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Global dynamics in a stochastic three species food-chain model with harvesting and distributed delays

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Abstract

This paper proposes a stochastic three species food-chain model with harvesting and distributed delays. Some criteria for the global dynamics of all positive solutions, including the existence of global positive solutions, stochastic boundedness, extinction, global asymptotic stability in the mean, and the probability distribution, are established by using the stochastic integral inequalities, Lyapunov function method, and the inequality estimation technique. Furthermore, the effects of harvesting are discussed, the optimal harvesting strategy and the maximum of expectation of sustainable yield (MESY for short) are obtained. Finally, numerical examples are carried out to illustrate our main results.

Keywords: Stochastic food-chain model; Distributed delay; Inequality estimation; Extinction; Global stability; Harvesting

1 Introduction

The notion of food-chain was first postulated by Eiton in 1927 (see [1]). As he said, he proposed this idea due to the Chinese folk-adage: big fish eat small fish, small fish eat shrimps, shrimps eat mud. We see that food-chain models have been extensively studied because of their academic and pragmatic implication. The following deterministic three species food-chain model has been investigated by many scholars (see [2–5]):

$$\begin{cases} \frac{dx_1(t)}{dt} = x_1(t)[r_1 - a_{11}x_1(t) - a_{12}x_2(t)], \\ \frac{dx_2(t)}{dt} = x_2(t)[-r_2 + a_{21}x_1(t) - a_{22}x_2(t) - a_{23}x_3(t)], \\ \frac{dx_3(t)}{dt} = x_3(t)[-r_3 + a_{32}x_2(t) - a_{33}x_3(t)], \end{cases}$$

where $x_i(t)$ $(i = 1, 2, 3)$ represents population sizes of prey, intermediate predator, and top predator at time t, respectively.

Nevertheless, in the real world, it is hard to protect population systems from environmental noise (see [6–15]). Taking the influence of white noises into the above model, Liu and Bai in [16] proposed the following stochastic three species food-chain model:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)[r_1 - h_1 - a_{11}x_1(t) - a_{12}x_2(t)]\,\mathrm{d}t + \sigma_1 x_1(t)\,\mathrm{d}B_1(t), \\ \mathrm{d}x_2(t) = x_2(t)[-r_2 - h_2 + a_{21}x_1(t) - a_{22}x_2(t) - a_{23}x_3(t)]\,\mathrm{d}t + \sigma_2 x_2(t)\,\mathrm{d}B_2(t), \\ \mathrm{d}x_3(t) = x_3(t)[-r_3 - h_3 + a_{32}x_2(t) - a_{33}x_3(t)]\,\mathrm{d}t + \sigma_3 x_3(t)\,\mathrm{d}B_3(t). \end{cases}$$

Time-delay is common and inevitable in nature, and often makes the system property decline or even causes instability. However, any species in nature will not always react at once to variation in its own population size or that of an interacting species, but will do so after a time lag preferably. In other words, it is essential to investigate the effect of delays on the food-chain model. Thus, Li and Wang in [17] proposed a delayed food-chain system with the Beddington–DeAngelis functional response, and they found that delays affect the stability of equilibrium points and the existence of Hopf bifurcation.

From [18, 19], we obtain that systems with distributed time delays include those not only with the discrete time delays but also the continuously distributed time delays. To the best of our knowledge to date, the problem of a stochastic food-chain model with harvesting and distributed delays has not been studied in the past research. Motivated by the above discussion, considering distributed time delays and white noises, in this paper, we establish the following stochastic three species food-chain model:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)[r_1 - h_1 - a_{11}x_1(t) - a_{12}\int_{-\tau_{12}}^{0} x_2(t+\theta)\,\mathrm{d}\mu_{12}(\theta)]\,\mathrm{d}t \\ \qquad\qquad + \sigma_1 x_1(t)\,\mathrm{d}B_1(t), \\ \mathrm{d}x_2(t) = x_2(t)[-r_2 - h_2 + a_{21}\int_{-\tau_{21}}^{0} x_1(t+\theta)\,\mathrm{d}\mu_{21}(\theta) - a_{22}x_2(t) \\ \qquad\qquad - a_{23}\int_{-\tau_{23}}^{0} x_3(t+\theta)\,\mathrm{d}\mu_{23}(\theta)]\,\mathrm{d}t + \sigma_2 x_2(t)\,\mathrm{d}B_2(t), \\ \mathrm{d}x_3(t) = x_3(t)[-r_3 - h_3 + a_{32}\int_{-\tau_{32}}^{0} x_2(t+\theta)\,\mathrm{d}\mu_{32}(\theta) - a_{33}x_3(t)]\,\mathrm{d}t \\ \qquad\qquad + \sigma_3 x_3(t)\,\mathrm{d}B_3(t), \end{cases} \tag{1}$$

where $r_1 > 0$ is intrinsic growth rate of species x_1, $r_i > 0$ ($i = 2, 3$) stand for death rates of species x_i, $a_{ii} > 0$ ($i = 1, 2, 3$) are intraspecific competition coefficients of species x_i, $a_{12} \geq 0$ and $a_{23} \geq 0$ are capture rates, $a_{21} \geq 0$ and $a_{32} \geq 0$ measure efficiency of food conversion, $h_i \geq 0$ ($i = 1, 2, 3$) stands for the harvesting effort of species x_i, $\mu_{ij}(\theta)$ ($i, j = 1, 2, 3$) are non-negative variation functions defined on $[-\tau_{ij}, 0]$ satisfying $\int_{-\tau_{ij}}^{0} \mathrm{d}\mu_{ij}(\theta) = 1$, $B_i(t)$ ($i = 1, 2, 3$) are standard independent Brownian motions defined on the complete probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, and σ_i^2 ($i = 1, 2, 3$) is the intensity of $B_i(t)$.

In this paper we firstly investigate the global dynamics of model (1), including the existence of global positive solutions, stochastic boundedness, extinction, global asymptotic stability in the mean, and the probability distribution, by using the stochastic integrals inequalities, Lyapunov function method, and the inequality estimation technique. Next, we discuss the effects of harvesting for the extinction and persistence of species of model (1), and establish the optimal harvesting effort $H^* = (h_1^*, h_2^*, h_3^*)$ such that all the species are not extinct and the maximal expectation of sustained yield $Y(H^*) = \lim_{t\to\infty} \sum_{i=1}^{3} E(h_i^* x_i(t))$.

The organization of this paper is as follows. In Sect. 2, we propose some useful lemmas which will be used in the proofs of main results. We also obtain the existence and stochastic boundedness of unique global positive solution with any positive initial value.

In Sect. 3, the global dynamics of positive solutions are investigated. A whole criterion for the extinction and global asymptotic stability in the mean with probability one is established. Furthermore, the criterion for the global asymptotic stability in the probability distribution is also established. In Sect. 4, the effects of harvesting for the extinction and persistence of species are discussed, and the sufficient conditions for the existence and non-existence of optimal harvesting are obtained. We also offer the numerical examples to illustrate our main results in Sect. 5. Lastly, in Sect. 6 we give a brief conclusion and propose some interesting open problems.

2 Preliminaries

Firstly, for convenience of the statements, we denote $b_1 = r_1 - h_1 - \frac{1}{2}\sigma_1^2$, $b_2 = r_2 + h_2 + \frac{1}{2}\sigma_2^2$, $b_3 = r_3 + h_3 + \frac{1}{2}\sigma_3^2$, $\Delta_{11} = b_1$, $\Delta_{21} = b_1a_{22} + b_2a_{12}$, $\Delta_{22} = b_1a_{21} - b_2a_{11}$, $\Delta_{31} = b_1(a_{22}a_{33} + a_{32}a_{23}) + b_2a_{33}a_{12} - b_3a_{12}a_{23}$, $\Delta_{32} = a_{33}(b_1a_{21} - b_2a_{11}) + b_3a_{11}a_{23}$, $\Delta_{33} = (b_1a_{21} - b_2a_{11})a_{32} - b_3(a_{11}a_{22} + a_{12}a_{21})$, $H_1 = a_{11}$, $H_2 = a_{11}a_{22} + a_{12}a_{21}$, and $H_3 = a_{11}a_{22}a_{33} + a_{33}a_{12}a_{21} + a_{11}a_{32}a_{23}$. Obviously, when $b_1 \geq 0$, we have $\Delta_{21} \geq 0$. Furthermore, we have the following.

Lemma 1 *If* $\Delta_{33} > 0$, *then* $\Delta_{31} > 0$ *and* $\Delta_{32} > 0$.

Proof Let $x_1^* = \frac{\Delta_{31}}{H_3}$, $x_2^* = \frac{\Delta_{32}}{H_3}$, and $x_3^* = \frac{\Delta_{33}}{H_3}$. Then $x_3^* > 0$. By calculating, we can obtain

$$a_{32}x_2^* = b_3 + a_{33}x_3^* > 0, \qquad a_{21}x_1^* = b_2 + a_{22}x_2^* + a_{23}x_3^* > 0.$$

Therefore, we have $\Delta_{31} > 0$ and $\Delta_{32} > 0$. This completes the proof. □

Lemma 2 *For any real numbers* $A \geq 0$, $B \geq 0$, $A_i \geq 0$ $(1 \leq i \leq n)$, *and* $p > 0$, $q > 0$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *one has*

$$\left(\sum_{i=1}^{n} A_i\right)^p \leq n^p \sum_{i=1}^{n} A_i^p, \qquad AB \leq \frac{A^p}{p} + \frac{B^q}{q}.$$

Let $\gamma = \max\{\tau_{12}, \tau_{21}, \tau_{23}, \tau_{32}\}$. The initial condition for model (1) is given by

$$x_1(\theta) = \xi(\theta), \qquad x_2(\theta) = \eta(\theta), \qquad x_3(\theta) = \varsigma(\theta), \quad -\gamma \leq \theta \leq 0. \tag{2}$$

On the existence and the ultimate boundedness of the global positive solution for model (1), we have the following results.

Lemma 3 *For any* $(\xi(\theta), \eta(\theta), \varsigma(\theta)) \in C([-\gamma, 0], R_+^3))$, *model* (1) *with condition* (2) *has a unique global solution* $x(t) = (x_1(t), x_2(t), x_3(t)) \in R_+^3$ *a.s. for all* $t \geq 0$. *Moreover, for any* $p > 0$, *there exist constants* $K_1(p) > 0$, $K_2(p) > 0$, *and* $K_3(p) > 0$ *such that*

$$\limsup_{t\to\infty} E\bigl[x_1^p(t)\bigr] \leq K_1(p), \qquad \limsup_{t\to\infty} E\bigl[x_2^p(t)\bigr] \leq K_2(p), \qquad \limsup_{t\to\infty} E\bigl[x_3^p(t)\bigr] \leq K_3(p).$$

Proof Since the coefficients of model (1) are locally Lipschitz, from [14, 20] we obtain that, for any initial data $(\xi(\theta), \eta(\theta), \varsigma(\theta)) \in C([-\gamma, 0], R_+^3))$, model (1) has a unique solution

$x(t) = (x_1(t), x_2(t), x_3(t)) \in R^3_+$ for all $t \in [-\gamma, \tau_e)$, where τ_e is the explosion time. We need to prove $\tau_e = \infty$ *a.s.* Let $k_0 > 0$ be an enough large integer such that $\xi(0), \eta(0), \varsigma(0) \in (\frac{1}{k_0}, k_0)$. For each integer $k > k_0$, define stopping times as follows:

$$\tau_k = \inf\left\{t \in [0, \tau_e) : x_1(t) \notin \left(\frac{1}{k}, k\right), x_2(t) \notin \left(\frac{1}{k}, k\right), x_3(t) \notin \left(\frac{1}{k}, k\right)\right\}. \tag{3}$$

It is clear that τ_k is increasing with k. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$. We have $\tau_\infty \leq \tau_e$ *a.s.* Thus, we only need to prove $\tau_\infty = \infty$ *a.s.*

If the conclusion is false, then there exist $T > 0$ and $\varepsilon \in (0, 1)$ such that $P(\tau_\infty \leq T) > \varepsilon$. Hence, there exists an integer $k_1 > k_0$ such that, for any $k > k_1$,

$$P(\tau_k \leq T) > \varepsilon. \tag{4}$$

Define $V_i(x_i) = x_i - 1 - \ln x_i$ $(i = 1, 2, 3)$. Using Itô's formula, we obtain

$$\begin{aligned}
\mathrm{d}V_1(x_1) &= \mathcal{L}\big[V_1(x_1)\big]\,\mathrm{d}t + \sigma_1(x_1 - 1)\,\mathrm{d}B_1(t),\\
\mathrm{d}V_2(x_2) &= \mathcal{L}\big[V_2(x_2)\big]\,\mathrm{d}t + \sigma_2(x_2 - 1)\,\mathrm{d}B_2(t),\\
\mathrm{d}V_3(x_3) &= \mathcal{L}\big[V_3(x_3)\big]\,\mathrm{d}t + \sigma_3(x_3 - 1)\,\mathrm{d}B_3(t),
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
\mathcal{L}\big[V_1(x_1)\big] &= (x_1 - 1)\left(r_1 - h_1 - a_{11}x_1(t) - a_{12}\int_{-\tau_{12}}^{0} x_2(t+\theta)\,\mathrm{d}\mu_{12}(\theta)\right) + \frac{1}{2}\sigma_1^2,\\
\mathcal{L}\big[V_2(x_2)\big] &= (x_2 - 1)\bigg(-r_2 - h_2 + a_{21}\int_{-\tau_{21}}^{0} x_1(t+\theta)\,\mathrm{d}\mu_{12}(\theta) - a_{22}x_2(t)\\
&\qquad - a_{23}\int_{-\tau_{23}}^{0} x_3(t+\theta)\,\mathrm{d}\mu_{23}(\theta)\bigg) + \frac{1}{2}\sigma_2^2,\\
\mathcal{L}\big[V_3(x_3)\big] &= (x_3 - 1)\left(-r_3 - h_3 - a_{33}x_3(t) + a_{32}\int_{-\tau_{32}}^{0} x_2(t+\theta)\,\mathrm{d}\mu_{32}(\theta)\right) + \frac{1}{2}\sigma_3^2.
\end{aligned} \tag{6}$$

