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Bilinear approach to soliton and periodic wave solutions of two nonlinear evolution equations of Mathematical Physics

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Abstract

In the present paper, the potential Kadomtsev–Petviashvili equation and (3 + 1)-dimensional potential-YTSF equation are investigated, which can be used to describe many mathematical and physical backgrounds, e.g., fluid dynamics and communications. Based on Hirota bilinear method, the bilinear equation for the (3 + 1)-dimensional potential-YTSF equation is obtained by applying an appropriate dependent variable transformation. Then N-soliton solutions of nonlinear evolution equation are derived by the perturbation technique, and the periodic wave solutions for potential Kadomtsev–Petviashvili equation and (3 + 1)-dimensional potential-YTSF equation are constructed by employing the Riemann theta function. Furthermore, the asymptotic properties of periodic wave solutions show that soliton solutions can be derived from periodic wave solutions.

Keywords: Potential Kadomtsev–Petviashvili equation; (3 + 1)-dimensional potential-YTSF equation; N-soliton solution; Periodic wave solution; Hirota method

1 Introduction

The construction of analytic solutions for nonlinear evolution equations (NLEEs) is a key topic in the study of nonlinear phenomena [1–7]. Analytic solutions can help one to well understand the mechanism of physical phenomena modeled by a nonlinear evolution equation. With the development of soliton theory and computer algebraic system like Maple, much attention has been paid to finding analytic solutions of nonlinear evolution equations, including soliton solutions, periodic wave solutions, shock wave solutions, and so on. Up to now, many powerful methods of searching for exact solutions to NLEEs have been proposed and developed. For example, Biswas and Bhrawy [8] employed the extended Jacobi elliptic function expansion method to study the Zakharov equation and the Davey–Stewartson equation and obtained cnoidal and snoidal wavesolutions. Ma and Lee et al. [9] investigated a $3+1$ dimensional Jimbo–Miwa equation via a transformed rational function method and obtained exact solutions. Other methods, such as Bäcklund transformation [10, 11], Darboux transformation [12, 13], Hirota bilinear method [14], the inverse scattering transform, the variable separation method [15], the sine–cosine method, the tanh-function method [16], the auxiliary equation method [17], the trial function method [18], etc., were also employed. Based on these methods, a variety of nonlinear equations have been investigated and solved.

Among them, the Hirota method is one of the most effective methods of constructing multiple soliton solutions of NLEEs. It can transform the given nonlinear evolution equations to the corresponding bilinear forms through the dependent variable transformation. Then employing the perturbation expansion method, multi-soliton solutions with exponential function are derived. Also through the bilinear Bäcklund transformation, Lax pairs are obtained. In recent years, the Hirota method has been developed to construct the Wronskian solutions, Pfaffian solutions, and periodic wave solutions by use of the Riemann theta functions [19–24]. By means of this method, Tian et al. have investigated the HS equation for shallow water waves and BLMP equation [19]. And Ma, Zhang et al. have constructed periodic wave solutions of (2 + 1)-dimensional Hirota bilinear equations and Ito equation [20, 21]. The advantage of this method lies in the fact that we obtain the periodic wave solutions in a direct method without algebraic-geometric theory. Furthermore, the soliton solutions can be derived from the periodic wave solutions via asymptotic analysis.

With a motivation to further expand the area of applications of this method, in the present paper, we study the potential Kadomtsev–Petviashvili equation and (3 + 1)-dimensional potential-YTSF equation to illustrate the efficiency of using the combination of the Hirota method and the Riemann theta function. To the best of our knowledge, these results are up to date and have not been reported.

The rest of the paper is organized as follows. In Sect. 2, the bilinear form of (3 + 1)-dimensional potential-YTSF equation is given by applying the Hirota bilinear method. In Sect. 3, N-soliton solutions are presented by using the perturbation approach. In Sect. 4, by virtue of the Riemann theta function, periodic wave solutions are derived successfully, and the asymptotic properties of periodic wave solutions show that periodic wave solution degenerate to soliton solution. Finally, the concluding remarks are presented in Sect. 5.

2 Bilinear form for (3 + 1)-dimensional potential-YTSF equation

In this section, we will give the bilinear form for the (3 + 1)-dimensional potential-YTSF equation by applying the Hirota direct method and the dependent variable transformation.

