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# Infinitely many solutions via critical points for a fractional $p$-Laplacian equation with perturbations

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**Abstract**

In this paper, we use variant fountain theorems to study the existence of infinitely many solutions for the fractional $p$-Laplacian equation

$$(-\Delta)_p^\alpha u + \lambda V(x)|u|^{p-2}u = f(x,u) - \mu g(x)|u|^{q-2}u, \quad x \in \mathbb{R}^N,$$

where $\lambda, \mu$ are two positive parameters, $N, p \geq 2$, $q \in (1,p)$, $\alpha \in (0,1)$, $(-\Delta)_p^\alpha$ is the fractional $p$-Laplacian, and $V, g, u : \mathbb{R}^N \to \mathbb{R}$, $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$.

**Keywords:** Fractional $p$-Laplacian equation; Infinitely many solutions; Variant fountain theorems

## 1 Introduction

In this paper we investigate the existence of infinitely many solutions for the fractional $p$-Laplacian equation

$$(-\Delta)_p^\alpha u + \lambda V(x)|u|^{p-2}u = f(x,u) - \mu g(x)|u|^{q-2}u, \quad x \in \mathbb{R}^N, \tag{1}$$

where $\lambda, \mu$ are two positive parameters, $N, p \geq 2$, $\alpha \in (0,1)$, $(-\Delta)_p^\alpha$ is the fractional $p$-Laplacian, and the potential function $V : \mathbb{R}^N \to \mathbb{R}$ satisfies the following conditions:

- (V1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0 > 0$, where $V_0$ is a positive constant.
- (V2) There exists $b > 0$ such that $\operatorname{meas}\{x \in \mathbb{R}^N : V(x) \leq b\}$ is finite, where meas denotes the Lebesgue measures.

The functions $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}^N \to \mathbb{R}$ satisfy the conditions:

- (f1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and $\lim_{|u|\to 0} \frac{f(x,u)}{|u|^{p-2}u} = 0$ uniformly in $x \in \mathbb{R}^N$.
- (f2) $F(x,u) = \int_0^u f(x,s)\,\mathrm{d}s \geq 0$ and $\mathscr{F}(x,u) = \frac{1}{p}f(x,u)u - F(x,u) \geq 0$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$.
- (f3) $\lim_{|u|\to\infty} \frac{f(x,u)u}{|u|^p} = +\infty$ uniformly in $x \in \mathbb{R}^N$.
- (f4) There exist $d_1, r_0 > 0$ and $\tau > \frac{p_\alpha^*}{p_\alpha^* - p}$ with $p_\alpha^* = \begin{cases} \frac{Np}{N-\alpha p} & \text{if } \alpha p < N, \\ \infty & \text{if } \alpha p \geq N, \end{cases}$ such that

$$|f(x,u)|^\tau \leq d_1 \mathscr{F}(x,u)|u|^{(p-1)\tau} \quad \text{for all } x \in \mathbb{R}^N \text{ and } |u| \geq r_0.$$

(f5) $f(x,-u) = -f(x,u)$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$.

(g) $g \in L^{q'}(\mathbb{R}^N)$ and $g(x) \geq 0$ ($\not\equiv 0$) for a.e. $x \in \mathbb{R}^N$, where $q' \in (\frac{p_\alpha^*}{p_\alpha^*-q}, \frac{p}{p-q}], q \in (1,p)$.

Fractional systems arise for example in phase transitions, chaos, diffusion, finance, flame propagation, and wave propagation. In [1], the authors introduced a fractional order modified Duffing system

$$\begin{cases} \frac{\mathrm{d}^{q_1}x}{\mathrm{d}t^{q_1}} = y, & \frac{\mathrm{d}^{q_2}y}{\mathrm{d}t^{q_2}} = -x - x^3 - ay + bz, \\ \frac{\mathrm{d}z}{\mathrm{d}t} = w, & \frac{\mathrm{d}w}{\mathrm{d}t} = -cz - dz^3, \end{cases}$$

where $\frac{\mathrm{d}^{q_1}x}{\mathrm{d}t^{q_1}}, \frac{\mathrm{d}^{q_2}y}{\mathrm{d}t^{q_2}}$ are fractional derivatives, and via phase portraits and bifurcation diagrams, they studied chaotic behaviors for this system; we also refer the reader to the books [2–4] and the papers [5–23]. Variational methods and critical point theory were used to study fractional Schrödinger equations in the literature [24–37]; for results on Schrödinger equations, we refer the reader to [38–66]. In [24, 25], Ambrosio and Torres used the mountain pass theorem and a variant of the fountain theorem to obtain the existence of nontrivial solutions for (1) with $\lambda = 1, \mu = 0$, where $f$ is $p$-superlinear at infinity. In [27], Tang et al. obtained infinitely many solutions for the following fractional $p$-Laplacian equations of Schrödinger–Kirchhoff type:

$$\left(a + b \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+p\alpha}} \,\mathrm{d}x\,\mathrm{d}y\right)^{p-1} (-\Delta)_p^\alpha u + V(x)|u|^{p-2}u = f(x,u), \tag{2}$$

where they used the condition:

(Tang) There exist $c_0 > 0, r_0 > 0$, and $\kappa > \{1, \frac{N}{p\alpha}\}$ such that

$$\left|F(x,t)\right|^\kappa \leq c_0|t|^{p\kappa}\left[\frac{1}{p^2}f(x,t)t - F(x,t)\right], \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}, |t| \geq r_0,$$

to ensure that the energy functional satisfies the Palais–Smale condition, i.e., (PS) sequence has a convergent subsequence; this condition can also be found in [26, 28, 40–42]. There are only a few papers on (1) with a sublinear perturbation. For example, in [29] the authors used the famous Ambrosetti–Rabinowitz condition:

(AR) There exists $\mu > p^2$ such that

$$0 < \mu F(x,t) \leq f(x,t)t, \quad \forall x \in \mathbb{R}^N, t \in \mathbb{R}\backslash\{0\},$$

to obtain nontrivial solutions for (2) with a perturbation $g$ ($g \in L^{\frac{p}{p-1}}(\mathbb{R}^N)$). In [30–32, 38, 39] similar methods were used to study various Schrödinger equations with perturbations.

