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# Fractional hybrid differential equations with three-point boundary hybrid conditions

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**Abstract**

In this paper, we study the existence of solutions for hybrid fractional differential equations involving fractional Caputo derivative of order $1 < \alpha \leq 2$. Our results rely on a hybrid fixed point theorem for a sum of three operators due to Dhage. An example is provided to illustrate the theory.

**MSC:** 34A08; 34A12

**Keywords:** Fractional differential equation; Caputo fractional derivative; Hybrid; Fixed point theorem

## 1 Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary noninteger order. Applications of fractional differential equations can be found in various fields of science and engineering. Indeed, there are numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetism, and so on [23, 26, 28, 30]. Recent developments of fractional differential and integral equations are given in [1–3, 36–40].

Many authors have studied the existence of solutions of fractional boundary value problems under various boundary conditions and by different approaches. We refer the readers to the papers [4, 5, 7, 16, 17, 19, 22, 24, 29, 33] and references therein.

In recent years, hybrid fractional differential equations have achieved a great deal of interest and attention of several researchers. For some developments on the existence results for hybrid fractional differential equations, we refer to [6, 8–15, 20, 21, 25, 27, 31, 32, 34, 35] and es references therein.

This paper deals with the existence and uniqueness of solutions for boundary-value problem of the fractional differential equations

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{\alpha}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right] = h(t,x(t)), \quad 1<\alpha\leq 2, t\in J=[0,T], \\ a_1\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=0} + b_1\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=T} = \lambda_1, \\ a_2{}^{c}\mathcal{D}_{0^+}^{\beta}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=\eta} + b_2{}^{c}\mathcal{D}_{0^+}^{\beta}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=T} = \lambda_2, \quad 0<\eta<T, \end{cases} \tag{1.1}$$

where ${}^{c}\mathcal{D}_{0^+}^{\alpha}$ and ${}^{c}\mathcal{D}_{0^+}^{\beta}$ denote the Caputo fractional derivatives of orders $\alpha$ and $\beta$, respectively, $0<\beta\leq 1$, $a_i, b_i, c_i$, $i=1,2$, are real constants such that $a_1+b_1\neq 0$, $a_2\eta^{1-\beta}+b_2T^{1-\beta}\neq 0$, $g\in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$, and $f,h\in C(J\times\mathbb{R},\mathbb{R})$.

This paper can be considered as a generalization of [19]. For example, if we choose $f(t,x(t)) = 0$ and $g(t,x(t)) = 1$ as constant functions, then our problem (1.1) reduces to the boundary value problem

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{\alpha}x(t) = h(t,x(t)), \quad 1 < \alpha \le 2, t \in J = [0,T], \\ a_1 x(0) + b_1 x(T) = \lambda_1, \\ a_2 {}^{c}\mathcal{D}_{0^+}^{\beta}x(\eta) + b_2 {}^{c}\mathcal{D}_{0^+}^{\beta}x(T) = \lambda_2, \quad 0 < \eta < T. \end{cases} \tag{1.2}$$

The paper is organized as follows. In Sect. 2, we introduce some notations, definitions. and lemmas. Then, in Sect. 3, we prove existence results for problems (1.1) by employing the hybrid fixed point theorem for three operators in a in Banach algebra due to Dhage. Finally, we illustrate the obtained results by an example.

## 2 Preliminaries

In this section, we recall some basic definitions of fractional calculus [23, 30] and present some auxiliary lemmas.

**Definition 2.1** The Riemann–Liouville fractional integral of order $\alpha > 0$ for a continuous function $f : [0,\infty) \to \mathbb{R}$ is defined as

$$I_{0^+}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,\mathrm{d}s, \quad \alpha > 0,$$

where $\Gamma$ is the Euler gamma function.

**Definition 2.2** Let $\alpha > 0$ and $n = [\alpha] + 1$. If $f \in C^n([a,b])$, then the Caputo fractional derivative of order $\alpha$ defined by

$${}^{c}\mathcal{D}_{0^+}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)\,\mathrm{d}s$$

exists almost everywhere on $[a,b]$ ($[\alpha]$ is the integer part of $\alpha$).

