+\sum_{i=1}^{l}I_i\bigl(u(t_i)\bigr)\\&\quad+\sum_{j=1}^{m}H_j\bigl(v(s_j)\bigr)-\int_0^T W\bigl(t,u(t),v(t)\bigr)\,dt\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\rho^-}{p}\int_0^T \left|{}_0^cD_t^\alpha u(t)\right|^p dt + \frac{\gamma^-}{q}\int_0^T \left|{}_0^cD_t^\beta v(t)\right|^q dt - \sum_{i=1}^{l}\left(k_1\left|u(t_i)\right|^{\nu_1} + k_2\right) \\
&\quad - \sum_{j=1}^{m}\left(k_3\left|v(s_j)\right|^{\nu_2} + k_4\right) - \int_0^T \left(d_1\left|u(t)\right|^p + d_2\left|v(t)\right|^q + g_1(t)\left|u(t)\right|^{p-\gamma_1}\right. \\
&\quad \left. + g_2(t)\left|v(t)\right|^{q-\gamma_2} + g_3(t)\right) dt \\
&\geq \frac{\rho^-}{p}\left\|{}_0^cD_t^\alpha u\right\|_{L^p}^p + \frac{\gamma^-}{q}\left\|{}_0^cD_t^\beta v\right\|_{L^q}^q - k_1 l\|u\|_\infty^{\nu_1} - k_2 l - k_3 m\|v\|_\infty^{\nu_2} - k_4 m \\
&\quad - d_1 C_\alpha^p\left\|{}_0^cD_t^\alpha u\right\|_{L^p}^p - d_2 C_\beta^q\left\|{}_0^cD_t^\beta v\right\|_{L^q}^q - \int_0^T g_1(t)\,dt\|u\|_\infty^{p-\gamma_1} \\
&\quad - \int_0^T g_2(t)\,dt\|v\|_\infty^{q-\gamma_2} - \int_0^T g_3(t)\,dt \\
&\geq \left(\frac{\rho^-}{p} - d_1 C_\alpha^p\right)\left\|{}_0^cD_t^\alpha u\right\|_{L^p}^p + \left(\frac{\gamma^-}{q} - d_2 C_\beta^q\right)\left\|{}_0^cD_t^\beta v\right\|_{L^q}^q \\
&\quad - k_1 l C_{\alpha,p,\infty}^{\nu_1}\left\|{}_0^cD_t^\alpha u\right\|_{L^p}^{\nu_1} - k_3 m C_{\beta,q,\infty}^{\nu_2}\left\|{}_0^cD_t^\beta v\right\|_{L^q}^{\nu_2} \\
&\quad - C_{\alpha,p,\infty}^{p-\gamma_1}\int_0^T g_1(t)\,dt\left\|{}_0^cD_t^\alpha u\right\|_{L^p}^{p-\gamma_1} - C_{\beta,q,\infty}^{q-\gamma_2}\int_0^T g_2(t)\,dt\left\|{}_0^cD_t^\beta v\right\|_{L^q}^{q-\gamma_2} \\
&\quad - \int_0^T g_3(t)\,dt - k_2 l - k_4 m.
\end{aligned} \tag{9}
$$

Note that $d_1 \in [0, \frac{\rho^-}{pC_\alpha^p})$, $d_2 \in [0, \frac{\gamma^-}{qC_\beta^q})$, $\nu_1 \in [0,p)$, $\nu_2 \in [0,q)$, $\gamma_1 \in (0,p)$ and $\gamma_2 \in (0,q)$. Moreover, by Proposition 2.1, it is easy to see that $\|{}_0^cD_t^\alpha u\|_{L^p}$ and $\|{}_0^cD_t^\beta v\|_{L^q}$ are equivalent to $\|u\|_p$ and $\|v\|_q$, respectively, which shows that $\|(u,v)\|_E$ is equivalent to the norm $\|(u,v)\|_L := \|u\|_{L^p} + \|v\|_{L^q}$. Thus (9) and Proposition 2.1 imply that $\Phi(u,v) \to +\infty$ as $\|(u,v)\|_E \to \infty$ and so Φ is bounded from below on E. □

Lemma 3.2 *Assume that* (W0), ($\mathcal{A}$), (W1), (I1) *and* (H1) *hold. Then* Φ *satisfies the Palais–Smale condition.*

