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Multiple positive solutions for a coupled system of nonlinear impulsive fractional differential equations with parameters

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Abstract

We consider the multiplicity of positive solutions (PSs) for a coupled system involving nonlinear impulsive fractional differential equations with parameters. By employing the classical Guo–Krasnosel'skii fixed point theorem, some sufficient criteria for the existence of multiple PSs in terms of different values of parameters are derived. As an application, an example is given to illustrate the theoretical results.

MSC: 26A33; 35A15; 34A08; 58E05

Keywords: Impulsive; Caputo fractional differential; Coupled system; Multiplicity

1 Introduction

The fractional calculus is an extension of the traditional integer calculus, which has the properties of an infinity memory and is hereditary. In recent decades, fractional calculus has aroused much attention and has been extensively applied to establish mathematical models in the fields of signals, viscoelastic theory, fluid dynamics, computer networking, electrical circuits, control theory and so on [1–9]. As a consequence, the subject of fractional differential equations (FDEs) is very popular and of importance. Especially, the investigation of the existence of the solution for FDEs has received considerable attention, the reader may refer to [10–22] and the references therein.

Though the theory of positive solutions (PSs) for ordinary differential equations with parameters is mature, not much has been done for FDEs with parameters [12, 13, 17, 20]. By using the Guo–Krasnosel'skii fixed point theorem on cones, some sufficient conditions for the existence of multiple PSs and eigenvalue intervals are established in [17] for the following FDEs with parameter:

$$\begin{cases} {}^cD_{0^+}^{\alpha}u(t) = \lambda f(u(t)), & t \in (0,1), \alpha \in (1,2), \\ u(0) + u'(0) = 0, & u(1) + u'(1) = 0. \end{cases}$$

It should be emphasized that much work focuses on the BVPs of nonlinear FDEs with impulses [11, 16, 18, 19, 21, 22]. The authors in [21] consider the following generalized antiperiodic BVPs for impulsive FDEs:

$$\begin{cases} {}^cD^q_{0^+}u(t) = f(t,u(t)), & t \in J = [0,1], t \neq t_k, \\ \Delta u(t_k) = I_k, \quad \Delta u'(t_k) = J_k, & k = 1,\dots,m, \\ au(0) + bu(1) = 0, \quad au'(0) + bu'(1) = 0, \end{cases}$$

where $q \in (1,2)$ and $a \geq b > 0$. Some new existence theorems of at least one solution are established via fixed point methods.

For the BVPs of a nonlinear coupled fractional differential system with parameters, the existence of PSs is considered in [20]. Some multiplicity theorems of PSs for nonlinear impulsive FDEs are presented in [16]. However, as far as we know, there is no paper to investigate the multiplicity of PSs for impulsive fractional differential coupled system with parameters. The above-mentioned work and observation inspire us to address the following coupled system of nonlinear impulsive FDEs with parameters (abbreviated by BVPs (1)):

$$\begin{cases} {}^cD^\alpha_{0^+}u(t) + \lambda f(t,u(t),v(t)) = 0, & t \in J = [0,1], t \neq t_k, \\ {}^cD^\beta_{0^+}v(t) + \mu g(t,u(t),v(t)) = 0, & t \in J = [0,1], t \neq t_k, \\ \Delta u(t_k) = I_k(u(t_k)), \quad \Delta u'(t_k) = J_k(u(t_k)), \\ \Delta v(t_k) = P_k(v(t_k)), \quad \Delta v'(t_k) = Q_k(v(t_k)), & k = 1,\dots,m, \\ u(0) = au(1), \quad u'(1) = bu'(0), \quad v(0) = cv(1), \quad v'(1) = dv'(0), \end{cases} \tag{1}$$

where $\alpha, \beta \in (1,2]$, $a,b,c,d \in (1,+\infty)$, $\lambda, \mu \in (0,+\infty)$ are parameters, ${}^cD^\alpha_{0^+}$ (${}^cD^\beta_{0^+}$) is the standard Caputo fractional derivative of order $\alpha(\beta)$, $f,g : J \times R^+ \times R^+ \to R^+$ are jointly continuous, $I_k, J_k, P_k, Q_k \in C(R^+, R^-)$, $R^+ = [0,+\infty)$, $R^- = (-\infty,0]$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, in which $u(t_k^-) = \lim_{\theta\to 0^-} u(t_k+\theta)$ and $u(t_k^+) = \lim_{\theta\to 0^+} u(t_k+\theta)$ indicate the left and right limits of $u(t)$ at $t = t_k$, respectively, and the impulsive point set $\{t_k\}_{k=1}^m$ satisfies $0 < t_1 < \cdots < t_m < t_{m+1} = 1$. Let us set $J_0 = [0,t_1]$ and $J_k = (t_k, t_{k+1}]$, where $k = 1,\dots,m$. So $J = \bigcup_{k=0}^m J_k$.

Due to the existence of impulsiveness in the nonlinear coupled system (1), it is challenging to deal with the existence of multiple PSs for BVPs (1). We first give the natural formulas of PSs for the nonlinear coupled system by constructing the associated Green's function. Based on the properties of the Green's function and some assumptions on the nonlinear functions, some sufficient criteria for the multiplicity of PSs are obtained. Meanwhile, the ranges of the parameters λ and μ of the existence for PSs are also given. The multiplicity theorems of this paper are established by applying the Guo–Krasnosel'skii fixed point theorem. Finally, an example is provided to illustrate the validity of our main results.

2 Preliminaries

Definition 2.1 ([1, 23]) The Riemann–Liouville fractional integral of the function $f(t) \in C^n([0,\infty),R)$ is defined as

$$I^\alpha_{0^+}f(t) = I^\alpha\big[f(\cdot)\big](t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)\,d\tau,$$

where $n-1<\alpha\leq n$, $n\in\{1,2,\ldots\}$ and $\Gamma(\cdot)$ is the well-known Gamma function, defined as $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,dt$.

Definition 2.2 ([1, 23]) The Caputo fractional derivative of the function $f(t)\in C^n([0,\infty),R)$ is defined as

$$ {}^cD_{0^+}^\alpha f(t)=I^{n-\alpha}f^{(n)}(t)=\int_0^t\frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)}f^{(n)}(\tau)\,d\tau, $$

where $n-1<\alpha\leq n$, $n\in\{1,2,\ldots\}$.

Lemma 2.1 ([1, 23]) *If $u\in C^n(J)$ and ${}^cD_{0^+}^\alpha u\in L^1(J)$ with a Caputo fractional derivative of order $\alpha>0$, then*

$$ I_{0^+}^\alpha {}^cD_{0^+}^\alpha u(t)=u(t)+\sum_{k=0}^{n-1}c_kt^k,\quad t\in J, $$

where $c_k\in R$ and n is the smallest integer not less than α.