For any integer $n > 0$, using Lemma 2 we can obtain

$$\begin{aligned}
\mathcal{L}\big[V_1(x_1)\big] &\leq \frac{\sigma_1^2}{2} - (r_1 - h_1) + \frac{n^2}{2}a_{12} + (r_1 - h_1)x_1 + a_{11}x_1 - a_{11}x_1^2\\
&\qquad + \frac{1}{2n^2}a_{12}\int_{-\tau_{12}}^{0} x_2^2(t+\theta)\,\mathrm{d}\mu_{12}(\theta),\\
\mathcal{L}\big[V_2(x_2)\big] &\leq \frac{\sigma_2^2}{2} + (r_2 + h_2) + \frac{n}{2}a_{21}\int_{-\tau_{21}}^{0} x_1^2(t+\theta)\,\mathrm{d}\mu_{21}(\theta) - (r_2 + h_2)x_2 + a_{22}x_2\\
&\qquad - a_{22}x_2^2 + \frac{x_2^2}{2n}a_{21} + \frac{n^2}{2}a_{23} + \frac{1}{2n^2}a_{23}\int_{-\tau_{23}}^{0} x_3^2(t+\theta)\,\mathrm{d}\mu_{23}(\theta),\\
\mathcal{L}\big[V_3(x_3)\big] &\leq \frac{\sigma_3^2}{2} + (r_3 + h_3) + \frac{x_3^2}{2n}a_{32} - (r_3 + h_3)x_3 + a_{33}x_3 - a_{33}x_3^2\\
&\qquad + \frac{n}{2}a_{32}\int_{-\tau_{32}}^{0} x_2^2(t+\theta)\,\mathrm{d}\mu_{32}(\theta).
\end{aligned} \tag{7}$$

Define $V_0(x_1,x_2,x_3) = \alpha V_1(x_1) + V_2(x_2) + \eta V_3(x_3) + V_4(t)$, where

$$V_4(t) = \frac{\alpha}{2n^2}a_{12}\int_{-\tau_{12}}^{0}\int_{t+\theta}^{t} x_2^2(s)\,\mathrm{d}s\,\mathrm{d}\mu_{12}(\theta) + \left(\frac{n}{2}a_{21}\int_{-\tau_{21}}^{0}\int_{t+\theta}^{t} x_1^2(s)\,\mathrm{d}s\,\mathrm{d}\mu_{21}(\theta) + \frac{1}{2n^2}a_{23}\int_{-\tau_{23}}^{0}\int_{t+\theta}^{t} x_3^2(s)\,\mathrm{d}s\,\mathrm{d}\mu_{23}(\theta)\right) + \eta\frac{n}{2}a_{32}\int_{-\tau_{32}}^{0}\int_{t+\theta}^{t} x_2^2(s)\,\mathrm{d}s\,\mathrm{d}\mu_{32}(\theta).$$

Choose the positive constants α, η and integer $n > 0$ such that

$$\begin{aligned}&\left(-a_{22} + \frac{1}{2n}a_{21}\right) + \frac{n\eta}{2}a_{32} + \frac{\alpha}{2n^2}a_{12} < 0,\\ &\left(-a_{33} + \frac{1}{2n}a_{32}\right)\eta + \frac{1}{2n^2}a_{23} < 0, -a_{11}\alpha + \frac{n}{2}a_{21} < 0.\end{aligned} \tag{8}$$

In fact, from $(-a_{33} + \frac{1}{2n}a_{32})\eta + \frac{1}{2n^2}a_{23} = 0$ and $-a_{11}\alpha + \frac{n}{2}a_{21} = 0$, we have $\eta = \frac{a_{23}}{n(2na_{33}-a_{32})}$ and $\alpha = \frac{na_{21}}{2a_{11}}$. Substituting η and α into the left of the first inequality of (8), we can obtain that there is enough large $n > 0$ such that $2na_{33} - a_{32} > 0$ and $-a_{22} + \frac{a_{21}}{2n} + \frac{a_{32}a_{23}}{2(2na_{33}-a_{32})} + \frac{a_{12}a_{21}}{4na_{11}} < -\frac{1}{2}a_{22}$. From this, we further choose positive constants $\eta > \frac{a_{23}}{n(2na_{33}-a_{32})}$ and $\alpha > \frac{na_{21}}{2a_{11}}$ such that (8) holds.

Using Itô's formula, from (5) we have

$$d\big[V_0(x_1,x_2,x_3)\big] = \mathcal{L}V_0(x_1,x_2,x_3)\,\mathrm{d}t + \alpha\sigma_1(x_1-1)\,\mathrm{d}B_1(t) + \sigma_2(x_2-1)\,\mathrm{d}B_2(t) + \eta\sigma_3(x_3-1)\,\mathrm{d}B_3(t).$$

From (6) and (7), we obtain

$$\begin{aligned}\mathcal{L}\big[V_0(x_1,x_2,x_3)\big] &= \alpha\mathcal{L}V_1(x_1) + \mathcal{L}V_2(x_2) + \eta\mathcal{L}V_3(x_3) + \frac{d}{dt}V_4(t)\\ &\le \frac{\alpha\sigma_1^2}{2} - \alpha(r_1-h_1) + \frac{\alpha n^2}{2}a_{12} + \alpha(r_1-h_1)x_1 + \alpha a_{11}x_1 - \alpha a_{11}x_1^2\\ &\quad + \frac{\sigma_2^2}{2} + (r_2+h_2) - (r_2+h_2)x_2 + a_{22}x_2 - a_{22}x_2^2 + \frac{x_2^2}{2n}a_{21}\\ &\quad + \frac{n^2}{2}a_{23} + \frac{\eta\sigma_3^2}{2} + (r_3+h_3)\eta + \frac{\eta x_3^2}{2n}a_{32} - \eta(r_3+h_3)x_3 + \eta a_{33}x_3\\ &\quad - \eta a_{33}x_3^2 + \frac{\alpha}{2n^2}x_2^2a_{12} + \frac{n}{2}x_1^2a_{21} + \frac{1}{2n^2}x_3^2a_{23} + \frac{n\eta}{2}x_2^2a_{32}.\end{aligned}$$

From (8) we can obtain that there exists a constant $K > 0$ such that

$$d\big[V_0(x_1,x_2,x_3)\big] \le K\,\mathrm{d}t + \alpha\sigma_1(x_1-1)\,\mathrm{d}B_1(t) + \sigma_2(x_2-1)\,\mathrm{d}B_2(t) + \eta\sigma_3(x_3-1)\,\mathrm{d}B_3(t). \tag{9}$$

Then, from (4) and (9), a similar argument as in [21] we can get the following contradiction:

$$\infty > V_0\big(x_1(0),x_2(0),x_3(0)\big) + KT \ge \infty.$$

Thus, we obtain $\tau_\infty = \infty$ *a.s.*, and hence, $\tau_e = \infty$ *a.s.*

For any $p > 0$, let $Q_1(t) = e^t x_1^p(t)$. By Itô's formula, we have

$$dQ_1(t) = \mathcal{L}Q_1(t)\,\mathrm{d}t + pe^t x_1^p \sigma_1\,\mathrm{d}B_1(t), \tag{10}$$

where

$$\begin{aligned}\mathcal{L}Q_1(t) &= e^t x_1^p\left\{1 + \frac{p(p-1)\sigma_1^2}{2} + p\left[r_1 - h_1 - a_{11}x_1 - a_{12}\int_{-\tau_{12}}^{0} x_2(t+\theta)\,\mathrm{d}\mu_{12}(\theta)\right]\right\}\\ &\le K_1(p)e^t \end{aligned}\tag{11}$$

with

$$K_1(p) = \max_{x_1\ge 0}\left\{\left[p(r_1 - h_1) + 1 + \frac{p(p-1)\sigma_1^2}{2}\right]x_1^p - pa_{11}x_1^{p+1}\right\}.$$

Integrating both sides of (10) and then taking expectations lead to

$$E\left[e^t x_1^p\right] - \xi^p(0) \le K_1(p)\left(e_1^t - 1\right), \tag{12}$$

which implies $\limsup_{t\to\infty} E[x_1^p(t)] \le K_1(p)$.

For any constant $p > 0$ and integer $n > 0$ with $a_{22} - a_{21}\frac{p}{p+1}n^{-\frac{p+1}{p}} > 0$, we define $Q_2(t)$ as follows:

$$Q_2(t) = C_1^* Q_1(t) + e^t x_2^p(t) + e^{\tau_{21}}\frac{pn^{p+1}}{p+1}a_{21}\int_{-\tau_{21}}^{0}\int_{t+\theta}^{t} e^s x_1^{p+1}(s)\,\mathrm{d}s\,\mathrm{d}\mu_{21}(\theta), \tag{13}$$

where $C_1^* = a_{11}^{-1}e^{\tau_{21}}n^{p+1}a_{21}$. We have by Itô's formula

$$dQ_2(t) = \mathcal{L}Q_2(t)\,\mathrm{d}t + C_1^* pe^t x_1^p\sigma_1\,\mathrm{d}B_1(t) + pe^t x_2^p\sigma_2\,\mathrm{d}B_2(t). \tag{14}$$

From (11), we have

$$\begin{aligned}\mathcal{L}Q_2(t) &= C_1^*\mathcal{L}Q_1(t) + \mathcal{L}\left(e^t x_2^p(t)\right) + \frac{d}{dt}\left(e^{\tau_{21}}\frac{pn^{p+1}}{p+1}a_{21}\int_{-\tau_{21}}^{0}\int_{t+\theta}^{t} e^s x_1^{p+1}(s)\,\mathrm{d}s\,\mathrm{d}\mu_{21}(\theta)\right)\\ &= C_1^* e^t x_1^p\left\{1 + \frac{p(p-1)\sigma_1^2}{2} + p\left[r_1 - h_1 - a_{11}x_1 - a_{12}\int_{-\tau_{12}}^{0} x_2(t+\theta)\,\mathrm{d}\mu_{12}(\theta)\right]\right\}\\ &\quad + e^t x_2^p\left\{1 + \frac{p(p-1)\sigma_2^2}{2} + p\left[-r_2 - h_2 + a_{21}\int_{-\tau_{21}}^{0} x_1(t+\theta)\,\mathrm{d}\mu_{21}(\theta)\right.\right.\\ &\quad \left.\left. - a_{22}x_2(t) - a_{23}\int_{-\tau_{23}}^{0} x_3(t+\theta)\,\mathrm{d}\mu_{23}(\theta)\right]\right\}\\ &\quad + e^{\tau_{21}}\frac{pn^{p+1}}{p+1}a_{21}\left(e^t x_1^{p+1}(t) - \int_{-\tau_{21}}^{0} e^{t+\theta}x_1^{p+1}(t+\theta)\,\mathrm{d}\mu_{21}(\theta)\right)\\ &\le C_1^* e^t\left\{\left[1 + \frac{p(p-1)\sigma_1^2}{2} + p(r_1 - h_1)\right]x_1^p - pa_{11}x_1^{p+1}\right\}\\ &\quad + e^t\left\{\left[1 + \frac{p(p-1)\sigma_2^2}{2} - p(r_2 + h_2)\right]x_2^p - p\left[a_{22} - a_{21}\frac{p}{p+1}n^{-\frac{p+1}{p}}\right]x_2^{p+1}\right.\end{aligned}$$

$$
\begin{aligned}
&+ \frac{p}{p+1} n^{p+1} a_{21} \int_{-\tau_{21}}^{0} x_1^{p+1}(t+\theta)\,\mathrm{d}\mu_{21}(\theta)\bigg\} \\
&+ e^{\tau_{21}} \frac{pn^{p+1}}{p+1} a_{21} \left(e^t x_1^{p+1}(t) - e^{-\tau_{21}} \int_{-\tau_{21}}^{0} e^t x_1^{p+1}(t+\theta)\,\mathrm{d}\mu_{21}(\theta) \right) \\
\le{}& e^t \left\{ \left[1 + \frac{p(p-1)\sigma_2^2}{2} - p(r_2+h_2) \right] x_2^p - p\left[a_{22} - a_{21} \frac{p}{p+1} n^{-\frac{p+1}{p}} \right] x_2^{p+1} \right. \\
&\left. + C_1^* \left[1 + \frac{p(p-1)\sigma_1^2}{2} + p(r_1 - h_1) \right] x_1^p - e^{\tau_{21}} \frac{p^2}{p+1} n^{p+1} a_{21} x_1^{p+1} \right\}.
\end{aligned} \tag{15}
$$

Obviously, there is a constant $K_2(p) > 0$ such that $\mathcal{L}Q_2(t) \le K_2(p)e^t$. According to (13) and (14), we obtain

$$
E\big[e^t x_2^p\big] \le EQ_2(t) \le EQ_2(0) + K_2(p)\big(e^t - 1\big),
$$

which implies $\limsup_{t\to\infty} E[x_2^p(t)] \le K_2(p)$.

For any constant $p > 0$ and integer $n > 0$ with $a_{33} - a_{32}\frac{p}{p+1} n^{-\frac{p+1}{p}} > 0$, we define $Q_3(t)$ as follows:

$$
Q_3(t) = C_2^* Q_2(t) + e^t x_3^p + e^{\tau_{32}} \frac{pn^{p+1}}{p+1} a_{32} \int_{-\tau_{32}}^{0} \int_{t+\theta}^{t} e^s x_2^{p+1}(s)\,\mathrm{d}s\,\mathrm{d}\mu_{32}(\theta), \tag{16}
$$

where $C_2^* = a_{22}^{-1} e^{\tau_{32}} n^{p+1} a_{32}$.