Proof The proof is standard (see, for example, [29, 30] and [31]). For any sequence $\{(u_n,v_n)\}_{n=1}^\infty \subset E$, suppose that there is a positive constant C_1 such that

$$
\left|\Phi(u_n,v_n)\right| \leq C_1, \qquad \left(1 + \left\|(u_n,v_n)\right\|_E\right)\left\|\Phi'(u_n,v_n)\right\|_{E^*} \leq C_1, \quad \text{for all } n \in \mathbb{N}, \tag{10}
$$

where E^* is the dual space of E. Then (9) implies that $\|(u,v)\|_L$ is bounded and then $\|(u,v)\|_E$ is bounded. So there is a subsequence (u_n,v_n) such that $u_n \rightharpoonup u$ in $E_0^{\alpha,p}(0,T)$ and $v_n \rightharpoonup v$ in $E_0^{\beta,q}(0,T)$. Then Proposition 2.2 implies that $\|u_n - u\|_\infty \to 0$ and $\|v_n - v\|_\infty \to 0$, and then $\|u_n - u\|_{L^p} \to 0$ and $\|v_n - v\|_{L^q} \to 0$. With a similar proof to Lemma 3.1 in [29], we have $\|{}_0^cD_t^\alpha u_n(t) - {}_0^cD_t^\alpha u(t)\|_{L^p} \to 0$ and $\|{}_0^cD_t^\alpha v_n(t) - {}_0^cD_t^\alpha v(t)\|_{L^q} \to 0$. Hence $\|u_n - u\|_p \to 0$ and $\|v_n - v\|_q \to 0$ as $n \to \infty$. □

Proof of Theorem 1.1 By combining Lemma 3.1, Lemma 3.2 with Lemma 2.1, the proof is easy to complete. □

Proof of Theorem 1.2 We follow the basic idea of the proof in [33]. We first investigate the following modified system:

$$\begin{cases} {}_tD_T^{\alpha}(\rho(t)\Phi_p({}_0^cD_t^{\alpha}u(t))) = \nabla_u\hat{W}(t,u(t),v(t)), & \text{a.e. } t\in[0,T], \\ {}_tD_T^{\beta}(\gamma(t)\Phi_q({}_0^cD_t^{\beta}v(t))) = \nabla_v\hat{W}(t,u(t),v(t)), & \text{a.e. } t\in[0,T], \\ \Delta({}_tD_T^{\alpha-1}(\rho\Phi_p({}_0^cD_t^{\alpha}u)))(t_i) = \nabla\hat{I}_i(u(t_i)), & i=1,2,\ldots,l, \\ \Delta({}_tD_T^{\beta-1}(\gamma\Phi_q({}_0^cD_t^{\beta}v)))(s_j) = \nabla\hat{H}_j(v(s_j)), & j=1,2,\ldots,m, \\ u(0)=u(T)=0, \qquad v(0)=v(T)=0, \end{cases} \tag{11}$$

where $\hat{I}_i,\hat{H}_j:\mathbb{R}^N\to\mathbb{R}$ are continuously differentiable, where $i=1,\ldots,l$ and $j=1,\ldots,m$, and $\hat{W}:[0,T]\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ satisfies (W0) and the following conditions:

(i) $\hat{W}(t,x,y)=\hat{W}(t,-x,-y)$ for a.e. $t\in[0,T]$ and all $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$, $\hat{W}(t,x,y)=W(t,x,y)$ for a.e. $t\in[0,T]$ and all $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$ with $|x|\le\delta_1$ and $|y|\le\delta_2$, and $\hat{W}(t,x,y)\equiv 0$ for a.e. $t\in[0,T]$ and all $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$ with $|x|>\delta_1$ and $|y|>\delta_2$;

(ii) $\hat{I}_i(x)=\hat{I}_i(-x)$ for a.e. $t\in[0,T]$ and all $x\in\mathbb{R}^N$, $\hat{I}_i(x)=I_i(x)$, $i=1,\ldots,l$ for all $x\in\mathbb{R}^N$ with $|x|\le\delta_3$, and $\hat{I}_i(x)\equiv 0$ for all $x\in\mathbb{R}^N$ with $|x|>\delta_3$;

(iii) $\hat{H}_j(y)=\hat{H}_j(-y)$, $j=1,\ldots,m$ for all $y\in\mathbb{R}^N$, $\hat{H}_j(y)=H_j(y)$, $j=1,\ldots,m$ for all $y\in\mathbb{R}^N$ with $|y|\le\delta_4$, and $\hat{H}_i(x)\equiv 0$ for all $y\in\mathbb{R}^N$ with $|y|>\delta_4$.