Lemma 2.2 (Guo–Krasnosel'skii fixed point theorem [24]) *Let $P\subseteq E$ be a cone, K_1 and K_2 be two bounded open balls of the Banach space E centered at the origin with $0\in K_1$ and $\overline{K}_1\subset K_2$. Assume that $T:P\cap(\overline{K}_2\setminus K_1)\to P$ is a completely continuous operator such that either*

(i) $\|Tv\|\leq\|v\|$, $v\in P\cap\partial K_1$ *and* $\|Tv\|\geq\|v\|$, $v\in P\cap\partial K_2$, *or*

(ii) $\|Tv\|\geq\|v\|$, $v\in P\cap\partial K_1$ *and* $\|Tv\|\leq\|v\|$, $v\in P\cap\partial K_2$ *hold.*

Then T has at least one fixed point in $P\cap(\overline{K}_2\setminus K_1)$.

Lemma 2.3 *Given $h\in C(J)$ and $\alpha\in(1,2]$, the unique solution of*

$$ \begin{cases} {}^cD_{0^+}^\alpha u(t)+h(t)=0, & t\in J, t\neq t_k,\\ \Delta u(t_k)=I_k(u), \quad \Delta u'(t_k)=J_k(u), & k=1,\ldots,m,\\ u(0)=au(1), \quad u'(1)=bu'(0), & a,b>1, \end{cases} \tag{2} $$

is $u(t)=\int_0^1 G_\alpha(t,s)h(s)\,ds-\sum_{i=1}^m G_{a,b}(t,t_i)J_i(u)-\sum_{i=1}^m G_a(t,t_i)I_i(u)$, $t\in J$, where

$$ G_\alpha(t,s)=\begin{cases} -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{a(1-s)^{\alpha-1}}{(a-1)\Gamma(\alpha)}+\frac{(a+(1-a)t)(1-s)^{\alpha-2}}{(a-1)(b-1)\Gamma(\alpha-1)}, & 0\leq s\leq t\leq 1,\\ \frac{a(1-s)^{\alpha-1}}{(a-1)\Gamma(\alpha)}+\frac{(a+(1-a)t)(1-s)^{\alpha-2}}{(a-1)(b-1)\Gamma(\alpha-1)}, & 0\leq t\leq s\leq 1, \end{cases} \tag{3} $$

$$ G_{a,b}(t,t_i)=\begin{cases} \frac{ab+(1-b)t_i+(1-a)bt}{(a-1)(b-1)}, & 0\leq t_i<t\leq 1, i=1,\ldots,m,\\ \frac{ab+a(1-b)t_i+(1-a)t}{(a-1)(b-1)}, & 0\leq t\leq t_i\leq 1, i=1,\ldots,m, \end{cases} \tag{4} $$

$$ G_a(t,t_i)=\begin{cases} \frac{1}{a-1}, & 0\leq t_i<t\leq 1, i=1,\ldots,m,\\ \frac{a}{a-1}, & 0\leq t\leq t_i\leq 1, i=1,\ldots,m \end{cases} \tag{5} $$

Proof By applying Lemma 2.1, the solution of impulsive BVPs (2) can be uniquely expressed as

$$ u(t)=-I_{0^+}^\alpha h(t)-c_k-d_kt,\quad t\in J_k, k=0,1,\ldots,m. \tag{6} $$

From (6), one has

$$u'(t) = -I_{0^+}^{\alpha-1}h(t) - d_k, \quad t \in J_k, k = 0, 1, \ldots, m. \tag{7}$$

Applying the boundary value conditions of BVPs (2), we can see from (6) and (7) that

$$-aI_{0^+}^{\alpha}h(1) + c_0 - ac_m - ad_m = 0, \tag{8}$$

$$-I_{0^+}^{\alpha-1}h(1) + bd_0 - d_m = 0. \tag{9}$$

It can be derived from the impulsive condition of BVPs (2) that

$$c_{k-1} - c_k + J_k(u)t_k = I_k(u), \tag{10}$$

$$d_{k-1} - d_k = J_k(u), \quad k = 1, \ldots, m. \tag{11}$$

It thus follows from (9) and (11) that

$$d_0 = \frac{1}{b-1}I_{0^+}^{\alpha-1}h(1) - \frac{1}{b-1}\sum_{i=1}^{m} J_i(u), \tag{12}$$

$$d_m = \frac{1}{b-1}I_{0^+}^{\alpha-1}h(1) - \frac{b}{b-1}\sum_{i=1}^{m} J_i(u). \tag{13}$$

In the light of (11) and (12), we have

$$\begin{aligned} d_k &= d_0 - \sum_{i=1}^{k} J_i(u) \\ &= \frac{1}{b-1}I_{0^+}^{\alpha-1}h(1) - \frac{1}{b-1}\sum_{i=1}^{m} J_i(u) - \sum_{i=1}^{k} J_i(u). \end{aligned} \tag{14}$$

We can see from (8), (10) and (13) that

$$\begin{aligned} c_0 &= \frac{a}{a-1}\left(-I_{0^+}^{\alpha}h(1) - d_m + \sum_{i=1}^{m}\bigl(I_i(u) - J_i(u)t_i\bigr)\right) \\ &= \frac{a}{a-1}\left(-I_{0^+}^{\alpha}h(1) - \frac{I_{0^+}^{\alpha-1}h(1)}{b-1} + \frac{b}{b-1}\sum_{i=1}^{m} J_i(u) + \sum_{i=1}^{m}\bigl(I_i(u) - J_i(u)t_i\bigr)\right), \end{aligned} \tag{15}$$

this together with (10) implies that

$$\begin{aligned} c_k = {} & \frac{a}{a-1}\left(-I_{0^+}^{\alpha}h(1) - \frac{I_{0^+}^{\alpha-1}h(1)}{b-1} + \frac{b}{b-1}\sum_{i=1}^{m} J_i(u) + \sum_{i=1}^{m}\bigl(I_i(u) - J_i(u)t_i\bigr)\right) \\ & - \sum_{i=1}^{k}\bigl(I_i(u) - J_i(u)t_i\bigr), \end{aligned} \tag{16}$$

where $k = 1, 2, \ldots, m$. It thus follows from (14) and (16) that

$$\begin{aligned} &c_k + d_k t \\ &\quad = \frac{a}{a-1}\left(-I_{0^+}^{\alpha} h(1) - \frac{I_{0^+}^{\alpha-1} h(1)}{b-1} + \frac{b}{b-1}\sum_{i=1}^{m} J_i(u) + \sum_{i=1}^{m}\bigl(I_i(u) - J_i(u)t_i\bigr)\right) \\ &\qquad + \frac{t}{b-1} I_{0^+}^{\alpha-1} h(1) - \frac{t}{b-1}\sum_{i=1}^{m} J_i(u) - \sum_{i=1}^{k}\bigl[I_i(u) + J_i(u)(t-t_i)\bigr]. \end{aligned} \tag{17}$$

