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Nonhomogeneous initial and boundary value problem for the Caputo-type fractional wave equation

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Abstract

In this paper, we give an analytical solution of a fractional wave equation for a vibrating string with Caputo time fractional derivatives. We obtain the exact solution in terms of three parameter Mittag-Leffler function. Furthermore, some examples of the main result are exhibited.

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Keywords: Fractional wave equation; Caputo time fractional derivative; Mittag-Leffler function; Laplace transform

1 Introduction

In recent years, fractional calculus has been one of the most popular topics in research [1–5]. There are many different definitions and representations of fractional integrals and derivatives in the literature, for instance, Riemann–Liouville integral, Riemann–Liouville derivatives, Caputo derivative, Hilfer derivative, and so on (see [6–16]).

The fractional calculus has been used effectively to solve different kinds of problems such as fractional relaxation and oscillation process, time fractional diffusive and wave processes [4, 17–21], generalized Langevin and fractional Fokker–Planck equations [22–33]. Furthermore, many authors have obtained the solutions of time fractional diffusion-wave equations in a bounded domain in terms of the Mittag-Leffler type functions (see [22, 31, 34–44]).

Here, we solve the following wave equation for a vibrating string:

$$C_*^\gamma w(x,t) = \frac{\partial^2 w(x,t)}{\partial x^2} - bC_*^\alpha w(x,t) + g(x,t), \tag{1}$$

with Caputo time fractional derivatives C_*^γ and C_*^α of orders $1 < \gamma < 2$ and $0 < \alpha < 1$, respectively, using the conditions

$$w(x,t)|_{x=0} = h_1(t), \qquad w(x,t)|_{x=l} = h_2(t) \tag{2}$$

and

$$w(x,t)|_{t=0_+} = \Theta(x), \qquad \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0_+} = \Phi(x), \tag{3}$$

where $t > 0$, $0 \le x \le l$, $g(x,t)$, $h_1(t)$, $h_2(t)$, $\Theta(x)$ and $\Phi(x)$ are sufficiently well-behaved functions, b is a positive constant, τ is the memory time, and $g(x,t)$ is the external force.

This problem has the solutions for $x \in [0,l]$ and in $L(0,\infty)$ such that

$$L(0,\infty) = \left\{ f : \|f\|_1 = \int_0^\infty \left|f(t)\right| dt < \infty \right\},$$

where $L(0,\infty)$the Lebesgue integrable function deals with t.

This paper is organized as follows. In Sect. 2, definitions and properties of Mittag-Leffler functions and fractional integrals and derivatives are presented. In Sect. 3, we consider the fractional wave equation (1) and solve this problem by using the separation of variables and Fourier expansion method. Also, some examples under the conditions are presented in Sect. 4. Finally, in Sect. 5, we give a concluding remark.

2 Mathematical background

2.1 The Mittag-Leffler functions

The Mittag-Leffler functions [45] were studied and introduced in the following series:

$$\begin{aligned} &E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + 1)}, \\ &\bigl(z \in \mathbb{C}, \operatorname{Re}(\alpha) > 0\bigr). \end{aligned} \tag{4}$$

A more general form of (4) was given by Wiman [46] in the form

$$\begin{aligned} &E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \\ &\bigl(z,\beta \in \mathbb{C}, \operatorname{Re}(\alpha) > 0\bigr). \end{aligned} \tag{5}$$

It is obvious that, by using (4) and (5), we have $E_{\alpha,1}(z) = E_\alpha(z)$. The Mittag-Leffler functions are a generalization of the exponential, hyperbolic, and trigonometric functions since $E_{1,1}(z) = e^z$, $E_{2,1}(z^2) = \cosh(z)$, $E_{2,1}(-z^2) = \cos(z)$, and $E_{2,2}(-z^2) = \sin(z)/z$.

The generalized Mittag-Leffler functions were defined by Praphakar [47], that is,

$$\begin{aligned} &E^\gamma_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \\ &\bigl(\alpha,\beta,\gamma \in \mathbb{C}, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0; z \in \mathbb{C}\bigr), \end{aligned} \tag{6}$$

where $(\gamma)_k$ is the Pochhammer symbol [48] defined by

$$(\gamma)_k = \frac{\Gamma(\gamma + k)}{\Gamma(\gamma)} = \begin{cases} 1; & k = 0, \gamma \neq 0, \\ \gamma(\gamma+1)\cdots(\gamma+k-1); & k = 1,2,\ldots. \end{cases}$$

Note that $E^{1}_{\alpha,\beta}(x) = E_{\alpha,\beta}(x)$. The four parameter Mittag-Leffler function [49] was defined by

$$E^{\gamma,\kappa}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}, \quad (\alpha,\beta,\gamma \in \mathbb{C}, \operatorname{Re}(\alpha) > \max\{0, \operatorname{Re}(\kappa)-1\}, \operatorname{Re}(\kappa) > 0). \tag{7}$$

From (6) and (7), we see that $E^{\gamma,1}_{\alpha,\beta}(z) = E^{\gamma}_{\alpha,\beta}(z)$.