Motivated by the above papers, in this paper we use variant fountain theorems to study the existence of nontrivial solutions for the fractional $p$-Laplacian equation (1). The novelty is two-fold: (i) the condition (Tang) is adopted to ensure that bounded sequences have convergent subsequences, (ii) we consider the influence of parameters and perturbation terms on the existence of solutions.

Now, we state our main result.

**Theorem 1.1** *Suppose that* (V1)–(V2), (f1)–(f5), *and* (g) *hold. Then, for sufficiently small* $\mu > 0$, *there exists* $\Lambda > 0$ *such that system* (1) *possesses infinitely many solutions when* $\lambda \geq \Lambda$.

*Remark* 1.2 Note that (f1), (f2), and (f4) imply that $f$ has subcritical growth. From (f2), (f4), for all $x \in \mathbb{R}^N$, $|u| \geq r_0$, we find

$$\begin{aligned} |f(x,u)|^{\tau} &\leq d_1 \mathscr{F}(x,u)|u|^{(p-1)\tau} = d_1\left(\frac{1}{p}f(x,u)u - F(x,u)\right)|u|^{(p-1)\tau} \\ &\leq \frac{d_1}{p}|f(x,u)||u|^{(p-1)\tau+1}. \end{aligned}$$

This shows that

$$|f(x,u)|^{\tau-1} \leq \frac{d_1}{p}|u|^{(p-1)\tau+1} \quad \text{and} \quad |f(x,u)| \leq \sqrt[\tau-1]{\frac{d_1}{p}}|u|^{\frac{(p-1)\tau+1}{\tau-1}}.$$

Let $\frac{(p-1)\tau+1}{\tau-1} = s-1$. Then $s = \frac{p\tau}{\tau-1} \in (p, p_\alpha^*)$. On the other hand, from (f1) for all $\varepsilon > 0$, we have

$$|f(x,u)| \leq \varepsilon |u|^{p-1} \quad \text{for } x \in \mathbb{R}^N, |u| \leq r_0,$$

and hence, there exists $c_\varepsilon = \sqrt[\tau-1]{\frac{d_1}{p}} > 0$ such that

$$|f(x,u)| \leq \varepsilon|u|^{p-1} + c_\varepsilon|u|^{s-1}, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}, \tag{3}$$

and from $F(x,u) = \int_0^u f(x,s)\,ds$ we have

$$|F(x,u)| \leq \frac{\varepsilon}{p}|u|^p + \frac{c_\varepsilon}{s}|u|^s, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{4}$$

*Remark* 1.3 Consider the Ambrosetti–Rabinowitz condition (see [29–32, 38, 39]):

(AR) There exists $\theta > p$ such that

$$0 < \theta F(x,u) \leq f(x,u)u \quad \text{for all } x \in \mathbb{R}^N, u \in \mathbb{R}\backslash\{0\}.$$

Let $F(x,u) = |\sin x||u|^p \ln(1+|u|), \forall x \in \mathbb{R}^N, u \in \mathbb{R}$. Then $f(x,u) = |\sin x|(p|u|^{p-2}u\ln(1+|u|) + \frac{|u|^{p-1}u}{1+|u|})$. Consequently, for all $x \in \mathbb{R}^N$, we have

$$\theta F(x,u) - f(x,u)u = |\sin x|(\theta-p)|u|^p \ln(1+|u|) - |\sin x|\frac{|u|^{p+1}}{1+|u|} \leq 0,$$

and this is impossible for large $|u|$. However, this function satisfies conditions (f1)–(f5).

## 2 Preliminaries

We first discuss the space $W^{\alpha,p}(\mathbb{R}^N)$ (for more details, we refer the reader to [67]). When $u: \mathbb{R}^N \to \mathbb{R}$ is a measurable function, we define the Gagliardo seminorm as follows:

$$[u]_{\alpha,p} := \left[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+\alpha p}}\,dx\,dy\right]^{\frac{1}{p}}, \quad p \geq 2.$$

Now, the fractional Sobolev space is given by

$$W^{\alpha,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : u \text{ is measurable and } [u]_{\alpha,p} < \infty\},$$

with the norm

$$\|u\|_{\alpha,p} = \left([u]_{\alpha,p}^p + \|u\|_p^p\right)^{\frac{1}{p}},$$

where $\|u\|_p$ is the norm for the usual Lebesgue space $L^p(\mathbb{R}^N)$, denoted by

$$\|u\|_p = \left(\int_{\mathbb{R}^N} |u(x)|^p \, \mathrm{d}x\right)^{\frac{1}{p}}.$$

For the potential function $V$, we consider the following fractional Sobolev space:

$$E := \left\{u \in W^{\alpha,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u(x)|^p \, \mathrm{d}x < \infty\right\},$$

with the norm

$$\|u\|_E := \left([u]_{\alpha,p}^p + \int_{\mathbb{R}^N} V(x)|u(x)|^p \, \mathrm{d}x\right)^{\frac{1}{p}}.$$

Note that the parameter $\lambda$ can be chosen large enough, so this norm can be replaced by

$$\|u\| := \left([u]_{\alpha,p}^p + \int_{\mathbb{R}^N} \lambda V(x)|u(x)|^p \, \mathrm{d}x\right)^{\frac{1}{p}}.$$

In summary, throughout our paper we use the space $(E, \|\cdot\|)$.

