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Stability of a stochastic discrete mutualism system

Xinyuan Liao[1*] and Yuming Chen[2]

*Correspondence:
xyl98@hotmail.com
[1]School of Mathematics and Physics, University of South China, Hengyang, China
Full list of author information is available at the end of the article

Abstract
Under the assumption that stochastic white noise perturbations are directly proportional to the deviation of the state from the equilibrium states of a continuous mutualistic model, we use the Euler–Maruyama discretization method to obtain a two-species stochastic discrete mutualism model. For this stochastic model, we establish conditions on the asymptotic mean square stability of the positive equilibrium state and the almost sure asymptotic stability of the three boundary equilibrium states. The theoretical results are supported with numerical simulations.

Keywords: Asymptotic mean square stability; Stochastic discrete mutualism model; Linearized method; Lyapunov functional method

1 Introduction

Any species in nature is not isolated, always related to other species in the community. The relationships among them are generally divided into four types: mutualism, parasitism, competition, and predation. Mutualism is one of the ubiquitous interactions among species, which is beneficial for all the species involved [1]. Therefore mutualism plays an important role in all ecosystems; in theoretical biology, it has received much attention of many scholars (see, e.g., [2–10]).

The simplest mutualistic model was proposed by May [11]. May's equations for two species can be written as

$$\begin{aligned} \frac{du_1}{dt} &= r_1 u_1\left(1-\frac{u_1}{K_1}\right)+r_1 u_1 \beta_{12}\frac{u_2}{K_1}, \\ \frac{du_2}{dt} &= r_2 u_2\left(1-\frac{u_2}{K_2}\right)+r_2 u_2 \beta_{21}\frac{u_1}{K_2}, \end{aligned} \tag{1}$$

where u_i, r_i, and K_i are respectively the density, intrinsic growth rate, and carrying capacity of species i $(= 1, 2)$, β_{12} is the coefficient that embodies the benefit for species 1 of each interaction with species 2, whereas β_{21} is the coefficient that embodies the benefit for species 2 of each interaction with species 1. Lots of modifications of May's model have been proposed to better understand mutualism.

Denote

$$a_{11}=\frac{r_1}{K_1}, \qquad a_{12}=\frac{r_1\beta_{12}}{K_1}, \qquad a_{21}=\frac{r_2}{K_2}, \qquad a_{12}=\frac{r_2\beta_{21}}{K_2}.$$

Then (1) can be rewritten as

$$\begin{aligned} du_1(t) &= u_1(t)\bigl(r_1 - a_{11}u_1(t) + a_{12}u_2(t)\bigr)dt, \\ du_2(t) &= u_2(t)\bigl(r_2 + a_{21}u_1(t) - a_{22}u_2(t)\bigr)dt. \end{aligned} \tag{2}$$

It is well known that, in a species ecosystem, the disturbance of various random factors is ubiquitous, which has a great influence on the evolution of biological species. In the study of practical species problems, deterministic models are established, where relatively small random disturbances are ignored in many cases. However, when there is a high demand for the dynamic behavior of the system or strong random interference, ignoring the action of random factors may lead to a considerable deviation, and the effect will not be satisfactory (see, e.g., [12–15]). In fact, it is also the commonest phenomenon in nature that species are disturbed by environmental noises. In recent decades, many scholars have established stochastic ecological models by adding the stochastic driving force, a Brownian motion, as a random factor to deterministic species models. Then they use the theory of stochastic differential equations to study the stochastic dynamic behavior. As pointed out by Mao [16], "a reasonable mathematical interpretation for the noise is the so-called white noise $\dot{B}(t)$, which is formally regarded as the derivative of the Brownian motion $B(t)$, that is, $\dot{B}(t) = dB(t)/dt$" (also see [17]).

Using the idea of Shaikhet [18, 19], we assume that system (2) is influenced by stochastic white noise perturbations that are directly proportional to the deviation of the state of the system $(u_1(t), u_2(t))$ from the equilibrium state (u_1^*, u_2^*), that is, if the deviation from the equilibrium state increases, then the stochastic perturbations increase accordingly. When the state of the system is at the equilibrium state, the stochastic perturbations are zero. This is a common phenomenon in ecological mutualism system. Such perturbations were first proposed by Beretta et al. [20] and now have been well accepted by many other researchers (see [18, 19, 21] and references therein).