The Laplace transform of the Mittag-Leffler functions (6) is represented by (see [47, 50])

$$\mathbb{L}\bigl[t^{\beta-1}E^{\gamma}_{\alpha,\beta}\bigl(ut^{\alpha}\bigr)\bigr](s) = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}-u)^{\gamma}}, \tag{8}$$

where $|\frac{u}{s^{\alpha}}| < 1$.

Now, we give basic definitions and properties that will be used throughout the paper.

Definition 2.1 (Riemann–Liouville integral (see [5])) Let $\Omega = [a,b]$ be a finite interval of the real axis. The Riemann–Liouville fractional integral of order $\mu \in \mathbb{C}$ ($\operatorname{Re}(\mu) > 0$) is defined by

$${}_{x}I^{\gamma}_{a^+}[g] = \frac{1}{\Gamma(\gamma)} \int_a^x \frac{g(t)\,dt}{(x-t)^{1-\gamma}} \quad \bigl(x > a, \operatorname{Re}(\mu) > 0\bigr). \tag{9}$$

Definition 2.2 (Caputo derivative (see [4] and [51])) Let $\gamma > 0$, $n = \lceil\gamma\rceil$, and $g \in AC^n[a,b]$. The Caputo derivative of $\gamma > 0$ is defined as

$$C^{\gamma}_{*}g(t) = \begin{cases} \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{g^{(n)}(s)}{(t-s)^{\gamma+1-n}}\,ds; & n-1 < \gamma < n, \\ \frac{d^n g(t)}{dt^n}; & \gamma = n. \end{cases} \tag{10}$$

Note that the following expression gives us the relationship between the Caputo fractional derivative (10) and the Riemann–Liouville fractional integral operator (9) (see [4])

$$C^{\gamma}_{*}g(t) = I^{n-\gamma}_{0^+}g^{(n)}(t),$$

where $g^{(n)}$ is denoted by n-order derivative.

We give the Laplace transform for the Caputo fractional derivative in the following formula (see [5, 52]):

$$\mathbb{L}\bigl[C^{\gamma}_{*}g(t)\bigr](s) = \int_0^{\infty} e^{-st}C^{\gamma}_{*}g(t)\,dt = s^{\gamma}G(s) - \sum_{k=0}^{n-1} g^{(k)}(0_+)s^{\gamma-1-k}, \tag{11}$$

where $\gamma \in (n-1,n)$ and $G(s)$ is the representation of Laplace transform for the function $g(t)$. Clearly, $C^{\gamma}_{*}1 \equiv 0$ for $\gamma > 0$.

Definition 2.3 The integral operator $\mathcal{E}^{u;\gamma,\kappa}_{a^+;\alpha,\beta}\varphi$ (see [49]) was introduced by Srivastava and Tomovski in the following form:

$$\begin{aligned}&\big(\mathcal{E}^{u;\gamma,\kappa}_{a^+;\alpha,\beta}\varphi\big)(x)=\int_a^x(x-t)^{\beta-1}E^{\gamma,\kappa}_{\alpha,\beta}\big[u(x-t)^{\alpha}\big]\varphi(t)\,dt,\\&\big(\rho,\mu,u,\gamma\in\mathbb{C},\operatorname{Re}(\rho),\operatorname{Re}(\mu)>0\big),\end{aligned}\tag{12}$$

where $E^{\gamma,\kappa}_{\alpha,\beta}(z)$ is the four parameter Mittag-Leffler function given in (7).

When $u=0$ and $a=0$, the integral operator (12) coincides with the Riemann–Liouville integral operator (9) such that

$$\big(\mathcal{E}^{0;\gamma,\kappa}_{0^+;\alpha,\beta}\varphi\big)(x)=\big(I^{\beta}_{0^+}\varphi\big)(x).$$

3 Analytical results for the problem

In this section, we investigate the analytical solution of the proposed problem (1)–(3). In order to obtain the solution, we need the following lemmas.

