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Meromorphic functions that share four or three small functions with their difference operators

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Abstract

In this paper, we prove that non-constant meromorphic functions of finite order and their difference operators are identical, if they share four small functions "IM", or share two small functions and ∞ CM. Our results show that a conjecture posed by Chen–Yi in 2013 is still valid for shared small functions, and improve some earlier results obtained by Li–Yi, Lü et al. We also study the uniqueness of a meromorphic function partially sharing three small functions with their difference operators.

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1 Introduction and main results

In this paper, a meromorphic function always means meromorphic in the complex plane. We adopt the standard notations in Nevanlinna theory; see, e.g. [11, 21]. In addition, we use the notations $\sigma(f), \sigma_2(f)$ to denote the order and the hyper-order of $f(z)$, respectively, where

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}, \qquad \sigma_2(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r,f)}{\log r}.$$

A meromorphic function $\alpha(\not\equiv \infty)$ is called a small function of f provided that $T(r,\alpha) = o(T(r,f))$ as $r \to \infty$, possibly outside a set of r of finite logarithmic measure. We use $S(f)$ to denote the family of all meromorphic functions which are small functions of f, and denote $\hat{S}(f) = S(f) \cup \{\infty\}$.

Let f and g be two non-constant meromorphic functions, and let α be a meromorphic function. We say that f and g share α CM (IM), provided that $f - \alpha$ and $g - \alpha$ have the same zeros counting multiplicities (ignoring multiplicities). If $\frac{1}{f}$ and $\frac{1}{g}$ share 0 CM (IM), then we say that f and g share ∞ CM (IM).

Nevanlinna's four-value theorem shows that if two non-constant meromorphic functions f and g share four distinct values CM, then f is a Möbius transformation of g. In [4], Gundersen constructed a counterexample to show that four-value theorem is not valid if 4 CM is replaced by 4 IM. But when g is the derivative of f, Gundersen and Mues–Steinmetz, respectively, obtained the following result.

Theorem A ([5, 18]) *If a non-constant meromorphic function f and its derivative f' share three distinct finite values a_1, a_2, a_3 IM, then $f \equiv f'$.*

Remark 1.1 Observe that a meromorphic function f and f' trivially share ∞ IM. So, in this sense, the four-value theorem is valid for f and f' sharing four values IM.

Furthermore, Gundersen and Mues–Steinmetz improved Theorem A as follows.

Theorem B ([6, 19]) *If a non-constant meromorphic function f and its derivative f' share two distinct finite values a_1, a_2 CM, then $f \equiv f'$.*

Recently, the difference analog of Nevanlinna theory has been established; see, e.g. [2, 7–10, 14]. Many researchers ([1, 12, 13, 15–17], etc.) started to consider the uniqueness of meromorphic functions sharing values with their shifts or their difference operators. For a nonzero finite value η, $f(z+\eta)$ is called a shift of $f(z)$, its difference operators are defined as

$$\Delta_\eta f(z) = f(z+\eta) - f(z) \quad \text{and} \quad \Delta_\eta^n f(z) = \Delta_\eta^{n-1}\bigl(\Delta_\eta f(z)\bigr), \quad n \in \mathbb{N}, n \geq 2.$$

It is well known that $\Delta_\eta f$ can be regarded as the difference counterpart of f'. So, considering the difference analog of Theorems A and B, the following results are obtained.

Theorem C ([15]) *Let f be a non-constant meromorphic function of $\sigma(f) < \infty$. If f and $\Delta_\eta f$ share four distinct values a_1, a_2, a_3, a_4 IM, then $f \equiv \Delta_\eta f$.*

Theorem D ([1]) *Let f be a transcendental meromorphic function such that $\sigma(f)$ is finite but not an integer. If f and $\Delta_\eta f (\not\equiv 0)$ share three distinct values a_1, a_2, ∞ CM, then $f \equiv \Delta_\eta f$.*

In [1], the authors conjecture that the condition "order of growth $\sigma(f)$ is not an integer or infinite" can be removed. Lü [17] considered this conjecture and obtained the following result.

Theorem E ([17]) *Let f be a transcendental meromorphic function of $\sigma(f) < \infty$. If f and $\Delta_\eta f$ share three distinct values a_1, a_2, ∞ CM, then $f \equiv \Delta_\eta f$.*

It is natural to pose the question: what can be said on replacing shared values in Theorems C–E by shared small functions. Concerning this question, we obtain the following results which extend Theorems C–E. For the convenience of statement, we need the following definition; see [21].