A new (3 + 1)-dimensional nonlinear evolution equation, called the potential YTSF equation, was first introduced by Yu, Toda, Sasa and Fukuyama (YTSF) [25]. The (3 + 1)-dimensional potential-YTSF equation can be written as

$$-4u_{xt} + u_{xxxz} + 3u_x u_{xz} + 3u_{xx}u_z + 3u_{yy} = 0. \tag{2.1}$$

Setting $u = \frac{3}{4}w_x$, substituting it into Eq. (2.1) and integrating with respect to x yields

$$-4w_{xt} + w_{xxxz} + 3w_{xx}w_{xz} + 3w_{yy} + \lambda = 0, \tag{2.2}$$

which is transformed into the bilinear representation

$$\left[-4D_xD_t + D_x^3D_z + 3D_y^2\right]F \cdot F + \lambda F^2 = 0, \tag{2.3}$$

under the dependent variable transformation $w = 2\ln F$, where $\lambda = \lambda(y,z,t)$ is an integration constant; D_x, D_y and D_t are the well-known Hirota operators defined by [21]

$$D_x^m D_t^n f \cdot g = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x,t) \times g\left(x',t'\right)|_{x'=x,t'=t}.$$

The D-operators have the following nice property when acting on exponential functions:

$$D_x^m D_y^s D_t^n \exp\xi_1 \cdot \exp\xi_2 = (P_1 - P_2)^m (Q_1 - Q_2)^s (\Omega_1 - \Omega_2)^n \exp(\xi_1 + \xi_2), \tag{2.4}$$

where $\xi_i = P_i x + Q_i y + \Omega_i t + \xi_i^0$ $(i = 1, 2)$.

More generally, we get

$$G(D_x, D_y, D_t) \exp\xi_1 \cdot \exp\xi_2 = G(P_1 - P_2, Q_1 - Q_2, \Omega_1 - \Omega_2) \exp(\xi_1 + \xi_2). \tag{2.5}$$

Remark 1 D operates on a product of two functions like the Leibniz rule, except for a crucial sign difference. For example,

$$\begin{aligned} &D_x G \cdot F = G_x F - G F_x, \\ &D_x D_t G \cdot F = G_{xt} F - G_x F_t + G F_{xt} - G_t F_x, \\ &D_{xx} G \cdot F = G_{xx} F - 2G_x F_x + G F_{xx}. \end{aligned}$$

3 N-soliton solutions for (3 + 1)-dimensional potential-YTSF equation

In the following, we will give N-soliton solutions for the (3 + 1)-dimensional potential-YTSF equation by virtue of the Hirota method and the perturbation expansion and truncation technique, as well as property (2.5).

Expanding F into the power series with respect to a small parameter ε gives

$$F = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \cdots. \tag{3.1}$$

Substituting (3.1) into bilinear equation (2.3) and setting the coefficients of the same power of ε to zero, we obtain the recursion relations for f_i.

Single-soliton solution For $n = 1$, Eq. (3.1) becomes

$$f_1 = \exp(\eta_1), \quad \eta_1 = k_1 x + l_1 y + m_1 z + \omega_1 t + \eta_1^0, \tag{3.2}$$

where k_1, l_1, m_1, ω_1 are arbitrary constants, and $-4k_1\omega_1 + k_1^3 m_1 + 3l_1^2 = 0$ is the dispersion relation.

Substituting (3.2) into bilinear equation (2.3), a single-soliton solution of Eq. (2.1) is given by

$$u(x, y, z, t) = \frac{3}{2}\left[\ln\left(1 + e^{\eta_1}\right)\right]_x. \tag{3.3}$$

Two-soliton solution A two-soliton solution is given by

$$u(x, y, z, t) = \frac{3}{2}\left[\ln\left(1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1 + \eta_2 + A_{12}}\right)\right]_x, \tag{3.4}$$

where $-4k_i\omega_i + k_i^3 m_i + 3l_i^2 = 0$ $(i = 1, 2)$ are the dispersion relations and the phase shift term is

$$e^{A_{12}} = -\frac{4(k_1 - k_2)(\omega_1 - \omega_2) - (k_1 - k_2)^3 (m_1 - m_2) - 3(l_1 - l_2)^2}{4(k_1 + k_2)(\omega_1 + \omega_2) - (k_1 + k_2)^3 (m_1 + m_2) - 3(l_1 + l_2)^2}.$$