Applying Itô's formula to $Q_3(t)$, we obtain

$$
dQ_3(t) = \mathcal{L}Q_3(t)\,\mathrm{d}t + C_2^*\big(C_1^* p e^t x_1^p \sigma_1\,\mathrm{d}B_1(t) + pe^t x_2^p \sigma_2\,\mathrm{d}B_2(t)\big) + pe^t x_3^p \sigma_3\,\mathrm{d}B_3(t), \tag{17}
$$

where

$$
\begin{aligned}
\mathcal{L}Q_3(t) ={}& C_2^* \mathcal{L}Q_2(t) + \mathcal{L}\big[e^t x_3^p\big] + a_{32} e^{\tau_{32}} e^t x_2^{p+1} \frac{pn^{p+1}}{p+1} \\
&- a_{32} e^{\tau_{32}} \frac{pn^{p+1}}{p+1} \int_{-\tau_{32}}^{0} e^t x_2^{p+1}(t+\theta)\,\mathrm{d}\mu_{32}(\theta) \\
\le{}& C_2^* \mathcal{L}Q_2(t) + \mathcal{L}\big[e^t x_3^p\big] + a_{32} e^{\tau_{32}} e^t x_2^{p+1} \frac{pn^{p+1}}{p+1} \\
&- a_{32} \frac{pn^{p+1}}{p+1} \int_{-\tau_{32}}^{0} x_2^{p+1}(t+\theta)\,\mathrm{d}\mu_{32}(\theta).
\end{aligned}
$$

Since

$$
\begin{aligned}
\mathcal{L}\big[e^t x_3^p\big] ={}& e^t \left\{ \left[1 + \frac{p(p-1)\sigma_3^2}{2} + p(-r_3 - h_3) \right] x_3^p \right. \\
&\left. + a_{32} p x_3^p \int_{-\tau_{32}}^{0} x_2(t+\theta)\,\mathrm{d}\mu_{32}(\theta) - a_{33} p x_3^{p+1}(t) \right\} \\
\le{}& e^t \left\{ \left[1 - p(r_3+h_3) + \frac{p(p-1)\sigma_3^2}{2} \right] x_3^p + a_{32} \frac{p}{p+1} n^{p+1} \int_{-\tau_{32}}^{0} x_2^{p+1}(t+\theta)\,\mathrm{d}\mu_{32}(\theta) \right. \\
&\left. - p\left[a_{33} - a_{32} \frac{p}{p+1} n^{-\frac{p+1}{p}} \right] x_3^{p+1} \right\},
\end{aligned}
$$

from (15) we further obtain

$$\begin{aligned}\mathcal{L}Q_3(t) \le e^t\bigg\{&\bigg[1-p(r_3+h_3)+\frac{p(p-1)\sigma_3^2}{2}\bigg]x_3^p - p\bigg[a_{33}-a_{32}\frac{p}{p+1}n^{-\frac{p+1}{p}}\bigg]x_3^{p+1}\\ &+\bigg(1-p(r_2+h_2)+\frac{p(p-1)\sigma_2^2}{2}\bigg)x_2^p C_2^*\\ &-\frac{p^2}{p+1}\big(n^{p+1}a_{32}e^{\tau_{32}}+n^{-\frac{p+1}{p}}a_{21}C_2^*\big)x_2^{p+1}\\ &-C_2^*e^{\tau_{21}}\frac{p^2}{p+1}n^{p+1}a_{21}x_1^{p+1}+C_1^*C_2^*\bigg[1+p(r_1-h_1)+\frac{p(p-1)\sigma_1^2}{2}\bigg]x_1^p\bigg\}.\end{aligned}$$

Obviously, there is a constant $K_3(p)>0$ such that $\mathcal{L}Q_3(t)\le K_3(p)e^t$. Hence, from (16) and (17) we obtain

$$E\big[e^t x_3^p\big] \le E\big[Q_3(t)\big] \le E\big[Q_3(0)\big] + K_3(p)\big(e^t-1\big).$$

Consequently, $\limsup_{t\to\infty} E[x_3^p(t)] \le K_3(p)$. This completes the proof. □

Lemma 4 *Assume that functions $Y\in C(R_+\times\Omega,R_+)$ and $Z\in C(R_+\times\Omega,R)$ satisfy* $\lim_{t\to\infty}\frac{Z(t)}{t}=0$ *a.s.*

(1) *If there are three positive constants T, β, and β_0 such that, for all $t\ge T$,*

$$\ln Y(t) = \beta t - \beta_0\int_0^t Y(s)\,\mathrm{d}s + Z(t) \quad a.s.,$$

then $\lim_{t\to\infty}\langle Y(t)\rangle = \frac{\beta}{\beta_0}$ *a.s., and* $\lim_{t\to\infty}\frac{\ln Y(t)}{t}=0$ *a.s.*

(2) *If there exist two positive constants β_0 and T, and a constant $\beta\in R$ such that, for $t\ge T$,*

$$\ln Y(t) \le \beta t - \beta_0\int_0^t Y(s)\,\mathrm{d}s + Z(t) \quad a.s.,$$

then $\limsup_{t\to\infty}\langle Y(t)\rangle \le \frac{\beta}{\beta_0}$ *a.s. if* $\beta\ge 0$, *and* $\lim_{t\to\infty} Y(t)=0$ *a.s. if* $\beta<0$.

(3) *If there exist three positive constants T, β, and β_0 such that, for all $t\ge T$,*

$$\ln Y(t) \ge \beta t - \beta_0\int_0^t Y(s)\,\mathrm{d}s + Z(t) \quad a.s.,$$

then $\liminf_{t\to\infty}\langle Y(t)\rangle \ge \frac{\beta}{\beta_0}$ *a.s.*

Lemma 4 can be found in [22]. We consider the following auxiliary system:

$$\begin{cases}\mathrm{d}Y_1(t) = Y_1(t)[r_1-h_1-a_{11}Y_1(t)]\,\mathrm{d}t+\sigma_1Y_1(t)\,\mathrm{d}B_1(t),\\ \mathrm{d}Y_2(t) = Y_2(t)[-r_2-h_2+a_{21}\int_{-\tau_{21}}^0 Y_1(t+\theta)\,\mathrm{d}\mu_{21}(\theta)-a_{22}Y_2(t)]\,\mathrm{d}t\\ \qquad\quad +\sigma_2Y_2(t)\,\mathrm{d}B_2(t),\\ \mathrm{d}Y_3(t) = Y_3(t)[-r_3-h_3+a_{32}\int_{-\tau_{32}}^0 Y_2(t+\theta)\,\mathrm{d}\mu_{32}(\theta)-a_{33}Y_3(t)]\,\mathrm{d}t\\ \qquad\quad +\sigma_3Y_3(t)\,\mathrm{d}B_3(t)\end{cases} \tag{18}$$

with the initial condition

$$Y_1(\theta) = \xi(\theta), \qquad Y_2(\theta) = \eta(\theta), \qquad Y_3(\theta) = \zeta(\theta), \quad -r \le \theta \le 0. \tag{19}$$

Firstly, by a similar argument as in the proof of Lemma 3, we can obtain that for any condition (19) system (18) has a unique global solution $(Y_1(t), Y_2(t), Y_3(t)) \in R^3_+$ *a.s.* for all $t \ge 0$. We have the following results.

Lemma 5 *Assume that* $(Y_1(t), Y_2(t), Y_3(t))$ *is a global positive solution of system* (18). *Then we have*:

(1) *If* $\Delta_{11} < 0$, *then* $\lim_{t\to\infty} Y_i(t) = 0$ *a.s. for* $i = 1, 2, 3$.
(2) *If* $\Delta_{11} = 0$, *then* $\lim_{t\to\infty}\langle Y_1(t)\rangle = 0$ *and* $\lim_{t\to\infty} Y_i(t) = 0$ *a.s. for* $i = 2, 3$.
(3) *If* $\Delta_{11} > 0$ *and* $\Delta_{22} < 0$, *then* $\lim_{t\to\infty}\langle Y_1(t)\rangle = \frac{\Delta_{11}}{a_{11}}$ *and* $\lim_{t\to\infty} Y_i(t) = 0$ *a.s. for* $i = 2, 3$.
(4) *If* $\Delta_{22} = 0$, *then* $\lim_{t\to\infty}\langle Y_1(t)\rangle = \frac{\Delta_{11}}{a_{11}}$, $\lim_{t\to\infty}\langle Y_2(t)\rangle = 0$, *and* $\lim_{t\to\infty} Y_3(t) = 0$ *a.s.*
(5) *If* $\Delta_{22} > 0$ *and* $\Delta_{33} < 0$, *then* $\lim_{t\to\infty}\langle Y_1(t)\rangle = \frac{\Delta_{11}}{a_{11}}$, $\lim_{t\to\infty}\langle Y_2(t)\rangle = \frac{\Delta_{22}}{a_{11}a_{22}}$, *and* $\lim_{t\to\infty} Y_3(t) = 0$ *a.s.*
(6) *If* $\Delta_{33} = 0$, *then* $\lim_{t\to\infty}\langle Y_1(t)\rangle = \frac{\Delta_{11}}{a_{11}}$, $\lim_{t\to\infty}\langle Y_2(t)\rangle = \frac{\Delta_{22}}{a_{11}a_{22}}$, *and* $\lim_{t\to\infty}\langle Y_3(t)\rangle = 0$ *a.s.*
(7) *If* $\Delta_{33} > 0$, *then*

$$\lim_{t\to\infty}\langle Y_1(t)\rangle = \frac{\Delta_{11}}{a_{11}}, \qquad \lim_{t\to\infty}\langle Y_2(t)\rangle = \frac{\Delta_{22}}{a_{11}a_{22}}, \qquad \lim_{t\to\infty}\langle Y_3(t)\rangle = \frac{\Delta_{33}}{a_{11}a_{22}a_{33}} \quad a.s.$$

(8) $\limsup_{t\to\infty} \frac{\ln Y_i(t)}{t} \le 0$ *a.s. for* $i = 1, 2, 3$.

Proof Applying Itô's formula to system (18), we have

$$\ln Y_1(t) = b_1 t - a_{11}\int_0^t Y_1(s)\,\mathrm{d}s + \sigma_1 B_1(t) + \ln Y_1(0), \tag{20}$$

$$\begin{aligned}\ln Y_2(t) &= -b_2 t + a_{21}\int_0^t\int_{-\tau_{21}}^0 Y_1(s+\theta)\,\mathrm{d}\mu_{21}(\theta)\,\mathrm{d}s \\ &\quad - a_{22}\int_0^t Y_2(s)\,\mathrm{d}s + \sigma_2 B_2(t) + \ln Y_2(0) \\ &= -b_2 t + a_{21}\int_0^t Y_1(s)\,\mathrm{d}s - a_{22}\int_0^t Y_2(s)\,\mathrm{d}s + \psi_1(t),\end{aligned} \tag{21}$$

and

$$\begin{aligned}\ln Y_3(t) &= -b_3 t + a_{32}\int_0^t\int_{-\tau_{32}}^0 Y_2(s+\theta)\,\mathrm{d}\mu_{32}(\theta)\,\mathrm{d}s \\ &\quad - a_{33}\int_0^t Y_3(s)\,\mathrm{d}s + \sigma_3 B_3(t) + \ln Y_3(0) \\ &= -b_3 t + a_{32}\int_0^t Y_2(s)\,\mathrm{d}s - a_{33}\int_0^t Y_3(s)\,\mathrm{d}s + \psi_2(t),\end{aligned} \tag{22}$$

where

$$\psi_1(t) = \sigma_2 B_2(t) + \ln Y_2(0) + a_{21}\int_{-\tau_{21}}^{0}\int_{\theta}^{0} Y_1(s)\,\mathrm{d}s\,\mathrm{d}\mu_{21}(\theta) - a_{21}\int_{-\tau_{21}}^{0}\int_{t+\theta}^{t} Y_1(s)\,\mathrm{d}s\,\mathrm{d}\mu_{21}(\theta),$$

$$\psi_2(t) = \sigma_3 B_3(t) + \ln Y_3(0) + a_{32}\int_{-\tau_{32}}^{0}\int_{\theta}^{0} Y_2(s)\,\mathrm{d}s\,\mathrm{d}\mu_{32}(\theta) - a_{32}\int_{-\tau_{32}}^{0}\int_{t+\theta}^{t} Y_2(s)\,\mathrm{d}s\,\mathrm{d}\mu_{32}(\theta).$$

Assume $\Delta_{11} \le 0$. From Lemma 4 and (20) we have $\lim_{t\to\infty} Y_1(t) = 0$ *a.s.* or $\lim_{t\to\infty}\langle Y_1(t)\rangle = 0$ *a.s.* Thus, $\lim_{t\to\infty}\frac{1}{t}\psi_1(t) = 0$ *a.s.* From (21), we have $\lim_{t\to\infty} Y_2(t) = 0$, then $\lim_{t\to\infty}\frac{1}{t}\psi_2(t) = 0$ *a.s.* From (22), we further have $\lim_{t\to\infty} Y_3(t) = 0$ *a.s.*

Assume $\Delta_{11} > 0$ and $\Delta_{22} < 0$. From Lemma 4 and (20) we obtain $\lim_{t\to\infty}\langle Y_1(t)\rangle = \frac{\Delta_{11}}{a_{11}}$ *a.s.* Thus, $\int_0^t Y_1(s)\,\mathrm{d}s = \frac{\Delta_{11}}{a_{11}}t + \alpha_1(t)$ for any $t \ge 0$, where $\lim_{t\to\infty}\frac{\alpha_1(t)}{t} = 0$ *a.s.* From (21), we obtain

$$\ln Y_2(t) = \frac{\Delta_{22}}{a_{11}}t - a_{22}\int_0^t Y_2(s)\,\mathrm{d}s + \psi_1(t) + a_{21}\alpha_1(t). \tag{23}$$

Since $\lim_{t\to\infty}\frac{1}{t}\psi_1(t) = 0$ *a.s.*, from Lemma 4 we obtain $\lim_{t\to\infty} Y_2(t) = 0$ *a.s.* Further, we also have $\lim_{t\to\infty} Y_3(t) = 0$ *a.s.*

Assume $\Delta_{22} = 0$. Then we have $\Delta_{11} > 0$. By a similar argument we obtain $\lim_{t\to\infty}\langle Y_1(t)\rangle = \frac{\Delta_{11}}{a_{11}}$ *a.s.*, $\lim_{t\to\infty}\langle Y_2(t)\rangle = 0$ *a.s.*, and $\lim_{t\to\infty} Y_3(t) = 0$ *a.s.*