Lemma 3.1 *Let $s,b,\alpha,\lambda_n\in\mathbb{R}^+$ and $u\in\mathbb{R}$. Then the inverse Laplace transform of the function*

$$f(s)=\frac{s^{\gamma-1}+bs^{\alpha-1}+us^{\gamma-2}}{s^{\gamma}+bs^{\alpha}+\lambda_n}\tag{13}$$

deals with

$$\begin{aligned}\big(\mathbb{L}^{-1}f\big)(t)&=\mathbb{L}^{-1}\big[f(s)\big](t)\\&=\sum_{p=0}^{\infty}(-b)^pt^{(\gamma-\alpha)p}E^{(p+1)}_{\gamma,(\gamma-\alpha)p+1}\big(-\lambda_nt^{\gamma}\big)\\&\quad+b\sum_{p=0}^{\infty}(-b)^pt^{(\gamma-\alpha)(p+1)}E^{(p+1)}_{\gamma,(\gamma-\alpha)(p+1)+1}\big(-\lambda_nt^{\gamma}\big)\\&\quad+u\sum_{p=0}^{\infty}(-b)^pt^{(\gamma-\alpha)p+1}E^{(p+1)}_{\gamma,(\gamma-\alpha)p+2}\big(-\lambda_nt^{\gamma}\big),\end{aligned}\tag{14}$$

where $0<\frac{\lambda_n}{s^{\gamma}+bs^{\alpha}}<1$ and $0<\frac{b}{s^{\gamma-\alpha}}<1$,

Proof Since $0<\frac{\lambda_n}{s^{\gamma}+bs^{\alpha}}<1$ and $0<\frac{b}{s^{\gamma-\alpha}}<1$, we rewrite relation (13) in the following way:

$$\begin{aligned}f(s)&=\big(s^{\gamma-1}+bs^{\alpha-1}+us^{\gamma-2}\big).\frac{s^{-\alpha}}{s^{\gamma-\alpha}+b}.\frac{1}{1+\frac{\lambda_ns^{-\alpha}}{s^{\gamma-\alpha}+b}}\\&=\sum_{j=0}^{\infty}(-\lambda_n)^j\left\{\frac{s^{-\alpha(j+1)+\gamma-1}}{(s^{\gamma-\alpha}+b)^{j+1}}+b\frac{s^{-\alpha j-1}}{(s^{\gamma-\alpha}+b)^{j+1}}+u\frac{s^{-\alpha(j+1)+\gamma-2}}{(s^{\gamma-\alpha}+b)^{j+1}}\right\}.\end{aligned}$$

By using relation (8), we get

$$\begin{aligned}
&\mathbb{L}^{-1}\big[f(s)\big](t)\\
&= \sum_{j=0}^{\infty}(-\lambda_n)^j t^{\gamma j} E^{(j+1)}_{\gamma-\alpha,\gamma j+1}\big(-bt^{\gamma-\alpha}\big)\\
&\quad + b\sum_{j=0}^{\infty}(-\lambda_n)^j t^{\gamma(j+1)-\alpha} E^{(j+1)}_{\gamma-\alpha,\gamma(j+1)-\alpha+1}\big(-bt^{\gamma-\alpha}\big)\\
&\quad + u\sum_{j=0}^{\infty}(-\lambda_n)^j t^{\gamma j+1} E^{j+1}_{\gamma-\alpha,\gamma j+2}\big(-bt^{\gamma-\alpha}\big)\\
&= \sum_{j=0}^{\infty}\sum_{p=0}^{\infty}(-\lambda_n)^j t^{\gamma j}\frac{(j+1)_p}{\Gamma((\gamma-\alpha)p+\gamma j+1)}\frac{(-bt^{\gamma-\alpha})^p}{p!}\\
&\quad + b\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}(-\lambda_n)^j t^{\gamma j+1-\alpha}\frac{(j+1)_p}{\Gamma((\gamma-\alpha)p+\gamma(j+1)-\alpha+1)}\frac{(-bt^{\gamma-\alpha})^p}{p!}\\
&\quad + u\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}(-\lambda_n)^j t^{\gamma j+1}\frac{(j+1)_p}{\Gamma((\gamma-\alpha)p+\gamma j+2)}\frac{(-bt^{\gamma-\alpha})^p}{p!}\\
&= \sum_{j=0}^{\infty}\sum_{p=0}^{\infty}(-b)^p t^{(\gamma-\alpha)p}\frac{(p+1)_j}{\Gamma((\gamma-\alpha)p+\gamma j+1)}\frac{(-\lambda_n t^{\gamma})^j}{j!}\\
&\quad + b\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}(-b)^p t^{(\gamma-\alpha)(p+1)}\frac{(p+1)_j}{\Gamma(\gamma j+(\gamma-\alpha)(p+1)+1)}\frac{(-\lambda_n t^{\gamma})^j}{j!}\\
&\quad + u\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}(-b)^p t^{(\gamma-\alpha)p+1}\frac{(p+1)_j}{\Gamma((\gamma-\alpha)p+\gamma j+2)}\frac{(-\lambda_n t^{\gamma})^j}{j!}\\
&= \sum_{p=0}^{\infty}(-b)^p t^{(\gamma-\alpha)p} E^{(p+1)}_{\gamma,(\gamma-\alpha)p+1}\big(-\lambda_n t^{\gamma}\big)\\
&\quad + b\sum_{p=0}^{\infty}(-b)^p t^{(\gamma-\alpha)(p+1)} E^{(p+1)}_{\gamma,(\gamma-\alpha)(p+1)+1}\big(-\lambda_n t^{\gamma}\big)\\
&\quad + u\sum_{p=0}^{\infty}(-b)^p t^{(\gamma-\alpha)p+1} E^{(p+1)}_{\gamma,(\gamma-\alpha)p+2}\big(-\lambda_n t^{\gamma}\big).
\end{aligned}$$