**Lemma 2.1** (see [67, Theorem 6.5] and [25, Lemma 2.1]) *The embedding $E \hookrightarrow L^t(\mathbb{R}^N)$ is continuous if $t \in [p, p_\alpha^*]$ and compact if $t \in [p, p_\alpha^*)$.*

Hence, there exists $C_t > 0$ such that

$$\|u\|_t \le C_t \|u\|, \quad \forall t \in [p, p_\alpha^*]. \tag{5}$$

Let $X$ be a reflexive and separable Banach space and $X^*$ be its dual space. Then there are (see [68, Sect. 17]) $\{\phi_n\}_{n\in\mathbb{N}} \subset X$ and $\{\phi_n^*\}_{n\in\mathbb{N}} \subset X^*$ such that $X = \overline{\operatorname{span}\{\phi_n : n \in \mathbb{N}\}}$, $X^* = \overline{\operatorname{span}\{\phi_n^* : n \in \mathbb{N}\}}$, and $\langle \phi_n, \phi_m \rangle = \begin{cases} 1, & n = m, \\ 0, & n \ne m. \end{cases}$ For $k = 1, 2, \ldots$, let $Y_k = \operatorname{span}\{\phi_1, \ldots, \phi_k\}$ and $Z_k = \overline{\operatorname{span}\{\phi_k, \phi_{k+1}, \ldots\}}$.

**Lemma 2.2** (see [69]) *Let $X$ be a Banach space, and $X = \overline{\bigoplus_{j\in\mathbb{N}} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \bigoplus_{j=0}^k X_j$, $Z_k = \overline{\bigoplus_{j=k+1}^\infty X_j}$. Consider the following $C^1$ functional $\Phi_\lambda : X \to \mathbb{R}$ defined by*

$$\Phi_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in [1, 2].$$

*Suppose that*

(Z1) *$\Phi_\lambda$ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$. Furthermore, $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for $(\lambda, u) \in [1, 2] \times X$;*

(Z2) *$B(u) \geq 0$; $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of $X$;*

(Z3) *There exist $\rho_k > r_k > 0$ such that*
$a_k(\lambda) = \inf_{u\in Z_k,\|u\|=\rho_k} \Phi_\lambda(u) \geq 0 > b_k(\lambda) = \max_{u\in Y_k,\|u\|=r_k} \Phi_\lambda(u)$ *for* $\lambda \in [1,2]$,
$d_k(\lambda) = \inf_{u\in Z_k,\|u\|\leq\rho_k} \Phi_\lambda(u) \to 0$ *as* $k \to \infty$, *uniformly for* $\lambda \in [1,2]$.

*Then there exist $\lambda_n \to 1$, $u(\lambda_n) \in Y_n$ such that $\Phi'_{\lambda_n}|_{Y_n}(u(\lambda_n)) = 0$, $\Phi_{\lambda_n}(u(\lambda_n)) \to c_k \in [d_k(2), b_k(1)]$ as $n \to \infty$. In particular, if $\{u(\lambda_n)\}$ has a convergent subsequence for every $k$, then $\Phi_1$ has infinitely many nontrivial critical points $\{u_k\} \subset X\backslash\{0\}$ satisfying $\Phi_1(u_k) \to 0^-$ as $k \to \infty$.*

## 3 Main results

Now, we can define the energy functional $J$ on $E$ as follows:

$$J(u) = \frac{1}{p}\|u\|^p - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x + \frac{\mu}{q}\int_{\mathbb{R}^N} g(x)|u|^q\,\mathrm{d}x \quad \text{for } x \in \mathbb{R}^N, u \in E. \tag{6}$$

From (4), (V1)–(V2), and (g) we have that $J$ is well defined and of class $C^1$. Moreover,

$$\begin{aligned}\langle J'(u),\varphi\rangle &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+\alpha p}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^N} \lambda V(x)|u|^{p-2}u\varphi\,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} f(x,u)\varphi\,\mathrm{d}x + \mu\int_{\mathbb{R}^N} g(x)|u|^{q-2}u\varphi\,\mathrm{d}x \quad \text{for } x\in\mathbb{R}^N, u,\varphi\in E. \end{aligned} \tag{7}$$

From the definition of $J'$, we see that the critical points of $J$ are weak solutions for (1). From [30], we know that the space $E$ can be decomposed as $X$ in Lemma 2.2, so we can consider the family of functionals $J_\nu : E \to \mathbb{R}$ defined by

$$J_\nu(u) = \frac{1}{p}\|u\|^p + \frac{\mu}{q}\int_{\mathbb{R}^N} g(x)|u|^q\,\mathrm{d}x - \nu\int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x := A(u) - \nu B(u) \quad \text{for } \nu \in [1,2].$$

Then $B(u) \geq 0$ for $u \in E$, and $J_\nu(-u) = J_\nu(u)$ for $(\nu,u) \in [1,2]\times E$. Also, it is easy to see that $J_\nu$ maps bounded sets to bounded sets uniformly on $\nu \in [1,2]$.

**Lemma 3.1** *Suppose that the assumptions of Theorem 1.1 hold. Then $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of $E$.*

*Proof* For any finite dimensional subspace $\widetilde{E} \subset E$, there exists $\varepsilon_1 > 0$ such that

$$\operatorname{meas}\{x\in\mathbb{R}^N : |u(x)|^p \geq \varepsilon_1\|u\|^p\} \geq \varepsilon_1, \quad \forall u \in \widetilde{E}\backslash\{0\}. \tag{8}$$

If (8) is not true, then for all $n \in \mathbb{N}$, there exists $u_n \in \widetilde{E}\backslash\{0\}$ such that

$$\operatorname{meas}\left\{x\in\mathbb{R}^N : |u_n(x)|^p \geq \frac{1}{n}\|u_n\|^p\right\} < \frac{1}{n}.$$

Define $v_n(x) = \frac{u_n(x)}{\|u_n\|} \in \widetilde{E}\backslash\{0\}$, then for all $n \in \mathbb{N}$, $\|v_n\| = 1$, and we obtain

$$\operatorname{meas}\left\{x\in\mathbb{R}^N : |v_n(x)|^p \geq \frac{1}{n}\right\} < \frac{1}{n}. \tag{9}$$

Since $\dim\widetilde{E} < \infty$, passing to a subsequence if necessary, we may assume that $v_n \to v_0$ in $\widetilde{E}$. Moreover, $\|v_0\| = 1$. From the equivalence of all norms on the finite dimensional space $\widetilde{E}$, we have

$$\int_{\mathbb{R}^N} |v_n - v_0|^p \, \mathrm{d}x \to 0, \quad \text{as } n \to \infty. \tag{10}$$

Thus, there exist $\xi_1, \xi_2 > 0$ such that

$$\operatorname{meas}\{x \in \mathbb{R}^N : |v_0(x)|^p \geq \xi_1\} \geq \xi_2. \tag{11}$$

If not, for all $n \in \mathbb{N}$, we obtain

$$\operatorname{meas}\left\{x \in \mathbb{R}^N : |v_0(x)|^p \geq \frac{1}{n}\right\} = 0.$$

This implies that

$$0 \leq \int_{\mathbb{R}^N} |v_0(x)|^{2p} \, \mathrm{d}x < \frac{1}{n}\|v_0\|_p^p \leq \frac{C_p^p}{n}\|v_0\|^p = \frac{C_p^p}{n} \to 0, \quad \text{as } n \to \infty, \text{ for some } C_p > 0.$$

Hence, $v_0 = 0$, contradicting $\|v_0\| = 1$, and then (11) holds.

