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Multiple positive solutions for nonlinear mixed fractional differential equation with p-Laplacian operator

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Abstract
In this article, multiple positive solutions are considered for nonlinear mixed fractional differential equations with a p-Laplacian operator. Using the Avery–Peterson fixed point theorem, we conclude to the existence of positive solutions for the fractional boundary value problem. An example is also presented to illustrate the effectiveness of the main result.

MSC: 34B15; 34B18

Keywords: Caputo fractional derivative; Riemann–Liouville fractional derivative; Positive solutions; p-Laplacian; Avery–Peterson fixed point theorem

1 Introduction

The differential equation arises in the modeling of different physical and natural phenomena: nonlinear flow laws, control systems and many other branches of engineering. In these years, integer order differential equations and fractional differential equations have found wide applications. There are many papers concerning integer order differential equations [1–8], Caputo fractional differential equations [9–14], Riemann–Liouville fractional differential equations [15–20] and mixed fractional differential equations [21].

By means of the Avery–Peterson fixed point theorem, Shen et al. [1] established the existence result of at least triple positive solutions for the following problems:

$$\begin{aligned} &-u''(t) = f\bigl(t, u(t), u'(t)\bigr), \quad t \in (0,1), \\ &u'(0) = 0, \qquad \beta u'(1) + u(\eta) = 0, \end{aligned}$$

or

$$u(0) = 0, \qquad \beta u'(1) + u(\eta) = 0,$$

where $f \in C([0,1] \times [0,\infty) \times (\infty,+\infty), (0,+\infty))$, $\eta \in (0,1)$, $\beta > 0$, with $\beta + \eta > 1$.

In [21], Liu et al. discussed the four-point problem for a class of fractional differential equations with mixed fractional derivative and with a p-Laplacian operator,

$$\begin{cases} D_{0^+}^{\alpha}(\varphi_p({}^cD_{0^+}^{\beta}u(t))) = f(t,u(t),{}^cD_{0^+}^{\beta}u(t)), & t\in(0,1), \\ {}^cD_{0^+}^{\beta}u(0) = u'(0) = 0, \\ u(1) = r_1 u(\eta), \qquad {}^cD_{0^+}^{\beta}u(1) = r_2^c D_{0^+}^{\beta}u(\xi). \end{cases}$$

Here $1 < \alpha, \beta \leq 2, r_1, r_2 \geq 0, f \in C([0,1]\times[0,\infty)\times(-\infty,0],[0,+\infty))$. Based on the method of lower and upper solutions, they studied the existence of positive solutions of the above boundary problem.

Motivated by the aforementioned work, this work discusses the existence of positive solutions for fractional differential equation:

$$\begin{cases} {}^cD_{0^+}^{\beta}[\phi_p(D_{0^+}^{\alpha}u(t))] + f(t,u(t)) = 0, & t\in(0,1), \\ [\phi_p(D_{0^+}^{\alpha}u(0))]^{(i)} = 0, & i = 1,2,\ldots,m, \\ \phi_p(D_{0^+}^{\alpha}u(0)) = \sum_{i=1}^{l-2} b_i[\phi_p(D_{0^+}^{\alpha}u(\xi_i))], \\ (u(0))^{(j)} = 0, & j = 0,1,2,\ldots,n-1, \\ D_{0^+}^{\alpha-1}u(1) = \sum_{i=1}^{l-2} a_i D_{0^+}^{\alpha}u(\xi_i), \end{cases} \tag{1.1}$$

where $2 \leq n < \alpha \leq n+1, 1 \leq m < \beta \leq m+1$ and $m+n+1 < \alpha+\beta \leq m+n+2, \phi_p(u) = |u|^{p-2}u$, $p > 1, D_{0^+}^{\alpha}$ is the Riemann–Liouville fractional derivatives and ${}^cD_{0^+}^{\beta}$ is the Caputo fractional derivatives. Using the Avery–Peterson fixed point theorem, we obtain the existence of positive solutions for the fractional boundary value problem. A function $u(t)$ is a positive solution of the boundary value problem (1.1) if and only if $u(t)$ satisfies the boundary value problem (1.1), and $u(t) \geq 0$ for $t \in [0,1]$.