Based on our discussion and (2), the following stochastic Lotka–Volterra mutualism system is obtained:

$$\begin{aligned} du_1(t) &= u_1(t)\bigl(r_1 - a_{11}u_1(t) + a_{12}u_2(t)\bigr)dt + \sigma_1\bigl(u_1(t) - u_1^*\bigr)dB_1(t), \\ du_2(t) &= u_2(t)\bigl(r_2 + a_{21}u_1(t) - a_{22}u_2(t)\bigr)dt + \sigma_2\bigl(u_2(t) - u_2^*\bigr)dB_2(t), \end{aligned} \tag{3}$$

where σ_i^2 $(i = 1, 2)$ denotes the intensity of the noise $\dot{B}_i(t)$, and $B_i(t)$ is a Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\{\mathcal{F}_t\}_{t\in R^+}$ satisfying the usual conditions [17].

We should mention that most of the studies on stochastic Lotka–Volterra systems are continuous models derived from differential equations (see [22–24] and references therein). Moreover, many authors have argued that the discrete-time models governed by difference equations are more appropriate than the continuous-time ones when the populations have nonoverlapping generations. Discrete-time models can also provide efficient computational models of continuous models for numerical simulations [25, 26]. Also, there is little research on the stability of the stochastic discrete models described by difference equations. With the Euler–Maruyama discretization method, the discretization method mentioned in the introduction of [27] (also see 219–220 pages in [28]), we obtain the following stochastic discrete mutualism model:

$$\begin{aligned} &u_1(n+1) \\ &\quad = u_1(n) + u_1(n)\bigl(r_1 - a_{11}u_1(n) + a_{12}u_2(n)\bigr)h + \sigma_1\sqrt{h}\bigl(u_1(n) - u_1^*\bigr)\xi_1(n+1), \\ &u_2(n+1) \\ &\quad = u_2(n) + u_2(n)\bigl(r_2 + a_{21}u_1(n) - a_{22}u_2(n)\bigr)h + \sigma_2\sqrt{h}\bigl(u_2(n) - u_2^*\bigr)\xi_2(n+1), \\ &n \in Z = \{0,1,2,\ldots\}, u_1(0) > 0, u_2(0) > 0, \end{aligned} \tag{4}$$

where h is the step size, $\xi_i(n+1)$ $(i = 1,2)$ is a mutually independent sequence of $\mathcal{F}_n$-adapted random variables satisfying $\mathbf{E}\xi_i(n) = 0$, $\mathbf{E}\xi_i^2(n) = 1$, and $\mathbf{E}\xi_i(n)\xi_j(n) = 0$ $(i \neq j)$ with $\mathbf{E}$ denoting the expectation (see [19] and references therein).

Note that (4) always has three boundary equilibrium states $E_0 = (0,0)$, $E_1 = (\frac{r_1}{a_{11}}, 0)$, and $E_2 = (0, \frac{r_2}{a_{22}})$. Furthermore, if $a_{11}a_{22} - a_{12}a_{21} > 0$, then there is another unique positive equilibrium state $E_3 \triangleq (u_{31}^*, u_{32}^*) = (\frac{r_1a_{22}+r_2a_{12}}{a_{11}a_{22}-a_{12}a_{21}}, \frac{r_2a_{11}+r_1a_{21}}{a_{11}a_{22}-a_{12}a_{21}})$. The purpose of this paper is to consider the stability of these equilibrium states. We first study the asymptotic mean square stability of the positive equilibrium state in Sect. 2, followed by the asymptotic stability of the boundary equilibrium states in Sect. 3. The paper ends with a brief conclusion.

2 The asymptotic mean square stability of the positive equilibrium state

For two symmetric real number matrices P and Q, we write $P > Q$ if the matrix $P - Q$ is a positive definite matrix.

For an arbitrary functional $V_i = V(i, z(0), z(1), \ldots, z(i))$, $i \in Z$, the operator ΔV_i is defined by

$$\Delta V_i = V\bigl(i+1, z(0), z(1), \ldots, z(i+1)\bigr) - V\bigl(i, z(0), z(1), \ldots, z(i)\bigr).$$

The linearized system of (4) at an equilibrium state $E^* = (u_1^*, u_2^*)$ is of the form

$$\begin{aligned} &z(n+1) = \bigl(A + \Theta\bigl(\xi(n+1)\bigr)\bigr)z(n), \\ &n \in Z, z(0) \in (0,\infty)^2, \end{aligned} \tag{5}$$

where $z(n) = (u_1(n), u_2(n))'$ (the transpose of $(u_1(n), u_2(n))$), A is a 2×2 constant matrix, and

$$\Theta\bigl(\xi(n+1)\bigr) = \begin{pmatrix} \sigma_1\sqrt{h}\xi_1(n+1) & 0 \\ 0 & \sigma_2\sqrt{h}\xi_2(n+1) \end{pmatrix}.$$

Now, we adopt the concept of asymptotic mean square stability and a result on Lyapunov functionals from Shaikhet [21].