Now let

$$\Omega_0 = \{x \in \mathbb{R}^N : |v_0(x)|^p \geq \xi_1\}, \qquad \Omega_n = \left\{x \in \mathbb{R}^N : |v_n(x)|^p < \frac{1}{n}\right\} \quad \text{and}$$
$$\Omega_n^c = \mathbb{R}^N \backslash \Omega_n = \left\{x \in \mathbb{R}^N : |v_n(x)|^p \geq \frac{1}{n}\right\}.$$

From (9) and (11), we have

$$\operatorname{meas}(\Omega_n \cap \Omega_0) \geq \operatorname{meas}(\Omega_0) - \operatorname{meas}(\Omega_n^c \cap \Omega_0) \geq \xi_2 - \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

For $n$ large enough (for example, taking $n$ such that $\xi_2 - \frac{1}{n} \geq \frac{1}{2}\xi_2$, $\frac{1}{2^{p-1}}\xi_1 - \frac{1}{n} \geq \frac{1}{2^p}\xi_1$), using the inequality $|v_n|^p = |v_n - v_0 + v_0|^p \leq 2^{p-1}|v_n - v_0|^p + 2^{p-1}|v_0|^p$, for $p \geq 2$, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |v_n - v_0|^p \, \mathrm{d}x &\geq \int_{\Omega_n \cap \Omega_0} |v_n - v_0|^p \, \mathrm{d}x \\
&\geq \frac{1}{2^{p-1}} \int_{\Omega_n \cap \Omega_0} |v_0(x)|^p \, \mathrm{d}x - \int_{\Omega_n \cap \Omega_0} |v_n(x)|^p \, \mathrm{d}x \\
&\geq \left(\frac{1}{2^{p-1}}\xi_1 - \frac{1}{n}\right) \operatorname{meas}(\Omega_n \cap \Omega_0) \\
&\geq \left(\frac{1}{2^{p-1}}\xi_1 - \frac{1}{n}\right)\left(\xi_2 - \frac{1}{n}\right) \geq \frac{\xi_1 \xi_2}{2^{p+1}} > 0.
\end{aligned}$$

This contradicts (10). As a result, (8) holds. For $\varepsilon_1$ in (8), let

$$\Omega_u = \{x \in \mathbb{R}^N : |u(x)|^p \geq \varepsilon_1 \|u\|^p\}, \quad \forall u \in \widetilde{E} \backslash \{0\}.$$

Then we have $\operatorname{meas}(\Omega_u) \geq \varepsilon_1$. On the other hand, from L'Hospital rule and (f3) we have

$$\lim_{|u|\to\infty} \frac{F(x,u)}{|u|^p} = +\infty \quad \text{uniformly in } x \in \mathbb{R}^N.$$

Hence, there exists sufficiently large $d_2 > 0$ such that

$$F(x,u) \geq d_2|u|^p \quad \text{for } x \in \mathbb{R}^N, |u| > r_1, \text{ for some } r_1 > 0.$$

From (4) with $s \in (p, p_\alpha^*)$, we have

$$F(x,u) \leq |u|^p\left(\frac{c_1}{p} + \frac{c_2}{s}|u|^{s-p}\right) \leq \left(\frac{c_1}{p} + \frac{c_2}{s}r_1^{s-p}\right)|u|^p \quad \text{for } x \in \mathbb{R}^N, |u| \leq r_1.$$

As a result, there exists $d_3 \in (0, d_2)$ such that

$$F(x,u) \geq (d_2 - d_3)|u|^p \quad \text{for } x \in \mathbb{R}^N. \tag{12}$$

This, together with (8), implies that

$$\begin{aligned} B(u) &= \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x \geq (d_2 - d_3)\int_{\mathbb{R}^N} \big|u(x)\big|^p\,\mathrm{d}x \geq (d_2 - d_3)\int_{\Omega_u} \big|u(x)\big|^p\,\mathrm{d}x \\ &\geq \varepsilon_1(d_2 - d_3)\|u\|^p \operatorname{meas}(\Omega_u) \geq \varepsilon_1^2(d_2 - d_3)\|u\|^p. \end{aligned} \tag{13}$$

Thus $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of $E$. This completes the proof. □

**Lemma 3.2** *Suppose that the assumptions of Theorem 1.1 hold. Then there exists a sequence $\rho_k \to 0^+$ as $k \to \infty$ such that*

$$a_k(\nu) = \inf_{u \in Z_k, \|u\| = \rho_k} J_\nu(u) \geq 0, \tag{14}$$

*and*

$$d_k(\nu) = \inf_{u \in Z_k, \|u\| \leq \rho_k} J_\nu(u) \to 0, \quad \textit{as } k \to \infty, \textit{ uniformly for } \nu \in [1,2], \tag{15}$$

*where $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$ for all $k \in \mathbb{N}$.*