We will always suppose the following conditions are satisfied:

(H_1) $0 < \xi_1 < \xi_2 < \cdots < \xi_{l-2} < 1$, $a_i > 0$, $b_i > 0$, $i = 1,2,\ldots,l-2$ are constants and $\sum_{i=1}^{l-2} a_i < 1$, $\sum_{i=1}^{l-2} b_i < 1$;

(H_2) $f(t,u) : [0,1]\times[0,\infty) \to [0,\infty)$ is continuous.

2 Preliminaries

To show the main result of this work, we give the following basic definitions, which can be found in [22, 23].

Definition 2.1 The fractional integral of order $\alpha > 0$ of a function $y : (0,+\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,\mathrm{d}s,$$

provided that the right side is pointwise defined on $(0,+\infty)$, where

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-x}x^{\alpha-1}\,\mathrm{d}x.$$

Definition 2.2 For a continuous function $y:(0,+\infty)\to\mathbb{R}$, the Caputo derivative of fractional order $\alpha>0$ is defined as

$$D_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}y^{(n)}(s)\,\mathrm{d}s,$$

where $n=[\alpha]+1$, provided that the right side is pointwise defined on $(0,+\infty)$.

Definition 2.3 For a continuous function $y:(0,+\infty)\to\mathbb{R}$, the Riemann–Liouville derivative of fractional order $\alpha>0$ is defined as

$$D_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_0^t (t-s)^{n-\alpha-1}y(s)\,\mathrm{d}s,$$

where $n=[\alpha]+1$, provided that the right side is pointwise defined on $(0,+\infty)$.

Let P be a cone in real Banach space E; γ, θ be nonnegative continuous convex functionals on P; ω be nonnegative continuous concave functionals on P and ψ be nonnegative continuous functionals on P. Then, for positive real numbers h, r, c and d, we define the following sets:

$$\begin{aligned}
&P(\gamma,d)=\bigl\{x\in P|\gamma(x)<d\bigr\},\\
&P(\gamma,\omega,r,d)=\bigl\{x\in P|r\le\omega(x),\gamma(x)\le d\bigr\},\\
&P(\gamma,\theta,\omega,r,c,d)=\bigl\{x\in P|r\le\omega(x),\theta(x)\le c,\gamma(x)\le d\bigr\},\\
&Q(\gamma,\psi,h,d)=\bigl\{x\in P|h\le\psi(x),\gamma(x)\le d\bigr\}.
\end{aligned}$$

Theorem 2.1 ([24]) *Let P be a cone in real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, ω be a nonnegative continuous concave functionals on P, and ψ be a nonnegative continuous functionals on P satisfying $\psi(\lambda x)\le\lambda\psi(x)$ for $0\le\lambda\le1$ such that, for some positive numbers d and M,*

$$\omega(x)\le\psi(x),\quad \textit{and}\quad \|x\|\le M\gamma(x),\quad \textit{for all } x\in\overline{P(\gamma,d)}.$$

Suppose further that $T:\overline{P(\gamma,d)}\to\overline{P(\gamma,d)}$ is completely continuous and there exist positive numbers h, r and c with $h<r$ such that:
(C_1) *$\{x\in P(\gamma,\theta,\omega,r,c,d)|\omega(x)>r\}\neq\emptyset$ and $\omega(Tx)>r$ for $x\in P(\gamma,\theta,\omega,r,c,d)$;*
(C_2) *$\omega(Tx)>r$ for $x\in P(\gamma,\omega,r,d)$ with $\theta(Tx)>c$;*
(C_3) *$0\notin Q(\gamma,\psi,h,d)$ and $\psi(Tx)<h$ for $x\in Q(\gamma,\psi,h,d)$ with $\psi(x)=h$.*
Then T has at least three fixed points $x_1,x_2,x_3\in\overline{P(\gamma,d)}$ such that

$$\begin{aligned}
&\gamma(x_i)\le d,\quad i=1,2,3;\\
&r<\omega(x_1);\\
&h<\psi(x_2)\quad \textit{with } \omega(x_2)<r,
\end{aligned}$$

and

$$\omega(x_3)<h.$$

3 Useful lemmas

Lemma 3.1 *The boundary value problem* (1.1) *is equivalent to the following equation*:

$$u(t) = c_1 t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} w(s)\,\mathrm{d}s, \tag{3.1}$$

where

$$c_1 = \frac{\int_0^1 w(s)\,\mathrm{d}s - \sum_{i=1}^{l-2} a_i \int_0^{\xi_i} w(s)\,\mathrm{d}s}{(1-\sum_{i=1}^{l-2} a_i)\Gamma(\alpha)}, \tag{3.2}$$