Definition 2.1 (See [21, Definition 1.2]) The zero solution of (4) is said to be mean square stable if, for each $\epsilon > 0$, there exists $\delta > 0$ such that $\mathbf{E}|z(n)|^2 < \epsilon$, $n \in Z$, if $\mathbf{E}|z(0)|^2 < \delta$. It is asymptotically mean square stable if it is mean square stable and $\lim_{n\to\infty} \mathbf{E}|z(n)|^2 = 0$.

Lemma 2.1 (See [21, Theorem 1.1]) *Let $V_i = V(i, z(0), z(1), \ldots, z(i))$ be a nonnegative functional satisfying the conditions $\mathbf{E}V(0,\phi) \le c_1\|\phi\|^2$ and $\mathbf{E}\Delta V_i \le -c_2\mathbf{E}|z(i)|^2$, $i \in Z$, where c_1 and c_2 are positive constants. Then the zero solution of (5) is asymptotically mean square stable.*

Theorem 2.1 *Suppose that there exists a positive definite matrix P such that the matrix equation*

$$A'DA - D = -P \tag{6}$$

has a positive semidefinite solution

$$D = \begin{pmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{pmatrix}$$

satisfying

$$P > \begin{pmatrix} d_{11}\sigma_1^2 h & 0 \\ 0 & d_{22}\sigma_2^2 h \end{pmatrix}. \tag{7}$$

Then the zero solution of (5) is asymptotically mean square stable.

Proof Consider the Lyapunov functional $V(\phi) = \phi' D\phi$. We have

$$\Delta V(n) = z'(n+1)Dz(n+1) - z'(n)Dz(n).$$

Then

$$\begin{aligned} \mathbf{E}\Delta V(n) &= \mathbf{E}\bigl(z'(n)\bigl(A + \Theta\bigl(\xi(n+1)\bigr)\bigr)' D\bigl(A + \Theta\bigl(\xi(n+1)\bigr)\bigr)z(n) - z'(n)Dz(n)\bigr) \\ &= \mathbf{E}\bigl(z'(n)\bigl(A' + \Theta'\bigl(\xi(n+1)\bigr)\bigr)D\bigl(A + \Theta\bigl(\xi(n+1)\bigr) - D\bigr)z(n)\bigr) \\ &= \mathbf{E}\bigl(z'(n)\bigl(A'DA - D + \Theta'\bigl(\xi(n+1)\bigr)D\Theta\bigl(\xi(n+1)\bigr)\bigr)z(n)\bigr). \end{aligned}$$

Moreover, since $\mathbf{E}\xi_i(n+1) = 0$, $\mathbf{E}\xi_i^2(n+1) = 1$, and $\mathbf{E}\xi_1(n+1)\xi_2(n+1) = 0$, we see that

$$\begin{aligned} &\mathbf{E}\Theta'\bigl(\xi(n+1)\bigr)D\Theta\bigl(\xi(n+1)\bigr) \\ &\quad= \mathbf{E}\begin{pmatrix} \sigma_1\sqrt{h}\xi_1(n+1) & 0 \\ 0 & \sigma_2\sqrt{h}\xi_2(n+1) \end{pmatrix} \\ &\qquad\times \begin{pmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{pmatrix} \begin{pmatrix} \sigma_1\sqrt{h}\xi_1(n+1) & 0 \\ 0 & \sigma_2\sqrt{h}\xi_2(n+1) \end{pmatrix} \\ &\quad= \mathbf{E}\begin{pmatrix} d_{11}\sigma_1^2 h\xi_1^2(n+1) & d_{12}\sigma_1\sigma_2 h\xi_1(n+1)\xi_2(n+1) \\ d_{12}\sigma_1\sigma_2 h\xi_1(n+1)\xi_2(n+1) & d_{22}\sigma_2^2 h\xi_2^2(n+1) \end{pmatrix} \\ &\quad= \begin{pmatrix} d_{11}\sigma_1^2 h & 0 \\ 0 & d_{22}\sigma_2^2 h \end{pmatrix}. \end{aligned}$$

Now, according to (7),

$$\begin{aligned}\mathbf{E}\Delta V(n) &= \mathbf{E}\left(z'(n)\left(-P + \begin{pmatrix} d_{11}\sigma_1^2 h & 0 \\ 0 & d_{22}\sigma_2^2 h\end{pmatrix}\right) z(n)\right) \\ &\le -c\mathbf{E}\bigl|z(n)\bigr|^2\end{aligned}$$

for some $c > 0$. A direct application of Lemma 2.1 immediately finishes the proof. □