*Proof* Let $\beta_s(k) = \sup_{u \in Z_k, \|u\|=1} \|u\|_s$ with $s \in (p, p_\alpha^*)$. Then from Lemma 3.8 of [70] and Lemma 2.1, we have $\beta_s(k) \to 0, k \to \infty$. Now, for $u \in Z_k$, from (4), (5), we obtain

$$\begin{aligned} J_\nu(u) &= \frac{1}{p}\|u\|^p - \nu\int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x + \frac{\mu}{q}\int_{\mathbb{R}^N} g(x)|u|^q\,\mathrm{d}x \\ &\geq \frac{1}{p}\|u\|^p - 2\int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x \\ &\geq \frac{1}{p}\|u\|^p - \frac{2\varepsilon}{p}\|u\|_p^p - \frac{2c_\varepsilon}{s}\|u\|_s^s \end{aligned}$$

$$\geq \frac{1}{p}\|u\|^p - \frac{2\varepsilon}{p}C_p^p\|u\|^p - \frac{2c_\varepsilon}{s}\beta_s^s(k)\|u\|^s.$$

Let $\|u\| = \rho_k = \beta_s(k), u \in Z_k$, note that $\beta_s(k)$ can be chosen arbitrarily small when $k$ is large, and if $\varepsilon = \frac{p}{2C_p^p}[\frac{1}{p} - \frac{3c_\varepsilon}{s}\beta_s^s(k)]$, we have

$$J_\nu(u) \geq \left[\frac{1}{p} - \frac{2\varepsilon}{p}C_p^p - \frac{2c_\varepsilon}{s}\beta_s^s(k)\right]\|u\|^s = \frac{c_\varepsilon}{s}\beta_s^{s+1}(k) \geq 0 \quad \text{for large } k.$$

On the other hand, for any $u \in Z_k$ with $\|u\| \leq \rho_k$, we have

$$J_\nu(u) \geq -\frac{2c_\varepsilon}{s}\beta_s^s(k)\|u\|^s.$$

Hence,

$$0 \geq \inf_{u\in Z_k,\|u\|\leq\rho_k} J_\nu(u) \geq -\frac{2c_\varepsilon}{s}\beta_s^s(k)\|u\|^s.$$

Since, $\rho_k \to 0$ as $k \to \infty$, we have

$$d_k(\nu) = \inf_{u\in Z_k,\|u\|\leq\rho_k} J_\nu(u) \to 0, \quad \text{as } k \to \infty \text{ uniformly for } \nu \in [1,2].$$

This completes the proof. □

**Lemma 3.3** *Suppose that all the assumptions of Theorem 1.1 hold (and $\mu$ is sufficiently small). For the sequence $\{\rho_k\}_{k\in\mathbb{N}}$ in Lemma 3.2, there exists $r_k \in (0,\rho_k)$ for $k \in \mathbb{N}$ such that*

$$b_k(\nu) = \max_{u\in Y_k,\|u\|=r_k} J_\nu(u) < 0 \quad \textit{for } \nu \in [1,2], \tag{16}$$

*where $Y_k = \overline{\bigoplus_{j=1}^k X_j}$ for $k \in \mathbb{N}$.*

*Proof* For $u \in Y_k$, from (13) and (5) we have

$$\begin{aligned}
J_\nu(u) &= \frac{1}{p}\|u\|^p - \nu\int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x + \frac{\mu}{q}\int_{\mathbb{R}^N} g(x)|u|^q\,\mathrm{d}x \\
&\leq \frac{1}{p}\|u\|^p - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x + \frac{\mu}{q}\int_{\mathbb{R}^N} g(x)|u|^q\,\mathrm{d}x \\
&\leq \frac{1}{p}\|u\|^p - \int_{\Omega_u} F(x,u)\,\mathrm{d}x + \frac{\mu}{q}\|g\|_{q'}C^q_{\frac{qq'}{q'-1}}\|u\|^q \\
&\leq \frac{1}{p}\|u\|^p - \varepsilon_1^2(d_2 - d_3)\|u\|^p + \frac{\mu}{q}\|g\|_{q'}C^q_{\frac{qq'}{q'-1}}\|u\|^q.
\end{aligned}$$

Note that we can take sufficiently large $d_2$ (and $\mu$ sufficiently small) such that

$$\max_{u\in Y_k,\|u\|=r_k} J_\nu(u) < 0, \quad \forall k \in \mathbb{N}, \text{ if } \|u\| = r_k < \rho_k \text{ small enough}.$$

This completes the proof. □

From Lemmas 3.1–3.3, we see (Z1)–(Z3) of Lemma 2.2 hold. Therefore, there exist $\nu_n \to 1$, $u(\nu_n) \in Y_n$ such that

$$J'_{\nu_n}|_{Y_n}\big(u(\nu_n)\big) = 0, \qquad J_{\nu_n}\big(u(\nu_n)\big) \to c_k \in \big[d_k(2), b_k(1)\big], \quad \text{as } n \to \infty. \tag{17}$$

For convenience, we denote $u_n = u(\nu_n)$ for all $n \in \mathbb{N}$.

**Lemma 3.4** *Suppose that all the assumptions of Theorem 1.1 hold. Then the sequence* $\{u_n\}$ *is bounded in* $E$.

*Proof* Note that $J_{\nu_n}(u(\nu_n))$ is bounded, and we have

$$\begin{aligned} c + 1 &\geq J_{\nu_n}(u_n) - \frac{1}{p}\langle J'_{\nu_n}(u_n), u_n\rangle \\ &= \frac{1}{p}\nu_n \int_{\mathbb{R}^N} f(x,u_n)u_n \,\mathrm{d}x - \nu_n \int_{\mathbb{R}^N} F(x,u_n)\,\mathrm{d}x \\ &\quad + \frac{\mu}{q}\int_{\mathbb{R}^N} g(x)|u_n|^q \,\mathrm{d}x - \frac{\mu}{p}\int_{\mathbb{R}^N} g(x)|u_n|^q\,\mathrm{d}x \\ &\geq \int_{\mathbb{R}^N} \mathscr{F}(x,u_n)\,\mathrm{d}x. \end{aligned} \tag{18}$$

We will argue by contradiction. If $\|u_n\|$ is unbounded in $E$, we assume that $\|u_n\| \to \infty$. Put $v_n = \frac{u_n}{\|u_n\|}$, and then $\|v_n\| = 1$. Passing to a subsequence, there exists $v \in E$ such that $v_n \rightharpoonup v$ weakly in $E$, $v_n \to v$ strongly in $L^r(\mathbb{R}^N)$ with $r \in [p, p^*_\alpha)$, $v_n(x) \to v(x)$ for a.e. $x \in \mathbb{R}^N$. For $0 \leq a < b$, let $\Omega_n(a,b) = \{x \in \mathbb{R}^N : a \leq |u_n(x)| < b\}$. Next we consider two cases.