N-soliton solution Now we derive N-soliton solutions as

$$u(x,y,z,t) = \frac{3}{2}\left[\ln \sum_{\mu=0,1} e^{\sum_{i=1}^{N}\mu_i\eta_i+\sum_{i<j}^{(N)}\mu_i\mu_j A_{ij}}\right]_x, \quad 1 \le i,j \le N, \tag{3.5}$$

where the phase shift term is

$$e^{A_{ij}} = -\frac{4(k_i-k_j)(\omega_i-\omega_j)-(k_i-k_j)^3(m_i-m_i)-3(l_i-l_j)^2}{4(k_i+k_j)(\omega_i+\omega_j)-(k_i+k_j)^3(m_i+m_j)-3(l_i+l_j)^2}.$$

Here $\sum_{\mu=0,1}$ means a summation over all possible combinations of $\mu_j = 0,1$ ($j = 1,2,\ldots,N$) and $\sum_{i<j}^{N}$ is a summation over all possible pairs (i,j) ($i = 1,\ldots,N$, $j = 1,\ldots,N$) with the condition that $i<j$.

4 Periodic wave solutions of two equations

In this section, we will construct periodic wave solutions for the potential Kadomtsev–Petviashvili equation and the (3 + 1)-dimensional potential-YTSF equation by employing the Hirota method and the Riemann theta function, as well as property (2.5).

4.1 Potential Kadomtsev–Petviashvili equation

The following (2 + 1)-dimensional potential Kadomtsev–Petviashvili equation [26] is considered:

$$u_t + \frac{3}{4}u_x^2 + \frac{1}{4}u_{xxx} + \frac{3}{4}\partial_x^{-1}u_{yy} = 0, \tag{4.1}$$

which is transformed into the bilinear form

$$\left[4D_xD_t + D_x^4 + 3D_y^2\right]F\cdot F = 0, \tag{4.2}$$

under the dependent variable transformation $u = 2(\ln F)_x$.

We introduce the Riemann theta function solution of Eq. (4.1) as

$$F = \sum_{n=-\infty}^{\infty} e^{2\pi i n\zeta + \pi i n^2\tau}, \tag{4.3}$$

where $n \in \mathcal{Z}$, $\tau \in \mathcal{C}$, $\operatorname{Im}\tau > 0$ and $\zeta = kx + ly + \omega t$.

Substituting (4.3) into (4.2), we get

$$\begin{aligned} &G(D_x,D_y,D_t)F\cdot F \\ &\quad = G(D_x,D_y,D_t)\sum_{n=-\infty}^{\infty} e^{2\pi i n\zeta+\pi i n^2\tau}\sum_{m=-\infty}^{\infty} e^{2\pi i m\zeta+\pi i m^2\tau} \\ &\quad = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} G(D_x,D_y,D_t)e^{2\pi i n\zeta+\pi i n^2\tau}e^{2\pi i m\zeta+\pi i m^2\tau} \\ &\quad = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} G\left[2\pi i(n-m)k, 2\pi i(n-m)l, 2\pi i(n-m)\omega\right]e^{2\pi i(n+m)\zeta+\pi i(n^2+m^2)\tau} \end{aligned}$$

$$= \sum_{p=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} G\big[2\pi i(2n-p)k, 2\pi i(2n-p)l, 2\pi i(2n-p)\omega\big] e^{\pi i(n^2+(p-n)^2)\tau} e^{2\pi i p\zeta}$$

$$= \sum_{p=-\infty}^{\infty} \bar{G}(p) e^{2\pi i p\zeta},$$

where $n+m=p$. Noting that

$$\begin{aligned}
\bar{G}(p) &= \sum_{n=-\infty}^{\infty} G\big(2\pi i(2n-p)k, 2\pi i(2n-p)l, 2\pi i(2n-p)\omega\big] e^{\pi i(n^2+(p-n)^2)\tau} \\
&= \sum_{N=-\infty}^{\infty} G\big[2\pi i\big(2N-(p-2)\big)k, 2\pi i\big(2N-(p-2)\big)l, 2\pi i \times \big(2N-(p-2)\big)\omega)\big] \\
&\quad \times e^{\pi i((N+1)^2+(p-N-1)^2)\tau} \\
&= \sum_{N=-\infty}^{\infty} G\big[2\pi i\big(2N-(p-2)\big)k, 2\pi i\big(2N-(p-2)\big)l, 2\pi i\big(2N-(p-2)\big)\omega)\big] \\
&\quad \times e^{\pi i(N^2+(p-N-2)^2)\tau} e^{2\pi i(p-1)\tau} \\
&= \bar{G}(p-2) e^{2\pi i(p-1)\tau},
\end{aligned}$$