**Lemma 2.3** *Let $\alpha > \beta > 0$ and $f \in L^1([a,b])$. Then for all $t \in [a,b]$, we have*:
- $I_{0^+}^{\alpha} I_{0^+}^{\beta} f(t) = I_{0^+}^{\alpha+\beta} f(t)$,
- ${}^{c}\mathcal{D}_{0^+}^{\alpha} I_{0^+}^{\alpha} f(t) = f(t)$,
- ${}^{c}\mathcal{D}_{0^+}^{\beta} I_{0^+}^{\alpha} f(t) = I_{0^+}^{\alpha-\beta} f(t)$.

**Lemma 2.4** *Let $\alpha > 0$. Then the differential equation*

$$\left({}^{c}\mathcal{D}_{0^+}^{\alpha}f\right)(t) = 0$$

*has a solution*

$$f(t) = \sum_{j=0}^{m-1} c_j t^j, \quad c_j \in \mathbb{R}, j = 0,\ldots,m-1,$$

*where $m - 1 < \alpha < m$.*

**Lemma 2.5** *Let $\alpha > 0$. Then*

$$I_{0^+}^{\alpha}\left({}^{c}D_{0^+}^{\alpha}f(t)\right) = f(t) + \sum_{j=0}^{m-1} c_j t^j$$

*for some $c_j \in \mathbb{R}$, $j = 0, 1, 2, \ldots, m-1$, where $m = [\alpha] + 1$.*

Define the supremum norm $\|\cdot\|$ in $E = C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t\in J} \left|x(t)\right|$$

and the multiplication in $E$ by

$$(xy)(t) = x(t)y(t).$$

Clearly, $E$ is a Banach algebra with respect to the supremum norm and multiplication in it.

To prove the existence result for the nonlocal boundary value problem (1.1), we will use the following hybrid fixed point theorem for three operators in a Banach algebra $E$ due to Dhage [18].

**Lemma 2.6** *Let $S$ be a closed convex bounded nonempty subset of a Banach algebra $E$, and let $A, C : E \longrightarrow E$ and $B : S \longrightarrow E$ be three operators such that*:

(a) *$A$ and $C$ are Lipschitzian with a Lipschitz constants $\delta$ and $\rho$, respectively*;
(b) *$B$ is compact and continuous*;
(c) *$x = AxBy + Cx \Rightarrow x \in S$ for all $y \in S$,*
(d) *$\delta M + \rho < 1$, where $M = \|B(S)\|$.*

*Then the operator equation $AxBx + Cx = x$ has a solution in $S$.*

## 3 Main results

In this section, we prove the existence results for the boundary value problems for hybrid differential equations with fractional order on the closed bounded interval $J = [0, T]$.

**Lemma 3.1** *Let $h$ be continuous function on $J := [0, T]$. Then the solution of the boundary value problem*

$$ {}^{c}D_{0^+}^{\alpha}\left[\frac{x(t) - f(t, x(t))}{g(t, x(t))}\right] = h(t), \quad t \in J, 1 < \alpha \le 2, \tag{3.1}$$

*with boundary conditions*

$$\begin{aligned} &a_1\left[\frac{x(t) - f(t, x(t))}{g(t, x(t))}\right]_{t=0} + b_1\left[\frac{x(t) - f(t, x(t))}{g(t, x(t))}\right]_{t=T} = \lambda_1, \\ &a_2\,{}^{c}D_{0^+}^{\beta}\left[\frac{x(t) - f(t, x(t))}{g(t, x(t))}\right]_{t=\eta} + b_2\,{}^{c}D_{0^+}^{\beta}\left[\frac{x(t) - f(t, x(t))}{g(t, x(t))}\right]_{t=T} = \lambda_2, \quad 0 < \eta < T, \end{aligned} \tag{3.2}$$