*Case 1*: Suppose $v = 0$.

Then $v_n \to 0$ in $L^r(\mathbb{R}^N)$ with $r \in [p, p^*_\alpha)$, and $v_n(x) \to 0$ for a.e. $x \in \mathbb{R}^N$. Let $r_0$ be as in (f4), and from (3) we have

$$\begin{aligned} \int_{\Omega_n(0,r_0)} \frac{f(x,u_n)u_n}{\|u_n\|^p}\,\mathrm{d}x &= \int_{\Omega_n(0,r_0)} \frac{f(x,u_n)u_n}{|u_n|^p}|v_n|^p\,\mathrm{d}x \\ &\leq \big(\varepsilon + c_\varepsilon r_0^{s-p}\big)\int_{\Omega_n(0,r_0)} |v_n|^p\,\mathrm{d}x \\ &\leq \big(\varepsilon + c_\varepsilon r_0^{s-p}\big)\int_{\mathbb{R}^N} |v_n|^p\,\mathrm{d}x \to 0. \end{aligned} \tag{19}$$

From (f4), we know $\tau > \frac{p^*_\alpha}{p^*_\alpha - p}$. Thus, if we set $\tau' = \tau/(\tau-1)$, then $p\tau' \in (p, p^*_\alpha)$. From the Hölder inequality and (18), we obtain

$$\begin{aligned} \int_{\Omega_n(r_0,\infty)} \frac{f(x,u_n)u_n}{\|u_n\|^p}\,\mathrm{d}x &= \int_{\Omega_n(r_0,\infty)} \frac{f(x,u_n)u_n}{|u_n|^p}|v_n|^p\,\mathrm{d}x \\ &\leq \left(\int_{\Omega_n(r_0,\infty)} \left(\frac{f(x,u_n)u_n}{|u_n|^p}\right)^\tau \mathrm{d}x\right)^{\frac{1}{\tau}} \left(\int_{\Omega_n(r_0,\infty)} |v_n|^{p\tau'}\,\mathrm{d}x\right)^{\frac{1}{\tau'}} \\ &\leq \left(\int_{\Omega_n(r_0,\infty)} \frac{|f(x,u_n)|^\tau}{|u_n|^{(p-1)\tau}}\,\mathrm{d}x\right)^{\frac{1}{\tau}} \left(\int_{\Omega_n(r_0,\infty)} |v_n|^{p\tau'}\,\mathrm{d}x\right)^{\frac{1}{\tau'}} \\ &\leq \left(\int_{\Omega_n(r_0,\infty)} d_1\mathscr{F}(x,u)\,\mathrm{d}x\right)^{\frac{1}{\tau}} \left(\int_{\Omega_n(r_0,\infty)} |v_n|^{p\tau'}\,\mathrm{d}x\right)^{\frac{1}{\tau'}} \end{aligned}$$

$$\leq \left[d_1(c+1)\right]^{\frac{1}{\tau}} \left( \int_{\mathbb{R}^3} |v_n|^{p\tau'} \,\mathrm{d}x \right)^{\frac{1}{\tau'}} \to 0. \tag{20}$$

Combining (19) and (20), we have

$$\int_{\mathbb{R}^N} \frac{f(x,u_n)u_n}{\|u_n\|^p} \,\mathrm{d}x = \int_{\Omega_n(0,r_0)} \frac{f(x,u_n)u_n}{\|u_n\|^p} \,\mathrm{d}x + \int_{\Omega_n(r_0,\infty)} \frac{f(x,u_n)u_n}{\|u_n\|^p} \,\mathrm{d}x \to 0. \tag{21}$$

On the other hand, note that $v_n \to 1$, from (5) and (g) we have

$$\begin{aligned}
1 = \frac{\|u_n\|^p}{\|u_n\|^p} &= \frac{\langle J'_{v_n}(u_n), u_n\rangle}{\|u_n\|^p} + \frac{v_n}{\|u_n\|^p} \int_{\mathbb{R}^N} f(x,u_n)u_n \,\mathrm{d}x - \frac{\mu}{\|u_n\|^p} \int_{\mathbb{R}^N} g(x)|u_n|^q \,\mathrm{d}x \\
&\leq \frac{\langle J'_{v_n}(u_n), u_n\rangle}{\|u_n\|^p} + \frac{v_n}{\|u_n\|^p} \int_{\mathbb{R}^N} f(x,u_n)u_n \,\mathrm{d}x + \frac{\mu C^q_{\frac{qq'}{q'-1}}}{\|u_n\|^p} \|g\|_{q'} \|u_n\|^q \\
&\leq \limsup_{n\to\infty} \left[ \frac{\langle J'_{v_n}(u_n), u_n\rangle}{\|u_n\|^p} + \frac{v_n}{\|u_n\|^p} \int_{\mathbb{R}^N} f(x,u_n)u_n \,\mathrm{d}x + \frac{\|u_n\|^q}{\|u_n\|^p} \mu \|g\|_{q'} C^q_{\frac{qq'}{q'-1}} \right] \\
&\leq \limsup_{n\to\infty} \frac{v_n}{\|u_n\|^p} \int_{\mathbb{R}^N} f(x,u_n)u_n \,\mathrm{d}x,
\end{aligned}$$

which contradicts (21).

*Case* 2: Suppose $v \neq 0$.