In the remaining of this section, we apply Theorem 2.1 to the positive equilibrium state E_3 of (4). In this case,

$$A = \begin{pmatrix} 1 - a_{11}u_{31}^* h & a_{12}u_{31}^* h \\ a_{21}u_{32}^* h & 1 - a_{22}u_{32}^* h\end{pmatrix}.$$

Then (6) is equivalent to

$$\begin{aligned} p_{11} &= \bigl[1 - \bigl(1 - a_{11}u_{31}^* h\bigr)^2\bigr] d_{11} - \bigl(1 - a_{11}u_{31}^* h\bigr)(a_{12} + a_{21})u_{32}^* h d_{12} \\ &\quad - \bigl(a_{21}u_{32}^* h\bigr)^2 d_{22}, \\ p_{12} &= -\bigl(1 - a_{11}u_{31}^* h\bigr) a_{12}u_{31}^* h d_{11} - \bigl(1 - a_{22}u_{32}^* h\bigr) a_{21}u_{32}^* h d_{22} \\ &\quad + \bigl[a_{11}u_{31}^* + a_{22}u_{32}^* - (a_{11}a_{22} + a_{12} + a_{21})u_{31}^* u_{32}^* h\bigr] h d_{12}, \\ p_{22} &= -\bigl(a_{12}u_{31}^* h\bigr)^2 d_{11} - \bigl(1 - a_{22}u_{32}^* h\bigr)(a_{12} + a_{21})u_{31}^* h d_{12} \\ &\quad + \bigl[1 - \bigl(1 - a_{22}u_{32}^* h\bigr)^2\bigr] d_{22}, \end{aligned} \tag{8}$$

and (7) is equivalent to

$$\begin{aligned} & p_{11} - d_{11}\sigma_1^2 h > 0, \\ & \bigl(p_{11} - d_{11}\sigma_1^2 h\bigr)\bigl(p_{22} - d_{22}\sigma_2^2 h\bigr) - p_{12}^2 > 0, \end{aligned} \tag{9}$$

under the condition that

$$p_{11}p_{22} - p_{12}^2 > 0. \tag{10}$$

Corollary 2.2 *If the system of linear equations* (8) *has a solution* (d_{11}, d_{12}, d_{22}) *satisfying* (9) *and* (10), *then the zero solution of* (5) *is asymptotically mean square stable, or, equivalently, the positive equilibrium state* $E_3 = (u_{31}^*, u_{32}^*)$ *of* (4) *is locally asymptotically mean square stable.*

Theoretically, we can solve the system of linear equations (8) for (d_{11}, d_{12}, d_{22}) and then verify whether we can choose p_{11}, p_{12}, and p_{22} such that both (9) and (10) hold. However, this is tedious because of so many parameters involved. As a result, we just provide a concrete example. According to Kwon [29, 30], for the ant–aphid mutualism, we choose $a_{11} = 2$, $a_{12} = 3/8$, $a_{21} = 1$, $a_{22} = 1$, $h = 0.1$, $r_1 = 0.5$, $r_2 = 3$, $\sigma_1 = \sigma_2 = 0.6$. Using MATLAB,

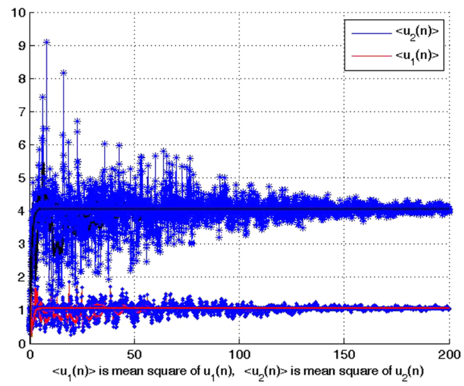


Figure 1 The positive equilibrium state E_3 of (4) is locally asymptotically mean square stable based on 50 trajectories of (4) with the initial condition $(u_1, u_2) = (0.5, 0.8)$, where $a_{11} = 2$, $a_{12} = 3/8$, $a_{21} = a_{22} = 1$, $r_1 = 0.5$, $r_2 = 3$, and $h = 0.1$

we get the following results: $E_3 = (1, 4)$ and positive definite matrices

$$D = \begin{pmatrix} 5.1043 & 2.0235 \\ 2.0235 & 1.0215 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 1.1258 & 0.9978 \\ 0.9978 & 1.0129 \end{pmatrix}.$$

By Corollary 2.2 the zero solution of (5) is asymptotically mean square stable, that is, the positive equilibrium state $E_3 = (1, 4)$ is locally asymptotically mean square stable (see Fig. 1).