For $t \in J_k = (t_k, t_{k+1}]$, $k = 1, \ldots, m$, by substituting (17) into (6), we obtain

$$\begin{aligned} u(t) &= -I_{0^+}^{\alpha} h(t) + \frac{a I_{0^+}^{\alpha} h(1)}{a-1} + \frac{a I_{0^+}^{\alpha-1} h(1)}{(a-1)(b-1)} - \frac{t}{b-1} I_{0^+}^{\alpha-1} h(1) \\ &\quad - \frac{ab}{(a-1)(b-1)}\sum_{i=1}^{m} J_i(u) - \frac{a}{a-1}\sum_{i=1}^{m}\bigl[I_i(u) - J_i(u)t_i\bigr] \\ &\quad + \frac{t}{b-1}\sum_{i=1}^{m} J_i(u) + \sum_{i=1}^{k} I_i(u) + \sum_{i=1}^{k}(t-t_i)J_i(u) \\ &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds + \frac{a}{a-1}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds \\ &\quad + \frac{a+(1-a)t}{(a-1)(b-1)}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)\,ds - \frac{a}{a-1}\sum_{i=1}^{m} I_i(u) \\ &\quad - \sum_{i=1}^{m}\frac{ab + a(1-b)t_i + (1-a)t}{(a-1)(b-1)} J_i(u) + \sum_{i=1}^{k}\bigl[I_i(u) + (t-t_i)J_i(u)\bigr] \\ &= \int_0^t \left[-\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{a(1-s)^{\alpha-1}}{(a-1)\Gamma(\alpha)} + \frac{(a+(1-a)t)(1-s)^{\alpha-2}}{(a-1)(b-1)\Gamma(\alpha-1)}\right] h(s)\,ds \\ &\quad + \int_t^1 \left[\frac{a(1-s)^{\alpha-1}}{(a-1)\Gamma(\alpha)} + \frac{(a+(1-a)t)(1-s)^{\alpha-2}}{(a-1)(b-1)\Gamma(\alpha-1)}\right] h(s)\,ds \\ &\quad - \sum_{i=1}^{k}\frac{ab + (1-b)t_i + (1-a)bt}{(a-1)(b-1)} J_i(u) \\ &\quad - \sum_{i=k+1}^{m}\frac{ab + a(1-b)t_i + (1-a)t}{(a-1)(b-1)} J_i(u) \\ &\quad - \frac{1}{a-1}\sum_{i=1}^{k} I_i(u) - \frac{a}{a-1}\sum_{i=k+1}^{m} I_i(u) \\ &= \int_0^1 G_\alpha(t,s)h(s)\,ds - \sum_{i=1}^{m} G_{a,b}(t,t_i)J_i(u) - \sum_{i=1}^{m} G_a(t,t_i)I_i(u), \end{aligned}$$

in which $G_\alpha(t,s)$, $G_{a,b}(t,t_i)$ and $G_a(t,t_i)$ are defined by (3), (4) and (5), respectively.

For $t \in J_0 = [0, t_1]$, substituting (12) and (15) into (6) yields

$$\begin{aligned} u(t) &= -I_{0^+}^{\alpha} h(t) - c_0 - d_0 t \\ &= -I_{0^+}^{\alpha} h(t) + \frac{a}{a-1} I_{0^+}^{\alpha} h(1) + \frac{a I_{0^+}^{\alpha-1} h(1)}{(a-1)(b-1)} - \frac{t}{b-1} I_{0^+}^{\alpha-1} h(1) \end{aligned}$$

$$
\begin{aligned}
&- \frac{ab\sum_{i=1}^{m} J_i(u)}{(a-1)(b-1)} - \frac{a}{a-1}\sum_{i=1}^{m}\bigl[I_i(u) - J_i(u)t_i\bigr] + \frac{t}{b-1}\sum_{i=1}^{m} J_i(u) \\
&= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds + \frac{a}{a-1}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds \\
&\quad + \frac{a+(1-a)t}{(a-1)(b-1)}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)\,ds \\
&\quad - \sum_{i=1}^{m} \frac{ab + a(1-b)t_i + (1-a)t}{(a-1)(b-1)} J_i(u) - \frac{a}{a-1}\sum_{i=1}^{m} I_i(u) \\
&= \int_0^1 G_\alpha(t,s)h(s)\,ds - \sum_{i=1}^{m} G_{a,b}(t,t_i)J_i(u) - \sum_{i=1}^{m} G_a(t,t_i)I_i(u),
\end{aligned}
$$

in which $G_\alpha(t,s)$, $G_{a,b}(t,t_i)$ and $G_a(t,t_i)$ are defined by (3), (4) and (5), respectively. The proof is thus completed. □

Lemma 2.4 *Let $a,b>1$, the functions $G_\alpha(t,s)$, $G_{a,b}(t,t_i)$ and $G_a(t,t_i)$ are continuous and satisfy the following properties*:

$$
\begin{aligned}
&G_\alpha(1,s) \le G_\alpha(t,s) \le aG_\alpha(1,s), \quad \forall t,s\in J, \\
&\frac{1}{(a-1)(b-1)} \le G_{a,b}(t,t_i) \le \frac{ab}{(a-1)(b-1)}, \quad \forall t,t_i \in J, \\
&\frac{1}{a-1} \le G_a(t,t_i) \le \frac{a}{a-1}, \quad \forall t,t_i\in J.
\end{aligned}
$$

Proof We can see from the expressions of $G_\alpha(t,s)$, $G_{a,b}(t,t_i)$ and $G_a(t,t_i)$ that $G_\alpha(t,s)$, $G_{a,b}(t,t_i)$, $G_a(t,t_i)\in C(J\times J)$. For $t,s\in J$, using (3) yields

$$
\frac{\partial}{\partial t}G_\alpha(t,s) = \begin{cases} -\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{(1-s)^{\alpha-2}}{(b-1)\Gamma(\alpha-1)}, & 0\le s\le t\le 1, \\ -\frac{(1-s)^{\alpha-2}}{(b-1)\Gamma(\alpha-1)}, & 0\le t\le s\le 1. \end{cases} \tag{18}
$$