Then the solutions of system (11) correspond to the critical points of the functional $\hat{\Psi}:E\to\mathbb{R}$ defined by

$$\begin{aligned} \hat{\Phi}(u,v) = {} & \frac{1}{p}\int_0^T \rho(t)\left|{}_0^cD_t^{\alpha}u(t)\right|^p dt + \frac{1}{q}\int_0^T \gamma(t)\left|{}_0^cD_t^{\beta}v(t)\right|^q dt + \sum_{i=1}^{l}\hat{I}_i\bigl(u(t_i)\bigr) \\ & + \sum_{j=1}^{m}\hat{H}_j\bigl(v(s_j)\bigr) + \int_0^T \hat{W}\bigl(t,u(t),v(t)\bigr)\,dt. \end{aligned} \tag{12}$$

By (W0) and (W1)′, it is easy to see that $\hat{\Phi}$ is well defined and $\hat{\Phi}\in C^1(E,\mathbb{R})$. It follows from (W1)′, (I1)′, (H1)′ and the definitions of $\hat{W}$, $\hat{I}$ and $\hat{H}$ that $\hat{W}$, $\hat{I}$ and $\hat{H}$ satisfy (W1), (I1) and (H1), respectively. Hence, by the argument of Lemma 3.1, $\hat{\Phi}(u,v)\to+\infty$ as $\|(u,v)\|_E\to\infty$ and then it is bounded from below. Similar to the argument of Lemma 3.2, $\hat{\Phi}$ satisfies the Palais–Smale condition.

Let X^k be a k-dimensional subspace of X for any $k\in\mathbb{N}$. Then all norms are equivalent in X^k. Hence, there exist positive constants C_2 and C_3 such that

$$\|u\|_{L^p}^p \ge C_2\|u\|_p^p, \qquad \|v\|_{L^q}^q \ge C_3\|v\|_q^q. \tag{13}$$

It follows from (W3), (I2) and (H2) that, for any given

$$M > \max\left\{\frac{1}{C_2}\left(\frac{\rho^+}{p}+b_1lC_{\alpha,p,\infty}^p\right), \frac{1}{C_3}\left(\frac{\gamma^+}{p}+b_2mC_{\beta,q,\infty}^q\right)\right\}, \tag{14}$$

where $\rho^+=\operatorname{esssup}_{[0,T]}\rho(t)$ and $\gamma^+=\operatorname{esssup}_{[0,T]}\gamma(t)$, there exists $\delta_0:=\delta_0(M)>0$ such that

$$W(t,x,y)\ge M\bigl(|x|^p+|y|^q\bigr), \tag{15}$$

for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|x| + |y| \leq \delta_0$, and

$$I_i(x) \leq b_1|x|^p, \qquad H_j(y) \leq b_2|y|^q, \tag{16}$$

for all $x \in \mathbb{R}^N$ with $|x| \leq \delta_0$ and all $y \in \mathbb{R}^N$ with $|y| \leq \delta_0$, where $i = 1,\ldots,l$ and $j = 1,\ldots,m$.

For any given $0 < \rho_k \leq \frac{\min\{\delta_0,\delta_1,\delta_2,\delta_3,\delta_4\}}{\max\{C_{\alpha,p\infty},C_{\beta,q,\infty}\}}$, define

$$S_{\rho_k} = \{(u,v) \in E | \|(u,v)\|_E = \rho_k\}.$$

Then, by Proposition 2.1, we have

$$\begin{aligned}\|u\|_\infty + \|v\|_\infty &\leq C_{\alpha,p,\infty}\|u\|_p + C_{\beta,q,\infty}\|v\|_q \leq \max\{C_{\alpha,p,\infty}, C_{\beta,q,\infty}\}\|(u,v)\|_E \\ &\leq \min\{\delta_0,\delta_1,\delta_2,\delta_3,\delta_4\}\end{aligned} \tag{17}$$

for all $(u,v) \in S_{\rho_k}$. Then (13), (15), (16), (17) and the definitions of $\hat{W}$, $\hat{I}_i$ and $\hat{H}_j$ ($i = 1,\ldots,l$, $j = 1,\ldots,m$) imply that