3 Asymptotic stability of the boundary equilibrium states

As we will see later, we cannot apply Theorem 2.1 to discuss the mean square stability of none of the three boundary equilibrium states. When such situations happen, Palmer [27] studied the almost sure asymptotic stability of a linear stochastic difference equation by a discretized Itô formula. Here we study the stochastic stability using this method and comparison theorems [31].

Definition 3.1 The zero solution of (4) is said to be almost sure asymptotically stable if

$$P\Bigl\{\lim_{n\to\infty} z(n) = 0\Bigr\} = 1;$$

we also write $\lim_{n\to\infty} z(n) = 0$ a.s. (the standard abbreviation for "almost surely").

We begin with the citation of a result whose proof is a direct consequence of [27, Thm. 5.3].

Lemma 3.1 *Let $\{u(n)\}_{n\in Z}$ be a solution of the equation*

$$u(n+1) = u(n) + rhu(n) + \sigma\sqrt{h}u(n)\xi(n+1), \quad n \in Z, r, \sigma \in R,$$

for $h > 0$ sufficiently small. Then the following two statements hold:

(i) $\lim_{n\to\infty} u(n) = 0$ *a.s. if and only if* $2r - \sigma^2 < 0$;
(ii) $\lim_{n\to\infty} u(n) = \infty$ *a.s. if and only if* $2r - \sigma^2 > 0$.

Now we use Lemma 3.1 to discuss the stochastic stability of the boundary equilibrium states one by one.

Firstly, we consider $E_0 = (0,0)$. For this case, we have $A = \left(\begin{smallmatrix} 1+r_1h & 0 \\ 0 & 1+r_2h \end{smallmatrix}\right)$ in the linearized system. Then (6) becomes

$$\begin{pmatrix} (2r_1h + r_1^2h^2)d_{11} & (1+r_1h)(1+r_2h)d_{12} \\ (1+r_1h)(1+r_2h)d_{12} & (2r_2h + r_2^2h^)d_{22} \end{pmatrix} = -\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix},$$

which clearly has no positive semidefinite solution D for any positive definite matrix P. This means that Theorem 2.1 is inapplicable. Recall that, in this case, (5) is

$$\begin{aligned} u_1(n+1) &= u_1(n) + r_1hu_1(n) + \sigma_1\sqrt{h}u_1(n)\xi_1(n+1), \\ u_2(n+1) &= u_2(n) + r_2hu_2(n) + \sigma_2\sqrt{h}u_2(n)\xi_2(n+1). \end{aligned} \tag{11}$$

By Lemma 3.1, for h sufficiently small, we must have:

(i) If $2r_i - \sigma_i^2 < 0$, then $\lim_{n\to\infty} u_i(n) = 0$ $(i = 1,2)$ a.s.;

(ii) If $2r_i - \sigma_i^2 > 0$, then $\lim_{n\to\infty} u_i(n) = \infty$ $(i = 1,2)$ a.s.

The above result is supported by Figs. 2 and 3. Here, for the purpose of simulation, we do not need the values of a_{ij}. For Fig. 2, $\sigma_1 = \sigma_2 = 0.9$, $r_1 = r_2 = 0.4$, and $h = 0.1$. Clearly, $2r_i - \sigma_i^2 < 0$, and hence $\lim_{n\to\infty}(u_1(n), u_2(n)) = (0,0)$ a.s. For Fig. 3, $2r_i - \sigma_i^2 > 0$ with $\sigma_1 = \sigma_2 = 0.6$, $r_1 = r_2 = 0.8$, and $h = 0.1$. It follows that $\lim_{n\to\infty}(u_1(n), u_2(n)) = (\infty,\infty)$ a.s.

Going back to the stability of E_0, we have the following:

Proposition 3.1 *For h sufficiently small, the equilibrium state E_0 of (4) is a.s. locally asymptotically stable if $2r_i - \sigma_i^2 < 0$ for $i = 1,2$.*

Figure 2 The boundary equilibrium state $E_0 = (0,0)$ of (4) is a.s. locally asymptotically stable when $2r_i - \sigma_i^2 < 0$ with $\sigma_1 = \sigma_2 = 0.9$, $r_1 = r_2 = 0.4$, and $h = 0.1$, where the initial condition is $(u_1(0), u_2(0)) = (0.5, 0.8)$