Let f, g and α be three distinct meromorphic functions, $\overline{N}_0(r,\alpha,f,g)$ denote the counting function of common *zeros of* $f(z)-\alpha(z)$ *and* $g(z)-\alpha(z)$, each counted only once. If

$$\overline{N}\left(r, \frac{1}{f-\alpha}\right) - \overline{N}_0(r,\alpha,f,g) = S(r,f)$$

and

$$\overline{N}\left(r, \frac{1}{g-\alpha}\right) - \overline{N}_0(r,\alpha,f,g) = S(r,g),$$

where $S(r,f) = o(T(r,f))$ as $r \to \infty$, possibly outside *a set of r of finite* logarithmic measure, then we say that f and g share α "IM". Obviously, if f and g share α IM, then f and g share α "IM". But the reverse is not true.

Theorem 1.1 *Let f be a transcendental meromorphic function of $\sigma_2(f) < 1$, $\alpha_j \in S(f)$ $(j = 1,2,3,4)$, and let η be a nonzero finite value. If f and $\Delta_\eta f$ share $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ "IM", then $f \equiv \Delta_\eta f$.*

Remark 1.2 Obviously, Theorem 1.1 is an improvement of Theorem C.

Theorem 1.2 *Let f be a non-constant meromorphic function of $\sigma(f) < \infty$, $\alpha_1,\alpha_2 \in S(f)$, and let η be a nonzero finite value. If f and $\Delta_\eta f$ share α_1,α_2,∞ CM, and if f and α_1,α_2 have no common poles with the same multiplicity, then $f \equiv \Delta_\eta f$.*

Remark 1.3 Obviously, Theorems D and E are direct results of Theorem 1.2.

By Theorem 1.2, we get the following corollary.

Corollary 1.1 *Let f be a non-constant entire function of $\sigma(f) < \infty$, $\alpha_1,\alpha_2 \in S(f)$, and let η be a nonzero finite value. If f and $\Delta_\eta f$ share α_1,α_2 CM, then $f \equiv \Delta_\eta f$.*

We do not know whether Theorem 1.2 is valid, if f and $\Delta_\eta f$ share three distinct functions $\alpha_1,\alpha_2,\alpha_3 \in S(f)$. But under some additional restriction on α_j, we get the following result.

Theorem 1.3 *Let f be a non-constant meromorphic function of $\sigma_2(f) < 1$, $\alpha_j \in S(f)$ $(j = 1,2,3)$, and let η be a nonzero finite value. If, for $j = 1,2,3$,*

$$E(\alpha_j,f) \subset E(\alpha_j,\Delta_\eta f), \qquad \Delta_\eta \alpha_j \equiv \alpha_j,$$

where $E(\alpha_j,f)$ is the set of zeros of $f - \alpha_j$, counting multiplicity, then $f \equiv \Delta_\eta f$.

Remark 1.4 The condition $\Delta_\eta \alpha_j \equiv \alpha_j$ $(j = 1,2,3)$ in Theorem 1.3 is necessary. For example, let $f(z) = \frac{1}{e^{\pi iz}+1}$, $\eta = 1$, $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_3 = \frac{3}{4}$, it is obvious that $E(\alpha_j,f) \subset E(\alpha_j,\Delta_\eta f)$ $(j = 1,2,3)$. But $\Delta_\eta \alpha_j \not\equiv \alpha_j$ $(j = 2,3)$, and $\Delta_\eta f(z) = \frac{2e^{\pi iz}}{1-e^{2\pi iz}} \not\equiv f(z)$.

2 Lemmas

Lemma 2.1 ([10]) *Let f be a non-constant meromorphic function, $\varepsilon > 0$, and η be a finite value. If f is of finite order, then there exists a set $E = E(f,\varepsilon)$ satisfying*

$$\limsup_{r\to\infty} \frac{\int_{E\cap[1,r)} dt/t}{\log r} \le \varepsilon,$$

i.e. of logarithmic density at most ε, such that

$$m\left(r,\frac{f(z+\eta)}{f(z)}\right) = O\left(\frac{\log r}{r}T(r,f)\right)$$

for all r outside the set E. If $\sigma_2(f) < 1$ and $\varepsilon > 0$, then

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\sigma_2(f)-\varepsilon}}\right)$$

for all r outside of a set of finite logarithmic measure.