*satisfies the equation*

$$x(t) = g\big(t,x(t)\big)\bigg[I_{0^+}^{\alpha}h(t) - \frac{b_1}{a_1+b_1}I_{0^+}^{\alpha}h(T) + \frac{\lambda_1}{a_1+b_1} + \frac{(b_1T-(a_1+b_1)t)\Gamma(2-\beta)(a_2I_{0^+}^{\alpha-\beta}h(\eta)+b_2I_{0^+}^{\alpha-\beta}h(T)-\lambda_2)}{(a_1+b_1)(a_2\eta^{1-\beta}+b_2T^{1-\beta})}\bigg] + f\big(t,x(t)\big). \quad (3.3)$$

*Proof* Applying the Riemann–Liouville fractional integral operator of order $\alpha$ to both sides of (3.1) and using Lemma 2.5, we have

$$\frac{x(t)-f(t,x(t))}{g(t,x(t))} = I_{0^+}^{\alpha}h(t) - c_0 - c_1t, \quad c_0,c_1 \in \mathbb{R}. \quad (3.4)$$

Consequently, the general solution of (3.1) is

$$x(t) = g\big(t,x(t)\big)\big(I_{0^+}^{\alpha}y(t) - c_0 - c_1t\big) + f\big(t,x(t)\big)c_0, \quad c_1 \in \mathbb{R}. \quad (3.5)$$

Applying the boundary conditions (3.2) in (3.4), we find that

$$-a_1c_0 + b_1\big(I_{0^+}^{\alpha}h(T) - c_0 - c_1T\big) = \lambda_1,$$
$$a_2I_{0^+}^{\alpha-\beta}h(\eta) + b_2I_{0^+}^{\alpha-\beta}h(T) - \frac{a_2\eta^{1-\beta}+b_2T^{1-\beta}}{\Gamma(2-\beta)}c_1 = \lambda_2.$$

Therefore we have

$$c_0 = -\frac{b_1T\Gamma(2-\beta)(a_2I_{0^+}^{\alpha-\beta}h(\eta)+b_2I_{0^+}^{\alpha-\beta}h(T)-\lambda_2)}{(a_1+b_1)(a_2\eta^{1-\beta}+b_2T^{1-\beta})} + \frac{b_1}{a_1+b_1}I_{0^+}^{\alpha}h(T) - \frac{\lambda_1}{a_1+b_1},$$
$$c_1 = \frac{\Gamma(2-\beta)(a_2I_{0^+}^{\alpha-\beta}h(\eta)+b_2I_{0^+}^{\alpha-\beta}h(T)-\lambda_2)}{a_2\eta^{1-\beta}+b_2T^{1-\beta}}.$$

Substituting the values of $c_0$, $c_1$ into (3.5), we get (3.3). □

Now we list the following hypotheses.

(H1) The functions $g: J\times\mathbb{R} \longrightarrow \mathbb{R}\setminus\{0\}$ and $h,f: J\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous.

(H2) There exist two positive functions $\phi_0$, $\phi_1$ with bounds $\|\phi_0\|$ and $\|\phi_0\|$, respectively, such that

$$\big|f(t,x)-f(t,y)\big| \le \phi_0(t)|x-y| \quad (3.6)$$

and

$$\big|g(t,x)-g(t,y)\big| \le \phi_1(t)|x-y|, \quad (3.7)$$

for all $(t,x,y)\in J\times\mathbb{R}\times\mathbb{R}$.

(H3) There exist a function $p\in L^{\infty}(J,\mathbb{R}_+)$ and a continuous nondecreasing function $\psi:[0,\infty)\longrightarrow(0,\infty)$ such that

$$\big|h(t,x)\big| \le p(t)\psi\big(|x|\big) \quad (3.8)$$

for all $t\in J$ and $x\in\mathbb{R}$.