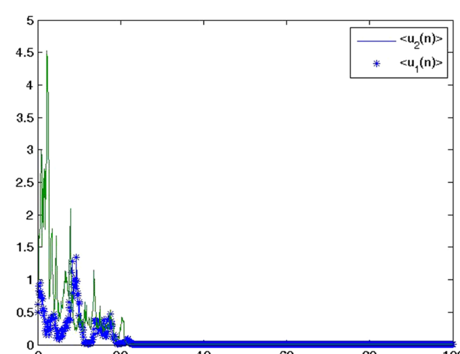
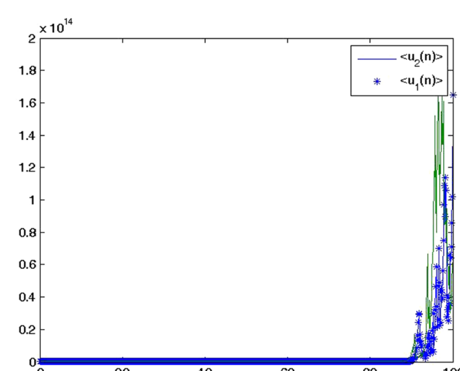


Figure 3 The boundary equilibrium state $E_0 = (0,0)$ of (4) is unstable when $2r_i - \sigma_i^2 > 0$ with $\sigma_1 = \sigma_2 = 0.6$, $r_1 = r_2 = 0.8$, and $h = 0.1$, where the initial condition is $(u_1(0,), u_2(0)) = (0.5, 0.8)$



Secondly, we consider $E_1 = (\frac{r_1}{a_{11}}, 0)$. This time,

$$A = \begin{pmatrix} 1 - r_1 h & \frac{a_{12}r_1}{a_{11}}h \\ 0 & 1 + (r_2 + \frac{r_1 a_{21}}{a_{11}})h \end{pmatrix}.$$

Then (6) is equivalent to the following system of linear equations:

$$\begin{aligned} p_{11} &= -(1 - r_1 h)^2 d_{11}, \\ p_{12} &= -(1 - r_1 h)\frac{a_{12}r_1}{a_{11}}h d_{11} - (1 - r_1 h)\left(1 + \left(r_2 + \frac{r_1 a_{21}}{a_{11}}\right)h\right)d_{12}, \\ p_{22} &= -\left(\frac{a_{12}r_1}{a_{11}}h\right)^2 d_{11} - 2\frac{a_{12}r_1}{a_{11}}h\left(1 + \left(r_2 + \frac{r_1 a_{21}}{a_{11}}\right)h\right)d_{12} \\ &\quad - \left(1 + \left(r_2 + \frac{r_1 a_{21}}{a_{11}}\right)h\right)^2 d_{22}. \end{aligned} \tag{12}$$

It is not difficult to see that, for any positive definite matrix P, (6) has no positive semidefinite solution D satisfying (7). This means that Theorem 2.1 cannot be applied. As a result, we study the almost sure asymptotic stability. Note that, in this case, (5) is

$$\begin{aligned} u_1(n+1) &= u_1(n) - r_1 h u_1(n) + \frac{a_{12}r_1}{a_{11}}h u_2(n) + \sigma_1\sqrt{h}u_1(n)\xi_1(n+1), \\ u_2(n+1) &= u_2(n) + \left(r_2 + \frac{a_{21}r_1}{a_{11}}\right)h u_2(n) + \sigma_2\sqrt{h}u_2(n)\xi_2(n+1). \end{aligned} \tag{13}$$

For h sufficiently small, from the second equation of (13), using Lemma 3.1, we have the following conclusion:

$$\lim_{n\to\infty} u_2(n) = 0 \quad \text{a.s. if and only if} \quad 2\left(r_2 + \frac{a_{21}r_1}{a_{11}}\right) - \sigma_2^2 < 0.$$

If $\lim_{n\to\infty} u_2(n) = 0$ a.s., then there is a sufficiently large N such that $P\{|\frac{a_{12}r_1}{a_{11}}hu_2(n)| \le \frac{r_1}{a_{11}}\} = 1$ for $n > N$. From the first equation of (13) we have

$$u_1(n+1) \le \frac{r_1}{a_{11}} + u_1(n) - r_1 h u_1(n) + \sigma_1\sqrt{h}u_1(n)\xi_1(n+1) \quad \text{a.s.}$$

By using the comparison theorem [31] and Lemma 3.1 we have $\lim_{n\to\infty} u_1(n) = \frac{r_1}{a_{11}}$ a.s. Figure 4 demonstrates the almost sure asymptotical stability of $(\frac{r_1}{a_{11}}, 0)$.

In summary, we have established the following result.