Thus, we get the desired result. □

Lemma 3.2 *Let $s, b, \alpha, \lambda_n \in \mathbb{R}^+$. We have*

$$\mathbb{L}^{-1}\left[\frac{1}{s^{\gamma}+bs^{\alpha}+\lambda_n}\mathbb{L}\big[\widetilde{g}_n(t)\big](s)\right](t) = \sum_{p=0}^{\infty}(-b)^p\big(\mathcal{E}^{-\lambda_n;p+1,1}_{0^+;\gamma,(\gamma-\alpha)p+\gamma}\widetilde{g}_n\big)(t),$$

$$\left(0<\frac{\lambda_n}{s^{\gamma}+bs^{\alpha}}<1, 0<\frac{b}{s^{\gamma-\alpha}}<1\right),$$

where $\mathcal{E}^{-\lambda_n;p+1,1}_{0^+;\gamma,(\gamma-\alpha)p+\gamma}$ is given in (12) and $\widetilde{g}_n(t)$ is a given function.

Proof Let

$$h(s)=\frac{1}{s^{\gamma}+bs^{\alpha}+\lambda_n}\mathbb{L}\bigl[\widetilde{g}_n(t)\bigr](s). \tag{15}$$

We rewrite relation (15) in the following form:

$$h(s)=\frac{s^{-\alpha}}{s^{\gamma-\alpha}+b}\cdot\frac{1}{1+\frac{\lambda_n s^{-\alpha}}{s^{\gamma-\alpha}+b}}\mathbb{L}\bigl[\widetilde{g}_n(t)\bigr](s).$$

Since $0<\frac{\lambda_n}{s^{\gamma}+bs^{\alpha}}<1$, we have

$$h(s)=\sum_{j=0}^{\infty}(-\lambda_n)^j\frac{s^{-\alpha(j+1)}}{(s^{\gamma-\alpha}+b)^{j+1}}\mathbb{L}\bigl[\widetilde{g}_n(t)\bigr](s).$$

With the help of relation (8), we obtain

$$\begin{aligned}h(s)&=\mathbb{L}\Biggl[\sum_{j=0}^{\infty}(-\lambda_n)^j t^{\gamma(j+1)-1}E^{j+1}_{\gamma-\alpha,\gamma(j+1)}\bigl(-bt^{\gamma-\alpha}\bigr)\Biggr](s)\mathbb{L}\bigl[\widetilde{g}_n(t)\bigr](s)\\&=\mathbb{L}\Biggl[\sum_{p=0}^{\infty}(-b)^p t^{(\gamma-\alpha)p+\gamma-1}E^{p+1}_{\gamma,(\gamma-\alpha)p+\gamma}\bigl(-\lambda_n t^{\gamma}\bigr)\Biggr](s)\mathbb{L}\bigl[\widetilde{g}_n(t)\bigr](s).\end{aligned}$$

Applying the Parseval theorem for the Laplace transform (see [53])

$$\mathbb{L}\biggl[\int_0^x k(x-t)\varphi(t)\,dt\biggr](s)=\mathbb{L}\bigl[k(x)\bigr](s)\mathbb{L}\bigl[\varphi(x)\bigr](s),$$

we have

$$h(s)=\sum_{p=0}^{\infty}(-b)^p\mathbb{L}\biggl[\int_0^t(t-\tau)^{(\gamma-\alpha)p+\gamma-1}E^{p+1}_{\gamma,(\gamma-\alpha)p+\gamma}\bigl(-\lambda_n[t-\tau]^{\gamma}\bigr)\widetilde{g}_n(\tau)\biggr](s). \tag{16}$$

Taking the inverse Laplace transform on both sides of (16), we get

$$\begin{aligned}&\mathbb{L}^{-1}\bigl[g(s)\bigr](t)\\&\quad=\mathbb{L}^{-1}\biggl[\frac{1}{s^{\gamma}+bs^{\alpha}+\lambda_n}\mathbb{L}\bigl[\widetilde{g}_n(t)\bigr](s)\biggr](t)=\sum_{p=0}^{\infty}(-b)^p\bigl(\mathcal{E}^{-\lambda_n;p+1,1}_{0^+;\gamma,(\gamma-\alpha)p+\gamma}\widetilde{g}_n\bigr)(t),\end{aligned}$$

which is the desired result. □

The solution of the problem given by (1)–(3) is given in the following theorem.