which indicates that if $\bar{G}(0)=\bar{G}(1)=0$, then

$$\bar{G}(p)=0, \quad p \in \mathcal{Z}. \tag{4.4}$$

Therefore, we may let

$$\bar{G}(0) = \sum_{n=-\infty}^{\infty} \big[-64\pi^2 n^2 k\omega + 256\pi^4 n^4 k^4 - 48\pi^2 n^2 l^2 + \mu\big] e^{2\pi i n^2 \tau} = 0, \tag{4.5}$$

$$\begin{aligned}
\bar{G}(1) &= \sum_{n=-\infty}^{\infty} \big[-16\pi^2(2n-1)^2 k\omega + 16\pi^4(2n-1)^4 k^4 - 12\pi^2(2n-1)^2 l^2 + \mu\big] \\
&\quad \times e^{\pi i(n^2+(n-1)^2)\tau} = 0.
\end{aligned} \tag{4.6}$$

Denote

$$\begin{aligned}
&\Delta_1(n) = e^{2\pi i n^2 \tau}, \qquad \Delta_2(n) = e^{\pi i(n^2+(n-1)^2)\tau}, \\
&A_{11} = -\sum_{n=-\infty}^{\infty} 64\pi^2 n^2 k \Delta_1(n), \qquad A_{12} = \sum_{n=-\infty}^{\infty} \Delta_1(n), \\
&A_{21} = -\sum_{n=-\infty}^{\infty} 16\pi^2(2n-1)^2 k \Delta_2(n), \qquad A_{22} = \sum_{n=-\infty}^{\infty} \Delta_2(n), \\
&B_1 = -\sum_{n=-\infty}^{\infty} \big(256\pi^4 n^4 k^4 - 48\pi^2 n^2 l^2\big) \Delta_1(n), \\
&B_2 = -\sum_{n=-\infty}^{\infty} \big[16\pi^4(2n-1)^4 k^4 - 12\pi^2(2n-1)^2 l^2\big] \Delta_2(n).
\end{aligned}$$

Then Eqs. (4.5) and (4.6) can be written as

$$A_{11}\omega + A_{12}\mu = B_1, \qquad A_{21}\omega + A_{22}\mu = B_2.$$

By solving this system, we get

$$\omega = \frac{B_1A_{22} - A_{12}B_2}{A_{11}A_{22} - A_{12}A_{21}}, \qquad \mu = \frac{A_{11}B_2 - A_{21}B_1}{A_{11}A_{22} - A_{12}A_{21}}. \tag{4.7}$$

Thus the periodic wave solution is given by

$$u = 2(\ln F)_x, \tag{4.8}$$

where ω and F are determined by Eqs. (4.7) and (4.3), respectively.

Next, we will demonstrate that the soliton solution can be obtained as a limit of a periodic wave solution. From Eq. (4.3), we rewrite F as

$$F = 1 + \alpha\left(e^{2\pi i\zeta} + e^{-2\pi i\zeta}\right) + \alpha^4\left(e^{4\pi i\zeta} + e^{-4\pi i\zeta}\right) + \cdots, \tag{4.9}$$

where $\alpha = e^{\pi i\tau}$.

Setting

$$K = 2\pi ik, \qquad L = 2\pi il, \qquad \Omega = 2\pi i\omega, \qquad \zeta' = Kx + Ly + \Omega t + \pi i\tau,$$

we get

$$\begin{aligned} F &= 1 + \alpha\left(e^{2\pi i\zeta} + e^{-2\pi i\zeta}\right) + \alpha^4\left(e^{4\pi i\zeta} + e^{-4\pi i\zeta}\right) + \cdots \\ &= 1 + e^{\zeta'} + \alpha^2\left(e^{-\zeta'} + e^{2\zeta'}\right) + \alpha^6\left(e^{-2\zeta'} + e^{3\zeta'}\right) + \cdots \\ &\to 1 + e^{\zeta'}, \quad \text{as } \alpha \to 0. \end{aligned} \tag{4.10}$$