(H4) There exists $r > 0$ such that

$$r \geq \frac{g_0 \Lambda + f_0}{1 - \|\phi_0\| \Lambda - \|\phi_1\|} \tag{3.9}$$

and

$$\|\phi_0\| \Lambda + \|\phi_1\| < 1, \tag{3.10}$$

where $f_0 = \sup_{t \in J} |f(t,0)|$, $g_0 = \sup_{t \in J} |g(t,0)|$, and

$$\begin{aligned} \Lambda = \psi(r)\|p\| \biggl( & \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{|b_1|T^\alpha}{|a_1+b_1|\Gamma(\alpha+1)} \\ & + \frac{(|b_1|T + (|a_1|+|b_1|)T)\Gamma(2-\beta)}{|a_1+b_1||a_2\eta^{1-\beta} + b_2 T^{1-\beta}|\Gamma(\alpha-\beta+1)} \\ & \times \bigl( \bigl( |a_2|\eta^{\alpha-\beta} + |b_2|T^{\alpha-\beta} \bigr) + |\lambda_2| \bigr) + \frac{|\lambda_1|}{|a_1+b_1|} \biggr). \end{aligned} \tag{3.11}$$

**Theorem 3.2** *Assume that conditions* (H1)–(H4) *hold. Then problem* (1.1) *has at least one solution defined on* $J$.

*Proof* Define the set

$$S = \bigl\{ x \in E : \|x\|_E \leq r \bigr\}.$$

Clearly, $S$ is a closed convex bounded subset of the Banach space $E$. By Lemma 3.1 the boundary value problem (1.1) is equivalent to the equation

$$\begin{aligned} x(t) = {} & f\bigl(t,x(t)\bigr) + g\bigl(t,x(t)\bigr) \biggl[ I_{0^+}^\alpha h\bigl(s,x(s)\bigr)(t) - \frac{b_1}{a_1+b_1} I_{0^+}^\alpha h\bigl(s,x(s)\bigr)(T) \\ & + \frac{\lambda_1}{a_1+b_1} + \frac{(b_1 T - (a_1+b_1)t)\Gamma(2-\beta)}{(a_1+b_1)(a_2\eta^{1-\beta} + b_2 T^{1-\beta})} \\ & \times \bigl( a_2 I_{0^+}^{\alpha-\beta} h\bigl(s,x(s)\bigr)(\eta) + b_2 I_{0^+}^{\alpha-\beta} h\bigl(s,x(s)\bigr)(T) - \lambda_2 \bigr) \biggr], \quad t \in J. \end{aligned} \tag{3.12}$$

Define three operators $A, C : E \longrightarrow E$ and $B : S \longrightarrow E$ by

$$\begin{aligned} Ax(t) &= g\bigl(t,x(t)\bigr), \quad t \in J, \\ Bx(t) &= I_{0^+}^\alpha h\bigl(s,x(s)\bigr)(t) - \frac{b_1}{a_1+b_1} I_{0^+}^\alpha h\bigl(s,x(s)\bigr)(T) + \frac{(b_1 T - (a_1+b_1)t)\Gamma(2-\beta)}{(a_1+b_1)(a_2\eta^{1-\beta} + b_2 T^{1-\beta})} \\ & \quad \times \bigl( a_2 I_{0^+}^{\alpha-\beta} h\bigl(s,x(s)\bigr)(\eta) + b_2 I_{0^+}^{\alpha-\beta} h\bigl(s,x(s)\bigr)(T) - \lambda_2 \bigr) + \frac{\lambda_1}{a_1+b_1}, \quad t \in J, \end{aligned}$$

and

$$Cx(t) = f\bigl(t,x(t)\bigr), \quad t \in J.$$

Then the integral equation (3.12) can be written in the operator form as

$$x(t) = Ax(t)Bx(t) + Cx(t), \quad t \in J.$$

We will show that the operators $A$, $B$, and $C$ satisfy all the conditions of Lemma 2.6. This will be achieved in the following series of steps.

*Step* 1: First, we show that $A$ and $C$ are Lipschitzian on $E$. Let $x, y \in E$. Then by (H2), for $t \in J$, we have

$$\bigl|Ax(t) - Ay(t)\bigr| = \bigl|g\bigl(t, x(t)\bigr) - g\bigl(t, y(t)\bigr)\bigr| \leq \phi_0(t)\bigl|x(t) - y(t)\bigr|$$

for all $t \in J$. Taking the supremum over $t$, we obtain

$$\|Ax - Ay\| \leq \|\phi_0\| \|x - y\|$$

for all $x, y \in E$. Therefore $A$ is Lipschitzian on $E$ with Lipschitz constant $\|\phi_0\|$.