Lemma 2.2 ([10]) *Let $T : [0, +\infty) \to [0, +\infty)$ be a non-decreasing continuous function and let $s > 0$. If*

$$\limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} = \varsigma < 1$$

and $\delta \in (0, 1 - \varsigma)$, then

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^{\delta}}\right),$$

where r runs to infinity outside of a set of finite logarithmic measure.

Let f be a meromorphic function, it is shown in [3], p. 66, that, for an arbitrary complex number $c \neq 0$, the inequalities

$$\bigl(1 + o(1)\bigr)T\bigl(r - |c|, f(z)\bigr) \le T\bigl(r, f(z+c)\bigr) \le \bigl(1 + o(1)\bigr)T\bigl(r + |c|, f(z)\bigr)$$

hold as $r \to \infty$. Similarly, we have

$$\bigl(1 + o(1)\bigr)N\bigl(r - |c|, f(z)\bigr) \le N\bigl(r, f(z+c)\bigr) \le \bigl(1 + o(1)\bigr)N\bigl(r + |c|, f(z)\bigr), \quad (r \to \infty).$$

So combining the above inequalities and Lemma 2.2, *we get the following result.*

Lemma 2.3 *Let f be a non-constant meromorphic function of $\sigma_2(f) < 1$. Then, for an arbitrary complex number $c \neq 0$,*

$$T\bigl(r, f(z+c)\bigr) = T\bigl(r, f(z)\bigr) + S(r,f), \qquad N\bigl(r, f(z+c)\bigr) = N\bigl(r, f(z)\bigr) + S(r,f).$$

Lemma 2.4 ([20]) *Let f be a transcendental meromorphic function and α_j $(j = 1, \ldots, q)$ be q distinct small functions of f. Then, for $\varepsilon > 0$,*

$$(q - 2 - \varepsilon)T(r,f) \le \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f - \alpha_j}\right) + o\bigl(T(r,f)\bigr)$$

as $r \notin E \to \infty$ for a set E of finite linear measure.

Remark 2.1 In [23], Zheng pointed out that the ε in the above inequality can be removed.

Using a similar argument to that *of* [21]*, Theorem 4.4,* we obtain the following result.

Lemma 2.5 *Let f and g be non-constant meromorphic functions, and share four distinct functions $\alpha_j \in S(f) \cap S(g)$ $(j = 1, 2, 3, 4)$ "IM". If $f \not\equiv g$, then*

(i) $T(r,f) = T(r,g) + S(r,f)$, $T(r,g) = T(r,f) + S(r,g)$.
(ii) $\sum_{j=1}^{4} \overline{N}(r, \frac{1}{f-\alpha_j}) = 2T(r,f) + S(r,f)$.

Lemma 2.6 ([22]) *Let f and g be non-constant meromorphic functions and let α_j ($j = 1,\ldots,5$) be five distinct elements in $\hat{S}(f) \cap \hat{S}(g)$. If $f \not\equiv g$, then*

$$\overline{N}_0(r,\alpha_5,f,g) \leq \sum_{j=1}^{4} \overline{N}_{12}(r,\alpha_j,f,g) + S(r,f) + S(r,g),$$

where $\overline{N}_{12}(r,\alpha,f,g) = \overline{N}(r,\frac{1}{f-\alpha}) + \overline{N}(r,\frac{1}{g-\alpha}) - 2\overline{N}_0(r,\alpha,f,g)$.

Lemma 2.7 ([21]) *Let $f_1,\ldots,f_n$ ($n \geq 2$) be meromorphic functions, and $g_1,\ldots,g_n$ be entire functions satisfying the following conditions.*
(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$.
(ii) *$g_j(z) - g_k(z)$ are not constants for $1 \leq j < k \leq n$.*
(iii) *For $1 \leq j \leq n$, $1 \leq t < k \leq n$, $T(r,f_j) = o(T(r,e^{g_t-g_k}))$ ($r \to \infty, r \notin E$), where $E \subset (1,\infty)$ has finite linear measure or finite logarithmic measure.*

Then $f_j(z) \equiv 0$ ($j = 1,\ldots,n$).