Clearly, for $t\in J$, $G_\alpha(t,s)$ is decreasing with respect to t. Therefore,

$$
G_\alpha(1,s) \le G_\alpha(t,s) \le G_\alpha(0,s) = aG_\alpha(1,s).
$$

In view of (4) and (5), it is obviously that

$$
\begin{aligned}
&\frac{1}{(a-1)(b-1)} = G_{a,b}(1,1) \le G_{a,b}(t,t_i) \le G_{a,b}(0,0) = \frac{ab}{(a-1)(b-1)}, \\
&\frac{1}{a-1} \le G_a(t,t_i) \le \frac{a}{a-1}, \quad \forall t,t_i\in J.
\end{aligned}
$$

The proof is thus completed. □

Similar results to Lemmas 2.3 and 2.4 can be formulated for the following BVPs (19):

$$
\begin{cases} {}^cD_{0^+}^{\beta} v(t) + \zeta(t) = 0, \quad t\in J\backslash\{t_1,\ldots,t_m\}, \\ \Delta v(t_k) = P_k(v(t_k)), \qquad \Delta v'(t_k) = Q_k(v(t_k)), \\ v(0) = cv(1), \qquad v'(1) = dv'(0), \quad c,d>1, \end{cases} \tag{19}
$$

where $\beta \in (1,2]$ and $\zeta(t) \in C(J)$, $k = 1,\dots,m$. We introduce $G_\beta(t,s)$, $G_{c,d}(t,t_i)$ and $G_c(t,t_i)$, the corresponding functions for the BVPs (19) defined in a similar manner to $G_\alpha(t,s)$, $G_{a,b}(t,t_i)$ and $G_a(t,t_i)$, respectively.

3 Main results

In this section, some sufficient criteria are derived to guarantee the multiplicity of PSs for BVPs (1).

Let $E = \{(u,v) : u,v \in C(J)\}$ be endowed with the norm $\|\cdot\|$ defined as $\|(u,v)\| = \|u\| + \|v\|$ for $(u,v) \in E$, where $\|u\| = \max_{t\in J} |u(t)|$ and $\|v\| = \max_{t\in J} |v(t)|$. Let the Banach space $PC(J)$ and the cone $K \in PC(J)$ be, respectively, defined as

$$PC(J) = \bigl\{(u,v) \in E : u,v \in C\bigl(J',R^+\bigr), u\bigl(t_k^-\bigr), u\bigl(t_k^+\bigr), v\bigl(t_k^-\bigr) \text{ and } v\bigl(t_k^+\bigr) \text{ exist with } u\bigl(t_k^-\bigr) = u(t_k) \text{ and } v\bigl(t_k^-\bigr) = v(t_k), J' = J\backslash\{t_1,\dots,t_m\}\bigr\}$$

and

$$K = \bigl\{(u,v) \in PC(J) : u + v \ge \gamma \bigl\|(u,v)\bigr\| \text{ with } \gamma = \min\bigl\{(ab)^{-1},(cd)^{-1}\bigr\}\bigr\}. \tag{20}$$

To begin with, we need the following assumptions to derive the main results.

(B_1) $a,b,c,d \in (1,+\infty)$ and $\sigma_1,\sigma_2 \in (0,+\infty)$ with $\sigma_1 = \int_0^1 G_\alpha(1,s)\,ds$ and $\sigma_2 = \int_0^1 G_\beta(1,s)\,ds$.

(B_2) $f(t,u,v), g(t,u,v) \in C(J \times R^+ \times R^+, R^+)$.

(B_3) $I_k(u), J_k(u), P_k(v), Q_k(v) \in C(R^+,R^-)$, $k = 1,\dots,m$.

For simplicity, some important notations and functions are introduced as follows:

$$\eta = \max\biggl\{-\frac{ab\sum_{i=1}^m J_i(u)}{(a-1)(b-1)} - \frac{a\sum_{i=1}^m I_i(u)}{a-1}, -\frac{cd\sum_{i=1}^m Q_i(v)}{(c-1)(d-1)} - \frac{c\sum_{i=1}^m P_i(v)}{c-1}\biggr\},$$

$$\Phi(r) = \max_{t\in J, u+v\in[\gamma r,r]}\bigl\{f(\cdot),g(\cdot)\bigr\}, \qquad \phi(r) = \min_{t\in J, u+v\in[\gamma r,r]}\bigl\{f(\cdot),g(\cdot)\bigr\},$$

$$f_\delta = \liminf_{u+v\to\delta}\min_{t\in J}\frac{f(\cdot)}{u+v}, \qquad g_\delta = \liminf_{u+v\to\delta}\min_{t\in J}\frac{g(\cdot)}{u+v},$$

where $f(\cdot) = f(t,u(t),v(t))$, $g(\cdot) = g(t,u(t),v(t))$ and δ denotes 0 or $+\infty$.

Define two operators $T_\alpha, T_\beta : PC(J) \to PC(J)$ as

$$T_\alpha(u,v) = \lambda\int_0^1 G_\alpha(t,s)f(\cdot)\,ds - \sum_{i=1}^m G_{a,b}(t,t_i)J_i(u) - \sum_{i=1}^m G_a(t,t_i)I_i(u),$$

$$T_\beta(u,v) = \mu\int_0^1 G_\beta(t,s)g(\cdot)\,ds - \sum_{i=1}^m G_{c,d}(t,t_i)Q_i(v) - \sum_{i=1}^m G_c(t,t_i)P_i(v),$$

and the operator $T : PC(J) \to PC(J)$ as

$$T(u,v) = \bigl(T_\alpha(u,v), T_\beta(u,v)\bigr).$$

It is obvious that (u,v) is a pair of PSs of BVPs (1) if (u,v) is a fixed point of T.

Lemma 3.1 *Assume that* (B_1)–(B_3) *hold, then* $T: K \to K$ *is completely continuous.*

Proof Due to the functions G_α, G_β, G_a, G_b, $G_{a,b}$, $G_{c,d}$, f, g, $-I_k$, $-J_k$, $-P_k$ and $-Q_k$ are nonnegative and continuous, λ and μ are positive parameters. It can be concluded that $T: K \to K$ is continuous. For every $(u,v) \in PC(J)$ we have

$$\begin{aligned} & T_\alpha(u,v) \\ &\quad \geq \lambda \int_0^1 G_\alpha(1,s) f(\cdot)\, ds - \frac{1}{(a-1)(b-1)} \sum_{i=1}^m J_i(u) - \frac{1}{a-1} \sum_{i=1}^m I_i(u) \\ &\quad \geq \frac{1}{ab} \left[a\lambda \int_0^1 G_\alpha(1,s) f(\cdot)\, ds - \frac{ab}{(a-1)(b-1)} \sum_{i=1}^m J_i(u) - \frac{a}{a-1} \sum_{i=1}^m I_i(u) \right] \\ &\quad \geq \frac{1}{ab} \max_{t \in J} \bigl| T_\alpha(u,v)(t) \bigr| = \frac{1}{ab} \bigl\| T_\alpha(u,v) \bigr\|. \end{aligned}$$