Proposition 3.2 *For h sufficiently small, the equilibrium state E_1 of (4) is a.s. locally asymptotically stable if* $2(r_2 + \frac{a_{21}r_1}{a_{11}}) - \sigma_2^2 < 0$.

Finally, the stability of $E_2 = (0, \frac{r_2}{a_{22}})$ can be discussed in a similar way as that for E_1, and the following result can be drawn.

Proposition 3.3 *For h sufficiently small, the equilibrium state E_2 of (4) is a.s. locally asymptotically stable if* $2(r_1 + \frac{a_{12}r_2}{a_{22}}) - \sigma_1^2 < 0$.

Figure 4 The boundary equilibrium state $E_1 = (1/8, 0)$ of (4) is a.s. locally asymptotically stable when $\sigma_1 = \sigma_2 = 0.9$, $r_1 = r_2 = 0.25$, $a_{11} = 2$, $a_{12} = 0.6$, $a_{21} = 1$, and $h = 0.1$ satisfy $2(r_2 + \frac{a_{21}r_1}{a_{11}}) - \sigma_2^2 < 0$, where the initial condition is $(u_1(0), u_2(0)) = (0.5, 0.8)$

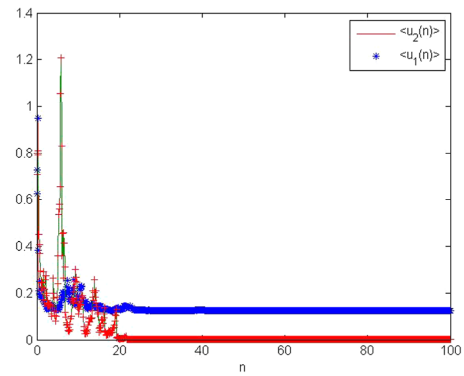
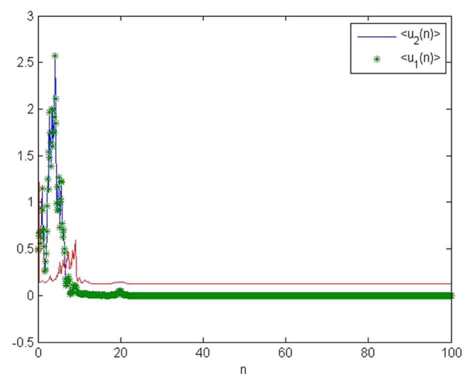


Figure 5 The boundary equilibrium state $E_2 = (0, 1/8)$ of (4) is a.s. locally asymptotically stable when $\sigma_1 = \sigma_2 = 0.9$, $r_1 = r_2 = 0.25$, $a_{22} = 2$, $a_{12} = 0.6$, $a_{21} = 1$, and $h = 0.1$ satisfy $2(r_1 + \frac{a_{12}r_2}{a_{22}}) - \sigma_1^2 < 0$, where the initial condition is $(u_1(0), u_2(0)) = (0.5, 0.8)$



The illustration of Proposition 3.3 is shown in Fig. 5.

4 Conclusion

In this paper, we proposed a two-species stochastic discrete mutualism model where the stochastic white noise perturbation is directly proportional to the deviation of the state from the equilibrium staes. For the positive equilibrium state, we established conditions for the asymptotic mean square stability by the Lyapunov functional approach. However, it seems not easy to apply this approach to obtain the mean square stability of the three boundary equilibrium states. As a result, we studied the almost sure asymptotic stability of them by employing a discretized Itô formula and comparison theorems. The obtained theoretical results are strongly supported by numerical simulations.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally to the writing of this paper. Both authors read and approved the final version of the manuscript.

Author details

[1] School of Mathematics and Physics, University of South China, Hengyang, China. [2] Department of Mathematics, Wilfrid Laurier University, Waterloo, Canada.