Assume $\Delta_{22} > 0$ and $\Delta_{33} < 0$. Then we have $\Delta_{11} > 0$. From Lemma 4, (20), and (23) we directly obtain $\lim_{t\to\infty}\langle Y_1(t)\rangle = \frac{\Delta_{11}}{a_{11}}$ *a.s.* and $\lim_{t\to\infty}\langle Y_2(t)\rangle = \frac{\Delta_{22}}{a_{11}a_{22}}$ *a.s.* Hence, $\int_0^t Y_2(s)\,\mathrm{d}s = \frac{\Delta_{22}}{a_{11}a_{22}}t + \alpha_2(t)$ for any $t \ge 0$, where $\lim_{t\to\infty}\frac{\alpha_2(t)}{t} = 0$ *a.s.* From (22), we obtain

$$\ln Y_3(t) = \frac{\Delta_{33}}{a_{11}a_{22}}t - a_{33}\int_0^t Y_3(s)\,\mathrm{d}s + \psi_2(t) + a_{32}\alpha_2(t). \tag{24}$$

Since $\lim_{t\to\infty}\frac{1}{t}\psi_2(t) = 0$ *a.s.*, from Lemma 4 we obtain $\lim_{t\to\infty} Y_3(t) = 0$ *a.s.*

Assume $\Delta_{33} = 0$ or $\Delta_{33} > 0$. Then we have $\Delta_{11} > 0$ and $\Delta_{22} > 0$. Hence, we obtain $\lim_{t\to\infty}\langle Y_1(t)\rangle = \frac{\Delta_{11}}{a_{11}}$ and $\lim_{t\to\infty}\langle Y_2(t)\rangle = \frac{\Delta_{22}}{a_{11}a_{22}}$ *a.s.* Then, from (24) and Lemma 4 we further obtain $\lim_{t\to\infty}\langle Y_3(t)\rangle = 0$ *a.s.* or $\lim_{t\to\infty}\langle Y_3(t)\rangle = \frac{\Delta_{33}}{a_{11}a_{22}a_{33}}$ *a.s.*

For any $i \in \{1,2,3\}$, from the above discussions we obtain that there is one of the following three cases: (a) $\lim_{t\to\infty} Y_i(t) = 0$ *a.s.*, (b) $\lim_{t\to\infty}\langle Y_i(t)\rangle = 0$ *a.s.*, (c) $\lim_{t\to\infty}\langle Y_i(t)\rangle = \alpha_i$ *a.s.*, where $\alpha_1 = \frac{\Delta_{11}}{a_{11}}$, $\alpha_2 = \frac{\Delta_{22}}{a_{11}a_{22}}$, and $\alpha_3 = \frac{\Delta_{33}}{a_{11}a_{22}a_{33}}$. For cases (a) and (b), we directly have $\limsup_{t\to\infty}\frac{\ln Y_i(t)}{t} \le 0$ *a.s.* For case (c), from (20) or (23), or (24) we can obtain $\limsup_{t\to\infty}\frac{\ln Y_i(t)}{t} = 0$ *a.s.* Therefore, conclusion (8) holds. This completes the proof. □

Lemma 6 *Assume that* $(x_1(t), x_2(t), x_3(t))$ *and* $(Y_1(t), Y_2(t), Y_3(t))$ *are the solutions of model* (1) *and system* (18), *respectively. If the initial values satisfy* $x_i(\theta) \le Y_i(\theta)$ *for all* $-r \le \theta \le 0$ *and* $i = 1,2,3$, *then*

(1) $x_i(t) \le Y_i(t)$ *for all* $t \ge 0$, $i = 1,2,3$,

(2) $\limsup_{t\to\infty}\frac{\ln x_i(t)}{t} \le 0$ *a.s.*, $i = 1,2,3$,

(3) *for any constant* $\tau > 0$, $\lim_{t\to\infty}\frac{1}{t}\int_{t-\tau}^{t} x_i(s)\,\mathrm{d}s = 0$ *a.s.*, $i = 1,2,3$.

Proof From model (1) we obtain

$$
\begin{aligned}
\mathrm{d}x_1(t) &\le x_1(t)\bigl[r_1 - h_1 - a_{11}x_1(t)\bigr]\,\mathrm{d}t + \sigma_1 x_1(t)\,\mathrm{d}B_1(t),\\
\mathrm{d}x_2(t) &\le x_2(t)\left[-r_2 - h_2 + a_{21}\int_{-\tau_{21}}^{0} x_1(t+\theta)\,\mathrm{d}\mu_{21}(\theta) - a_{22}x_2(t)\right]\mathrm{d}t + \sigma_2 x_2(t)\,\mathrm{d}B_2(t),\\
\mathrm{d}x_3(t) &= x_3(t)\left[-r_3 - h_3 + a_{32}\int_{-\tau_{32}}^{0} x_2(t+\theta)\,\mathrm{d}\mu_{32}(\theta) - a_{33}x_3(t)\right]\mathrm{d}t + \sigma_3 x_3(t)\,\mathrm{d}B_3(t).
\end{aligned}
$$

Using the comparison theorem and Theorem 2.1 given in Bao and Yuan [23], for any $t \ge 0$, we obtain $x_i(t) \le Y_i(t)$ $(i = 1,2,3)$. Then from Lemma 5 we obtain that $\limsup_{t\to\infty} \frac{\ln x_i(t)}{t} \le 0$ *a.s.* $(i = 1,2,3)$, and $\lim_{t\to\infty} \frac{1}{t}\int_{t-\tau}^{t} x_i(s)\,\mathrm{d}s = 0$ *a.s.* $(i = 1,2,3)$ for any constant $\tau > 0$. This completes the proof. □

3 Global dynamics

Firstly, on the extinction and persistence and global stability in the mean with probability one, we can establish the following integrated results.

Theorem 1 *Assume that* $(x_1(t), x_2(t), x_3(t))$ *is a global positive solution of model* (1). *Then we have*

(1) *If* $\Delta_{11} < 0$, *then* $\lim_{t\to\infty} x_i(t) = 0$ *a.s. for* $i = 1,2,3$.
(2) *If* $\Delta_{11} = 0$, *then* $\lim_{t\to\infty}\langle x_1(t)\rangle = 0$ *and* $\lim_{t\to\infty} x_i(t) = 0$ *a.s. for* $i = 2,3$.
(3) *If* $\Delta_{11} > 0$ *and* $\Delta_{22} < 0$, *then* $\lim_{t\to\infty}\langle x_1(t)\rangle = \frac{\Delta_{11}}{H_1}$ *and* $\lim_{t\to\infty} x_i(t) = 0$ *a.s. for* $i = 2,3$.
(4) *If* $\Delta_{22} = 0$, *then* $\lim_{t\to\infty}\langle x_1(t)\rangle = \frac{\Delta_{11}}{H_1}$, $\lim_{t\to\infty}\langle x_2(t)\rangle = 0$, *and* $\lim_{t\to\infty} x_3(t) = 0$ *a.s.*
(5) *If* $\Delta_{22} > 0$ *and* $\Delta_{33} < 0$, *then* $\lim_{t\to\infty}\langle x_1(t)\rangle = \frac{\Delta_{21}}{H_2}$, $\lim_{t\to\infty}\langle x_2(t)\rangle = \frac{\Delta_{22}}{H_2}$, *and* $\lim_{t\to\infty} x_3(t) = 0$ *a.s.*
(6) *If* $\Delta_{33} = 0$ *and* $a_{33}a_{22}(a_{11}a_{22} + a_{12}a_{21}) - a_{12}a_{21}a_{23}a_{32} > 0$, *then* $\lim_{t\to\infty}\langle x_1(t)\rangle = \frac{\Delta_{21}}{H_2}$, $\lim_{t\to\infty}\langle x_2(t)\rangle = \frac{\Delta_{22}}{H_2}$, *and* $\lim_{t\to\infty}\langle x_3(t)\rangle = 0$ *a.s.*
(7) *If* $\Delta_{33} > 0$ *and* $a_{33}a_{22}(a_{11}a_{22} + a_{12}a_{21}) - a_{12}a_{21}a_{23}a_{32} > 0$, *then*

$$
\lim_{t\to\infty}\bigl\langle x_1(t)\bigr\rangle = \frac{\Delta_{31}}{H_3}, \qquad \lim_{t\to\infty}\bigl\langle x_2(t)\bigr\rangle = \frac{\Delta_{32}}{H_3}, \qquad \lim_{t\to\infty}\bigl\langle x_3(t)\bigr\rangle = \frac{\Delta_{33}}{H_3} \quad \textit{a.s.}
$$

Proof Using Itô's formula to model (1), we obtain

$$
\begin{aligned}
\ln x_1(t) &= b_1 t - a_{11}\int_0^t x_1(s)\,\mathrm{d}s - a_{12}\int_0^t\int_{-\tau_{12}}^{0} x_2(s+\theta)\,\mathrm{d}\mu_{12}(\theta)\,\mathrm{d}s + \sigma_1 B_1(t) + \ln x_1(0)\\
&= b_1 t - a_{11}\int_0^t x_1(s)\,\mathrm{d}s - a_{12}\int_0^t x_2(s)\,\mathrm{d}s + \phi_1(t), \qquad (25)\\
\ln x_2(t) &= -b_2 t + a_{21}\int_0^t\int_{-\tau_{21}}^{0} x_1(s+\theta)\,\mathrm{d}\mu_{21}(\theta)\,\mathrm{d}s - a_{22}\int_0^t x_2(s)\,\mathrm{d}s\\
&\quad - a_{23}\int_0^t\int_{-\tau_{23}}^{0} x_3(s+\theta)\,\mathrm{d}\mu_{23}(\theta)\,\mathrm{d}s + \sigma_2 B_2(t) + \ln x_2(0)\\
&= -b_2 t + a_{21}\int_0^t x_1(s)\,\mathrm{d}s - a_{22}\int_0^t x_2(s)\,\mathrm{d}s - a_{23}\int_0^t x_3(s)\,\mathrm{d}s + \phi_2(t) \qquad (26)
\end{aligned}
$$

and

$$\begin{aligned}\ln x_3(t) &= -b_3 t + a_{32}\int_0^t\int_{-\tau_{32}}^0 x_2(s+\theta)\,\mathrm{d}\mu_{32}(\theta)\,\mathrm{d}s - a_{33}\int_0^t x_3(s)\,\mathrm{d}s + \sigma_3 B_3(t) + \ln x_3(0)\\ &= -b_3 t + a_{32}\int_0^t x_2(s)\,\mathrm{d}s - a_{33}\int_0^t x_3(s)\,\mathrm{d}s + \phi_3(t),\end{aligned} \tag{27}$$

where

$$\begin{aligned}\phi_1(t) &= \sigma_1 B_1(t) + \ln x_1(0) + a_{12}\int_{-\tau_{12}}^0\int_{t+\theta}^t x_2(s)\,\mathrm{d}s\,\mathrm{d}\mu_{12}(\theta)\\ &\quad - a_{12}\int_{-\tau_{12}}^0\int_\theta^0 x_2(s)\,\mathrm{d}s\,\mathrm{d}\mu_{12}(\theta),\\ \phi_2(t) &= \sigma_2 B_2(t) + \ln x_2(0) + a_{21}\int_{-\tau_{21}}^0\int_\theta^0 x_1(s)\,\mathrm{d}s\,\mathrm{d}\mu_{21}(\theta)\\ &\quad - a_{21}\int_{-\tau_{21}}^0\int_{t+\theta}^t x_1(s)\,\mathrm{d}s\,\mathrm{d}\mu_{21}(\theta)\\ &\quad + a_{23}\int_{-\tau_{23}}^0\int_{t+\theta}^t x_3(s)\,\mathrm{d}s\,\mathrm{d}\mu_{23}(\theta) - a_{23}\int_{-\tau_{23}}^0\int_\theta^0 x_3(s)\,\mathrm{d}s\,\mathrm{d}\mu_{23}(\theta),\\ \phi_3(t) &= \sigma_3 B_3(t) + \ln x_3(0) + a_{32}\int_{-\tau_{32}}^0\int_\theta^0 x_2(s)\,\mathrm{d}s\,\mathrm{d}\mu_{32}(\theta)\\ &\quad - a_{32}\int_{-\tau_{32}}^0\int_{t+\theta}^t x_2(s)\,\mathrm{d}s\,\mathrm{d}\mu_{32}(\theta).\end{aligned}$$

Further, we also obtain

$$\ln x_1(t) \le b_1 t - a_{11}\int_0^t x_1(s)\,\mathrm{d}s + \sigma_1 B_1(t) + \ln x_1(0) \tag{28}$$

and

$$\begin{aligned}\ln x_2(t) &\le -b_2 t + a_{21}\int_0^t\int_{-\tau_{21}}^0 x_1(s+\theta)\,\mathrm{d}\mu_{21}(\theta)\,\mathrm{d}s - a_{22}\int_0^t x_2(s)\,\mathrm{d}s + \sigma_2 B_2(t) + \ln x_2(0)\\ &= -b_2 t + a_{21}\int_0^t x_1(s)\,\mathrm{d}s - a_{22}\int_0^t x_2(s)\,\mathrm{d}s + \sigma_2 B_2(t) + \ln x_2(0)\\ &\quad + a_{21}\int_{-\tau_{21}}^0\int_\theta^0 x_1(s)\,\mathrm{d}s\,\mathrm{d}\mu_{21}(\theta) - a_{21}\int_{-\tau_{21}}^0\int_{t+\theta}^t x_1(s)\,\mathrm{d}s\,\mathrm{d}\mu_{21}(\theta).\end{aligned} \tag{29}$$

Assume $\Delta_{11} \le 0$. From (28), Lemmas 5 and 6, we can immediately obtain that conclusions (1) and (2) hold.