$$\begin{aligned}\hat{\Phi}(u,v) \leq{}& \frac{\rho^+}{p}\int_0^T \left|{}_0^cD_t^\alpha u(t)\right|^p dt + \frac{\gamma^+}{q}\int_0^T \left|{}_0^cD_t^\beta v(t)\right|^q dt + b_1\sum_{i=1}^l \left|u(t_i)\right|^p \\ &+ b_2\sum_{j=1}^m \left|v(s_j)\right|^q - M\int_0^T \left(\left|u(t)\right|^p + \left|v(t)\right|^q\right) dt \\ \leq{}& \frac{\rho^+}{p}\|u\|_p^p + \frac{\gamma^+}{q}\|v\|_q^q + b_1 l\|u\|_\infty^p + b_2 m\|v\|_\infty^q - M\|u\|_{L^p}^p - M\|v\|_{L^q}^q \\ \leq{}& \frac{\rho^+}{p}\|u\|_p^p + \frac{\gamma^+}{q}\|v\|_q^q + b_1 l C_{\alpha,p,\infty}^p\|u\|_p^p + b_2 m C_{\beta,q,\infty}^q\|v\|_q^q - MC_2\|u\|_p^p \\ &- MC_3\|v\|_q^q\end{aligned} \tag{18}$$

for any $(u,v) \in S_{\rho_k}$. So (14) and (18) imply that $\hat{\Phi}(u,v)|_{S_{\rho_k}\cap X^k} < 0$. The conditions (i)–(iii), (W2), (I3) and (H3) imply that $\hat{\Phi}$ is even and $\hat{\Phi}(0,0) = 0$. Hence, Lemma 2.2 and Lemma 3.2 show that system (11) has infinitely many solutions $\{(u_k,v_k)\}$ such that $\|(u_k,v_k)\|_E \to 0$ as $k \to \infty$. Then (17) implies that $\|u_k\|_\infty \to 0$ and $\|v_k\|_\infty \to 0$ as $k \to \infty$. So there exists a sufficiently large integer $k_0 > 0$ such that $|u_k(t)| + |v_k(t)| \leq \min\{\delta_0,\delta_1,\delta_2,\delta_3,\delta_4\}$ for all $k \geq k_0$. Then the definitions of $\hat{W}$, $\hat{I}_i$ ($i = 1,\ldots,l$) and $\hat{H}_j$ ($j = 1,\ldots,m$) imply that (u_k,v_k) are also solutions of system (6) for all $k \geq k_0$. □

Acknowledgements

The authors would like to express their sincerely thanks to the reviewers for the reviewers' valuable comments.

Funding

This project is supported by the National Natural Science Foundation of China (No: 11301235) and Candidate Talents Training Fund of Yunnan Province (Project No: 2016PY027).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have made equal contributions and they read and approved the final manuscript.

Author details
[1]Faculty of Science, Kunming University of Science and Technology, Kunming, P.R. China. [2]Faculty of Transportation Engineering, Kunming University of Science and Technology, Kunming, P.R. China. [3]School of Mathematics and Statistics, Central South University, Changsha, P.R. China.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 12 June 2018 Accepted: 21 February 2019 Published online: 08 March 2019