Lemma 2.8 *Let $\alpha(\not\equiv 0)$ be a meromorphic function, and let c,η be nonzero finite values. If $\alpha(z+\eta) = c\alpha(z)$, then $T(r,\alpha) \geq dr - O(1)$ holds for sufficiently large r, where d is a positive number.*

Proof It follows from $\alpha(z+\eta) = c\alpha(z)$ that $\alpha(z)$ is transcendental. If $0,\infty$ are the Picard exceptional values of $\alpha(z)$, then there exists a non-constant entire function $h(z)$, such that $\alpha(z) = e^{h(z)}$. This implies that $T(r,\alpha) \geq dr - O(1)$ holds for sufficiently large r and some positive number d. If $\alpha(z)$ has at least one zero or one pole z_0, then $z_0 + j\eta, j \in \mathbb{Z}$ are zeros or poles of $\alpha(z)$. This implies that $N(r,\frac{1}{\alpha}) \geq dr$ or $N(r,\alpha) \geq dr$ holds for sufficiently large r and some positive number d. So we get $T(r,\alpha) \geq dr - O(1)$ holds for sufficiently large r. □

Lemma 2.9 ([9]) *Let $\mathcal{M}$ be the set of all meromorphic functions in the complex plane, $\mathcal{N}$ be a subfield of $\mathcal{M}$, and let $f \in \mathcal{N} \setminus \ker(L)$, where $L : \mathcal{M} \to \mathcal{M}$ is a linear operator such that $m(r,\frac{L(f)}{f}) = S(r,f)$. If $a_1,\ldots,a_q$ are $q \geq 1$ different elements of $\ker(L) \cap S(f)$, then*

$$(q-1)T(r,f) + N_{L(f)}(r,f) \leq N(r,f) + \sum_{j=1}^{q} N\left(r,\frac{1}{f-a_j}\right) + S(r,f),$$

where $N_{L(f)}(r,f) = 2N(r,f) - N(r,L(f)) + N(r,\frac{1}{L(f)})$.

3 Proofs of the results

Proof of Theorem 1.1 Suppose that $f \not\equiv \Delta_\eta f$, from the fact that f and $\Delta_\eta f$ share $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ "IM" and Lemma 2.5, we get

$$T(r,f) = T(r,\Delta_\eta f) + S(r,f), \qquad T(r,\Delta_\eta f) = T(r,f) + S(r,\Delta_\eta f),$$

from which we deduce that $\Delta_\eta f$ is transcendental and

$$S(r,\Delta_\eta f)=S(r,f). \tag{1}$$

By Lemmas 2.4 and 2.5, we get

$$\begin{aligned}3T(r,f)&\le \overline{N}(r,f)+\sum_{j=1}^{4}\overline{N}\left(r,\frac{1}{f-\alpha_j}\right)+S(r,f)\\&\le \overline{N}(r,f)+2T(r,f)+S(r,f)\\&\le N(r,f)+2T(r,f)+S(r,f),\end{aligned}$$

from which we deduce that

$$T(r,f)=\overline{N}(r,f)+S(r,f)=N(r,f)+S(r,f). \tag{2}$$

Since $n_{(2)}(r,f)\le 2(n(r,f)-\overline{n}(r,f))$, it follows from (2) that

$$N_{(2)}(r,f)=S(r,f), \tag{3}$$

where $n_{(2)}(r,f)$ denotes *the number of multiple poles* of f in $|z|\le r$, counting multiplicity, $N_{(2)}(r,f)$ denotes its corresponding counting function. Similarly, we get

$$N_{(2)}(r,\Delta_\eta f)=S(r,\Delta_\eta f)=S(r,f). \tag{4}$$

On the other hand, from the fact that f and $\Delta_\eta f$ share $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ "IM" and Lemma 2.6, (1), we get

$$\overline{N}_0(r,\infty;f,\Delta_\eta f)\le \sum_{j=1}^{4}\overline{N}_{12}(r,\alpha_j;f,\Delta_\eta f)+S(r,f)+S(r,\Delta_\eta f)=S(r,f). \tag{5}$$

Let $\overline{N}(r,f(z)=a,g(z)\neq b)$ denote the reduced counting function of those points in $|z|\le r$, which are a-points of f, not b-points of $g(z)$, (5) and Lemma 2.3 imply that

$$\begin{aligned}\overline{N}(r,\Delta_\eta f-f)&\le \overline{N}\big(r,\Delta_\eta f=\infty,f(z)\neq\infty\big)+\overline{N}\big(r,f(z)=\infty,\Delta_\eta f\neq\infty\big)\\&\quad+\overline{N}_0(r,\infty;f,\Delta_\eta f)\\&=\overline{N}\big(r,f(z+\eta)=\infty,f(z)\neq\infty\big)+\overline{N}\big(r,f(z)=\infty,\Delta_\eta f\neq\infty\big)+S(r,f)\\&\le \overline{N}\big(r,f(z)=\infty,\Delta_\eta f\neq\infty\big)+S(r,f)\\&\le \overline{N}(r,f)+S(r,f).\end{aligned} \tag{6}$$