Now, for $C : E \longrightarrow E$, $x, y \in E$, we have

$$\bigl|Cx(t) - Cy(t)\bigr| = \bigl|f\bigl(t, x(t)\bigr) - f\bigl(t, y(t)\bigr)\bigr| \leq \phi_1(t)\bigl|x(t) - y(t)\bigr|$$

for all $t \in J$. Taking the supremum over $t$, we obtain

$$\|Cx - Cy\| \leq \|\phi_1\| \|x - y\|.$$

Hence $C : E \longrightarrow E$ is Lipschitzian on $E$ with Lipschitz constant $\|\phi_1\|$.

*Step* 2: We show that $B$ is is a completely continuous operator from $S$ into $E$. First, we show that $B$ is continuous on $S$. Let $\{x_n\}$ be a sequence in $S$ converging to a point $x \in S$. Then by the Lebesgue dominated convergence theorem we have

$$\begin{aligned}
\lim_{n\to\infty} Bx_n(t) &= \frac{1}{\Gamma(\alpha)} \lim_{n\to\infty} \int_0^t (t-s)^{\alpha-1} h\bigl(s, x_n(s)\bigr)\,\mathrm{d}s \\
&\quad - \frac{b_1}{(a_1+b_1)\Gamma(\alpha)} \lim_{n\to\infty} \int_0^T (T-s)^{\alpha-1} h\bigl(s, x_n(s)\bigr)\,\mathrm{d}s \\
&\quad + \frac{(b_1 T - (a_1+b_1)t)\Gamma(2-\beta)}{(a_1+b_1)(a_2\eta^{1-\beta} + b_2 T^{1-\beta})\Gamma(\alpha-\beta)} \\
&\quad \times \biggl(a_2 \lim_{n\to\infty} \int_0^\eta (\eta-s)^{\alpha-\beta-1} h\bigl(s, x_n(s)\bigr)\,\mathrm{d}s \\
&\quad + b_2 \lim_{n\to\infty} \int_0^T (T-s)^{\alpha-\beta-1} h\bigl(s, x_n(s)\bigr)\,\mathrm{d}s - \lambda_2\biggr) + \frac{\lambda_1}{a_1+b_1} \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lim_{n\to\infty} h\bigl(s, x_n(s)\bigr)\,\mathrm{d}s \\
&\quad - \frac{b_1}{(a_1+b_1)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \lim_{n\to\infty} h\bigl(s, x_n(s)\bigr)\,\mathrm{d}s \\
&\quad + \frac{(b_1 T - (a_1+b_1)t)\Gamma(2-\beta)}{(a_1+b_1)(a_2\eta^{1-\beta} + b_2 T^{1-\beta})\Gamma(\alpha-\beta)} \\
&\quad \times \biggl(a_2 \int_0^\eta (\eta-s)^{\alpha-\beta-1} \lim_{n\to\infty} h\bigl(s, x_n(s)\bigr)\,\mathrm{d}s \\
&\quad + b_2 \int_0^T (T-s)^{\alpha-\beta-1} \lim_{n\to\infty} h\bigl(s, x_n(s)\bigr)\,\mathrm{d}s - \lambda_2\biggr) + \frac{\lambda_1}{a_1+b_1}
\end{aligned}$$

$$\begin{aligned}
&= I_{0^+}^{\alpha}h\bigl(t,x(t)\bigr) - \frac{b_1}{a_1+b_1}I_{0^+}^{\alpha}h\bigl(T,x(T)\bigr) + \frac{\lambda_1}{a_1+b_1} \\
&\quad + \frac{(b_1T-(a_1+b_1)t)\Gamma(2-\beta)}{(a_1+b_1)(a_2\eta^{1-\beta}+b_2T^{1-\beta})\Gamma(\alpha-\beta)} \\
&\quad \times \bigl(a_2I_{0^+}^{\alpha-\beta}h\bigl(\eta,x(\eta)\bigr) + b_2I_{0^+}^{\alpha-\beta}h\bigl(T,x(T)\bigr) - \lambda_2\bigr) \\
&= Bx(t)
\end{aligned}$$

for all $t \in J$. This shows that $B$ is a continuous operator on $S$.