Similarly, one gets $T_\beta(u,v) \geq (cd)^{-1} \| T_\beta(u,v) \|$. Therefore

$$\begin{aligned} T_\alpha(u,v)(t) + T_\beta(u,v)(t) &\geq \min\bigl\{(ab)^{-1}, (cd)^{-1}\bigr\} \bigl(\bigl\| T_\alpha(u,v) \bigr\| + \bigl\| T_\beta(u,v) \bigr\| \bigr) \\ &= \min\bigl\{(ab)^{-1}, (cd)^{-1}\bigr\} \bigl\| \bigl(T_\alpha(u,v), T_\beta(u,v) \bigr) \bigr\| \\ &= \gamma \bigl\| T(u,v) \bigr\|, \end{aligned}$$

namely $T(K) \subset K$. We can further see from the Ascoli–Arzela theorem that $T: K \to K$ is completely continuous. □

Theorem 3.1 *Suppose that* (B_1)–(B_3) *hold and there exist two constants* ρ *and* δ *with* $\rho \geq 4\eta > 0$ *and* $\delta > 0$ *such that*

$$\Phi(\rho) < \frac{\rho}{4\delta} \min\bigl\{(a\sigma_1)^{-1}, (c\sigma_2)^{-1}\bigr\}. \tag{21}$$

Then, for each

$$\begin{aligned} &\lambda \in \bigl((2\gamma)^{-1} \max\bigl\{(\sigma_1 f_0)^{-1}, (\sigma_1 f_\infty)^{-1}\bigr\}, \delta\bigr], \\ &\mu \in \bigl((2\gamma)^{-1} \max\bigl\{(\sigma_2 g_0)^{-1}, (\sigma_2 g_\infty)^{-1}\bigr\}, \delta\bigr], \end{aligned}$$

BVPs (1) *have at least two pairs of PSs* (u_i, v_i), $i = 1, 2$, *which satisfy*

$$0 < \bigl\| (u_1, v_1) \bigr\| < \rho < \bigl\| (u_2, v_2) \bigr\|. \tag{22}$$

Proof We first choose two constants r and R such that $0 < r < \rho < R$. Considering the case when $\lambda > (2\gamma\sigma_1 f_0)^{-1}$ and $\mu > (2\gamma\sigma_2 g_0)^{-1}$. From the definitions of f_0 and g_0, we can conclude that there exists $r > 0$ such that $f(\cdot) \geq (f_0 - \varepsilon_1)(u+v)$ and $g(\cdot) \geq (g_0 - \varepsilon_2)(u+v)$ as $u + v \in [0, r]$ and $t \in J$, where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ satisfy $2\lambda\gamma\sigma_1(f_0 - \varepsilon_1) \geq 1$ and $2\mu\gamma\sigma_2(g_0 - \varepsilon_2) \geq 1$. Then for each $(u,v) \in \partial K_r = \{(u,v) \in K : \|(u,v)\| = r\}$ and $t \in J$, it can be derived from Lemma 2.4

that

$$\begin{aligned}
\|T(u,v)\| &= \max_{t\in J}\left|T_\alpha(u,v)(t)\right| + \max_{t\in J}\left|T_\beta(u,v)(t)\right| \\
&\geq \lambda\int_0^1 G_\alpha(t,s)f(\cdot)\,ds - \sum_{i=1}^m G_{a,b}(t,t_i)J_i(u) - \sum_{i=1}^m G_a(t,t_i)I_i(u) \\
&\quad + \mu\int_0^1 G_\beta(t,s)g(\cdot)\,ds - \sum_{i=1}^m G_{c,d}(t,t_i)Q_i(v) - \sum_{i=1}^m G_c(t,t_i)P_i(v) \\
&\geq \lambda\int_0^1 G_\alpha(1,s)f(\cdot)\,ds + \mu\int_0^1 G_\beta(1,s)g(\cdot)\,ds \\
&\geq \lambda\int_0^1 G_\alpha(1,s)(f_0-\varepsilon_1)(u+v)\,ds \\
&\quad + \mu\int_0^1 G_\beta(1,s)(g_0-\varepsilon_2)(u+v)\,ds \\
&\geq \lambda\int_0^1 G_\alpha(1,s)(f_0-\varepsilon_1)\gamma\|(u,v)\|\,ds \\
&\quad + \mu\int_0^1 G_\beta(1,s)(g_0-\varepsilon_2)\gamma\|(u,v)\|\,ds \\
&= \lambda\gamma\sigma_1(f_0-\varepsilon_1)\|(u,v)\| + \mu\gamma\sigma_2(g_0-\varepsilon_2)\|(u,v)\| \geq \|(u,v)\|.
\end{aligned} \tag{23}$$

Next, considering the case when $\lambda > (2\gamma\sigma_1 f_\infty)^{-1}$ and $\mu > (2\gamma\sigma_2 g_\infty)^{-1}$. In view of the definitions of f_∞ and g_∞, we can see that there exists $R > 0$ such that $f(\cdot) \geq (f_\infty - \varepsilon_3)(u+v)$ and $g(\cdot) \geq (g_\infty - \varepsilon_4)(u+v)$ as $u+v \in [R,\infty)$ and $t \in J$, where $\varepsilon_3, \varepsilon_4 > 0$ with $2\lambda\gamma\sigma_1(f_\infty - \varepsilon_3) \geq 1$ and $2\mu\gamma\sigma_2(g_\infty - \varepsilon_4) \geq 1$. Then, for $(u,v) \in \partial K_R$ and $t \in J$, it follows from (23) that

$$\begin{aligned}
\|T(u,v)\| &\geq \lambda\int_0^1 G_\alpha(1,s)f(\cdot)\,ds + \mu\int_0^1 G_\beta(1,s)g(\cdot)\,ds \\
&\geq \lambda\int_0^1 G_\alpha(1,s)(f_\infty-\varepsilon_3)(u+v)\,ds \\
&\quad + \mu\int_0^1 G_\beta(1,s)(g_\infty-\varepsilon_4)(u+v)\,ds \\
&\geq \lambda\gamma\sigma_1(f_\infty-\varepsilon_3)\|(u,v)\| + \mu\gamma\sigma_2(g_0-\varepsilon_4)\|(u,v)\| \geq \|(u,v)\|.
\end{aligned} \tag{24}$$

Finally, we can see from (21) that

$$f(\cdot) \leq \Phi(\rho) < \frac{\rho}{4a\sigma_1\delta}, \qquad g(\cdot) \leq \Phi(\rho) < \frac{\rho}{4c\sigma_2\delta}, \qquad u+v \in [\gamma\rho,\rho], \quad t \in J.$$