Theorem 3.3 *The problem given in* (1), (2), *and* (3) *has a summable solution in* $L(0,\infty)$ *with respect to* t *as follows*:

$$w(x,t)=\sum_{n=1}^{\infty}T_n(t)\sin\left(\frac{n\pi x}{l}\right)+\sum_{n=1}^{\infty}\sum_{p=0}^{\infty}(-b)^p\left(\mathcal{E}^{-\lambda_n;p+1,1}_{0^+;\gamma,(\gamma-\alpha)p+\gamma}\widetilde{g}_n\right)(t)\sin\left(\frac{n\pi x}{l}\right)+h_1(t)+\frac{x}{l}\left[h_2(t)-h_1(t)\right]$$

for $x\in[0,l]$, *where*

$$\begin{aligned}T_n(t)&=T_n^{(0)}(0_+)\Bigg[\sum_{p=0}^{\infty}(-b)^p t^{(\gamma-\alpha)p}E^{(p+1)}_{\gamma,(\gamma-\alpha)p+1}\left(-\frac{n^2\pi^2}{l^2}t^{\gamma}\right)\\&\quad+b\sum_{p=0}^{\infty}(-b)^p t^{(\gamma-\alpha)(p+1)}E^{(p+1)}_{\gamma,(\gamma-\alpha)(p+1)+1}\left(-\frac{n^2\pi^2}{l^2}t^{\gamma}\right)\\&\quad+\frac{T_n^{(1)}(0_+)}{T_n^{(0)}(0_+)}\sum_{p=0}^{\infty}(-b)^p t^{(\gamma-\alpha)p+1}E^{(p+1)}_{\gamma,(\gamma-\alpha)+2}\left(-\frac{n^2\pi^2}{l^2}t^{\gamma}\right)\Bigg],\\\widetilde{g}_n(t)&=\frac{2}{l}\int_0^l\Bigg[g(x,t)+\frac{\partial^2(h_1(t)+\frac{x}{l}[h_2(t)-h_1(t)])}{\partial x^2}\\&\quad-C^{\gamma}_{*}\left(h_1(t)+\frac{x}{l}\left[h_2(t)-h_1(t)\right]\right)-bC^{\alpha}_{*}w(x,t)\Bigg]\sin\left(\frac{n\pi x}{l}\right)dx,\\T_n^{(0)}(0_+)&=\frac{2}{l}\int_0^l\left(\widetilde{\Theta}(x)-\left(h_1(t)+\frac{x}{l}\left[h_2(t)-h_1(t)\right]\right)\bigg|_{t=0_+}\right)\sin\left(\frac{n\pi x}{l}\right)dx,\end{aligned}\tag{17}$$

and

$$T_n^{(1)}(0_+)=\frac{2}{l}\int_0^l\left(\widetilde{\Phi}(x)-\frac{\partial(h_1(t)+\frac{x}{l}[h_2(t)-h_1(t)])}{\partial t}\bigg|_{t=0_+}\right)\sin\left(\frac{n\pi x}{l}\right)dx.$$

Proof Suppose $w(x,t)$ is given as

$$w(x,t)=W(x,t)+\nu(x,t).\tag{18}$$

Clearly, conditions (2) satisfy $\nu(x,t)$ where

$$\nu(x,t)=h_1(t)+\frac{x}{l}\left[h_2(t)-h_1(t)\right].\tag{19}$$

From relations (18) and (19), we get

$$W(x,t)|_{x=0}=0,\qquad W(x,t)|_{x=l}=0.$$

By (3), we get

$$W(x,t)|_{t=0_+}=\Theta(x)-\nu(x,t)|_{t=0_+}=\widetilde{\Theta}(x),$$

$$\frac{\partial W(x,t)}{\partial t}\bigg|_{t=0_+} = \Phi(x) - \frac{\partial v(x,t)}{\partial x}\bigg|_{t=0_+} = \widetilde{\Phi}(x).$$

By representing

$$W(x,t) = W_1(x,t) + W_2(x,t)$$

and by using (1) and (18), we get

$$C_*^\gamma\big[W_1(x,t) + W_2(x,t)\big] = \frac{\partial^2}{\partial x^2}\big[W_1(x,t) + W_2(x,t)\big] - bC_*^\alpha\big[W_1(x,t) + W_2(x,t)\big] + \widetilde{g}(x,t),$$

where

$$\widetilde{g}(x,t) = g(x,t) + \frac{\partial^2 v(x,t)}{\partial x^2} - C_*^\gamma v(x,t) - bC_*^\alpha w(x,t). \tag{20}$$