Next, we will prove that the set $B(S)$ is a uniformly bounded in $S$. For any $x \in S$, we have

$$\begin{aligned}
\bigl|Bx(t)\bigr| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigl|h\bigl(s,x(s)\bigr)\bigr|\,\mathrm{d}s \\
&\quad + \frac{|b_1|}{|a_1+b_1|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\bigl|h\bigl(s,x(s)\bigr)\bigr|\,\mathrm{d}s \\
&\quad + \frac{(|b_1|T+(|a_1|+|b_1|)T)\Gamma(2-\beta)}{|a_1+b_1||a_2\eta^{1-\beta}+b_2T^{1-\beta}|\Gamma(\alpha-\beta)}\biggl(|a_2|\int_0^\eta (\eta-s)^{\alpha-\beta-1}\bigl|h\bigl(s,x(s)\bigr)\bigr|\,\mathrm{d}s \\
&\quad + |b_2|\int_0^T (T-s)^{\alpha-\beta-1}\bigl|h\bigl(s,x(s)\bigr)\bigr|\,\mathrm{d}s + |\lambda_2|\biggr) + \frac{|\lambda_1|}{|a_1+b_1|}.
\end{aligned}$$

Using (3.8), we can write

$$\begin{aligned}
\bigl|Bx(t)\bigr| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\psi(r)p(s)\,\mathrm{d}s + \frac{|b_1|}{|a_1+b_1|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\psi(r)p(s)\,\mathrm{d}s \\
&\quad + \frac{(|b_1|T+(|a_1|+|b_1|)T)\Gamma(2-\beta)}{|a_1+b_1||a_2\eta^{1-\beta}+b_2T^{1-\beta}|\Gamma(\alpha-\beta)}\biggl(|a_2|\int_0^\eta (\eta-s)^{\alpha-\beta-1}\psi(r)p(s)\,\mathrm{d}s \\
&\quad + |b_2|\int_0^T (T-s)^{\alpha-\beta-1}\psi(r)p(s)\,\mathrm{d}s + |\lambda_2|\biggr) + \frac{|\lambda_1|}{|a_1+b_1|} \\
&\le \psi(r)\|p\|\biggl(\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{|b_1|T^\alpha}{|a_1+b_1|\Gamma(\alpha+1)}\biggr) \\
&\quad + \frac{(|b_1|T+(|a_1|+|b_1|)T)\Gamma(2-\beta)}{|a_1+b_1||a_2\eta^{1-\beta}+b_2T^{1-\beta}|\Gamma(\alpha-\beta+1)}\bigl(|a_2|\eta^{\alpha-\beta}+|b_2|T^{\alpha-\beta}\bigr) \\
&\quad + \frac{(|b_1|T+(|a_1|+|b_1|)T)\Gamma(2-\beta)}{|a_1+b_1||a_2\eta^{1-\beta}+b_2T^{1-\beta}|\Gamma(\alpha-\beta+1)}|\lambda_2| + \frac{|\lambda_1|}{|a_1+b_1|}.
\end{aligned}$$

Thus $\|Bx\| \le \Lambda$ for all $x \in S$ with $\Lambda$ given in (3.11). This shows that $B$ is uniformly bounded on $S$.

Now, we will show that $B(S)$ is an equicontinuous set in $E$.

Let $t_1, t_2 \in J$. Then for any $x \in S$, by (3.8) we get

$$\begin{aligned}
\bigl|Bx(t_2)-Bx(t_1)\bigr| &\le \frac{\psi(r)\|p\|}{\Gamma(\alpha)}\int_0^{t_1}\bigl((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\bigr)\,\mathrm{d}s \\
&\quad + \frac{\psi(r)\|p\|}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\,\mathrm{d}s + \frac{|\lambda_2||a_1+b_1||t_1-t_2|}{|a_1+b_1||a_2\eta^{1-\beta}+b_2T^{1-\beta}|\Gamma(\alpha-\beta)}
\end{aligned}$$

$$+\frac{|a_1+b_1||t_1-t_2|\psi(r)\|p\|}{|a_1+b_1||a_2\eta^{1-\beta}+b_2T^{1-\beta}|\Gamma(\alpha-\beta)}\left(|a_2|\int_0^{\eta}(\eta-s)^{\alpha-\beta-1}\,ds\right.$$
$$\left.+|b_2|\int_0^{T}(T-s)^{\alpha-\beta-1}\,ds\right). \tag{3.13}$$

Obviously, the right-hand side of inequality (3.13) tends to zero independently of $x\in S$ as $t_2\to t_1$. As a consequence of the Ascoli–Arzelà theorem, $B$ is a completely continuous operator on $S$.