Assume $\Delta_{11} > 0$ and $\Delta_{22} \le 0$. From (28), Lemmas 5 and 6, we immediately obtain that $\limsup_{t\to\infty}\langle x_1(t)\rangle \le \frac{\Delta_{11}}{H_1}$, $\lim_{t\to\infty} x_2(t) = 0$ or $\lim_{t\to\infty}\langle x_2(t)\rangle = 0$, and $\lim_{t\to\infty} x_3(t) = 0$ *a.s.* For any $\varepsilon > 0$ with $b_1 - a_{12}\varepsilon > 0$, we have $\int_0^t x_2(s)\,\mathrm{d}s < \varepsilon t$ *a.s.* for enough large t, and from (25)

$$\ln x_1(t) \ge (b_1 - a_{12}\varepsilon)t - a_{11}\int_0^t x_1(s)\,\mathrm{d}s + \phi_1(t).$$

Since

$$\int_{-\tau_{12}}^{0}\int_{t+\theta}^{t} x_2(s)\,\mathrm{d}s\,\mathrm{d}\mu_{12}(\theta) \le \int_{-\tau_{12}}^{0}\mathrm{d}\mu_{12}(\theta)\int_{t-\tau_{12}}^{t} x_2(s)\,\mathrm{d}s,$$
$$\int_{-\tau_{12}}^{0}\int_{\theta}^{0} x_2(s)\,\mathrm{d}s\,\mathrm{d}\mu_{12}(\theta) \le \int_{-\tau_{12}}^{0}\mathrm{d}\mu_{12}(\theta)\int_{-\tau_{12}}^{0} x_2(s)\,\mathrm{d}s,$$

by Lemma 6 we obtain $\lim_{t\to\infty}\frac{1}{t}\int_{-\tau_{12}}^{0}\int_{t+\theta}^{t} x_2(s)\,\mathrm{d}s\,\mathrm{d}\mu_{12}(\theta) = 0$ and $\lim_{t\to\infty}\frac{1}{t}\times\int_{-\tau_{12}}^{0}\int_{\theta}^{0} x_2(s)\,\mathrm{d}s\,\mathrm{d}\mu_{12}(\theta) = 0$. Hence, $\lim_{t\to\infty}\frac{\phi_1(t)}{t} = 0$ *a.s.* Thus, from Lemma 4 and the arbitrariness of ε we have $\liminf_{t\to\infty}\langle x_1(t)\rangle \ge \frac{\Delta_{11}}{H_1}$. This shows that $\lim_{t\to\infty}\langle x_1(t)\rangle = \frac{\Delta_{11}}{H_1}$.

Assume $\Delta_{33} > 0$. From (25)–(27), we obtain

$$a_{32}\big[a_{21}\ln x_1(t) + a_{11}\ln x_2(t)\big] + H_2\ln x_3(t) = \Delta_{33}t - H_3\int_0^t x_3(s)\,\mathrm{d}s + \phi_4(t), \tag{30}$$

where $\phi_4(t) = a_{21}a_{32}\phi_1(t) + a_{11}a_{32}\phi_2(t) + H_2\phi_3(t)$. By a similar argument as in the above, for $\phi_1(t)$, we have $\lim_{t\to\infty}\frac{\phi_4(t)}{t} = 0$ *a.s.* For any $\varepsilon > 0$ with $\Delta_{33} - 2\varepsilon > 0$, by Lemma 6, $\ln x_1(t) < \frac{\varepsilon}{a_{32}a_{21}+1}t$ and $\ln x_2(t) < \frac{\varepsilon}{a_{32}a_{11}+1}t$ for t enough large. Then from (30) we further have

$$H_2\ln x_3(t) > (\Delta_{33} - 2\varepsilon)t - H_3\int_0^t x_3(s)\,\mathrm{d}s + \phi_4(t).$$

Hence, by Lemma 4 and the arbitrariness of ε, we further have

$$\liminf_{t\to\infty}\langle x_3(t)\rangle \ge \frac{\Delta_{33}}{H_3}. \tag{31}$$

From (25) and (26), we obtain

$$a_{22}\ln x_1(t) - a_{12}\ln x_2(t) = \Delta_{21}t - H_2\int_0^t x_1(s)\,\mathrm{d}s + a_{12}a_{23}\int_0^t x_3(s)\,\mathrm{d}s + \phi_5(t), \tag{32}$$

where $\phi_5(t) = a_{22}\phi_1(t) - a_{12}\phi_2(t)$. Similarly, as in the above for $\phi_1(t)$, we can obtain $\lim_{t\to\infty}\frac{\phi_5(t)}{t} = 0$ *a.s.* For any $\varepsilon > 0$, from Lemma 6 and the properties of superior limit, we have $\int_0^t x_3(s)\,\mathrm{d}s < (\limsup_{t\to\infty}\langle x_3(t)\rangle + \varepsilon)t$ and $\ln x_2(t) < \frac{\varepsilon}{a_{12}+1}t$ for enough large t. Then from (32) we further have

$$a_{22}\ln x_1(t) \le \Delta_{21}t + a_{12}a_{23}\Big(\limsup_{t\to\infty}\langle x_3(t)\rangle + \varepsilon\Big)t + \varepsilon t - H_2\int_0^t x_1(s)\,\mathrm{d}s + \phi_5(t).$$

From Lemma 4 and the arbitrariness of ε it follows that

$$\limsup_{t\to\infty}\langle x_1(t)\rangle \le \frac{\Delta_{21} + a_{12}a_{23}\limsup_{t\to\infty}\langle x_3(t)\rangle}{H_2} \quad a.s. \tag{33}$$

Combining (31), for any $\varepsilon > 0$ enough small, when t is enough large, we have

$$\int_0^t x_3(s)\,\mathrm{d}s > \Big(\frac{\Delta_{33}}{H_3} - \varepsilon\Big)t, \qquad \int_0^t x_1(s)\,\mathrm{d}s < \Big(\frac{\Delta_{21} + a_{12}a_{23}\limsup_{t\to\infty}\langle x_3(t)\rangle}{H_2} + \varepsilon\Big)t.$$

Hence, from (29) we further have

$$\begin{aligned}\ln x_2(t) \le{}& -b_2 t + \left(\frac{a_{21}(\Delta_{21} + a_{12}a_{23}\limsup_{t\to\infty}\langle x_3(t)\rangle)}{H_2} + \varepsilon\right)t \\ &- a_{23}\left(\frac{\Delta_{33}}{H_3} - \varepsilon\right)t - a_{22}\int_0^t x_2(s)\,\mathrm{d}s + \phi_2(t).\end{aligned} \tag{34}$$

We have $\lim_{t\to\infty}\frac{\phi_2(t)}{t} = 0$ *a.s.* by Lemma 6. From (31), we obtain

$$\begin{aligned}&-b_2 + \frac{a_{21}(\Delta_{21} + a_{12}a_{23}\limsup_{t\to\infty}\langle x_3(t)\rangle)}{H_2} - a_{23}\frac{\Delta_{33}}{H_3} \\ &\quad\ge -b_2 + a_{21}\frac{\Delta_{21}}{H_2} - a_{23}\frac{\Delta_{33}}{H_3} + \frac{a_{21}a_{12}a_{23}\Delta_{33}}{H_2H_3} = \frac{a_{22}\Delta_{32}}{H_3} > 0.\end{aligned}$$

Hence, from (34), Lemma 4, and the arbitrariness of ε, we have

$$\begin{aligned}a_{22}\limsup_{t\to\infty}\langle x_2(t)\rangle &\le \left(-b_2 + \frac{a_{21}(\Delta_{21} + a_{12}a_{23}\limsup_{t\to\infty}\langle x_3(t)\rangle)}{H_2} - a_{23}\frac{\Delta_{33}}{H_3}\right) \\ &\triangleq M \quad a.s.\end{aligned} \tag{35}$$

For any $\varepsilon > 0$, when t is enough large, we have $\int_0^t x_2(s)\,\mathrm{d}s < (\frac{M}{a_{22}} + \varepsilon)t$. Then from (27) it follows that

$$\begin{aligned}\ln x_3(t) &\le -b_3 t + a_{32}\left(\frac{M}{a_{22}} + \varepsilon\right)t - a_{33}\int_0^t x_3(s)\,\mathrm{d}s + \phi_3(t) \\ &\le -b_3 t + a_{32}\varepsilon t + \frac{a_{32}}{a_{22}}\left(-b_2 + \frac{a_{21}(\Delta_{21} + a_{12}a_{23}\limsup_{t\to\infty}\langle x_3(t)\rangle)}{H_2} - a_{23}\frac{\Delta_{33}}{H_3}\right)t \\ &\quad - a_{33}\int_0^t x_3(s)\,\mathrm{d}s + \phi_3(t).\end{aligned} \tag{36}$$

We have $\lim_{t\to\infty}\frac{\phi_3(t)}{t} = 0$ *a.s.* by Lemma 6. From (31), we also have

$$\begin{aligned}&-b_3 + \frac{a_{32}}{a_{22}}\left(-b_2 + \frac{a_{21}(\Delta_{21} + a_{12}a_{23}\limsup_{t\to\infty}\langle x_3(t)\rangle)}{H_2} - a_{23}\frac{\Delta_{33}}{H_3}\right) \\ &\quad\ge -b_3 + \frac{a_{32}}{a_{22}}\left(-b_2 + a_{21}\frac{\Delta_{21}}{H_2} - a_{23}\frac{\Delta_{33}}{H_3} + \frac{a_{21}a_{12}a_{23}\Delta_{33}}{H_2H_3}\right) = a_{33}\frac{\Delta_{33}}{H_3} > 0.\end{aligned}$$

Hence, from (36), Lemma 4, and the arbitrariness of ε, one can derive that

$$\begin{aligned}&a_{33}\limsup_{t\to\infty}\langle x_3(t)\rangle \\ &\quad\le -b_3 + \frac{a_{32}}{a_{22}}\left(-b_2 + a_{21}\frac{\Delta_{21}}{H_2} + \frac{a_{12}a_{21}a_{23}\limsup_{t\to\infty}\langle x_3(t)\rangle}{H_2} - a_{23}\frac{\Delta_{33}}{H_3}\right).\end{aligned}$$

That is equivalent to the following equation:

$$\begin{aligned}&\left[a_{33}a_{22}(a_{11}a_{22} + a_{12}a_{21}) - a_{12}a_{21}a_{23}a_{32}\right]\limsup_{t\to\infty}\langle x_3(t)\rangle \\ &\quad\le \left[a_{33}a_{22}(a_{11}a_{22} + a_{12}a_{21}) - a_{12}a_{21}a_{23}a_{32}\right] \times \frac{\Delta_{33}}{H_3}.\end{aligned}$$

Hence, we obtain $\limsup_{t\to\infty}\langle x_3(t)\rangle \le \frac{\Delta_{33}}{H_3}$ *a.s.* Combining (31), we finally obtain $\lim_{t\to\infty}\langle x_3(t)\rangle = \frac{\Delta_{33}}{H_3}$ *a.s.*

From (33) and (35) we can obtain

$$\limsup_{t\to\infty}\langle x_1(t)\rangle \le \frac{b_1(a_{22}a_{33}+a_{32}a_{23})+b_2a_{33}a_{12}-b_3a_{12}a_{23}}{a_{11}a_{22}a_{33}+a_{12}a_{21}a_{33}+a_{11}a_{32}a_{23}} = \frac{\Delta_{31}}{H_3} \quad a.s. \tag{37}$$

and

$$\limsup_{t\to\infty}\langle x_2(t)\rangle \le \frac{b_1a_{21}a_{33}-b_2a_{33}a_{11}+b_3a_{11}a_{23}}{a_{11}a_{22}a_{33}+a_{12}a_{21}a_{33}+a_{11}a_{32}a_{23}} = \frac{\Delta_{32}}{H_3} \quad a.s. \tag{38}$$

For any $\varepsilon > 0$, from Lemma 6 there is $T > 0$ such that, for any $t > T$,

$$\int_0^t x_3(s)\,ds < \left(\frac{\Delta_{33}}{H_3}+\varepsilon\right)t, \qquad \ln x_1(t) < \frac{\varepsilon}{a_{21}+1}t. \tag{39}$$

From (25) and (26), we obtain

$$a_{21}\ln x_1(t) + a_{11}\ln x_2(t) = \Delta_{22}t - H_2\int_0^t x_2(s)\,ds - a_{11}a_{23}\int_0^t x_3(s)\,ds + \phi_6(t), \tag{40}$$

where $\phi_6(t) = a_{21}\phi_1(t) + a_{11}\phi_2(t)$. We have $\lim_{t\to\infty}\frac{\phi_6(t)}{t} = 0$ *a.s.* by Lemma 6. Substituting (39) into (40), we have, when $t > T$,

$$a_{11}\ln x_2(t) \ge \Delta_{22}t - a_{11}a_{23}\left(\frac{\Delta_{33}}{H_3}+\varepsilon\right)t - \varepsilon t - H_2\int_0^t x_2(s)\,ds + \phi_6(t).$$

From Lemma 4 and the arbitrariness of ε, we have $\liminf_{t\to\infty}\langle x_2(t)\rangle \ge \frac{\Delta_{32}}{H_3}$ *a.s.* Combining (38), we finally obtain $\lim_{t\to\infty}\langle x_2(t)\rangle = \frac{\Delta_{32}}{H_3}$ *a.s.*

For any $\varepsilon > 0$, from (38) when t is enough large we have $\int_0^t x_2(s)\,ds < (\frac{\Delta_{32}}{H_3}+\varepsilon)t$. Then from (25) it follows that

$$\ln x_1(t) \ge b_1t - a_{11}\int_0^t x_1(s)\,ds - a_{12}\left(\frac{\Delta_{32}}{H_3}+\varepsilon\right)t + \phi_1(t).$$

From Lemma 4 and the arbitrariness of ε, we have $\liminf_{t\to\infty}\langle x_1(t)\rangle \ge \frac{\Delta_{31}}{H_3}$. Combining (37), we finally obtain $\lim_{t\to\infty}\langle x_1(t)\rangle = \frac{\Delta_{31}}{H_3}$ *a.s.*

Assume $\Delta_{33} = 0$. Then we can have $\Delta_{22} > 0$ and $\Delta_{11} > 0$. By a similar argument as in the above for case $\Delta_{33} > 0$, we can obtain

$$a_{33}\limsup_{t\to\infty}\langle x_3(t)\rangle \le -b_3 + \frac{a_{32}}{a_{22}}\left(-b_2 + a_{21}\frac{\Delta_{21}}{H_2} + \frac{a_{12}a_{21}a_{23}\limsup_{t\to\infty}\langle x_3(t)\rangle}{H_2}\right).$$