Thus, the periodic wave solution (4.8) turns to the soliton solution

$$u = 2(\ln F)_x, \qquad F = 1 + e^{\zeta'}, \qquad \zeta' = Kx + Ly + \Omega t + \pi i\tau, \tag{4.11}$$

if we can prove that

$$\Omega \to -\frac{K^3}{4} - \frac{3L^2}{4K}. \tag{4.12}$$

In fact, it is easy to known that

$$\begin{aligned} &A_{11} = -128\pi^2 k\left(\alpha^2 + 4\alpha^8 + \cdots\right), \qquad A_{12} = 1 + 2\alpha^2 + 2\alpha^8 + \cdots, \\ &A_{21} = -32\pi^2 k\left(\alpha + 9\alpha^5 + \cdots\right), \qquad A_{22} = 2\alpha + 2\alpha^5 + \cdots, \\ &B_1 = 2\left(256\pi^4k^4 - 48\pi^2l^2\right)\alpha^2 + 2\left(256\pi^4 2^4k^4 - 48\pi^2 2^2 l^2\right)\alpha^8 + \cdots, \\ &B_2 = -2\left(16\pi^4k^4 - 12\pi^2l^2\right)\alpha + 2\left(16\pi^4 3^4k^4 - 12\pi^2 3^2 l^2\right)\alpha^5 + \cdots, \end{aligned}$$

which lead to

$$\begin{aligned} B_1A_{22}-A_{12}B_2 &= 2\left(16\pi^4k^4-12\pi^2l^2\right)\alpha+o(\alpha), \\ A_{11}A_{22}-A_{12}A_{21} &= 32\pi^2k\alpha+o(\alpha), \end{aligned} \tag{4.13}$$

According to (4.7), we get

$$\omega\to\pi^2k^3-\frac{3l^2}{4k},\quad \text{as } \alpha\to 0, \tag{4.14}$$

which is equivalent to

$$\Omega\to-\frac{K^3}{4}-\frac{3L^2}{4K},\quad \text{as } \alpha\to 0.$$

4.2 (3 + 1)-dimensional potential-YTSF equation

By a similar analysis process as in Sect. 4.1, we have

$$\bar{G}(0)=\sum_{n=-\infty}^{\infty}\left[64\pi^2n^2k\omega+256\pi^4n^4k^3m-48\pi^2n^2l^2+\lambda\right]e^{2\pi in^2\tau}=0, \tag{4.15}$$

$$\begin{aligned} \bar{G}(1) = \sum_{n=-\infty}^{\infty}&\left[16\pi^2(2n-1)^2k\omega+16\pi^4(2n-1)^4k^3m-12\pi^2(2n-1)^2l^2+\lambda\right] \\ &\times e^{\pi i(n^2+(n-1)^2)\tau}=0. \end{aligned} \tag{4.16}$$

Denote

$$\begin{aligned} &\Delta_1(n)=e^{2\pi in^2\tau},\qquad \Delta_2(n)=e^{\pi i(n^2+(n-1)^2)\tau}, \\ &A_{11}=\sum_{n=-\infty}^{\infty}64\pi^2n^2k\Delta_1(n),\qquad A_{12}=\sum_{n=-\infty}^{\infty}\Delta_1(n), \\ &A_{21}=\sum_{n=-\infty}^{\infty}16\pi^2(2n-1)^2k\Delta_2(n),\qquad A_{22}=\sum_{n=-\infty}^{\infty}\Delta_2(n), \\ &B_1=-\sum_{n=-\infty}^{\infty}\left(256\pi^4n^4k^3m-48\pi^2n^2l^2\right)\Delta_1(n), \\ &B_2=-\sum_{n=-\infty}^{\infty}\left[16\pi^4(2n-1)^4k^3m-12\pi^2(2n-1)^2l^2\right]\Delta_2(n). \end{aligned}$$

Then Eqs. (4.15) and (4.16) can be written as

$$A_{11}\omega+A_{12}\lambda=B_1,\qquad A_{21}\omega+A_{22}\lambda=B_2.$$

By solving the system, we get

$$\omega=\frac{B_1A_{22}-A_{12}B_2}{A_{11}A_{22}-A_{12}A_{21}},\qquad \lambda=\frac{A_{11}B_2-A_{21}B_1}{A_{11}A_{22}-A_{12}A_{21}}. \tag{4.17}$$

Thus, the periodic wave solution is given by

$$u = \frac{3}{2}(\ln F)_x, \tag{4.18}$$

where ω and F are determined by Eqs. (4.17) and (4.3), respectively.