Then, for each $(u,v) \in \partial K_\rho$ with $\rho \geq 4\eta$, it follows from Lemma 2.4 that

$$\begin{aligned}
\|T(u,v)\| &\leq a\lambda\int_0^1 G_\alpha(1,s)f(\cdot)\,ds - \frac{ab\sum_{i=1}^m J_i(u)}{(a-1)(b-1)} - \frac{a\sum_{i=1}^m I_i(u)}{a-1} \\
&\quad + c\mu\int_0^1 G_\beta(1,s)g(\cdot)\,ds - \frac{cd\sum_{i=1}^m Q_i(v)}{(c-1)(d-1)} - \frac{c\sum_{i=1}^m P_i(v)}{c-1}
\end{aligned}$$

$$
\begin{aligned}
&< a\lambda \int_0^1 G_\alpha(1,s)\frac{\rho}{4a\sigma_1\delta}\,ds + c\mu \int_0^1 G_\beta(1,s)\frac{\rho}{4c\sigma_2\delta}\,ds + 2\eta \\
&= \frac{(\lambda+\mu)\rho}{4\delta} + 2\eta \le \frac{\rho}{2} + \frac{\rho}{2} = \|(u,v)\|.
\end{aligned}
$$

Hence,

$$
\|T(u,v)\| < \|(u,v)\|, \quad (u,v) \in \partial K_\rho. \tag{25}
$$

Thus, applying Lemma 2.2 to (23)–(25) shows that $T(u,v)$ has the fixed point $(u_1,v_1) \in K \cap (\overline{K}_\rho \setminus K_r)$ and the fixed point $(u_2,v_2) \in K \cap (\overline{K}_R \setminus K_\rho)$. In the light of (25) being a strict inequality, $\|(u_1,v_1)\| \neq \rho$ and $\|(u_2,v_2)\| \neq \rho$. Consequently, BVPs (1) have at least two pairs of PSs (u_i,v_i), $i = 1,2$, satisfying (22). The proof is thus completed. □

Theorem 3.2 *Suppose that* (B_1)–(B_3) *hold and there exist three constants* ξ_i $(i = 1,2,3)$ *with* $4\eta \le \xi_1 < \xi_2 < \xi_3$ *such that either*

(H_1) $\frac{\overline{\varphi}}{2\sigma_1} \le \lambda \le \frac{\xi_2}{4a\sigma_1\Phi(\xi_2)}$ *and* $\frac{\overline{\varphi}}{2\sigma_2} \le \mu \le \frac{\xi_2}{4c\sigma_2\Phi(\xi_2)}$, *or*

(H_2) $\frac{\xi_2}{2\sigma_1\phi(\xi_2)} < \lambda \le \frac{\underline{\varphi}}{4a\sigma_1}$ *and* $\frac{\xi_2}{2\sigma_2\phi(\xi_2)} < \mu \le \frac{\underline{\varphi}}{4c\sigma_2}$ *hold,*

where $\overline{\varphi} = \max\{\xi_1\phi^{-1}(\xi_1), \xi_3\phi^{-1}(\xi_3)\}$ *and* $\underline{\varphi} = \min\{\xi_1\Phi^{-1}(\xi_1), \xi_3\Phi^{-1}(\xi_3)\}$. *Then BVPs* (1) *have at least two pairs of PSs* (u_i,v_i), $i = 1,2$, *which satisfy*

$$
\xi_1 \le \|(u_1,v_1)\| < \xi_2 < \|(u_2,v_2)\| \le \xi_3. \tag{26}
$$

Proof Due to the proofs of case (H_1) and case (H_2) being similar, here we prove only case (H_1). We first consider the case when $\lambda \ge \xi_1(2\sigma_1\phi(\xi_1))^{-1}$ and $\mu \ge \xi_1(2\sigma_2\phi(\xi_1))^{-1}$. Note that $f(\cdot) \ge \phi(\xi_1)$ and $g(\cdot) \ge \phi(\xi_1)$ as $u + v \in [\gamma\xi_1, \xi_1]$ and $t \in J$. Then, for $(u,v) \in \partial K_{\xi_1}$ and $t \in J$, one has

$$
\begin{aligned}
\|T(u,v)\| &\ge \lambda \int_0^1 G_\alpha(1,s)f(\cdot)\,ds + \mu \int_0^1 G_\beta(1,s)g(\cdot)\,ds \\
&\ge \lambda\phi(\xi_1) \int_0^1 G_\alpha(1,s)\,ds + \mu\phi(\xi_1) \int_0^1 G_\beta(1,s)\,ds \\
&\ge \frac{\xi_1}{2\sigma_1\phi(\xi_1)}\phi(\xi_1)\sigma_1 + \frac{\xi_1}{2\sigma_2\phi(\xi_1)}\phi(\xi_1)\sigma_2 = \xi_1 = \|(u,v)\|.
\end{aligned} \tag{27}
$$

For the case when $\lambda \le \xi_2(4a\sigma_1\Phi(\xi_2))^{-1}$ and $\mu \le \xi_2(4c\sigma_2\Phi(\xi_2))^{-1}$, noting that $f(\cdot) \le \Phi(\xi_2)$ and $g(\cdot) \le \Phi(\xi_2)$ as $u + v \in [\gamma\xi_2, \xi_2]$ and $t \in J$. Then for $(u,v) \in \partial K_{\xi_2}$, $t \in J$, one obtains

$$
\begin{aligned}
\|T(u,v)\| &\le a\lambda \int_0^1 G_\alpha(1,s)f(\cdot)\,ds + c\mu \int_0^1 G_\beta(1,s)g(\cdot)\,ds + 2\eta \\
&\le \lambda a\Phi(\xi_2) \int_0^1 G_\alpha(1,s)\,ds + \mu c\Phi(\xi_2) \int_0^1 G_\beta(1,s)\,ds + 2\eta \\
&\le \frac{\xi_2}{4a\sigma_1\Phi(\xi_2)}a\Phi(\xi_2)\sigma_1 + \frac{\xi_2}{4c\sigma_2\Phi(\xi_2)}c\Phi(\xi_2)\sigma_2 + \frac{\xi_1}{2} \\
&< \frac{\xi_2}{4} + \frac{\xi_2}{4} + \frac{\xi_2}{2} = \xi_2 = \|(u,v)\|.
\end{aligned} \tag{28}
$$