That is equivalent to the following equation:

$$\left[a_{33}a_{22}(a_{11}a_{22}+a_{12}a_{21}) - a_{12}a_{21}a_{23}a_{32}\right]\limsup_{t\to\infty}\langle x_3(t)\rangle \le 0.$$

Therefore, we finally have $\lim_{t\to+\infty}\langle x_3(t)\rangle = 0$. Thus, for any $\varepsilon > 0$, there is $T > 0$ such that $\int_0^t x_3(s)\,ds < \varepsilon t$ for all $t > T$. Hence, from (39) and (40) we further obtain as $t > T$

$$a_{11}\ln x_2(t) \geq \Delta_{22}t - a_{11}a_{23}\varepsilon t - \varepsilon t - H_2\int_0^t x_2(s)\,ds + \phi_6(t).$$

From Lemma 4 and the arbitrariness of ε, we have

$$\liminf_{t\to\infty}\langle x_2(t)\rangle \geq \frac{\Delta_{22}}{H_2} \quad a.s. \tag{41}$$

Using the same method as in the proof of $\liminf_{t\to\infty}\langle x_1(t)\rangle \geq \frac{\Delta_{31}}{H_3}$ in the above, we can successively prove $\limsup_{t\to\infty}\langle x_1(t)\rangle \leq \frac{\Delta_{21}}{H_2}$ *a.s.*, $\limsup_{t\to\infty}\langle x_2(t)\rangle \leq \frac{\Delta_{22}}{H_2}$ *a.s.*, and $\liminf_{t\to\infty}\langle x_1(t)\rangle \geq \frac{\Delta_{21}}{H_2}$ *a.s.* Combining (41), we finally obtain $\lim_{t\to\infty}\langle x_2(t)\rangle = \frac{\Delta_{22}}{H_2}$ *a.s.* and $\lim_{t\to\infty}\langle x_1(t)\rangle = \frac{\Delta_{21}}{H_2}$ *a.s.*

Assume $\Delta_{22} > 0$ and $\Delta_{33} < 0$. From (30) we directly obtain

$$a_{32}\big[a_{21}\ln x_1(t) + a_{11}\ln x_2(t)\big] + H_2\ln x_3(t) \leq \Delta_{33}t + \phi_4(t).$$

Hence,

$$\limsup_{t\to\infty}\frac{1}{t}\big(a_{21}a_{32}\ln x_1(t) + a_{11}a_{32}\ln x_2(t) + H_2\ln x_3(t)\big) \leq \Delta_{33} < 0.$$

This shows $\lim_{t\to\infty}(x_1(t))^{a_{21}a_{32}}(x_2(t))^{a_{11}a_{32}}(x_3(t))^{H_2} = 0$, which implies that there is $i \in \{1,2,3\}$ such that

$$\lim_{t\to\infty} x_i(t) = 0. \tag{42}$$

For $1 \leq i \leq j \leq 3$, similarly to the above arguments for cases $\Delta_{11} \leq 0$, and $\Delta_{11} > 0$ and $\Delta_{22} \leq 0$, we can easily prove that if $\lim_{t\to\infty} x_i(t) = 0$ *a.s.*, then $\lim_{t\to+\infty} x_j(t) = 0$ *a.s.* Therefore, from (42) we finally obtain $\lim_{t\to\infty} x_3(t) = 0$ *a.s.* Consequently, $\lim_{t\to\infty}\langle x_3(t)\rangle = 0$ *a.s.* By a similar argument as in the above for case $\Delta_{33} = 0$, we also know $\lim_{t\to\infty}\langle x_1(t)\rangle = \frac{\Delta_{21}}{H_2}$ and $\lim_{t\to\infty}\langle x_2(t)\rangle = \frac{\Delta_{22}}{H_2}$. This completes the proof. □

Next, we can establish the following result on the global attractivity in the expectation for any global positive solutions of model (1).

Theorem 2 *Let* $(x_1(t;\phi), x_2(t;\phi), x_3(t;\phi))$ *and* $(y_1(t;\phi^*), y_2(t;\phi^*), y_3(t;\phi^*))$ *be two solutions of model* (1) *with initial values* $\phi, \phi^* \in C([-\gamma, 0], R_+^3))$. *Assume that there are positive constants* w_1, w_2, *and* w_3 *such that*

$$w_1a_{11} - w_2a_{21} > 0, \qquad w_2a_{22} - w_1a_{12} - w_3a_{32} > 0, \qquad w_3a_{33} - w_2a_{23} > 0.$$

Then

$$\lim_{t\to\infty} E\sqrt{\left|x_1(t;\phi) - x_1\left(t;\phi^*\right)\right|^2 + \left|x_2(t;\phi) - x_2\left(t;\phi^*\right)\right|^2 + \left|x_3(t;\phi) - x_3\left(t;\phi^*\right)\right|^2} = 0.$$

Proof We only need to show

$$\lim_{t\to\infty} E\left|x_i(t;\phi) - x_i\left(t;\phi^*\right)\right| = 0, \quad i = 1,2,3. \tag{43}$$

Define functions as follows:

$$V_i(x_i) = \left|\ln x_i(t;\phi) - \ln y_i\left(t;\phi^*\right)\right|, \quad i = 1,2,3.$$

Applying Itô's formula, we obtain

$$\begin{aligned} \mathcal{L}V_1(x_1) \le{}& -a_{11}\left|x_1(t;\phi) - y_1\left(t;\phi^*\right)\right| \\ &+ a_{12}\int_{-\tau_{12}}^{0}\left|x_2(t+\theta;\phi) - y_2\left(t+\theta;\phi^*\right)\right|\,\mathrm{d}\mu_{12}(\theta), \end{aligned} \tag{44}$$

$$\begin{aligned} \mathcal{L}V_2(x_2) \le{}& -a_{22}\left|x_2(t;\phi) - y_2\left(t;\phi^*\right)\right| + a_{21}\int_{-\tau_{21}}^{0}\left|x_1(t+\theta;\phi) - y_1\left(t+\theta;\phi^*\right)\right|\,\mathrm{d}\mu_{21}(\theta) \\ &+ a_{23}\int_{-\tau_{23}}^{0}\left|x_3(t+\theta;\phi) - y_3\left(t+\theta;\phi^*\right)\right|\,\mathrm{d}\mu_{23}(\theta), \end{aligned} \tag{45}$$

and

$$\begin{aligned} \mathcal{L}V_3(x_3) \le{}& -a_{33}\left|x_3(t;\phi) - y_3\left(t;\phi^*\right)\right| \\ &+ a_{32}\int_{-\tau_{32}}^{0}\left|x_2(t+\theta;\phi) - y_2\left(t+\theta;\phi^*\right)\right|\,\mathrm{d}\mu_{32}(\theta). \end{aligned} \tag{46}$$

Define function as follows:

$$V(t) = w_1V_1(x_1) + w_2V_2(x_2) + w_3V_3(x_3) + V_4(t), \tag{47}$$

where

$$\begin{aligned} V_4(t) ={}& w_1a_{12}\int_{-\tau_{12}}^{0}\int_{t+\theta}^{t}\left|x_2(s;\phi) - y_2\left(s;\phi^*\right)\right|\,\mathrm{d}s\,\mathrm{d}\mu_{12}(\theta) \\ &+ w_2a_{21}\int_{-\tau_{21}}^{0}\int_{t+\theta}^{t}\left|x_1(s;\phi) - y_1\left(s;\phi^*\right)\right|\,\mathrm{d}s\,\mathrm{d}\mu_{21}(\theta) \\ &+ w_2a_{23}\int_{-\tau_{23}}^{0}\int_{t+\theta}^{t}\left|x_3(s;\phi) - y_3\left(s;\phi^*\right)\right|\,\mathrm{d}s\,\mathrm{d}\mu_{23}(\theta) \\ &+ w_3a_{32}\int_{-\tau_{32}}^{0}\int_{t+\theta}^{t}\left|x_2(s;\phi) - y_2\left(s;\phi^*\right)\right|\,\mathrm{d}s\,\mathrm{d}\mu_{32}(\theta). \end{aligned} \tag{48}$$

From (44)–(48) we obtain

$$\begin{aligned} \mathcal{L}V(t) ={}& w_1\mathcal{L}V_1(x_1) + w_2\mathcal{L}V_2(x_2) + w_3\mathcal{L}V_3(x_3) + \frac{\mathrm{d}V_4(t;\phi,\phi^*)}{\mathrm{d}t} \\ \le{}& -(w_1a_{11} - w_2a_{21})\left|x_1(t;\phi) - y_1\left(t;\phi^*\right)\right| \end{aligned}$$

$$- (w_2a_{22} - w_1a_{12} - w_3a_{32})\left|x_2(t;\phi) - y_2\left(t;\phi^*\right)\right|$$
$$- (w_3a_{33} - w_2a_{23})\left|x_3(t;\phi) - y_3\left(t;\phi^*\right)\right|.$$

Hence, we have

$$\begin{aligned} E\left[V(t)\right] \le{}& E\left[V(0)\right] - (w_1a_{11} - w_2a_{21})\int_0^t E\left[\left|x_1(s;\phi) - y_1\left(s;\phi^*\right)\right|\right]\mathrm{d}s \\ &- (w_2a_{22} - w_1a_{12} - w_3a_{32})\int_0^t E\left[\left|x_2(s;\phi) - y_2\left(s;\phi^*\right)\right|\right]\mathrm{d}s \\ &- (w_3a_{33} - w_2a_{23})\int_0^t E\left[\left|x_3(s;\phi) - y_3\left(s;\phi^*\right)\right|\right]\mathrm{d}s, \end{aligned}$$

which implies

$$\int_0^t E\left[\left|x_i(s;\phi) - y_i\left(s;\phi^*\right)\right|\right]\mathrm{d}s < +\infty, \quad i = 1,2,3. \tag{49}$$

Define functions

$$F_i(t) = E\left[\left|x_i(t;\phi) - y_i\left(t;\phi^*\right)\right|\right], \quad i = 1,2,3.$$

Then, for any $t_1, t_2 \in [0,+\infty)$, we obtain, for each $i = 1,2,3$,

$$\begin{aligned} \left|F_i(t_2) - F_i(t_1)\right| &= \left|E\left[\left|x_i(t_2;\phi) - y_i\left(t_2;\phi^*\right)\right| - \left|x_i(t_1;\phi) - y_i\left(t_1;\phi^*\right)\right|\right]\right| \\ &\le E\left[\left|\left(x_i(t_2;\phi) - y_i\left(t_2;\phi^*\right)\right) - \left(x_i(t_1;\phi) - y_i\left(t_1;\phi^*\right)\right)\right|\right] \\ &\le E\left[\left|x_i(t_2;\phi) - x_i(t_1;\phi)\right|\right] + E\left[\left|y_i\left(t_2;\phi^*\right) - y_i\left(t_1;\phi^*\right)\right|\right]. \end{aligned} \tag{50}$$

From model (1), applying Itô's formula, we have

$$\begin{aligned} & x_1(t_2;\phi) - x_1(t_1;\phi) \\ &\quad = \int_{t_1}^{t_2} x_1(s;\phi)\left[r_1 - h_1 - a_{11}x_1(s;\phi) - a_{12}\int_{-\tau_{12}}^0 x_2(s+\theta;\phi)\,\mathrm{d}\mu_{12}(\theta)\right]\mathrm{d}s \\ &\qquad + \int_{t_1}^{t_2} \sigma_1 x_1(s;\phi)\,\mathrm{d}B_1(s), \\ & x_2(t_2;\phi) - x_2(t_1;\phi) \\ &\quad = \int_{t_1}^{t_2} x_2(s;\phi)\left[-r_2 - h_2 + a_{21}\int_{-\tau_{21}}^0 x_1(s+\theta;\phi)\,\mathrm{d}\mu_{21}(\theta) - a_{22}x_2(s;\phi)\right. \\ &\qquad \left. - a_{23}\int_{-\tau_{23}}^0 x_3(s+\theta;\phi)\,\mathrm{d}\mu_{23}(\theta)\right]\mathrm{d}s + \int_{t_1}^{t_2}\sigma_2 x_2(s;\phi)\,\mathrm{d}B_2(s), \\ & x_3(t_2;\phi) - x_3(t_1;\phi) \\ &\quad = \int_0^t x_3(s;\phi)\left[-r_3 - h_3 + a_{32}\int_{-\tau_{32}}^0 x_2(s+\theta;\phi)\,\mathrm{d}\mu_{32}(\theta) - a_{33}x_3(s)\right]\mathrm{d}s \\ &\qquad + \int_{t_1}^{t_2}\sigma_3 x_3(s;\phi)\,\mathrm{d}B_3(s). \end{aligned} \tag{51}$$

For any $t_2 > t_1$ and $p > 1$, using Hölder's inequality, from the first equation of (51), we have

$$\begin{aligned}
&\big(E\big[\big|x_1(t_2;\phi)-x_1(t_1;\phi)\big|\big]\big)^p \\
&\quad\le E\big[\big|x_1(t_2;\phi)-x_1(t_1;\phi)\big|^p\big] \\
&\quad\le E\biggl[\biggl(\int_{t_1}^{t_2} x_1(s;\phi)\biggl|r_1-h_1-a_{11}x_1(s;\phi)-a_{12}\int_{-\tau_{12}}^0 x_2(s+\theta;\phi)\,\mathrm{d}\mu_{12}(\theta)\biggr|\,\mathrm{d}s \\
&\qquad+\biggl|\int_{t_1}^{t_2}\sigma_1 x_1(s;\phi)\,\mathrm{d}B_1(s)\biggr|\biggr)^p\biggr] \\
&\quad\le 2^pE\biggl[\biggl(\int_{t_1}^{t_2} x_1(s;\phi)\biggl|r_1-h_1-a_{11}x_1(s;\phi)-a_{12}\int_{-\tau_{12}}^0 x_2(s+\theta;\phi)\,\mathrm{d}\mu_{12}(\theta)\biggr|\,\mathrm{d}s\biggr)^p\biggr] \\
&\qquad+2^pE\biggl[\biggl|\int_{t_1}^{t_2}\sigma_1 x_1(s;\phi)\,\mathrm{d}B_1(s)\biggr|^p\biggr].
\end{aligned}\tag{52}$$