Set $A = \{x \in \mathbb{R}^N : v(x) \neq 0\}$ and $\operatorname{meas}(A) > 0$. For $x \in A$, we have $\lim_{n\to\infty} |u_n(x)| = \infty$. Hence $A \subset \Omega_n(r_0,\infty)$ for large $n$. From (3) and (f3), note the nonnegativity of $f(x,u)u$, Fatou's lemma enables us to obtain

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{o(1)}{\|u_n\|^p} = \lim_{n\to\infty} \frac{\langle J'_{v_n}(u_n), u_n\rangle}{\|u_n\|^p} \\
&= \lim_{n\to\infty} \left[ \frac{\|u_n\|^p}{\|u_n\|^p} + \frac{\mu}{\|u_n\|^p} \int_{\mathbb{R}^N} g(x)|u_n|^q \,\mathrm{d}x - \frac{v_n}{\|u_n\|^p} \int_{\mathbb{R}^N} f(x,u_n)u_n \,\mathrm{d}x \right] \\
&\leq 1 + \lim_{n\to\infty} \left[ \frac{\|u_n\|^q}{\|u_n\|^p} \mu \|g\|_{q'} C^q_{\frac{qq'}{q'-1}} - \int_{\Omega_n(0,r_0)} \frac{f(x,u_n)u_n}{\|u_n\|^p} \,\mathrm{d}x \right. \\
&\qquad \left. - \int_{\Omega_n(r_0,\infty)} \frac{f(x,u_n)u_n}{|u_n|^p} |v_n|^p \,\mathrm{d}x \right] \\
&\leq 1 + \limsup_{n\to\infty} \int_{\Omega_n(0,r_0)} \frac{f(x,u_n)u_n}{\|u_n\|^p} \,\mathrm{d}x - \liminf_{n\to\infty} \int_{\Omega_n(r_0,\infty)} \frac{f(x,u_n)u_n}{|u_n|^p} |v_n|^p \,\mathrm{d}x \\
&\leq 1 + \limsup_{n\to\infty} \frac{\varepsilon r_0^p + c_\varepsilon r_0^s}{\|u_n\|^p} \cdot \operatorname{meas}\bigl(\Omega_n(0,r_0)\bigr) \\
&\qquad - \liminf_{n\to\infty} \int_{\Omega_n(r_0,\infty)} \frac{f(x,u_n)u_n}{|u_n|^p} \bigl[\chi_{\Omega_n(r_0,\infty)}(x)\bigr] |v_n|^p \,\mathrm{d}x \\
&\leq 1 - \int_{\Omega_n(r_0,\infty)} \liminf_{n\to\infty} \frac{f(x,u_n)u_n}{|u_n|^p} \bigl[\chi_{\Omega_n(r_0,\infty)}(x)\bigr] |v_n|^p \,\mathrm{d}x \to -\infty.
\end{aligned}$$

This is also a contradiction.

Thus $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $E$. This completes the proof. □

**Lemma 3.5** *Suppose that all the assumptions of Theorem 1.1 hold. For some $\Lambda > 0$, the sequence $\{u_n\}$ possesses a strong convergent subsequence in $E$.*

*Proof* From Lemma 3.4, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $E$. Then there exists $u \in E$ such that $u_n \rightharpoonup u$ weakly in $E$, $u_n \to u$ strongly in $L^r(\mathbb{R}^N)$ for $r \in [p, p_\alpha^*)$ and $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^N$ after passing to a subsequence if necessary. Next, we prove two claims.

*Claim* 1. $\langle J'_{v_n}(u_n - u), u_n - u\rangle = o(1)$ as $n \to \infty$.

Let $w_n = u_n - u$. Then $w_n \rightharpoonup 0$ weakly in $E$, $w_n \to 0$ strongly in $L^r(\mathbb{R}^N)$ for $r \in [p, p_\alpha^*)$, and $w_n(x) \to 0$ for a.e. $x \in \mathbb{R}^N$ after passing to a subsequence. Recall that $u_n \rightharpoonup u$ weakly in $E$, we have $\|w_n\| = \|u_n\| - \|u\| + o(1)$, and from (7) we only need to show

$$\int_{\mathbb{R}^N} f(x, w_n)w_n \,\mathrm{d}x = o(1) \quad \text{and} \quad \int_{\mathbb{R}^N} g(x)|w_n|^q \,\mathrm{d}x = o(1), \quad \text{as } n \to \infty.$$

In fact, from (3) we have

$$\left|\int_{\mathbb{R}^N} f(x, w_n)w_n \,\mathrm{d}x\right| \le \int_{\mathbb{R}^N} \big|f(x, w_n)\big||w_n| \,\mathrm{d}x \le \varepsilon \int_{\mathbb{R}^N} |w_n|^p \,\mathrm{d}x + c_\varepsilon \int_{\mathbb{R}^N} |w_n|^s \,\mathrm{d}x \to 0,$$
$$\text{as } n \to \infty \text{ with } s \in \big[p, p_\alpha^*\big),$$

and

$$\int_{\mathbb{R}^N} g(x)|w_n|^q \,\mathrm{d}x \le \|g\|_{q'} \|w_n\|^q_{\frac{qq'}{q'-1}} \to 0, \quad \text{as } n \to \infty \text{ with } \frac{qq'}{q'-1} \in \big[p, p_\alpha^*\big).$$

*Claim* 2. There is $M > 0$ such that

$$\int_{\mathbb{R}^N} \mathscr{F}(x, w_n) \,\mathrm{d}x \le M.$$

From Lemma A.1 of [70], there exists $\sigma(x) \in L^r(\mathbb{R}^N)$ with $r \in [p, p_\alpha^*)$ such that

$$\big|u_n(x)\big| \le \sigma(x), \qquad \big|u(x)\big| \le \sigma(x) \quad \text{for } x \in \mathbb{R}^N, n \in \mathbb{N}. \tag{22}$$

Note that $w_n = u_n - u$, by (3), (4), and (22) we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \mathscr{F}(x, w_n) \,\mathrm{d}x &= \int_{\mathbb{R}^N} \left(\frac{1}{p} f(x, w_n)w_n - F(x, w_n)\right) \mathrm{d}x \\
&\le \int_{\mathbb{R}^N} \left(\frac{2\varepsilon}{p}|w_n|^p + \frac{c_\varepsilon(p+s)}{ps}|w_n|^s\right) \mathrm{d}x \\
&\le \int_{\mathbb{R}^N} \left(\frac{2^{p+1}\varepsilon}{p}\sigma_1^p(x) + \frac{2^s c_\varepsilon(p+s)}{ps}\sigma_2^s(x)\right) \mathrm{d}x \\
&\le M,
\end{aligned}$$

where $M > 0$, $\sigma_1 \in L^p(\mathbb{R}^N), \sigma_2 \in L^s(\mathbb{R}^N)$ with $s \in (p, p_\alpha^*)$.