The problem now can be reduced as follows:

$$C_*^\gamma W_1(x,t) = \frac{\partial^2 W_1(x,t)}{\partial x^2} - bW_1(x,t),$$

$$W_1(x,t)|_{x=0} = 0, \qquad W_1(x,t)|_{x=l} = 0,$$

$$W_1(x,t)|_{t=0_+} = \widetilde{\Theta}(x), \qquad \frac{\partial W_1(x,t)}{\partial t}\bigg|_{t=0_+} = \widetilde{\Phi}(x),$$

and

$$C_*^\gamma W_2(x,t) = \frac{\partial^2 W_2(x,t)}{\partial x^2} - bC_*^\alpha W_2(x,t) + \widetilde{g}(x,t), \tag{21}$$

$$W_2(x,t)|_{x=0} = 0, \qquad W_2(x,t)|_{x=l} = 0, \tag{22}$$

$$W_2(x,t)|_{t=0_+} = 0, \qquad \frac{\partial W_2(x,t)}{\partial t}\bigg|_{t=0_+} = 0. \tag{23}$$

Letting $W_1(x,t) = X(x)T(t)$, the differential equations take the following forms:

$$C_*^\gamma T(t) + bC_*^\alpha T(t) + \lambda T(t) = 0, \tag{24}$$

$$\frac{d^2X(x)}{dx^2} + \lambda X(x) = 0, \tag{25}$$

where λ is called a separation constant. So, the solution of the Sturm–Liouville problem (25) deals with the function $X(x)$ with boundary conditions:

$$X(x)|_{x=0} = 0, \qquad X(x)|_{x=l} = 0. \tag{26}$$

The eigenfunctions of the problem are given in the form $X_n(x) = \sin(\sqrt{\lambda_n}x)$ where $\lambda_n = \frac{n^2\pi^2}{l^2}$, $(0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \cdots)$. The relation for the eigenfunctions is satisfied by

$$\int_0^l X_n^2(x)\,dx = \|X_n\|^2\delta_{nm},$$

where $\|X_n\|^2 = \frac{1}{2}$ is the norm of the eigenfunctions and δ_{nm} is the Kronecker delta.

By using the Laplace transform, (24) is solved in the space $L(0,\infty)$. So, we get

$$\begin{aligned} & s^{\gamma}\mathbb{L}\big[T_n(t)\big](s) - s^{\gamma-1}T_n^{(0)}(0_+) - s^{\gamma-2}T_n^{(1)}(0_+) \\ & \quad + b\big\{s^{\alpha}\mathbb{L}\big[T_n(t)\big](s) - s^{\alpha-1}T_n^{(0)}(0_+)\big\} \\ & \quad + \lambda_n\mathbb{L}\big[T_n(t)\big](s) = 0. \end{aligned} \tag{27}$$

From (27), we get

$$\mathbb{L}\big[T_n(t)\big](s) = T_n^{(0)}(0_+)\left[\frac{s^{\gamma-1} + bs^{\alpha-1} + \frac{T_n^{(1)}(0_+)}{T_n^{(0)}(0_+)}s^{\gamma-2}}{s^{\gamma} + bs^{\alpha} + \lambda_n}\right]. \tag{28}$$

By using (14) and Lemma 3.1, the inverse Laplace transform of relation (28) yields

$$\begin{aligned} & T_n(t) \\ & \quad = T_n^{(0)}(0_+)\Bigg[\sum_{p=0}^{\infty}(-b)^p t^{(\gamma-\alpha)p}E_{\gamma,(\gamma-\alpha)p+1}^{(p+1)}\big(-\lambda_n t^{\gamma}\big) \\ & \qquad + b\sum_{p=0}^{\infty}(-b)^p t^{(\gamma-\alpha)(p+1)}E_{\gamma,(\gamma-\alpha)(p+1)+1}^{(p+1)}\big(-\lambda_n t^{\gamma}\big) \\ & \qquad + \frac{T_n^{(1)}(0_+)}{T_n^{(0)}(0_+)}\sum_{p=0}^{\infty}(-b)^p t^{(\gamma-\alpha)p+1}E_{\gamma,(\gamma-\alpha)p+2}^{(p+1)}\big(-\lambda_n t^{\gamma}\big)\Bigg], \end{aligned}$$

so we obtain the solution of $W_1(x,t)$ such that

$$W_1(x,t) = \sum_{n=1}^{\infty}T_n(t)\sin\left(\frac{n\pi x}{l}\right). \tag{29}$$

By using the Fourier expansions, we find the solution of (21):

$$W_2(x,t) = \sum_{n=1}^{\infty}w_n(t)\sin\left(\frac{n\pi x}{l}\right), \tag{30}$$

$$\widetilde{g}(x,t) = \sum_{n=1}^{\infty}\widetilde{g}_n(t)\sin\left(\frac{n\pi x}{l}\right), \tag{31}$$

where $\widetilde{g}_n(t)$ is represented in (17). From (30), (31), and (21), we get

$$\sum_{n=1}^{\infty}\big[C_*^{\gamma}w_n(t) + bC_*^{\alpha}w_n(t) + \lambda_n w_n(t) - \widetilde{g}_n(t)\big]\sin\left(\frac{n\pi x}{l}\right) = 0$$

if

$$C_*^{\gamma}w_n(t) + bC_*^{\alpha}w_n(t) + \lambda_n w_n(t) - \widetilde{g}_n(t) = 0, \tag{32}$$

where $n \in \mathbb{N}$.