Considering $\lambda \geq \xi_3(2\sigma_1\phi(\xi_3))^{-1}$ and $\mu \geq \xi_3(2\sigma_2\phi(\xi_3))^{-1}$, for $(u,v) \in \partial K_{\xi_3}$, $t \in J$, we derive

$$\begin{aligned}\|T(u,v)\| &\geq \lambda \int_0^1 G_\alpha(1,s)f(\cdot)\,ds + \mu \int_0^1 G_\beta(1,s)g(\cdot)\,ds \\ &\geq \lambda\phi(\xi_3)\int_0^1 G_\alpha(1,s)\,ds + \mu\phi(\xi_3)\int_0^1 G_\beta(1,s)\,ds \\ &\geq \frac{\xi_3}{2\sigma_1\phi(\xi_3)}\phi(\xi_3)\sigma_1 + \frac{\xi_3}{2\sigma_2\phi(\xi_3)}\phi(\xi_3)\sigma_2 = \xi_3 = \|(u,v)\|. \end{aligned} \tag{29}$$

Thus, applying Lemma 2.2 to (27)–(29) shows that T has the fixed point $(u_1,v_1) \in K \cap (\overline{K}_{\xi_2} \setminus K_{\xi_1})$ and the fixed point $(u_2,v_2) \in K \cap (\overline{K}_{\xi_3} \setminus K_{\xi_2})$. In the light of (28), one gets $\|(u_1,v_1)\| \neq \xi_2$ and $\|(u_2,v_2)\| \neq \xi_2$. Therefore (26) holds, and the proof is thus completed. □

The following general theorem can be obtained by following a similar analysis to that of Theorem 3.2.

Theorem 3.3 *Suppose that (B_1)–(B_3) hold and there exist $n+1$ constants ξ_i $(i = 1,2,\ldots,n+1)$ with $4\eta \leq \xi_1 < \xi_2 < \cdots < \xi_{n+1}$ such that either*

(H_3) $\frac{\xi_{2j-1}}{2\sigma_1\phi(\xi_{2j-1})} < \lambda < \frac{\xi_{2j}}{4a\sigma_1\Phi(\xi_{2j})}$ *and* $\frac{\xi_{2j-1}}{2\sigma_2\phi(\xi_{2j-1})} < \mu < \frac{\xi_{2j}}{4c\sigma_2\Phi(\xi_{2j})}$, $j = 1,2,\ldots,[\frac{n+2}{2}]$, *or*

(H_4) $\frac{\xi_{2j}}{2\sigma_1\phi(\xi_{2j})} < \lambda < \frac{\xi_{2j-1}}{4a\sigma_1\Phi(\xi_{2j-1})}$ *and* $\frac{\xi_{2j}}{2\sigma_2\phi(\xi_{2j})} < \mu < \frac{\xi_{2j-1}}{4c\sigma_2\Phi(\xi_{2j-1})}$, $j = 1,2,\ldots,[\frac{n+2}{2}]$ *hold.*

Then BVPs (1) have at least n pairs of PSs (u_i,v_i), $i = 1,2,\ldots,n$, which satisfy

$$\xi_i < \|(u_i,v_i)\| < \xi_{i+1}. \tag{30}$$

Proof When $n = 1$, we can see from the case (H_3) that $\xi_1(2\sigma_1\phi(\xi_1))^{-1} < \lambda < \xi_2(4a\sigma_1\Phi(\xi_2))^{-1}$ and $\xi_1(2\sigma_2\phi(\xi_1))^{-1} < \mu < \xi_2(4c\sigma_2\Phi(\xi_2))^{-1}$. Then it follows from (27) and (28) that $\|T(u,v)\| > \|(u,v)\|$ for $(u,v) \in \partial K_{\xi_1}$ and $\|T(u,v)\| < \|(u,v)\|$ for $(u,v) \in \partial K_{\xi_2}$. This together with Lemma 2.2 implies that T has a fixed point (u_1,v_1) satisfies $\xi_1 < \|(u_1,v_1)\| < \xi_2$. Similarly, when $n = 2$ or $n = 3$, namely $j = 1,2$, we can further see that

$$\begin{cases} \|T(u,v)\| > \|(u,v)\|, & (u,v) \in \partial K_{\xi_1}, \\ \|T(u,v)\| < \|(u,v)\|, & (u,v) \in \partial K_{\xi_2}, \\ \|T(u,v)\| > \|(u,v)\|, & (u,v) \in \partial K_{\xi_3}, \\ \|T(u,v)\| < \|(u,v)\|, & (u,v) \in \partial K_{\xi_4}. \end{cases} \tag{31}$$

Thus, applying Lemma 2.2 to (31) shows that T has at least three fixed points (u_i,v_i), $i = 1,2,3$, satisfying

$$\xi_1 < \|(u_1,v_1)\| < \xi_2 < \|(u_2,v_2)\| < \xi_3 < \|(u_3,v_3)\| < \xi_4. \tag{32}$$

Therefore, by following the above analysis, we can see that (30) holds if (H_3) or (H_4) is satisfied. The proof is thus completed. □

4 Example

Consider the BVPs of the following nonlinear coupled system with impulses:

$$\begin{cases} {}^cD_{0^+}^{\frac{3}{2}}u(t) = -\lambda f(t,u(t),v(t)), & t \in J, t \neq \frac{1}{4}, \\ {}^cD_{0^+}^{\frac{3}{2}}v(t) = -\mu g(t,u(t),v(t)), & t \in J, t \neq \frac{1}{4}, \\ \Delta u(\frac{1}{4}) = I(u(\frac{1}{4})), \qquad \Delta u'(\frac{1}{4}) = J(u(\frac{1}{4})), \\ \Delta v(\frac{1}{4}) = P(v(\frac{1}{4})), \qquad \Delta v'(\frac{1}{4}) = Q(v(\frac{1}{4})), \\ u(0) = 2u(1), \qquad u'(1) = 3u'(0), \qquad v(0) = 2v(1), \qquad v'(1) = 3v'(0), \end{cases} \tag{33}$$

where

$$f = \frac{2}{1+t}\left[\frac{(1+\sin(\frac{\pi}{2}+u+v))(u+v)}{50e^{u+v}} + \frac{(u+v)^3+3(u+v)}{(u+v)^2+959(u+v)+1}\right],$$

$g = \frac{u+v}{20(5+t)}|\ln(u+v)|$, $I(u) = -\frac{u}{15(1+u)}$, $J(u) = -\frac{2u}{5(1+6u)}$, $P(v) = \frac{2\cos(5v)-3}{50}$ and $Q(v) = \frac{\sin(v)-1}{10}$. Obviously, (B_1)–(B_3) hold. By simple calculation, one can easily obtain $\eta = \frac{1}{2}$, $\gamma = \frac{1}{6}$, $\sigma_1 = \sigma_2 = \frac{10}{3\sqrt{\pi}}, f_0 = \frac{76}{25}, f_\infty = 1$ and $g_0 = g_\infty = +\infty$. We then further see that

$$\begin{aligned} &\frac{1}{2\gamma}\max\{(\sigma_1 f_0)^{-1},(\sigma_1 f_\infty)^{-1}\} = \frac{9\sqrt{\pi}}{10}, \\ &\frac{1}{2\gamma}\max\{(\sigma_2 g_0)^{-1},(\sigma_2 g_\infty)^{-1}\} = 0. \end{aligned}$$