Now, we prove that the sequence $\{u_n\}_{n\in\mathbb{N}}$ has a convergent subsequence. Note $V(x) < b$ on a set of finite measure and $w_n \to 0$ strongly in $L^r(\mathbb{R}^N)$, $r \in [p, p_\alpha^*)$, and we have

$$\|w_n\|_p^p = \int_{\mathbb{R}^N} |w_n|^p \,\mathrm{d}x \le \frac{1}{\lambda b}\int_{V \ge b} \lambda V(x)|w_n|^p \,\mathrm{d}x + \int_{V<b} |w_n|^p \,\mathrm{d}x \le \frac{1}{\lambda b}\|w_n\|^p + o(1).$$

Combining this and the Hölder inequality, for $s = \frac{p\tau}{\tau-1} \in [p, p_\alpha^*)$, fixed $\nu \in (s, p_\alpha^*)$, and we have

$$\begin{aligned}
\|w_n\|_s^s &= \int_{\mathbb{R}^N} |w_n|^s \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} |w_n|^{\frac{p(\nu-s)}{\nu-p}} |w_n|^{s-\frac{p(\nu-s)}{\nu-p}} \,\mathrm{d}x \\
&\leq \left( \int_{\mathbb{R}^N} |w_n|^{\frac{p(\nu-s)}{\nu-p}\frac{\nu-p}{\nu-s}} \,\mathrm{d}x \right)^{\frac{\nu-s}{\nu-p}} \left( \int_{\mathbb{R}^N} |w_n|^{(s-\frac{p(\nu-s)}{\nu-p})\frac{\nu-p}{s-p}} \,\mathrm{d}x \right)^{\frac{s-p}{\nu-p}} \\
&= \left( \int_{\mathbb{R}^N} |w_n|^p \,\mathrm{d}x \right)^{\frac{\nu-s}{\nu-p}} \left( \int_{\mathbb{R}^N} |w_n|^\nu \,\mathrm{d}x \right)^{\frac{s-p}{\nu-p}} \\
&\leq \left( \frac{1}{\lambda b} \right)^{\frac{\nu-s}{\nu-p}} C_\nu^{\frac{\nu(s-p)}{\nu-p}} \|w_n\|^{\frac{p(\nu-s)}{\nu-p}} \|w_n\|^{\frac{\nu(s-p)}{\nu-p}} \\
&= \left( \frac{1}{\lambda b} \right)^{\frac{\nu-s}{\nu-p}} C_\nu^{\frac{\nu(s-p)}{\nu-p}} \|w_n\|^s \quad \text{for } C_\nu > 0.
\end{aligned}$$

From (f1), for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $|f(x,u)| \leq \varepsilon |u|^{p-1}$ for $x \in \mathbb{R}^N$ and $|u| \leq \delta$. Moreover, (f4) is also satisfied for some suitable $\delta$. Therefore, we have

$$\int_{|w_n|\leq\delta} f(x,w_n)w_n \,\mathrm{d}x \leq \varepsilon \int_{|w_n|\leq\delta} |w_n|^p \,\mathrm{d}x \leq \frac{\varepsilon}{\lambda b} \|w_n\|^p + o(1),$$

and

$$\begin{aligned}
\int_{|w_n|\geq\delta} f(x,w_n)w_n \,\mathrm{d}x &= \int_{|w_n|\geq\delta} \frac{f(x,w_n)w_n}{|w_n|^p} |w_n|^p \,\mathrm{d}x \\
&\leq \left( \int_{|w_n|\geq\delta} \frac{|f(x,w_n)|^\tau}{|w_n|^{(p-1)\tau}} \,\mathrm{d}x \right)^{1/\tau} \left( \int_{|w_n|\geq\delta} |w_n|^{\frac{p\tau}{\tau-1}} \,\mathrm{d}x \right)^{(\tau-1)/\tau} \\
&\leq \left( \int_{|w_n|\geq\delta} d_1 \mathscr{F}(x,u) \,\mathrm{d}x \right)^{1/\tau} \|w_n\|_s^p \\
&\leq (d_1 M)^{1/\tau} \left( \frac{1}{\lambda b} \right)^{\frac{p(\nu-s)}{s(\nu-p)}} C_\nu^{\frac{p\nu(s-p)}{s(\nu-p)}} \|w_n\|^p + o(1).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
o(1) = \langle J'_{\nu_n}(w_n), w_n \rangle &= \|w_n\|^p + \mu \int_{\mathbb{R}^N} g(x)|w_n|^q \,\mathrm{d}x - \nu_n \int_{\mathbb{R}^N} f(x,w_n)w_n \,\mathrm{d}x \\
&\geq \|w_n\|^p - 2 \int_{\mathbb{R}^N} f(x,w_n)w_n \,\mathrm{d}x \\
&\geq \left[ 1 - \frac{2\varepsilon}{\lambda b} - 2(d_1 M)^{1/\tau} \left( \frac{1}{\lambda b} \right)^{\frac{p(\nu-s)}{s(\nu-p)}} C_\nu^{\frac{p\nu(s-p)}{s(\nu-p)}} \right] \|w_n\|^p + o(1).
\end{aligned}$$

Thus there exists $\Lambda > 0$ such that $w_n \to 0$ in $E$ when $\lambda > \Lambda$. This implies that $u_n \to u$ in $E$. This completes the proof. □

*Proof of Theorem* 1.1 From the last assertion of Lemma 2.2, we know that $J = J_1$ has infinitely many nontrivial critical points. Therefore, (1) possesses infinitely many small negative-energy solutions. This completes the proof. □

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