Hence by (3), (4) and (6), we get

$$\begin{aligned}N(r,\Delta_\eta f-f)&\le \overline{N}(r,\Delta_\eta f-f)+N_{(2)}(r,\Delta_\eta f-f)\\&\le \overline{N}(r,f)+N_{(2)}(r,\Delta_\eta f)+N_{(2)}(r,f)+S(r,f)\end{aligned}$$

$$\leq \overline{N}(r,f) + S(r,f). \tag{7}$$

Then, by Lemma 2.1, Lemma 2.4 and (7), we get

$$\begin{aligned}
3T(r,f) &\leq \overline{N}(r,f) + \sum_{j=1}^{4} \overline{N}\left(r, \frac{1}{f-\alpha_j}\right) + S(r,f) \\
&\leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f-\Delta_\eta f}\right) + S(r,f) \\
&\leq \overline{N}(r,f) + N(r,f-\Delta_\eta f) + m(r,f-\Delta_\eta f) + S(r,f) \\
&\leq 2\overline{N}(r,f) + m(r,f) + S(r,f) \\
&\leq 2T(r,f) + S(r,f),
\end{aligned}$$

which implies $T(r,f) = S(r,f)$. This is absurd. So we get $f \equiv \Delta_\eta f$. □

Proof of Theorem 1.2 It follows from Lemma 2.3 that $\Delta_\eta f$ is of finite order. Since f and $\Delta_\eta f$ share $\alpha_1, \alpha_2, \infty$ CM, we get

$$\frac{\Delta_\eta f - \alpha_1}{f - \alpha_1} = e^P, \qquad \frac{\Delta_\eta f - \alpha_2}{f - \alpha_2} = e^Q, \tag{8}$$

where P, Q are polynomials.

Suppose that $\Delta_\eta f \not\equiv f$, then $e^P \not\equiv 1, e^Q \not\equiv 1$ and $e^P \not\equiv e^Q$. By (8), we get

$$f(z) = \alpha_1(z) + \bigl(\alpha_2(z) - \alpha_1(z)\bigr) \frac{e^{Q(z)} - 1}{e^{Q(z)} - e^{P(z)}} \tag{9}$$

and

$$\Delta_\eta f(z) = \alpha_1(z) + \bigl(\alpha_2(z) - \alpha_1(z)\bigr) \frac{e^{P(z)+Q(z)} - e^{p(z)}}{e^{Q(z)} - e^{P(z)}}. \tag{10}$$

On the other hand, (9) also implies

$$\begin{aligned}
\Delta_\eta f(z) = {}& \Delta_\eta \alpha_1(z) + \bigl(\alpha_2(z+\eta) - \alpha_1(z+\eta)\bigr) \frac{e^{Q(z+\eta)} - 1}{e^{Q(z+\eta)} - e^{P(z+\eta)}} \\
& - \bigl(\alpha_2(z) - \alpha_1(z)\bigr) \frac{e^{Q(z)} - 1}{e^{Q(z)} - e^{P(z)}}.
\end{aligned} \tag{11}$$

Now we discuss the following three cases.

Case 1. Suppose that both e^P and e^Q are constants, then, by (9), we get $T(r,f) = S(r,f)$. This is absurd.

Case 2. Suppose that only one between e^P and e^Q is a constant, without loss of generality, we assume that $e^P \equiv c$, by (9), we get

$$T(r,f) = T\bigl(r, e^Q\bigr) + S(r,f), \qquad S(r,f) = S\bigl(r, e^Q\bigr). \tag{12}$$

Subcase 2.1. If $e^{Q(z+\eta)} \equiv e^{Q(z)}$, then $\deg Q = 1$. (10) and (11) imply that

$$\bigl\{(1-c)\alpha_1(z) + c\alpha_2(z) - \Delta_\eta \alpha_2(z)\bigr\} e^{Q(z)} = (1-c)\Delta_\eta \alpha_1(z) + c\alpha_2(z) - \Delta_\eta \alpha_2(z). \tag{13}$$