*Step* 3: Hypothesis (c) of Lemma 2.6 is satisfied.

Let $x\in E$ and $y\in S$ be arbitrary elements such that $x=AxBy+Cx$. Then we have

$$\begin{aligned}
|x(t)| &\le |Ax(t)||By(t)|+|Cx(t)|\\
&\le \left|g\bigl(t,x(t)\bigr)\right|\left\{\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left|h\bigl(s,y(s)\bigr)\right|\,ds\right.\\
&\quad+\frac{|b_1|}{|a_1+b_1|\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\left|h\bigl(s,y(s)\bigr)\right|\,ds\\
&\quad+\frac{(|b_1|T+(|a_1|+|b_1|)T)\Gamma(2-\beta)}{|a_1+b_1||a_2\eta^{1-\beta}+b_2T^{1-\beta}|\Gamma(\alpha-\beta)}\left(|a_2|\int_0^{\eta}(\eta-s)^{\alpha-\beta-1}\left|h\bigl(s,y(s)\bigr)\right|\,ds\right.\\
&\quad\left.\left.+|b_2|\int_0^T(T-s)^{\alpha-\beta-1}\left|h\bigl(s,y(s)\bigr)\right|\,ds+|\lambda_2|\right)+\frac{|\lambda_1|}{|a_1+b_1|}\right\}+\left|f\bigl(t,x(t)\bigr)\right|\\
&\le \Bigl(\left|g\bigl(t,x(t)\bigr)-g(t,0)\right|+\left|g(t,0)\right|\Bigr)\left\{\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\psi(r)p(s)\,ds\right.\\
&\quad+\frac{|b_1|}{|a_1+b_1|\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\psi(r)p(s)\,ds\\
&\quad+\frac{(|b_1|T+(|a_1|+|b_1|)T)\Gamma(2-\beta)}{|a_1+b_1||a_2\eta^{1-\beta}+b_2T^{1-\beta}|\Gamma(\alpha-\beta)}\left(|a_2|\int_0^{\eta}(\eta-s)^{\alpha-\beta-1}\psi(r)p(s)\,ds\right.\\
&\quad\left.\left.+|b_2|\int_0^T(T-s)^{\alpha-\beta-1}\psi(r)p(s)\,ds+|\lambda_2|\right)+\frac{|\lambda_1|}{|a_1+b_1|}\right\}\\
&\quad+\left|f\bigl(t,x(t)\bigr)-f(t,0))\right|+\left|f(t,0)\right|\\
&\le \bigl(\|\phi_0\||x(t)|+g_0\bigr)\Lambda+\|\phi_1\||x(t)|+f_0.
\end{aligned}$$

Thus

$$|x(t)|\le\frac{g_0\Lambda+f_0}{1-\|\phi_0\|\Lambda-\|\phi_1\|}.$$

Taking the supremum over $t$, we get

$$\|x\|\le\frac{g_0\Lambda+f_0}{1-\|\phi_0\|\Lambda-\|\phi_1\|}\le r.$$

*Step* 4: Finally, we show that $\delta M+\rho<1$, that is, (d) of Lemma 2.6 holds.