Choose $\delta = 2\sqrt{\pi}$ and $\rho = 6$, then, for $t \in J$ and $u+v \in [1,6]$, one gets

$$\begin{aligned} &\max_{t\in J,u+v\in[1,6]} f(t,u,v) < 2\times\left(\frac{2\times 1}{50e} + \frac{6^3+3\times 6}{6^2+959\times 6+1}\right) \approx 0.1102, \\ &\max_{t\in J,u+v\in[1,6]} g(t,u,v) < \frac{6\ln 6}{20(5+0)} \approx 0.1075. \end{aligned}$$

Thus, one has

$$\frac{\rho}{4\delta}\min\{(a\sigma_1)^{-1},(c\sigma_2)^{-1}\} = \frac{9}{80} \approx 0.1125 > \Phi(6) \approx 0.1102.$$

We can see from Theorem 3.1 that, for $\lambda \in (\frac{9\sqrt{\pi}}{10}, 2\sqrt{\pi}]$ and $\mu \in (0, 2\sqrt{\pi}]$, BVPs (33) have at least two pairs of PSs.

5 Conclusions

This paper has discussed the multiplicity of PSs of impulsive BVPs for a fractional-order coupled system involving parameters. Some sufficient conditions have been derived to guarantee the existence of multiple PSs for the considered fractional-order coupled system. An example has been provided to illustrate the obtained results. Note that only a two-point BVP is considered in this paper. Similar to the work in [25], an interesting topic for future research is to deal with the multi-point even nonlocal BVP. Another interesting topic is to consider the multiplicity of the solutions for impulsive FDEs on the half-line.

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The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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References

1. Kilbas, A., Srivastava, H., Trujillo, J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
2. Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000)
3. Rossikhin, Y.A., Shitikova, M.V.: Application of fractional calculus for dynamic problems of solid mechanics: novel trends and recent result. Appl. Mech. Rev. **63**, 010801 (2010)
4. Tarasov, V.E.: Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media. Springer, Berlin (2011)
5. Rahimy, M.: Applications of fractional differential equations. Appl. Math. Sci. **4**(50), 2453–2461 (2010)
6. Gong, P.: Distributed consensus of non-linear fractional-order multi-agent systems with directed topologies. IET Control Theory Appl. **10**(18), 2515–2525 (2016)
7. Gong, P.: Distributed tracking of heterogeneous nonlinear fractional-order multi-agent systems with an unknown leader. J. Franklin Inst. **354**(5), 2226–2244 (2017)
8. Wu, G., Baleanu, D., Huang, L.: Novel Mittag-Leffler stability of linear fractional delay difference equations with impulse. Appl. Math. Lett. **82**, 71–78 (2018)
9. Wu, G., Baleanu, D.: Stability analysis of impulsive fractional difference equations. Fract. Calc. Appl. Anal. **21**, 354–375 (2018)
10. Baleanu, D., Agarwal, R., Khan, H., Khan, R., Jafari, H.: On the existence of solution for fractional differential equations of order $3 < \delta_1 \leq 4$. Adv. Differ. Equ. **2015**, Article ID 362 (2015). https://doi.org/10.1186/s13662-015-0686-1
11. Anguraj, A., Karthikeyan, P., Rivero, M., Trujillo, J.: On new existence results for fractional integro-differential equations with impulsive and integral conditions. Comput. Math. Appl. **66**(12), 2587–2594 (2014)
12. Han, Z., Lu, H., Zhang, C.: Positive solutions for eigenvalue problems of fractional differential equation with generalized p-Laplacian. Appl. Math. Comput. **257**, 526–536 (2015)
13. Ma, T., Tian, Y., Huo, Q., Zhang, Y.: Boundary value problem for linear and nonlinear fractional differential equations. Appl. Math. Lett. **86**, 1–7 (2018)
14. Zhao, K., Gong, P.: Existence of positive solutions for a class of higher-order Caputo fractional differential equation. Qual. Theory Dyn. Syst. **14**(1), 157–171 (2015)
15. Zhao, K., Gong, P.: Positive solutions of Riemann–Stieltjes integral boundary problems for the nonlinear coupling system involving fractional-order differential. Adv. Differ. Equ. **2014**, Article ID 254 (2014). https://doi.org/10.1186/1687-1847-2014-254
16. Zhao, K., Gong, P.: Positive solutions for impulsive fractional differential equations with generalized periodic boundary value conditions. Adv. Differ. Equ. **2014**, Article ID 255 (2014). https://doi.org/10.1186/1687-1847-2014-255
17. Zhao, Y., Sun, S., Han, Z., Zhang, M.: Positive solutions for boundary value problems of nonlinear fractional differential equations. Appl. Math. Comput. **217**(16), 6950–6958 (2011)
18. Zhao, Y., Chen, H., Xu, C.: Nontrivial solutions for impulsive fractional differential equations via Morse theory. Appl. Math. Comput. **307**, 170–179 (2017)
19. Lin, Z., Wang, J., Wei, W.: Multipoint BVPs for generalized impulsive fractional differential equations. Appl. Math. Comput. **258**, 608–616 (2015)
20. Zhao, K., Gong, P.: Positive solutions of nonlocal integral BVPs for the nonlinear coupled system involving high-order fractional differential. Math. Slovaca **67**, 447–466 (2017)
21. Li, X., Chen, F., Li, X.: Generalized anti-periodic boundary value problems of impulsive fractional differential equations. Commun. Nonlinear Sci. Numer. Simul. **18**(1), 28–41 (2013)
22. Tian, Y., Bai, Z.: Impulsive boundary value problem for differential equations with fractional order. Differ. Equ. Dyn. Syst. **21**, 253–260 (2013)
23. Podlubny, I., Trujillo, J.: Fractional Differential Equations. Academic Press, New York (1993)
24. Guo, D., Lakshmikantham, V., Liu, X.: Nonlinear Integral Equations in Abstract Spaces. Mathematics and Its Applications, vol. 373. Kluwer Academic, Dordrecht (1996)
25. Zhao, K., Gong, P.: Positive solutions of m-point multi-term fractional integral BVP involving time-delay for fractional differential equations. Bound. Value Probl. **2015**, Article ID 19 (2015). https://doi.org/10.1186/s13661-014-0280-6