By (12) and (13), we get

$$(1-c)\alpha_1(z)+c\alpha_2(z)-\Delta_\eta\alpha_2(z)\equiv 0,\qquad (1-c)\Delta_\eta\alpha_1(z)+c\alpha_2(z)-\Delta_\eta\alpha_2(z)\equiv 0.\tag{14}$$

Solving (14) implies $\Delta_\eta\alpha_1(z)\equiv\alpha_1(z)$, *that is,*

$$\alpha_1(z+\eta)\equiv 2\alpha_1(z).\tag{15}$$

Then, by Lemma 2.8, (12), $\deg Q=1$ *and* (15), we get $\liminf_{r\to\infty}\frac{T(r,\alpha_1)}{T(r,f)}>0$, *which contradicts that* α_1 *is a small function of* f.

Subcase 2.2. If $e^{Q(z+\eta)}\not\equiv e^{Q(z)}$, let z_0 be a zero of $e^{Q(z)}-\frac{c}{e^{Q(z+\eta)-Q(z)}}$, then z_0 is a zero of $e^{Q(z+\eta)}-c$. So by (11), we know that one of the following cases must occur.

(i) z_0 is a pole of $\Delta_\eta f(z)$. Since $\Delta_\eta f$ and f share ∞ CM, by (9), we know that if z_0 is not a pole of α_1 or α_2, then z_0 must be a zero of $e^{Q(z)}-c$. This implies that z_0 is a zero of $e^{Q(z+\eta)-Q(z)}-1$.

(ii) z_0 is not a pole of $\Delta_\eta f(z)$. By (11), we know that if z_0 is not a pole of $\Delta_\eta\alpha_1$ or $\alpha_2-\alpha_1$, then z_0 is either a zero of $\alpha_2(z+\eta)-\alpha_1(z+\eta)$, or a zero of $e^{Q(z)}-c$. For the latter, z_0 must be a zero of $e^{Q(z+\eta)-Q(z)}-1$. *While if* z_0 *is a pole of* $\Delta_\eta\alpha_1$ *or* $\alpha_2-\alpha_1$, *then, by* (12), *we get*

$$\overline{N}\big(r,e^{Q(z+\eta)}=c,\Delta_\eta\alpha_1=\infty\big)\le\overline{N}(r,\Delta_\eta\alpha_1)=S(r,f)=S\big(r,e^{Q}\big),$$

where $\overline{N}(r,e^{Q(z+\eta)}=c,\Delta_\eta\alpha_1=\infty)$ *denotes the reduced counting function of those points in* $|z|\le r$, *which are c-points of* $e^{Q(z+\eta)}$ *and poles of* $\Delta_\eta\alpha_1(z)$. *Similarly, we have* $\overline{N}(r,e^{Q(z+\eta)}=c,\alpha_2-\alpha_1=\infty)=S(r,e^{Q})$.

From the above analyses, (12) and Lemma 2.3, we get

$$\begin{aligned}\overline{N}\left(r,\frac{1}{e^{Q(z)}-\frac{c}{e^{Q(z+\eta)-Q(z)}}}\right)&=\overline{N}\left(r,\frac{1}{e^{Q(z+\eta)}-c}\right)\\&\le\overline{N}\left(r,\frac{1}{e^{Q(z+\eta)-Q(z)}-1}\right)+S\big(r,e^{Q}\big)\\&=S\big(r,e^{Q}\big).\end{aligned}\tag{16}$$

So from the second main theorem related to small functions and (16), we get $T(r,e^{Q})=S(r,e^{Q})$. This is absurd.

Case 3. Suppose that both e^{P} and e^{Q} are not constants, by (10) and (11), we get

$$H_{2p}e^{2P}+H_{2p+q}e^{2P+Q}+H_{p+2q}e^{P+2Q}+H_{2q}e^{2Q}+H_{p+q}e^{P+Q}+H_pe^{P}+H_qe^{Q}=0,\tag{17}$$

where

$$\begin{aligned}&H_{2p}=(\alpha_2-\Delta_\eta\alpha_1)e^{\Delta_\eta P},\qquad H_{2p+q}=(\alpha_1-\alpha_2)e^{\Delta_\eta P},\qquad H_{p+2q}=(\alpha_2-\alpha_1)e^{\Delta_\eta Q},\\&H_{2q}=(\alpha_1-\Delta_\eta\alpha_2)e^{\Delta_\eta Q},\qquad H_{p+q}=(\Delta_\eta\alpha_2-\alpha_1)e^{\Delta_\eta Q}+(\Delta_\eta\alpha_1-\alpha_2)e^{\Delta_\eta P},\\&H_p(z)=\big(\alpha_2(z)-\alpha_1(z)\big)e^{\Delta_\eta P(z)}-\alpha_2(z+\eta)+\alpha_1(z+\eta),\\&H_q(z)=\big(\alpha_1(z)-\alpha_2(z)\big)e^{\Delta_\eta Q(z)}+\alpha_2(z+\eta)-\alpha_1(z+\eta).\end{aligned}\tag{18}$$