$$\begin{aligned} w(s) &= \phi_q\biggl(-d_0 + \frac{\int_0^s (s-\tau)^{\beta-1} f(\tau,u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)}\biggr) \\ &= \phi_q\biggl(\frac{\sum_{i=1}^{l-2} b_i \int_0^{\xi_i} (\xi_i-\tau)^{\beta-1} f(\tau,u(\tau))\,\mathrm{d}\tau}{(1-\sum_{i=1}^{l-2} b_i)\Gamma(\beta)} + \frac{\int_0^s (s-\tau)^{\beta-1} f(\tau,u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)}\biggr), \end{aligned} \tag{3.3}$$

$\phi_q(u)$ is the inverse function of $\phi_p(u)$, i.e. $\frac{1}{p}+\frac{1}{q}=1$.

Proof In view of ${}^cD_{0^+}^{\beta}[\phi_p(D_{0^+}^{\alpha} u(t))] + f(t,u(t)) = 0$, we have

$$\phi_p\bigl(D_{0^+}^{\alpha} u(t)\bigr) = d_0 + d_1 t + \cdots + d_m t^m - \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f\bigl(\tau,u(\tau)\bigr)\,\mathrm{d}\tau.$$

In view of $[\phi_p(D_{0^+}^{\alpha} u(0))]^{(i)} = 0$, $i = 1,2,\ldots,m$, we obtain

$$d_1 = d_2 = \cdots = d_m = 0,$$

that is,

$$\phi_p\bigl(D_{0^+}^{\alpha} u(t)\bigr) = d_0 - \frac{\int_0^t (t-\tau)^{\beta-1} f(\tau,u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)}. \tag{3.4}$$

In view of $\phi_p(D_{0^+}^{\alpha} u(0)) = \sum_{i=1}^{l-2} b_i[\phi_p(D_{0^+}^{\alpha} u(\xi_i))]$, we get

$$d_0 = -\frac{\sum_{i=1}^{l-2} b_i \int_0^{\xi_i} (\xi_i-\tau)^{\beta-1} f(\tau,u(\tau))\,\mathrm{d}\tau}{(1-\sum_{i=1}^{l-2} b_i)\Gamma(\beta)}.$$

By (3.4), we have

$$D_{0^+}^{\alpha} u(t) = \phi_q\biggl(d_0 - \frac{\int_0^t (t-\tau)^{\beta-1} f(\tau,u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)}\biggr).$$

For $t \in [0,1]$, integrating from 0 to t, we get

$$\begin{aligned} u(t) &= c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_{n+1} t^{\alpha-n-1} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q\biggl(d_0 - \frac{\int_0^s (s-\tau)^{\beta-1} f(\tau,u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)}\biggr)\,\mathrm{d}s. \end{aligned}$$

In view of $(D_{0^+}^{\alpha}u(0))^{(j)} = 0$, $j = 0, 1, 2, \dots, n-1$, we obtain

$$c_2 = c_3 = \cdots = c_{n+1} = 0,$$

that is,

$$\begin{aligned} u(t) &= c_1 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q\left(d_0 - \frac{\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)}\right) \mathrm{d}s \\ &= c_1 t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} w(s)\,\mathrm{d}s. \end{aligned}$$

By a straightforward calculation, we get

$$D_{0^+}^{\alpha-1} u(t) = c_1 \Gamma(\alpha) - \int_0^t w(s)\,\mathrm{d}s.$$

By use of $D_{0^+}^{\alpha-1}u(1) = \sum_{i=1}^{l-2} a_i D_{0^+}^{\alpha} u(\xi_i)$, we obtain

$$c_1 = \frac{\int_0^1 w(s)\,\mathrm{d}s - \sum_{i=1}^{l-2} a_i \int_0^{\xi_i} w(s)\,\mathrm{d}s}{(1 - \sum_{i=1}^{l-2} a_i)\Gamma(\alpha)}.$$

The proof is complete. □

Lemma 3.2 *Suppose that the conditions* (H_1) *and* (H_2) *hold, then* $u(t)$ *defined by* (3.1) *is a nonnegative nondecreasing function.*