References
1. Belmekki, M., Nieto, J.J., Rodríguez-López, R.: Existence of periodic solution for a nonlinear fractional differential equation. Bound. Value Probl. **2009**, Article ID 324561 (2009)
2. Benchohra, M., Cabada, A., Seba, D.: An existence result for nonlinear fractional differential equations on Banach spaces. Bound. Value Probl. **2009**, Article ID 628916 (2009)
3. Zhang, S.: Existence of a solution for the fractional differential equation with nonlinear boundary conditions. Comput. Math. Appl. **61**, 1202–1208 (2011)
4. Liang, S., Zhang, J.: Positive solutions for boundary value problems of nonlinear fractional differential equation. Nonlinear Anal. **71**, 5545–5550 (2009)
5. Jiao, F., Zhou, Y.: Existence of solutions for a class of fractional boundary value problems via critical point theory. Comput. Math. Appl. **62**, 1181–1199 (2011)
6. Zhao, Y., Chen, H., Qin, B.: Multiple solutions for a coupled system of nonlinear fractional differential equations via variational methods. Appl. Math. Comput. **257**, 417–427 (2015)
7. Chen, J., Tang, X.H.: Existence and multiplicity of solutions for some fractional boundary value problem via critical point theory. Abstr. Appl. Anal. **2012**, Article ID 648635 (2012)
8. Li, Y.N., Sun, H.R., Zhang, Q.G.: Existence of solutions to fractional boundary value problems with a parameter. Electron. J. Differ. Equ. **2013**, 141 (2013)
9. Zhang, Z., Li, J.: Variational approach to solutions for a class of fractional boundary value problems. Electron. J. Qual. Theory Differ. Equ. **2015**, 11 (2015)
10. Torres, C.: Mountain pass solution for a fractional boundary value problem. J. Fract. Calc. Appl. **5**, 1–10 (2014)
11. Li, P., Wang, H., Li, Z.: Infinitely many solutions to boundary value problems for a coupled system of fractional differential equations. J. Nonlinear Sci. Appl. **9**, 3433–3444 (2016)
12. Heidarkhani, S., Cabada, A., Afrouzi, G.A., Moradi, S., Caristi, G.: A variational approach to perturbed impulsive fractional differential equations. J. Comput. Appl. Math. **341**, 42–60 (2018)
13. Heidarkhani, S., Zhou, Y., Caristi, G., Afrouzi, G.A., Moradi, S.: Existence results for fractional differential systems through a local minimization principle. Comput. Math. Appl. (to appear)
14. Heidarkhani, S.: Infinitely many solutions for nonlinear perturbed fractional boundary value problems. An. Univ. Craiova, Ser. Mat. Inform. **41**(1), 88–103 (2014)
15. Heidarkhani, S.: Multiple solutions for a nonlinear perturbed fractional boundary value problem. Dyn. Syst. Appl. **23**, 317–332 (2014)
16. Nieto, J.J., O'Regan, D.: Variational approach to impulsive differential equations. Nonlinear Anal., Real World Appl. **10**, 680–690 (2009)
17. Tian, Y., Ge, W.G.: Applications of variational methods to boundary value problem for impulsive differential equ. Proc. Edinb. Math. Soc. **51**, 509–527 (2008)
18. D'Aguí, G., Di Bella, B., Tersian, S.: Multiplicity results for superlinear boundary value problems with impulsive effects. Math. Methods Appl. Sci. **39**, 1060–1068 (2016)
19. Zhang, Z., Yuan, R.: An application of variational methods to Dirichlet boundary value problem with impulses. Nonlinear Anal., Real World Appl. **11**(1), 155–162 (2010)
20. Zhang, X.: Subharmonic solutions for a class of second-order impulsive Lagrangian systems with damped term. Bound. Value Probl. **2013**(1), Article ID 218 (2013)
21. Xiao, J., Nieto, J.J.: Variational approach to some damped Dirichlet nonlinear impulsive differential equations. J. Franklin Inst. **348**, 369–377 (2011)
22. Rodríguez-López, R., Tersian, S.: Multiple solutions to boundary value problem for impulsive fractional differential equations. Fract. Calc. Appl. Anal. **17**, 1016–1038 (2014)
23. Bonanno, G., Rodríguez-López, R., Tersian, S.: Existence of solutions to boundary value problem for impulsive fractional differential equations. Fract. Calc. Appl. Anal. **17**(3), 717–744 (2014)
24. Zhao, Y., Zhao, Y.: Nontrivial solutions for a class of perturbed fractional differential systems with impulsive effects. Bound. Value Probl. **2016**, Article ID 129 (2016)
25. Zhao, Y., Chen, H., Xu, C.: Nontrivial solutions for impulsive fractional differential equations via Morse theory. Appl. Math. Comput. **307**, 170–179 (2017)
26. Nyamoradi, N., Rodríguez-López, R.: On boundary value problems for impulsive fractional differential equations. Appl. Math. Comput. **271**, 874–892 (2015)
27. Heidarkhani, S., Zhao, Y., Caristi, G., Afrouzi, G.A., Moradi, S.: Infinitely many solutions for perturbed impulsive fractional differential systems. Appl. Anal. **96**(8), 1401–1424 (2017)
28. Bonanno, G., Candito, P.: Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities. J. Differ. Equ. **244**, 3031–3059 (2008)
29. Zhao, Y., Tang, L.: Multiplicity results for impulsive fractional differential equations with p-Laplacian via variational methods. Bound. Value Probl. **2017**, 213 (2017)
30. Xie, J., Zhang, X.: Infinitely many solutions for a class of fractional impulsive coupled systems with (p,q)-Laplacian. Discrete Dyn. Nat. Soc. **2018**, Article ID 9256192 (2018)
31. Zhou, Y.: Basic Theory of Fractional Differential Equations. World Scientific, Singapore (2014)

32. Mawhin, J., Willem, M.: Critical Point Theory and Hamiltonian Systems. Springer, New York (1989)
33. Liu, Z., Wang, Z.Q.: On Clark's theorem and its applications to partially sublinear problems. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **32**, 1015–1037 (2015)
34. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)