Subcase 3.1. $\deg P > \deg Q$. By (9) we get

$$S(r,f) = S(r,e^P). \tag{19}$$

Equation (17) implies that

$$\psi_1 e^{2P} + \psi_2 e^P = \psi_3, \tag{20}$$

where

$$\psi_1 = H_{2p} + H_{2p+q}e^Q, \qquad \psi_2 = H_{p+2q}e^{2Q} + H_{p+q}e^Q + H_p, \qquad \psi_3 = -H_{2q}e^{2Q} - H_q e^Q,$$

such that $T(r,\psi_j) = S(r,e^P)$ $(j = 1,2,3)$. Then, by (20) and Lemma 2.7, we get $\psi_j \equiv 0$ $(j = 1,2,3)$. From this and (18), we get

$$\begin{cases} (\alpha_2 - \Delta_\eta\alpha_1) - (\alpha_2 - \alpha_1)e^Q = 0, \\ (\alpha_2 - \alpha_1)e^{\Delta_\eta Q + 2Q} + (\Delta_\eta\alpha_1 - \alpha_2)e^{\Delta_\eta P + Q} + (\alpha_2 - \alpha_1)e^{\Delta_\eta P} = (\alpha_2 - \alpha_1)e^{\Delta_\eta Q}. \end{cases} \tag{21}$$

Solving (21) deduce

$$(\alpha_1 - \Delta_\eta\alpha_1)(2\alpha_2 - \alpha_1 - \Delta_\eta\alpha_1)\left(e^{\Delta_\eta Q} - e^{\Delta_\eta P}\right) \equiv 0.$$

Since $\deg(\Delta_\eta P) = \deg P - 1 > \deg Q - 1 = \deg(\Delta_\eta Q)$, we get $\alpha_1 \equiv \Delta_\eta\alpha_1$ or $\Delta_\eta\alpha_1 \equiv 2\alpha_2 - \alpha_1$. From this and (21), we get $e^Q \equiv 1$ or $e^Q \equiv -1$, which contradicts that e^Q is not a constant.

Subcase 3.2. $\deg P < \deg Q$. By (9) we get $S(r,f) = S(r,e^Q)$. Using a similar argument to subcase 3.1, we get $e^P \equiv 1$ or $e^P \equiv -1$, which contradicts that e^P is not a constant.

Subcase 3.3. $\deg P = \deg Q = m \geq 1$. By (9), we get

$$S(r,f) = S\left(r,e^{z^m}\right). \tag{22}$$

Set

$$P(z) = az^m + a_{m-1}z^{m-1} + \cdots + a_0, \qquad Q(z) = bz^m + b_{m-1}z^{m-1} + \cdots + b_0, \tag{23}$$

where $a, a_{m-1}, \ldots, a_0, b, b_{m-1}, \ldots, b_0$ are constants such that $ab \neq 0$. By (17) and (23), we get

$$\sum_{j\in\Lambda} \varphi_j(z)e^{jz^m} = 0, \tag{24}$$

where

$$\Lambda = \{2a, 2a+b, a+2b, 2b, a+b, a, b\},$$
$$\varphi_{2a} = H_{2p}\gamma^2, \qquad \varphi_{2a+b} = H_{2p+q}\gamma^2\eta, \qquad \varphi_{a+2b} = H_{p+2q}\gamma\eta^2, \qquad \varphi_{2b} = H_{2q}\eta^2,$$
$$\varphi_{a+b} = H_{p+q}\gamma\eta, \qquad \varphi_a = H_p\gamma, \qquad \varphi_b = H_q\eta,$$
$$\gamma(z) = e^{P(z)-az^m}, \qquad \eta(z) = e^{Q(z)-bz^m}$$

such that

$$T(r,\varphi_j) = S\left(r,e^{z^m}\right), \quad (j \in \Lambda). \tag{25}$$

If $a \notin \{b, \frac{b}{2}, -b\}$, then $2a+b \notin \{2a, a+2b, 2b, a+b, a, b\}$. So by (24), (25) and Lemma 2.7, we get $\varphi_{2a+b} = H_{2p+q}\gamma^2\eta \equiv 0$. Combining this and (18), we get $\alpha_2 \equiv \alpha_1$. This is absurd.