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References

1. Cranshaw, W., Redak, R.: Bugs Rule, an Introduction to the World of Insects. Princeton University Press, Princeton (2013)
2. Bronstein, J.L.: Our current understanding of mutualism. Q. Rev. Biol. **69**, 31–51 (1994)
3. Holland, J.N., DeAngelis, D.L.: A consumer-resource approach to the density-dependent population dynamics of mutualism. Ecology **91**, 1286–1295 (2010)
4. Kang, Y., Clark, R., Makiyama, M., Fewell, J.: Mathematical modeling on obligate mutualism: interactions between leaf-cutter ants and their fungus garden. J. Theor. Biol. **289**, 116–127 (2011)
5. Liu, Q., Chen, Q., Hu, Y.: Analysis of a stochastic mutualism model. Commun. Nonlinear Sci. Numer. Simul. **29**, 188–197 (2015)
6. Liu, Q., Jiang, D., Shi, N., Hayat, T., Alsaedi, A.: Stochastic mutualism model with Lévy jumps. Commun. Nonlinear Sci. Numer. Simul. **43**, 78–90 (2017)
7. Takabayashi, J., Dicke, M.: Plant–carnivore mutualism through herbivore-induced carnivore attractants. Trends Plant Sci. **1**, 109–113 (1996)
8. Tegelaar, K., Hagman, M., Glinwood, R., Pettersson, J., Leimar, O.: Ant–aphid mutualism: the influence of ants on the aphid summer cycle. Oikos **121**, 61–66 (2012)
9. Yamamura, N., Higashi, M., Behera, N., Wakano, J.Y.: Evolution of mutualism through spatial effects. J. Theor. Biol. **226**, 421–428 (2004)
10. Zhang, S., Zhang, Y., Ma, K.: Mixed effects of ant-aphid mutualism on plants across different spatial scales. Basic Appl. Ecol. **16**, 452–459 (2015)
11. May, R.: Models for two interacting populations. In: Theoretical Ecology, Principles and Applications, 2nd edn., pp. 78–104. University of California Press, Berkeley (2012)
12. Gard, T.: Stability for multispecies population models in random environments. Nonlinear Anal., Theory Methods Appl. **10**, 1411–1419 (1986)
13. Gard, T.: Introduction to Stochastic Differential Equations. Dekker, New York (1988)
14. Lai, Y.C.: Beneficial role of noise in promoting species diversity through stochastic resonance. Phys. Rev. E, Stat. Nonlinear Soft Matter Phys. **72**, 042901 (2005)
15. May, R.M., Allen, P.M.: Stability and Complexity in Model Ecosystems. Princeton University Press, Princeton (2001)
16. Mao, X.: Stochastic Differential Equations and Applications. Horwood, Chichester (2007)
17. Gikhman, I.I., Skorokhod, A.V.: Stochastic Differential Equations. Springer, Berlin (1972)
18. Shaikhet, L.: Lyapunov Functionals and Stability of Stochastic Functional Differential Equations. Springer, London (2013)
19. Shaikhet, L.: Stability of equilibrium states for a stochastically perturbed exponential type system of difference equations. J. Comput. Appl. Math. **290**, 92–103 (2015)
20. Beretta, E., Kolmanovskii, V., Shaikhet, L.: Stability of epidemic model with time delays influenced by stochastic perturbations. Math. Comput. Simul. **45**(3–4), 269–277 (1998)
21. Shaikhet, L.: Lyapunov Functionals and Stability of Stochastic Difference Equations. Springer, London (2011)
22. Mao, X., Sabanis, S., Renshaw, E.: Asymptotic behaviour of the stochastic Lotka–Volterra model. J. Math. Anal. Appl. **287**, 141–156 (2003)
23. Liu, M., Wang, K.: A note on a delay Lotka–Volterra competitive system with random perturbations. Appl. Math. Lett. **26**, 589–594 (2013)
24. Silva-Dias, L., Lopez-Castillo, A.: Spontaneous symmetry breaking of population: stochastic Lotka–Volterra model for competition among two similar preys and predators. Math. Biosci. **300**, 36–46 (2018)
25. Luo, J., Shaikhet, L.: Stability in probability of nonlinear stochastic Volterra difference equations with continuous variable. Stoch. Anal. Appl. **25**, 1151–1165 (2007)
26. Liao, X., Zhou, S., Chen, Y.: Permanence for a discrete time Lotka–Volterra type food-chain model with delays. Appl. Math. Comput. **186**, 279–285 (2007)
27. Palmer, P.: Application of a discrete Itö formula to determine stability (instability) of the equilibrium of a scalar linear stochastic difference equation. Comput. Math. Appl. **64**, 2302–2311 (2012)
28. Hartung, F., Pituk, M.: Recent Advances in Delay Differential and Difference Equations. Springer, Switzerland (2014)
29. Kwon, T.: Ants foraging on grasses in South Korea: high diversity in the Jeju Island and negative correlation with aphids. J. Asia-Pac. Biodivers. **10**, 465–471 (2017)
30. Ngoc, D.N., Parra, R.B., Zavala, M.A., Auger, P.: Competition and species coexistence in a metapopulation model: can fast symmetric migration reverse the outcome of competition in a homogeneous environment. J. Theor. Biol. **266**, 256–263 (2010)
31. Wang, T.: Comparison theorems of stochastic difference equations. J. East China Norm. Univ. Natur. Sci. Ed. **3**, 59–66 (2008)