Proof In view of the conditions (H_1) and (H_2), we get

$$\begin{aligned} w(s) &= \phi_q\left(\frac{\sum_{i=1}^{l-2} b_i \int_0^{\xi_i} (\xi_i - \tau)^{\beta-1} f(\tau, u(\tau))\,\mathrm{d}\tau}{(1 - \sum_{i=1}^{l-2} b_i)\Gamma(\beta)} + \frac{\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)}\right) \\ &\geq \phi_q\left(\frac{\sum_{i=1}^{l-2} b_i \int_0^{\xi_i} (\xi_i - \tau)^{\beta-1} f(\tau, u(\tau))\,\mathrm{d}\tau}{(1 - \sum_{i=1}^{l-2} b_i)\Gamma(\beta)}\right) \\ &\geq 0. \end{aligned}$$

So

$$\begin{aligned} u(t) &= c_1 t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} w(s)\,\mathrm{d}s \\ &= \frac{\int_0^1 w(s)\,\mathrm{d}s - \sum_{i=1}^{l-2} a_i \int_0^{\xi_i} w(s)\,\mathrm{d}s}{(1 - \sum_{i=1}^{l-2} a_i)\Gamma(\alpha)} t^{\alpha-1} - \frac{\int_0^t (t-s)^{\alpha-1} w(s)\,\mathrm{d}s}{\Gamma(\alpha)} \\ &\geq \frac{\int_0^1 w(s)\,\mathrm{d}s - \sum_{i=1}^{l-2} a_i \int_0^1 w(s)\,\mathrm{d}s}{(1 - \sum_{i=1}^{l-2} a_i)\Gamma(\alpha)} t^{\alpha-1} - \frac{\int_0^t w(s)\,\mathrm{d}s}{\Gamma(\alpha)} t^{\alpha-1} \\ &= \frac{\int_0^1 w(s)\,\mathrm{d}s}{\Gamma(\alpha)} t^{\alpha-1} - \frac{\int_0^t w(s)\,\mathrm{d}s}{\Gamma(\alpha)} t^{\alpha-1} \\ &\geq 0. \end{aligned}$$

Therefore, we see that $u(t)$ is nonnegative.

It is similar to the proof of $u(t) \geq 0$, we can obtain $u'(t) \geq 0$, so $u(t)$ is nondecreasing. The proof is complete. □

4 Main result

Let the Banach space $E = C[0,1]$ be endowed with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Define the cone P by

$$P = \{u \in E, u(t) \text{ is nonnegative and nondecreasing for } t \in [0,1]\}.$$

Define the operator $T : P \to E$,

$$Tu(t) = c_1 t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} w(s)\, ds,$$

where c_1 and $w(s)$ are defined by (3.2) and (3.3). Obviously, $u(t)$ is a solution of problem (1.1) if and only if $u(t)$ is a fixed point of T. Now we introduce the following notations for convenience.

Let

$$M_1 = \left(1 - \sum_{i=1}^{l-2} a_i\right) \Gamma(\alpha) \left[\left(1 - \sum_{i=1}^{l-2} b_i\right) \Gamma(\beta+1)\right]^{q-1},$$

$$M_2 = \frac{(1 - \sum_{i=1}^{l-2} a_i)\Gamma(\alpha)[(1 - \sum_{i=1}^{l-2} b_i)\Gamma(\beta+1)]^{q-1}}{[\sum_{i=1}^{l-2} b_i \xi_i^{\beta}]^{q-1}(1-\xi_{l-2})\xi_{l-2}^{\alpha-1}}.$$

The following theorem is the main result in this paper.

Theorem 4.1 *Suppose that the conditions* (H_1) *and* (H_2) *hold. In addition, assume that there exist positive numbers* h, r, c *and* d *such that* $h < r < \frac{r}{\xi_{l-2}^{\alpha-1}} \leq c < d$, *and that* $f(t,u)$ *satisfies the following growth conditions*:

(H_3) $f(t,u) \leq (dM_1)^{p-1}$, *for* $(t,u) \in [0,1] \times [0,d]$,

(H_4) $f(t,u) > (rM_2)^{p-1}$, *for* $(t,u) \in [0,1] \times [r,c]$,

(H_5) $f(t,u) < (hM_1)^{p-1}$, *for* $(t,u) \in [0,1] \times [0,h]$.