From Eq. (4.3), we rewrite F as

$$F = 1 + \delta\left(e^{2\pi i\zeta} + e^{-2\pi i\zeta}\right) + \delta^4\left(e^{4\pi i\zeta} + e^{-4\pi i\zeta}\right) + \cdots, \tag{4.19}$$

where $\delta = e^{\pi i\tau}$.

Setting

$$K = 2\pi ik, \qquad L = 2\pi il, \qquad M = 2\pi im, \qquad \Omega = 2\pi i\omega,$$
$$\zeta' = Kx + Ly + Mz + \Omega t + \pi i\tau,$$

yields

$$\begin{aligned} F &= 1 + \delta\left(e^{2\pi i\zeta} + e^{-2\pi i\zeta}\right) + \delta^4\left(e^{4\pi i\zeta} + e^{-4\pi i\zeta}\right) + \cdots \\ &= 1 + e^{\zeta'} + \delta^2\left(e^{-\zeta'} + e^{2\zeta'}\right) + \delta^6\left(e^{-2\zeta'} + e^{3\zeta'}\right) + \cdots \\ &\to 1 + e^{\zeta'}, \quad \text{as } \delta \to 0. \end{aligned} \tag{4.20}$$

Thus, if we can prove that

$$\Omega \to \frac{K^2M}{4} + \frac{3L^2}{4K}, \quad \text{as } \delta \to 0, \tag{4.21}$$

the periodic wave solution (4.18) turns to the soliton solution

$$u = \frac{3}{2}(\ln F)_x, \qquad F = 1 + e^{\zeta'}, \qquad \zeta' = Kx + Ly + Mz + \Omega t + \pi i\tau. \tag{4.22}$$

In fact, it is easy to known that

$$A_{11} = 128\pi^2 k\left(\delta^2 + 4\delta^8 + \cdots\right),$$
$$A_{12} = 1 + 2\delta^2 + 2\delta^8 + \cdots, \qquad A_{21} = 32\pi^2 k\left(\delta + 9\delta^5 + \cdots\right), \qquad A_{22} = 2\delta + 2\delta^5 + \cdots,$$
$$B_1 = 2\left(256\pi^4 k^3 m - 48\pi^2 l^2\right)\delta^2 + 2\left(256\pi^4 2^4 k^3 m - 48\pi^2 2^2 l^2\right)\delta^8 + \cdots,$$
$$B_2 = -2\left(16\pi^4 k^3 m - 12\pi^2 l^2\right)\delta + 2\left(16\pi^4 3^4 k^3 m - 12\pi^2 3^2 l^2\right)\delta^5 + \cdots,$$

which lead to

$$B_1A_{22} - A_{12}B_2 = 2\left(16\pi^4 k^3 m - 12\pi^2 l^2\right)\delta + o(\delta),$$
$$A_{11}A_{22} - A_{12}A_{21} = -32\pi^2 k\delta + o(\delta).$$

According to (4.17), we get

$$\omega \to -\pi^2 k^2 m + \frac{3l^2}{4k}, \quad \text{as } \delta \to 0,$$

which is equivalent to

$$\Omega \to \frac{K^2 M}{4} + \frac{3L^2}{4K}, \quad \text{as } \delta \to 0.$$

5 Discussion and conclusion

In the present paper, we investigate the $(2+1)$-dimensional potential KP equation and $(3+1)$-dimensional potential-YTSF equation based on the Hirota method and the Riemann theta function. As a result, we obtain the bilinear form and N-soliton solutions of the $(3+1)$-dimensional potential-YTSF equation under constraint conditions. By virtue of the Hirota method and the Riemann theta function, periodic wave solutions have been presented. And via asymptotic analysis, classical soliton solutions have been derived from their periodic wave solutions. Finally, it is worthwhile to note that the Hirota direct method can be applied to other variable coefficient NLEEs in mathematical physics.

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Competing interests
The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' contributions
The authors carried out the calculation and conceived of the study. The authors read and approved the final manuscript.

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