Using Hölder's inequality again, we also have

$$\begin{aligned}
&E\biggl[\biggl(\int_{t_1}^{t_2} x_1(s;\phi)\biggl|r_1-h_1-a_{11}x_1(s;\phi)-a_{12}\int_{-\tau_{12}}^0 x_2(s+\theta;\phi)\,\mathrm{d}\mu_{12}(\theta)\biggr|\,\mathrm{d}s\biggr)^p\biggr] \\
&\quad\le E\biggl[\biggl(\int_{t_1}^{t_2}\biggl(|r_1-h_1|x_1(s;\phi)+a_{11}x_1^2(s;\phi) \\
&\qquad+a_{12}\int_{-\tau_{12}}^0 x_1(s;\phi)x_2(s+\theta;\phi)\,\mathrm{d}\mu_{12}(\theta)\biggr)\,\mathrm{d}s\biggr)^p\biggr] \\
&\quad\le (t_2-t_1)^{p-1}E\biggl[\int_{t_1}^{t_2}\biggl(|r_1-h_1|x_1(s;\phi)+a_{11}x_1^2(s;\phi) \\
&\qquad+a_{12}\int_{-\tau_{12}}^0 x_1(s;\phi)x_2(s+\theta;\phi)\,\mathrm{d}\mu_{12}(\theta)\biggr)^p\,\mathrm{d}s\biggr] \\
&\quad\le (t_2-t_1)^{p-1}E\biggl[\int_{t_1}^{t_2}3^p\biggl(|r_1-h_1|^px_1^p(s;\phi)+a_{11}^px_1^{2p}(s;\phi) \\
&\qquad+\biggl(a_{12}\int_{-\tau_{12}}^0 x_1(s;\phi)x_2(s+\theta;\phi)\,\mathrm{d}\mu_{12}(\theta)\biggr)^p\biggr)\,\mathrm{d}s\biggr] \\
&\quad=3^p(t_2-t_1)^{p-1}|r_1-h_1|^p\int_{t_1}^{t_2}E\big[x_1^p(s;\phi)\big]\,\mathrm{d}s+3^pa_{11}^p(t_2-t_1)^{p-1}\int_{t_1}^{t_2}E\big[x_1^{2p}(s;\phi)\big]\,\mathrm{d}s \\
&\qquad+3^p(t_2-t_1)^{p-1}E\biggl[\int_{t_1}^{t_2}\biggl(a_{12}\int_{-\tau_{12}}^0 x_1(s;\phi)x_2(s+\theta;\phi)\,\mathrm{d}\mu_{12}(\theta)\biggr)^p\,\mathrm{d}s\biggr]
\end{aligned}\tag{53}$$

and

$$\begin{aligned}
&E\biggl[\int_{t_1}^{t_2}\biggl(a_{12}\int_{-\tau_{12}}^0 x_1(s;\phi)x_2(s+\theta;\phi)\,\mathrm{d}\mu_{12}(\theta)\biggr)^p\,\mathrm{d}s\biggr] \\
&\quad\le E\biggl[\int_{t_1}^{t_2}\biggl(\frac{1}{2}a_{12}x_1^2(s;\phi)+\frac{1}{2}a_{12}\int_{-\tau_{12}}^0 x_2^2(s+\theta;\phi)\,\mathrm{d}\mu_{12}(\theta)\biggr)^p\,\mathrm{d}s\biggr] \\
&\quad\le E\biggl[\int_{t_1}^{t_2}\biggl(a_{12}^px_1^{2p}(s;\phi)+\biggl(a_{12}\int_{-\tau_{12}}^0 x_2^2(s+\theta;\phi)\,\mathrm{d}\mu_{12}(\theta)\biggr)^p\biggr)\,\mathrm{d}s\biggr]
\end{aligned}$$

$$
\begin{aligned}
&\leq E\left[\int_{t_1}^{t_2}\left(a_{12}^{p}x_1^{2p}(s;\phi)+a_{12}^{p-1}\int_{-\tau_{12}}^{0}x_2^{2p}(s+\theta;\phi)\,\mathrm{d}\mu_{12}(\theta)\right)\mathrm{d}s\right]\\
&=a_{12}^{p}\int_{t_1}^{t_2}E\left[x_1^{2p}(s;\phi)\right]\mathrm{d}s+a_{12}^{p-1}\int_{t_1}^{t_2}\int_{-\tau_{12}}^{0}E\left[x_2^{2p}(s+\theta;\phi)\right]\mathrm{d}\mu_{12}(\theta)\,\mathrm{d}s. \qquad (54)
\end{aligned}
$$

In view of Theorem 7.1 in [24], for any $t_2 > t_1$ and $1 < p \leq 2$, we obtain

$$
E\left[\left|\int_{t_1}^{t_2}\sigma_1 x_1(s;\phi)\,\mathrm{d}B_1(s)\right|^{p}\right]\leq\left|\sigma_1^{p}\right|\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}(t_2-t_1)^{\frac{p-2}{2}}\int_{t_1}^{t_2}E\left[x_1^{p}(s;\phi)\right]\mathrm{d}s. \qquad (55)
$$

From Lemma 3, there exist $K_1^{**}(p) > 0$, $K_2^{**}(p) > 0$, and $K_3^{**}(p) > 0$ such that $\sup_{t\geq-\gamma}E[x_1^p(t)]\leq K_1^{**}(p)$, $\sup_{t\geq-\gamma}E[x_2^p(t)]\leq K_2^{**}(p)$, and $\sup_{t\geq-\gamma}E[x_3^p(t)]\leq K_3^{**}(p)$. Therefore, from (52)–(55) there exists $\delta > 0$ such that, for any $t_1 \geq 0$, $t_2 \geq 0$, and $1 < p \leq 2$ with $|t_2 - t_1| \leq \delta$,

$$
\begin{aligned}
&\left(E\left[\left|x_1(t_2;\phi)-x_1(t_1;\phi)\right|\right]\right)^{p}\\
&\quad\leq 2^{p}\left[\left|\sigma_1^{p}\right|\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}(t_2-t_1)^{\frac{p}{2}}K_1^{**}(p)\right]+2^{p}\left[3^{p}(t_2-t_1)^{p}|r_1-h_1|^{p}K_1^{**}(p)\right.\\
&\qquad\left.+3^{p}a_{11}^{p}(t_2-t_1)^{p}K_1^{**}(2p)\right]+2^{p}3^{p}(t_2-t_1)^{p}a_{12}^{p}\left[K_1^{**}(2p)+K_2^{**}(2p)\right]\\
&\quad\leq M_1^{**}|t_2-t_1|^{\frac{p}{2}},
\end{aligned}
$$

where

$$
\begin{aligned}
M_1^{**}={}&\left|\sigma_1^{p}\right|\left(2p(p-1)\right)^{\frac{p}{2}}K_1^{**}(p)+[36\delta]^{\frac{p}{2}}\left[|r_1-h_1|^{p}K_1^{**}(p)+a_{11}^{p}K_1^{**}(2p)\right]\\
&+[36\delta]^{\frac{p}{2}}a_{12}^{p}\left[K_1^{**}(2p)+K_2^{**}(2p)\right].
\end{aligned}
$$

Similarly, we also obtain

$$
\left(E\left[\left|y_1(t_2;\phi^*)-y_1(t_1;\phi^*)\right|\right]\right)^{p}\leq M_1^{**}|t_2-t_1|^{\frac{p}{2}}
$$

for any $t_1 \geq 0$, $t_2 \geq 0$ with $|t_2 - t_1| \leq \delta$ and $1 < p \leq 2$. Thus, from (50), we obtain

$$
\begin{aligned}
\left|F_1(t_2)-F_1(t_1)\right|&\leq E\left[\left|x_1(t_2;\phi)-x_1(t_1;\phi)\right|\right]+E\left[\left|y_1(t_2;\phi^*)-y_1(t_1;\phi^*)\right|\right]\\
&\leq 2\left(M_1^{**}\right)^{\frac{1}{p}}\sqrt{|t_2-t_1|}. \qquad (56)
\end{aligned}
$$

Using a similar argument, for $F_2(t)$ and $F_3(t)$ we can also obtain that there is $\delta > 0$ for any $t_1 \geq 0$, $t_2 \geq 0$ with $|t_2 - t_1| \leq \delta$ and $1 < p \leq 2$

$$
\left|F_2(t_2)-F_2(t_1)\right|\leq 2\left(M_2^{**}\right)^{\frac{1}{p}}\sqrt{|t_2-t_1|} \qquad (57)
$$

and

$$
\left|F_3(t_2)-F_3(t_1)\right|\leq 2\left(M_3^{**}\right)^{\frac{1}{p}}\sqrt{|t_2-t_1|}, \qquad (58)
$$

where

$$M_2^{**} = \left|\sigma_2^p\right|\left(2p(p-1)\right)^{\frac{p}{2}} K_2^{**}(p) + [64\delta]^{\frac{p}{2}}\left[|r_2+h_2|^p K_2^{**}(p) + a_{22}^p K_2^{**}(2p)\right] + [64\delta]^{\frac{p}{2}} a_{23}^p\left[K_2^{**}(2p) + K_3^{**}(2p)\right] + [64\delta]^{\frac{p}{2}} a_{21}^p\left[K_1^{**}(2p) + K_3^{**}(2p)\right]$$

and

$$M_3^{**} = \left|\sigma_3^p\right|\left(2p(p-1)\right)^{\frac{p}{2}} K_3^{**}(p) + [36\delta]^{\frac{p}{2}}\left[|r_2+h_2|^p K_3^{**}(p) + a_{33}^p K_3^{**}(2p)\right] + [36\delta]^{\frac{p}{2}} a_{32}^p\left[K_3^{**}(2p) + K_2^{**}(2p)\right].$$

From (56)–(58), we obtain that $F_1(t)$, $F_2(t)$, and $F_3(t)$ for $t \in (0,\infty)$ are uniformly continuous. Therefore, from (48) and Barbalat lemma in [25] we can finally obtain (43). This completes the proof. □

Denote by $\mathcal{P}([-\gamma,0],R_+^3)$ the space of all probability measures on $C([-\gamma,0],R_+^3)$. For $P_1,P_2 \in \mathcal{P}([-\gamma,0],R_+^3)$, define

$$d_{\mathrm{BL}}(P_1,P_2) = \sup_{f\in \mathrm{BL}}\left|\int_{R_+^3} f(z)P_1(\mathrm{d}z) - \int_{R_+^3} f(z)P_2(\mathrm{d}z)\right|,$$

where set BL is defined as follows:

$$\mathrm{BL} = \left\{f : C\left([-\gamma,0],R_+^3\right) \to R : \left|f(z_1)-f(z_2)\right| \le \|z_1-z_2\|, \left|f(\cdot)\right| \le 1\right\}.$$

Denote by $p(t,\phi,\mathrm{d}x)$ the transition probability of process $x(t) = (x_1(t),x_2(t),x_3(t))$. We have the following results.

Theorem 3 *Assume that there are positive constants q_1, q_2, and q_3 such that*

$$q_1a_{11} - q_2a_{21} > 0, \qquad q_2a_{22} - q_1a_{12} - q_3a_{32} > 0, \qquad q_3a_{33} - q_2a_{23} > 0.$$

Then model (1) is asymptotically stable in distribution, i.e., there exists a unique probability measure $\nu(\cdot)$ such that, for any initial function $\phi \in C([-\gamma,0],R_+^3)$, the transition probability $p(t,\phi,\cdot)$ of $x(t,\phi) = (x_1(t,\phi),x_2(t,\phi),x_3(t,\phi))$ satisfies

$$\lim_{t\to\infty} d_{\mathrm{BL}}\left(p(t,\phi,\cdot),\nu(\cdot)\right) = 0.$$

This theorem can be proved using a standard argument as in [15, 16] by using Lemma 1 and Theorem 2. Hence, we here omit it.

4 Effect of harvesting

In model (1), $h_i \ge 0$ ($i = 1,2,3$) denotes the harvesting rates of species x_i, respectively. Firstly, based on Theorem 1, we discuss the effects of harvesting for the persistence and extinction of species in model (1).

From $\Delta_{11} = 0$, the critical value of harvesting rate h_1 for prey x_1 is determined by $h_1' = r_1 - \frac{\sigma_1^2}{2}$. When $h_1 \ge h_1'$, all species x_1 ($i = 1,2,3$) will die out from conclusions (1) and (2) of

Theorem 1. This shows that the excessive harvesting for the prey will lead to the extinction of all species in a food-chain system.

When $h_1 < h_1'$, from $\Delta_{22} = 0$, the critical value of harvesting rate h_2 for middle predator x_2 is determined by $h_2' = \frac{a_{21}}{a_{11}}(r_1 - \frac{\sigma_1^2}{2} - h_1) - (r_2 + \frac{\sigma_2^2}{2})$. When $h_2 \geq h_2'$, from conclusions (3) and (4) of Theorem 1 we see that prey x_1 will be permanent in the mean, but two predators x_2 and x_3 will die out. This shows that the excessive harvesting for the middle predator will lead to the extinction of all top species. Furthermore, we see that h_2' decreasingly depends on the harvesting rate h_1 for prey x_1. This shows that when we increase the harvest for the prey, then the harvest for the middle predator must decrease to just guarantee the non-extinction of the whole food-chain system.