Then the boundary value problem (1.1) *has at least three positive solutions* u_1, u_2 *and* u_3 *such that*

$$\gamma(u_i) < d, \quad i = 1,2,3;$$

$$r < \omega(u_1);$$

$$h < \psi(u_2) \quad \text{with } \omega(u_2) < r;$$

and

$$\omega(u_3) < h.$$

Proof First of all, we show $T : P \to P$ is a completely continuous operator.

For $u \in P$, in view of Lemma 3.2, we see that $Tu(t)$ is nonnegative and nondecreasing, consequently, we have $T : P \to P$. By using the continuity of $f(t,u)$, we obtain the operator T is continuous.

Let $\Omega \subset P$ be bounded, that is, there exists a positive constant l for any $u \in \Omega$, and let $L = \max_{0\leq t\leq 1, 0\leq u\leq l} f(t,u)$, then, for any $u \in \Omega$, we have

$$\begin{aligned}
w(s) &= \phi_q\left(\frac{\sum_{i=1}^{l-2} b_i \int_0^{\xi_i}(\xi_i-\tau)^{\beta-1}f(\tau,u(\tau))\,\mathrm{d}\tau}{(1-\sum_{i=1}^{l-2} b_i)\Gamma(\beta)} + \frac{\int_0^s (s-\tau)^{\beta-1}f(\tau,u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)}\right) \\
&\leq \phi_q\left(\frac{L\sum_{i=1}^{l-2} b_i \int_0^{\xi_i}(\xi_i-\tau)^{\beta-1}\,\mathrm{d}\tau}{(1-\sum_{i=1}^{l-2} b_i)\Gamma(\beta)} + \frac{L\int_0^1 (1-\tau)^{\beta-1}\,\mathrm{d}\tau}{\Gamma(\beta)}\right) \\
&= \phi_q\left(\frac{L\sum_{i=1}^{l-2} b_i\xi_i^{\beta}}{(1-\sum_{i=1}^{l-2} b_i)\Gamma(\beta+1)} + \frac{L}{\Gamma(\beta+1)}\right) \\
&\leq \left(\frac{L}{(1-\sum_{i=1}^{l-2} b_i)\Gamma(\beta+1)}\right)^{q-1}. \qquad (4.1)
\end{aligned}$$

So we get

$$\begin{aligned}
Tu(t) &= c_1 t^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}w(s)\,\mathrm{d}s \\
&\leq \frac{\int_0^1 w(s)\,\mathrm{d}s}{(1-\sum_{i=1}^{l-2} a_i)\Gamma(\alpha)} \leq \frac{L^{q-1}}{(1-\sum_{i=1}^{l-2} a_i)\Gamma(\alpha)[(1-\sum_{i=1}^{l-2} b_i)\Gamma(\beta+1)]^{q-1}}.
\end{aligned}$$

Hence, $T(\Omega)$ is uniformly bounded.

Now, we will prove that $T(\Omega)$ is equicontinuous. For each $u \in \Omega$, $0 \leq t_1 < t_2 \leq 1$, we have

$$\begin{aligned}
&\bigl|(Tu)(t_2)-(Tu)(t_1)\bigr| \\
&\quad= \left|c_1 t_2^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}w(s)\,\mathrm{d}s - c_1 t_1^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}w(s)\,\mathrm{d}s\right| \\
&\quad\leq c_1\bigl|t_2^{\alpha-1}-t_1^{\alpha-1}\bigr| + \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}(t_2-s)^{\alpha-1}w(s)\,\mathrm{d}s - \int_0^{t_1}(t_1-s)^{\alpha-1}w(s)\,\mathrm{d}s\right| \\
&\quad\leq \frac{1}{(1-\sum_{i=1}^{l-2} a_i)\Gamma(\alpha)}\left(\frac{L}{(1-\sum_{i=1}^{l-2} b_i)\Gamma(\beta+1)}\right)^{q-1}\bigl(t_2^{\alpha-1}-t_1^{\alpha-1}\bigr) \\
&\qquad+ \frac{1}{\Gamma(\alpha+1)}\left(\frac{L}{(1-\sum_{i=1}^{l-2} b_i)\Gamma(\beta+1)}\right)^{q-1}\bigl[\bigl(t_2^{\alpha}-t_1^{\alpha}\bigr)+(t_2-t_1)^{\alpha}\bigr] \\
&\qquad+ \frac{1}{\Gamma(\alpha+1)}\left(\frac{L}{(1-\sum_{i=1}^{l-2} b_i)\Gamma(\beta+1)}\right)^{q-1}(t_2-t_1)^{\alpha}.
\end{aligned}$$

Therefore, $T(\Omega)$ is equicontinuous. Applying the Arzelá–Ascoli theorem, we conclude that T is a completely continuous operator.