If $a = b$, then, by (24), we get

$$(\varphi_{2a+b} + \varphi_{a+2b})e^{3bz^m} + (\varphi_{2a} + \varphi_{2b} + \varphi_{a+b})e^{2bz^m} + (\varphi_a + \varphi_b)e^{bz^m} = 0. \tag{26}$$

Combining with (26) and Lemma 2.7, we get

$$H_{2p+q}\gamma^2\eta + H_{p+2q}\gamma\eta^2 \equiv 0, \qquad H_{2p}\gamma^2 + H_{2q}\eta^2 + H_{p+q}\gamma\eta \equiv 0, \qquad H_p\gamma + H_q\eta \equiv 0.$$

Then, by (18), we get

$$\begin{cases} e^{\Delta_\eta P}\gamma = e^{\Delta_\eta Q}\eta, \\ \{(\alpha_2 - \alpha_1)e^{\Delta_\eta P} - \alpha_2(z+\eta) + \alpha_1(z+\eta)\}\gamma = \{(\alpha_2 - \alpha_1)e^{\Delta_\eta Q} - \alpha_2(z+\eta) + \alpha_1(z+\eta)\}\eta. \end{cases}$$

Solving the above equation, we get $\{\alpha_2(z+\eta) - \alpha_1(z+\eta)\}(\gamma - \eta) \equiv 0$, which implies $\alpha_2 \equiv \alpha_1$ or $e^P \equiv e^Q$. This is absurd.

If $a = \frac{b}{2}$, then, by (24), we get

$$\varphi_{a+2b}e^{\frac{5}{2}bz^m} + (\varphi_{2a+b} + \varphi_{2b})e^{2bz^m} + \varphi_{a+b}e^{\frac{3}{2}bz^m} + (\varphi_{2a} + \varphi_b)e^{bz^m} + \varphi_a e^{\frac{b}{2}z^m} = 0. \tag{27}$$

Combining with (27) and Lemma 2.7, we get $\varphi_{a+2b} \equiv 0$. Then, by (18), we get $\alpha_2 \equiv \alpha_1$. This is absurd.

If $a = -b$, then, by (24), we get

$$\varphi_{2a}e^{-2bz^m} + (\varphi_{2a+b} + \varphi_a)e^{-bz^m} + (\varphi_{a+2b} + \varphi_b)e^{bz^m} + \varphi_{2b}e^{2bz^m} = -\varphi_{a+b}. \tag{28}$$

Combining with (28) and Lemma 2.7, we get $\varphi_{2a} \equiv 0, \varphi_{2b} \equiv 0$. Then, by (18), we get $\alpha_2 \equiv \Delta_\eta\alpha_1$ and $\alpha_1 \equiv \Delta_\eta\alpha_2$, which implies $\alpha_2 \equiv \alpha_1$. This is absurd. Theorem 1.2 is thus proved. □

Proof of Theorem 1.3 Suppose that $f \not\equiv \Delta_\eta f$, let $L(f(z)) = f(z+\eta) - 2f(z)$, then, by Lemma 2.3, we get

$$N\big(r,L(f)\big) \le N\big(r,f(z+\eta)\big) + N\big(r,f(z)\big) = 2N(r,f) + S(r,f). \tag{29}$$

Then, by (29) and Lemma 2.9, we get

$$\begin{aligned} 2T(r,f) \le{}& N(r,f) + \sum_{j=1}^{3} N\left(r,\frac{1}{f-\alpha_j}\right) - \big(2N(r,f) - N\big(r,L(f)\big)\big) \\ & - N\left(r,\frac{1}{L(f)}\right) + S(r,f) \end{aligned}$$

$$\le N(r,f) + N\left(r, \frac{1}{f - \Delta_\eta f}\right) - N\left(r, \frac{1}{L(f)}\right) + S(r,f)$$
$$\le N(r,f) + S(r,f),$$

which implies $T(r,f) = S(r,f)$. This is absurd. Theorem 1.3 is thus proved. □

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The authors declare that they have no competing interests.

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