Using the Laplace transform method (11) to (32), we get

$$\begin{aligned} & s^{\gamma}\mathbb{L}\big[w_n(t)\big](s) - s^{\gamma-1}w_n^{(0)}(0_+) - s^{\alpha-2}w_n^{(1)}(0_+)\big] \\ & \quad + b\big\{s^{\alpha}\mathbb{L}\big[w_n(t)\big](s) - s^{\alpha-1}w_n(0_+)\big\} \\ & \quad + \lambda_n\mathbb{L}\big[w_n(t)\big](s) - \mathbb{L}\big[\widetilde{g}_n(t)\big](s) = 0. \end{aligned} \tag{33}$$

From conditions (23), it follows that $\frac{\partial^p w_n(x,t)}{\partial t^p}|_{t=0^+} = 0$ for $p = 0, 1$. From (33), we get

$$\mathbb{L}\big[w_n(t)\big](s) = \frac{1}{s^{\gamma} + bs^{\alpha} + \lambda_n}\mathbb{L}\big[\widetilde{g}_n(t)\big](s). \tag{34}$$

Finally, we get the inverse Laplace transform of (34) and use Lemma 3.2 to obtain the following result:

$$w_n(t) = \sum_{p=0}^{\infty}(-b)^p\big(\mathcal{E}_{0^+;\gamma,(\gamma-\alpha)p+\gamma}^{-\lambda_n;p+1,1}\widetilde{g}_n\big)(t). \tag{35}$$

Thus, the proof is completed. □

4 Some applications of the main problem

In this section, we give some applications for time fractional wave equation (1)–(3) by considering special cases of the external force, conditions given in (2) and (3).

Example 4.1 Let $g(x,t) = 0$, $\Theta(x) = x(1-x)$, $\Phi(x) = 0$, $h_1(t) = h_2(t) = 0$, $x \in [0,1]$ in the above theorem. The time fractional wave equation takes the form as follows:

$$C_*^{\gamma}w(x,t) = \frac{\partial^2 w(x,t)}{\partial x^2} - bC_*^{\alpha}w(x,t),$$

where $1 < \gamma < 2$ and $0 < \alpha < 1$, with the conditions

$$w(x,t)|_{x=0} = 0, \qquad w(x,t)|_{x=1} = 0$$

and

$$w(x,t)|_{t=0_+} = 0, \qquad \frac{\partial w(x,t)}{\partial t}\bigg|_{t=0_+} = 0$$

has the following solution

$$w(x,t) = \sum_{n=1}^{\infty}T_n(t)\sin(n\pi x), \tag{36}$$

where

$$T_n(t) = 4\frac{1-(-1)^n}{n^3\pi^3}\Bigg[\sum_{p=0}^{\infty}(-b)^p t^{(\gamma-\alpha)p} E^{(p+1)}_{\gamma,(\gamma-\alpha)p+1}\big(-n^2\pi^2 t^\gamma\big) + b\sum_{p=0}^{\infty}(-b)^p t^{(\gamma-\alpha)(p+1)} E^{(p+1)}_{\gamma,(\gamma-\alpha)(p+1)+1}\big(-n^2\pi^2 t^\gamma\big) + u\sum_{p=0}^{\infty}(-b)^p t^{(\gamma-\alpha)p+1} E^{(p+1)}_{\gamma,(\gamma-\alpha)p+2}\big(-n^2\pi^2 t^\gamma\big)\Bigg]. \tag{37}$$

When $n = 2r$ $(r = 1, 2, \ldots)$, we have $T_{2r}(t) = 0$. Therefore, we have just the odd terms $T_{2r-1}(t)$. Hence, the solution of (36) is $w(x,t) = \sum_{r=1}^{\infty} T_{2r-1}(t)\sin[(2r-1)\pi x]$.