For $u \in P$, let

$$\gamma(u) = \theta(u) = \psi(u) = \max_{t\in[0,1]}\bigl|u(t)\bigr|, \qquad \omega(u) = \min_{t\in[\xi_{l-2},1]}\bigl|u(t)\bigr|. \qquad (4.2)$$

Secondly, we prove $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$.

For $u \in \overline{P(\gamma,d)}$, in view of (4.1), we get

$$w(s) \leq \left(\frac{(dM_1)^{p-1}}{(1-\sum_{i=1}^{l-2} b_i)\Gamma(\beta+1)}\right)^{q-1} = \frac{dM_1}{[(1-\sum_{i=1}^{l-2} b_i)\Gamma(\beta+1)]^{q-1}},$$

then

$$\begin{aligned}
Tu(t) &= c_1 t^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} w(s)\,\mathrm{d}s \\
&= \frac{\int_0^1 w(s)\,\mathrm{d}s - \sum_{i=1}^{l-2} a_i \int_0^{\xi_i} w(s)\,\mathrm{d}s}{(1-\sum_{i=1}^{l-2} a_i)\Gamma(\alpha)} t^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} w(s)\,\mathrm{d}s \\
&\leq \frac{\int_0^1 w(s)\,\mathrm{d}s}{(1-\sum_{i=1}^{l-2} a_i)\Gamma(\alpha)} \leq \frac{dM_1}{(1-\sum_{i=1}^{l-2} a_i)\Gamma(\alpha)[(1-\sum_{i=1}^{l-2} b_i)\Gamma(\beta+1)]^{q-1}} \\
&= d.
\end{aligned}$$

So we obtain $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$.

Finally, we show conditions (C_1)–(C_3) in Theorem 2.1 are satisfied for T. To prove that the second part of condition (C_1) holds, taking $u_0(t) = 0.5(r+c)$, in view of (4.1), we obtain

$$\gamma(u_0) = \theta(u_0) = 0.5(r+c), \qquad \omega(u_0) = 0.5(r+c).$$

So $\{u_0 \in P(\gamma,\theta,\omega,r,c,d) | \omega(u_0) > r\}$, which shows that

$$\bigl\{x \in P(\gamma,\theta,\omega,r,c,d) | \omega(x) > r\bigr\} \neq \emptyset$$

for all $u \in P(\gamma,\theta,\omega,r,c,d)$, then

$$\begin{aligned}
\omega(Tu) &= \min_{t\in[\xi_{l-2},1]} \bigl|Tu(t)\bigr| = \bigl|Tu(\xi_{l-2})\bigr| \\
&= c_1 \xi_{l-2}^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_0^{\xi_{l-2}} (\xi_{l-2}-s)^{\alpha-1} w(s)\,\mathrm{d}s \\
&= \frac{\int_0^1 w(s)\,\mathrm{d}s - \sum_{i=1}^{l-2} a_i \int_0^{\xi_i} w(s)\,\mathrm{d}s}{(1-\sum_{i=1}^{l-2} a_i)\Gamma(\alpha)} \xi_{l-2}^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_0^{\xi_{l-2}} (\xi_{l-2}-s)^{\alpha-1} w(s)\,\mathrm{d}s \\
&\geq \frac{\int_0^1 w(s)\,\mathrm{d}s - \sum_{i=1}^{l-2} a_i \int_0^{\xi_{l-2}} w(s)\,\mathrm{d}s}{(1-\sum_{i=1}^{l-2} a_i)\Gamma(\alpha)} \xi_{l-2}^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\xi_{l-2}^{\alpha-1}\int_0^{\xi_{l-2}} w(s)\,\mathrm{d}s \\
&= \frac{\int_{\xi_{l-2}}^1 w(s)\,\mathrm{d}s}{(1-\sum_{i=1}^{l-2} a_i)\Gamma(\alpha)} \xi_{l-2}^{\alpha-1} \\
&= \frac{\int_{\xi_{l-2}}^1 \phi_q\bigl(\frac{\sum_{i=1}^{l-2} b_i \int_0^{\xi_i} (\xi_i-\tau)^{\beta-1} f(\tau,u(\tau))\,\mathrm{d}\tau}{(1-\sum_{i=1}^{l-2} b_i)\Gamma(\beta)} + \frac{\int_0^s (s-\tau)^{\beta-1} f(\tau,u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)}\bigr)\,\mathrm{d}s}{(1-\sum_{i=1}^{l-2} a_i)\Gamma(\alpha)} \xi_{l-2}^{\alpha-1} \\
&\geq \frac{\int_{\xi_{l-2}}^1 \phi_q\bigl(\frac{\sum_{i=1}^{l-2} b_i \int_0^{\xi_i} (\xi_i-\tau)^{\beta-1} f(\tau,u(\tau))\,\mathrm{d}\tau}{(1-\sum_{i=1}^{l-2} b_i)\Gamma(\beta)}\bigr)\,\mathrm{d}s}{(1-\sum_{i=1}^{l-2} a_i)\Gamma(\alpha)} \xi_{l-2}^{\alpha-1}
\end{aligned}$$