Since

$$M=\|B(S)\|=\sup_{x\in S}\Bigl\{\sup_{t\in J}\left|Bx(t)\right|\Bigr\}\le\Lambda,$$

we have

$$\|\phi_0\|M + \|\phi_1\| \leq \|\phi_0\|\Lambda + \|\phi_1\| < 1$$

with $\delta = \|\phi_0\|$ and $\rho = \|\phi_1\|$ Thus all the conditions of Lemma 2.6 are satisfied, and hence the operator equation $x = AxBx + Cx$ has a solution in $S$. As a result, problem (1.1) has a solution on $J$. □

*Example* 3.3 Let us consider the following boundary value problem:

$$\begin{cases} {}^{\mathrm{c}}D_{0^+}^{\frac{3}{2}}[\frac{x(t)-f(t,x(t))}{g(t,x(t))}] = \frac{e^{-2t}}{\sqrt{(9+t)}}\sin x(t), \quad t \in J = [0,1], \\ [\frac{x(t)-f(t,x(t))}{g(t,x(t))}]_{t=0} + 2[\frac{x(t)-f(t,x(t))}{g(t,x(t))}]_{t=1} = \frac{1}{2}, \\ 3\,{}^{\mathrm{c}}D_{0^+}^{\frac{1}{2}}[\frac{x(t)-f(t,x(t))}{g(t,x(t))}]_{t=\frac{1}{2}} + 0.25\,{}^{\mathrm{c}}D_{0^+}^{\frac{1}{2}}[\frac{x(t)-f(t,x(t))}{g(t,x(t))}]_{t=1} = 1. \end{cases} \tag{3.14}$$

We take

$$\begin{aligned} &\alpha = \frac{3}{2}, \qquad \beta = \frac{1}{2}, \qquad a_1 = 1, \qquad a_2 = 3, \qquad \lambda_1 = \frac{1}{2}, \\ &b_1 = 2, \qquad b_2 = \frac{1}{4}, \qquad \lambda_2 = 1, \qquad \eta = \frac{1}{2}, \qquad T = 1, \\ &f\bigl(t,x(t)\bigr) = \frac{t^2}{100}\left(\frac{1}{2}\bigl(x(t) + \sqrt{x^2+1}\bigr) + e^{-t}\right), \\ &g\bigl(t,x(t)\bigr) = \frac{\sqrt{\pi}e^{-2\pi t}\cos(\pi t)}{(7\pi + 15e^t)^2}\frac{x(t)}{1+x(t)} + \frac{t}{10}, \\ &h\bigl(t,x(t)\bigr) = \frac{e^{-2t}}{\sqrt{(9+t)}}\sin x(t). \end{aligned}$$

We can show that

$$\begin{aligned} &\bigl|f(t,x) - f(t,y)\bigr| \leq \frac{t^2}{100}|x-y|, \\ &\bigl|g(t,x) - g(t,y)\bigr| \leq \frac{\sqrt{\pi}e^{-2\pi t}}{(7\pi + 15e^t)^2}|x-y|, \\ &\bigl|h(t,x)\bigr| \leq p(t)\psi\bigl(|x|\bigr), \end{aligned}$$

where

$$\psi\bigl(|x|\bigr) = |x|, \qquad p(t) = e^{-2t}.$$

Hence we have

$$\phi_0(t) = \frac{t^2}{100}, \qquad \phi_1(t) = \frac{\sqrt{\pi}e^{-2\pi t}}{(7\pi + 15e^t)^2}.$$

Then

$$\|\phi_0\| = \frac{1}{100}, \qquad \|\phi_1\| = \frac{\sqrt{\pi}}{(7\pi + 15)^2}, \qquad \|p\| = 1,$$

and

$$f_0 = \sup_{t\in J}\bigl|f(t,0)\bigr| = \frac{1}{100}, \qquad g_0 = \sup_{t\in J}\bigl|g(t,0)\bigr| = \frac{1}{10}.$$

Using the Matlab program, it follows by (3.9) and (3.10) that the constant $r$ satisfies the inequality $0.0146 < r < 21.8589$. As all the conditions of Theorem 3.2 are satisfied, problem (3.14) has at least one solution on $J$.

## 4 Concluding remarks

In this paper, we have provided some sufficient conditions guaranteeing the existence of solutions for a class of hybrid fractional differential equations involving fractional Caputo derivative of order $1 < \alpha \leq 2$. Our results rely on a hybrid fixed point theorem for a sum of three operators due to Dhage. Our results extend and complete those in the literature.

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