Example 4.2 Let $g(x,t) = ct^{\kappa-1}E^{\zeta}_{\alpha,\kappa}(-t^\alpha)$, $b = 1$, $\tau = 1$, $\Theta(x) = x(1-x)$, $\Phi(x) = 0$, $h_1(t) = h_2(t) = 0$, $x \in [0,1]$ in the above theorem. The time fractional wave equation takes the following form:

$$C^{\gamma}_{*} w(x,t) = \frac{\partial^2 w(x,t)}{\partial x^2} - bC^{\alpha}_{*} w(x,t) + ct^{\kappa-1}E^{\zeta}_{\alpha,\kappa}\big(-t^\alpha\big),$$

where $1 < \gamma < 2$, $1 < \alpha < 2$, c is any constant, with the conditions

$$w(x,t)|_{x=0} = 0, \qquad w(x,t)|_{x=1} = 0$$

and

$$w(x,t)|_{t=0_+} = 0, \qquad \frac{\partial w(x,t)}{\partial t}\bigg|_{t=0_+} = 0$$

has the following solution

$$w(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x) + \sum_{n=1}^{\infty} w_n(t)\sin(n\pi x),$$

where $T_n(t)$ is given by (37), and

$$w_n(t) = 2c\frac{\{1-(-1)^n\}}{n\pi}\sum_{p=0}^{\infty}(-b)^p t^{\mu+\kappa-1}E^{\gamma+\zeta}_{\rho,\mu+\kappa}\big(-t^\rho\big).$$

Note that only odd terms $T_{2r-1}(t)$ and $U_{2r-1}(t)$ are not equal to zero for $r = 1, 2, \ldots$

5 Concluding remark

For $\gamma \to 2$, (1) becomes

$$\frac{\partial^2 w(x,t)}{\partial t^2} = \frac{\partial^2 w(x,t)}{\partial x^2} - bC^{\alpha}_{*} w(x,t) + g(x,t), \tag{38}$$

with conditions

$$w(x,t)|_{x=0} = h_1(t), \qquad w(x,t)|_{x=l} = h_2(t), \tag{39}$$

and

$$w(x,t)|_{t=0_+} = \Theta(x), \qquad \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0_+} = \Phi(x) \tag{40}$$

which is considered in [43].

The above problem has the solution (see p. 1558, [43], (27)–(29)) $w(x,t) = W_1(x,t) + W_2(x,t) + v(x,t)$, with

$$\begin{aligned} W_1(x,t) = \sum_{n=1}^{\infty} \Bigg\{ & \sum_{p=0}^{\infty} (-b)^p t^{(2-\alpha)p} E_{2,(2-\alpha)p+1}^{(p+1)}\left(-\lambda_n t^2\right) \\ & + b \sum_{p=0}^{\infty} (-b)^p t^{(2-\alpha)(p+1)} E_{2,(2-\alpha)(p+1)+1}^{(p+1)}\left(-\lambda_n t^2\right) \\ & + u \sum_{p=0}^{\infty} (-b)^p t^{(2-\alpha)p+1} E_{2,(2-\alpha)p+2}^{(p+1)}\left(-\lambda_n t^2\right) \Bigg\} T_n^{(0)}(0_+) \sin\left(\frac{n\pi x}{l}\right), \end{aligned} \tag{41}$$

$$W_2(x,t) = \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} (-b)^p \left(\mathcal{E}_{0^+;2,(2-\alpha)p+2}^{-\lambda_n;p+1,1} \widetilde{g}_n\right)(t) \sin\left(\frac{n\pi x}{l}\right), \tag{42}$$

$$v(x,t) = h_1(t) + \frac{x}{l}\left[h_2(t) - h_1(t)\right]. \tag{43}$$

$$\widetilde{g}_n(t) = \frac{2}{l} \int_0^l \widetilde{g}(x,t) \sin\left(\frac{n\pi x}{l}\right) dx, \tag{44}$$

$$\widetilde{g}(x,t) = g(x,t) + \frac{\partial^2 v(x,t)}{\partial x^2} - \frac{\partial^2 v(x,t)}{\partial t^2} - bC_*^{\alpha} w(x,t), \tag{45}$$

where $\lambda_n = \frac{n^2\pi^2}{l^2}$ are eigenvalues of the problem, $u = T_n^{(1)}(0_+)/T_n^{(0)}(0_+)$, $T_n^{(0)}(0_+) = \frac{2}{l}\int_0^l \widetilde{\Theta}(x)\sin(\frac{n\pi x}{l})\,dx$, $T_n^{(1)}(0_+) = \frac{2}{l}\int_0^l \widetilde{\Phi}(x)\sin(\frac{n\pi x}{l})\,dx$ are Fourier coefficients, $\widetilde{\Theta}(x) = \Theta(x) - v(x,t)|_{t=0_+}$, and $\widetilde{\Phi}(x) = \Phi(x) - \frac{\partial v(x,t)}{\partial t}|_{t=0_+}$.

It is easily observed that for $\gamma \to 2$, the solution which is given in Theorem 3.3 coincides with (41)–(43).

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