$$
\begin{aligned}
&> \frac{(\frac{(rM_2)^{p-1}\sum_{i=1}^{l-2} b_i\xi_i^\beta}{(1-\sum_{i=1}^{l-2} b_i)\Gamma(\beta+1)})^{q-1}(1-\xi_{l-2})}{(1-\sum_{i=1}^{l-2} a_i)\Gamma(\alpha)}\xi_{l-2}^{\alpha-1} \\
&= \frac{rM_2[\sum_{i=1}^{l-2} b_i\xi_i^\beta]^{q-1}(1-\xi_{l-2})\xi_{l-2}^{\alpha-1}}{(1-\sum_{i=1}^{l-2} a_i)\Gamma(\alpha)[(1-\sum_{i=1}^{l-2} b_i)\Gamma(\beta+1)]^{q-1}} \\
&= r.
\end{aligned}
$$

Hence, we shall verify the condition (C_2). If $u \in P(\gamma,\omega,r,d)$ with $\theta(Tu) > c$, in view of (4.1), we have

$$
\begin{aligned}
\omega(Tu) &= \min_{t\in[\xi_{l-2},1]}\left|Tu(t)\right| = \left|Tu(\xi_{l-2})\right| \\
&= c_1\xi_{l-2}^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_0^{\xi_{l-2}}(\xi_{l-2}-s)^{\alpha-1}w(s)\,\mathrm{d}s
\end{aligned}
$$

and

$$
\begin{aligned}
\theta(Tu) &= \max_{t\in[0,1]}\left|Tu(t)\right| = Tu(1) \\
&= c_1 - \frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}w(s)\,\mathrm{d}s,
\end{aligned}
$$

then

$$
\begin{aligned}
&\omega(Tu) - \xi_{l-2}^{\alpha-1}\theta(Tu) \\
&\quad= c_1\xi_{l-2}^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_0^{\xi_{l-2}}(\xi_{l-2}-s)^{\alpha-1}w(s)\,\mathrm{d}s - c_1\xi_{l-2}^{\alpha-1} + \frac{\xi_{l-2}^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}w(s)\,\mathrm{d}s \\
&\quad= \frac{\xi_{l-2}^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}w(s)\,\mathrm{d}s - \frac{1}{\Gamma(\alpha)}\int_0^{\xi_{l-2}}(\xi_{l-2}-s)^{\alpha-1}w(s)\,\mathrm{d}s \\
&\quad= \frac{1}{\Gamma(\alpha)}\int_0^1(\xi_{l-2}-s\xi_{l-2})^{\alpha-1}w(s)\,\mathrm{d}s - \frac{1}{\Gamma(\alpha)}\int_0^{\xi_{l-2}}(\xi_{l-2}-s)^{\alpha-1}w(s)\,\mathrm{d}s \\
&\quad\geq 0.
\end{aligned}
$$

So we get $\omega(Tu) \geq \xi_{l-2}^{\alpha-1}\theta(Tu) > \xi_{l-2}^{\alpha-1}c \geq r$, that is, $\omega(Tu) > r$. So we finished the proof of (C_2).

Lastly, we shall prove the condition (C_3). It is easy to see that $0 \notin Q(\gamma,\psi,h,d)$, then we shall prove for $u \in Q(\gamma,\psi,h,d)$ with $\psi(u) = h$, we get $\psi(Tu) < h$.

For $u \in Q(\gamma,\psi,h,d)$ with $\psi(u) = h$, in view of (4.1), we have

$$
w(s) < \left(\frac{(hM_1)^{p-1}}{(1-\sum_{i=1}^{l-2} b_i)\Gamma(\beta+1)}\right)^{q-1} = \frac{hM_1}{[(1-\sum_{i=1}^{l-2} b_i)\Gamma(\beta+1)]^{q-1}},
$$

so we get

$$
\begin{aligned}
\psi(Tu) &= \max_{t\in[0,1]}\left|Tu(t)\right| \\
&= Tu(1) = c_1 - \frac{\int_0^1(1-s)^{\alpha-1}w(s)\,\mathrm{d}s}{\Gamma(\alpha)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\int_0^1 w(s)\,\mathrm{d}s - \sum_{i=1}^{l-2} a_i \int_0^{\xi_i} w(s)\,\mathrm{d}s}{(1-\sum_{i=1}^{l-2} a_i)\Gamma(\alpha)} - \frac{\int_0^1 (1-s)^{\alpha-1} w(s)\,\mathrm{d}s}{\Gamma(\alpha)} \\
&\leq \frac{\int_0^1 w(s)\,\mathrm{d}s}{(1-\sum_{i=1}^{l-2} a_i)\Gamma(\alpha)} \\
&< \frac{hM_1}{(1-\sum_{i=1}^{l-2} a_i)\Gamma(\alpha)[(1-\sum_{i=1}^{l-2} b_i)\Gamma(\beta+1)]^{q-1}} = h.
\end{aligned}
$$

Consequently, the boundary value problem (1.1) has at least three positive solutions u_1, u_2 and u_3 such that

$$
\begin{aligned}
&\gamma(u_i) < d, \quad i = 1, 2, 3; \\
&r < \omega(u_1); \\
&h < \psi(u_2) \quad \text{with } \omega(u_2) < r;
\end{aligned}
$$

and

$$
\omega(u_3) < h.
$$

The proof is complete. □

Remark 4.1 Assume that $f(t,0) \neq 0$ on a compact set, then u_3 is a nontrivial solution.

5 Example

In this section, we give a simple example to explain the main theorem.

Example 5.1 For the problem (1.1), let $\alpha = 2.8$, $\beta = 1.8$, $a_1 = 0.1$, $a_2 = 0.3$, $b_1 = 0.1$, $b_2 = 0.5$, $\xi_1 = 0.2$, $\xi_2 = 0.4$, $p = 3.0$ and

$$
f(t,u) = \begin{cases}
0.5t, & 0 \leq t \leq 1, 0 \leq u \leq 1, \\
0.5t + 246(u-1), & 0 \leq t \leq 1, 1 < u \leq 2, \\
0.5t + 246 + 2(u-2), & 0 \leq t \leq 1, 2 < u \leq 12, \\
0.5t + 266, & 0 \leq t \leq 1, u > 12.
\end{cases}
$$

In addition, if we take $h = 1$, $r = 2$, $c = 12$ and $d = 22$, then $f(t,u)$ satisfies the following growth conditions:

$$
\begin{aligned}
&f(t,u) \leq (dM_1)^{p-1} \approx 328.408680, \quad (t,u) \in [0,1] \times [0,22], \\
&f(t,u) > (rM_2)^{p-1} \approx 231.653244, \quad (t,u) \in [0,1] \times [2,12], \\
&f(t,u) < (hM_1)^{p-1} \approx 0.678530, \quad (t,u) \in [0,1] \times [0,1].
\end{aligned}
$$

Then all the conditions of Theorem 4.1 are satisfied. Hence, by Theorem 4.1, we see that the aforementioned problem has at least three positive solutions u_1, u_2 and u_3 such that

$$\gamma(u_i) < 22, \quad i = 1, 2, 3;$$

$$2 < \omega(u_1);$$

$$1 < \psi(u_2) \quad \text{with } \omega(u_2) < 2;$$

and

$$\omega(u_3) < 1.$$

6 Conclusions

The Avery–Peterson fixed point theorem is used to solve the problem of a kind of nonlinear mixed fractional differential equation with a p-Laplacian operator. Under certain nonlinear growth conditions of the nonlinearity, we get the existence of multiple positive solutions for the boundary value problem. Finally, an example is presented to illustrate the effectiveness of the main result.

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