When $h_2 < h_2'$, from $\Delta_{33} = 0$, we further obtain that the critical value of harvesting rate h_3 for top predator x_3 is

$$h_3' = \frac{[(r_1 - \frac{\sigma_1^2}{2} - h_1)a_{21} - (r_2 + \frac{\sigma_2^2}{2} + h_2)a_{11}]a_{32}}{H_2} - \left(r_3 + \frac{\sigma_3^2}{2}\right).$$

When $h_3 \geq h_3'$, then from conclusions (5) and (6) of Theorem 1 we see that prey x_1 and middle predator x_2 will be permanent in the mean, but top predator x_3 will die out; whereas when $h_3 < h_3'$, from conclusion (7) of Theorem 1 we see that all species x_1 $(i = 1, 2, 3)$ will be permanent in the mean. This shows that only temperate harvesting for all species can ensure the persistence of all species and a continuous income. Furthermore, we also see that h_3' decreasingly depends on the harvesting rates h_1 and h_2 for prey and middle predators x_1 and x_2. This shows that when there exist the harvests for prey and middle predator, then the harvest for the top predator must decrease; if not, then top predator will die out.

Next, we discuss the optimal harvesting problem under the harvesting rates h_1, h_2, and h_3 for species x_1, x_2, and x_3, respectively. We can establish the following comparatively integrated results.

Theorem 4 *Assume that there are positive constants m_1, m_2, and m_3 such that*

$$m_1 a_{11} - m_2 a_{21} > 0, \qquad m_2 a_{22} - m_1 a_{12} - m_3 a_{32} > 0, \qquad m_3 a_{33} - m_2 a_{23} > 0.$$

Let

$$\begin{aligned} h_1^* ={}& \frac{-a_{11}(a_{32} - a_{23})^2 + 2a_{33}a_{21}(a_{12} - a_{21}) + 4a_{11}a_{22}a_{33}}{2[4a_{11}a_{22}a_{33} - a_{33}(a_{12} - a_{21})^2 - a_{11}(a_{23} - a_{32})^2]}\left(r_1 - \frac{\sigma_1^2}{2}\right) \\ &+ \frac{a_{11}a_{33}(a_{12} + a_{21})}{4a_{11}a_{22}a_{33} - a_{33}(a_{12} - a_{21})^2 - a_{11}(a_{23} - a_{32})^2}\left(r_2 + \frac{\sigma_2^2}{2}\right) \\ &+ \frac{a_{11}(a_{12} + a_{21})(a_{32} - a_{23})}{2[4a_{11}a_{22}a_{33} - a_{33}(a_{12} - a_{21})^2 - a_{11}(a_{23} - a_{32})^2]}\left(r_3 + \frac{\sigma_3^2}{2}\right), \\ h_2^* ={}& \left\{\frac{2a_{22}a_{33}(a_{12} + a_{21})}{2[4a_{11}a_{22}a_{33} - a_{33}(a_{12} - a_{21})^2 - a_{11}(a_{23} - a_{32})^2]}\right. \\ &\left.+ \frac{(a_{32} - a_{23})(a_{23}a_{12} - a_{32}a_{21})}{2[4a_{11}a_{22}a_{33} - a_{33}(a_{12} - a_{21})^2 - a_{11}(a_{23} - a_{32})^2]}\right\}\left(r_1 - \frac{\sigma_1^2}{2}\right) \\ &+ \frac{a_{33}a_{12}(a_{12} - a_{21}) - a_{11}a_{32}(a_{23} - a_{32}) - 4a_{11}a_{22}a_{33}}{4a_{11}a_{22}a_{33} - a_{33}(a_{12} - a_{21})^2 - a_{11}(a_{23} - a_{32})^2}\left(r_2 + \frac{\sigma_2^2}{2}\right) \end{aligned} \tag{59}$$

$$
\begin{aligned}
&+\left\{\frac{(a_{12}-a_{21})(a_{21}a_{32}-a_{12}a_{23})}{2[4a_{11}a_{22}a_{33}-a_{33}(a_{12}-a_{21})^2-a_{11}(a_{23}-a_{32})^2]}\right.\\
&\left.+\frac{2a_{11}a_{22}(a_{23}+a_{32})}{2[4a_{11}a_{22}a_{33}-a_{33}(a_{12}-a_{21})^2-a_{11}(a_{23}-a_{32})^2]}\right\}\left(r_3+\frac{\sigma_3^2}{2}\right),\\
h_3^* = {} & \frac{a_{33}(a_{21}-a_{12})(a_{23}+a_{32})}{2[4a_{11}a_{22}a_{33}-a_{33}(a_{12}-a_{21})^2-a_{11}(a_{23}-a_{32})^2]}\left(r_1-\frac{\sigma_1^2}{2}\right)\\
&-\frac{2a_{11}a_{33}(a_{23}+a_{32})}{4a_{11}a_{22}a_{33}-a_{33}(a_{12}-a_{21})^2-a_{11}(a_{23}-a_{32})^2}\left(r_2+\frac{\sigma_2^2}{2}\right)\\
&+\frac{a_{33}(a_{12}-a_{21})^2+2a_{11}a_{23}(a_{23}-a_{32})-4a_{11}a_{22}a_{33}}{2[4a_{11}a_{22}a_{33}-a_{33}(a_{12}-a_{21})^2-a_{11}(a_{23}-a_{32})^2]}\left(r_3+\frac{\sigma_3^2}{2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
Y^*(H) = {} & -(a_{22}a_{33}+a_{23}a_{32})h_1^2+(a_{33}a_{12}-a_{33}a_{21})h_1h_2\\
&-a_{11}a_{33}h_2^2+(a_{11}a_{23}-a_{11}a_{32})h_2h_3-(a_{11}a_{22}+a_{12}a_{21})h_3^2\\
&-(a_{12}a_{23}+a_{21}a_{32})h_1h_3+\left[\left(r_1-\frac{\sigma_1^2}{2}\right)(a_{22}a_{33}+a_{23}a_{32})\right.\\
&\left.+\left(r_2+\frac{\sigma_2^2}{2}\right)a_{33}a_{12}-\left(r_3+\frac{\sigma_3^2}{2}\right)a_{12}a_{23}\right]h_1\\
&+\left[\left(r_1-\frac{\sigma_1^2}{2}\right)a_{33}a_{21}-\left(r_2+\frac{\sigma_2^2}{2}\right)a_{11}a_{33}+\left(r_3+\frac{\sigma_3^2}{2}\right)a_{11}a_{23}\right]h_2\\
&+\left[\left(r_1-\frac{\sigma_1^2}{2}\right)a_{21}a_{32}-\left(r_2+\frac{\sigma_2^2}{2}\right)a_{11}a_{32}\right.\\
&\left.-\left(r_3+\frac{\sigma_3^2}{2}\right)(a_{11}a_{22}+a_{12}a_{21})\right]h_3.
\end{aligned}
\tag{60}
$$

We have the following conclusions.

($\mathcal{A}_1$) *If* $h_1^*\geq 0$, $h_2^*\geq 0$, *and* $h_3^*\geq 0$, *and*

$$
\begin{aligned}
&\Delta_{33}|_{h_1=h_1^*,h_2=h_2^*,h_3=h_3^*}>0,\\
&4a_{11}a_{22}a_{33}-a_{33}(a_{12}-a_{21})^2-a_{11}(a_{23}-a_{32})^2>0.
\end{aligned}
\tag{61}
$$

Then there is an optimal harvesting strategy $H^*=(h_1^*,h_2^*,h_3^*)$ *for model* (1), *and*

$$
MESY=\frac{Y^*(H^*)}{H_3}.
\tag{62}
$$

($\mathcal{A}_2$) *If one of the following conditions holds, then there is not the optimal harvesting strategy for model* (1).

($\mathcal{B}_1$) $b_1|h_1=h_1^*\leq 0$;

($\mathcal{B}_2$) $\Delta_{33}|_{h_1=h_1^*,h_2=h_2^*,h_3=h_3^*}\leq 0$;

($\mathcal{B}_3$) $h_1^*<0$ *or* $h_2^*<0$ *or* $h_3^*<0$;

($\mathcal{B}_4$) $4a_{11}a_{22}a_{33}-a_{33}(a_{12}-a_{21})^2-a_{11}(a_{23}-a_{32})^2<0$.

Proof Define a set as follows:

$$\mathcal{U} = \left\{H = (h_1, h_2, h_3)^T \in R^3 : \Delta_{33} > 0, h_i \geq 0, i = 1, 2, 3\right\}.$$

It is clear that for any $H \in \mathcal{U}$ conclusion (7) of Theorem 1 holds. From the condition of conclusion $(\mathcal{A}_1)$, we see that if optimal harvesting strategy H^* exists, then $H^* \in \mathcal{U}$.

Proof of conclusion $(\mathcal{A}_1)$. Based on condition (61) we obtain that $\mathcal{U}$ is not empty. From Theorem 3, we obtain that there exists a unique invariant measure $\nu(\cdot)$ for model (1). From Corollary 3.4.3 in Prato and Zbczyk [26], we obtain that $\nu(\cdot)$ is strong mixing. By Theorem 3.2.6 in [26], we further obtain that measure $\nu(\cdot)$ is also ergodic. Let $x(t) = (x_1(t), x_2(t), x_3(t))$ be any global positive solution of model (1) with initial value $(\xi(\theta), \eta(\theta), \varsigma(\theta)) \in C([-\gamma, 0], R_+^3))$. Based on Theorem 3.3.1 in [26], for $H = (h_1, h_2, h_3)^T \in \mathcal{U}$, we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t H^T x(s)\,\mathrm{d}s = \int_{R_+^3} H^T x \nu(\mathrm{d}x). \tag{63}$$

Let $\varrho(z)$ be the stationary probability density of model (1), then we get

$$Y(H) = \lim_{t\to\infty} E\left[\sum_{i=1}^{3} h_i x_i(t)\right] = \lim_{t\to\infty} E\left[H^T x(t)\right] = \int_{R_+^3} H^T x \varrho(x)\,\mathrm{d}x. \tag{64}$$

Note that the invariant measure of model (1) is unique and there exists a one-to-one correspondence between $\varrho(z)$ and its corresponding invariant measure. We deduce

$$\int_{R_+^3} H^T x \varrho(x)\,\mathrm{d}x = \int_{R_+^3} H^T x \nu(\mathrm{d}x). \tag{65}$$

Therefore, from conclusion (5) of Theorem 1, (59), and (63)–(65), we have

$$\begin{aligned} Y(H) &= \lim_{t\to+\infty} \frac{1}{t} \int_0^t H^T x(s)\,\mathrm{d}s \\ &= h_1 \lim_{t\to+\infty} \frac{1}{t} \int_0^t x_1(s)\,\mathrm{d}s + h_2 \lim_{t\to+\infty} \frac{1}{t} \int_0^t x_2(s)\,\mathrm{d}s + h_3 \lim_{t\to+\infty} \frac{1}{t} \int_0^t x_3(s)\,\mathrm{d}s \\ &= \frac{Y^*(H)}{H_3}. \end{aligned}$$

By calculating we obtain

$$\begin{aligned} \frac{\partial Y^*(H)}{\partial h_1} &= -2(a_{22}a_{33} + a_{23}a_{32})h_1 + (a_{33}a_{12} - a_{33}a_{21})h_2 - (a_{12}a_{23} + a_{21}a_{32})h_3 \\ &\quad + \left(r_1 - \frac{\sigma_1^2}{2}\right)(a_{22}a_{33} + a_{23}a_{32}) + \left(r_2 + \frac{\sigma_2^2}{2}\right)a_{33}a_{12} - \left(r_3 + \frac{\sigma_3^2}{2}\right)a_{12}a_{23}, \\ \frac{\partial Y^*(H)}{\partial h_2} &= -2a_{11}a_{33}h_2 + (a_{33}a_{12} - a_{33}a_{21})h_1 + (a_{11}a_{23} - a_{11}a_{32})h_3 \\ &\quad + \left(r_1 - \frac{\sigma_1^2}{2}\right)a_{33}a_{21} - \left(r_2 + \frac{\sigma_2^2}{2}\right)a_{11}a_{33} + \left(r_3 + \frac{\sigma_3^2}{2}\right)a_{11}a_{23}, \end{aligned}$$

$$\frac{\partial Y^*(H)}{\partial h_3} = -2(a_{11}a_{22} + a_{12}a_{21})h_3 + (a_{11}a_{23} - a_{11}a_{32})h_2 - (a_{12}a_{23} + a_{21}a_{32})h_1 + \left(r_1 - \frac{\sigma_1^2}{2}\right)a_{21}a_{32} - \left(r_2 + \frac{\sigma_2^2}{2}\right)a_{11}a_{32} - \left(r_3 + \frac{\sigma_3^2}{2}\right)(a_{11}a_{22} + a_{12}a_{21}).$$

Solving equations $\frac{\partial Y^*(H)}{\partial h_1} = 0$, $\frac{\partial Y^*(H)}{\partial h_2} = 0$, and $\frac{\partial Y^*(H)}{\partial h_3} = 0$, we can obtain $h_1 = h_1^*$, $h_2 = h_2^*$, and $h_3 = h_3^*$, which are given in (59). Let $H^* = (h_1^*, h_2^*, h_3^*)$, by calculating we further obtain

$$\begin{aligned}
&\frac{\partial^2 Y^*(H^*)}{\partial h_1^2} = -2(a_{22}a_{33} + a_{23}a_{32}), \qquad \frac{\partial^2 Y^*(H^*)}{\partial h_1 \partial h_2} = a_{33}(a_{12} - a_{21}),\\
&\frac{\partial^2 Y^*(H^*)}{\partial h_1 \partial h_3} = -(a_{12}a_{23} + a_{21}a_{32}), \qquad \frac{\partial^2 Y^*(H^*)}{\partial h_2^2} = -2a_{11}a_{33},\\
&\frac{\partial^2 Y^*(H^*)}{\partial h_2 \partial h_1} = a_{33}(a_{12} - a_{21}), \qquad \frac{\partial^2 Y^*(H^*)}{\partial h_2 \partial h_3} = a_{11}(a_{23} - a_{32}),\\
&\frac{\partial^2 Y^*(H^*)}{\partial h_3^2} = -2(a_{11}a_{22} + a_{12}a_{21}), \qquad \frac{\partial^2 Y^*(H^*)}{\partial h_3 \partial h_1} = -(a_{12}a_{23} + a_{21}a_{32}),\\
&\frac{\partial^2 Y^*(H^*)}{\partial h_3 \partial h_2} = a_{11}(a_{23} - a_{32}).
\end{aligned}$$

Define matrix $M = (\frac{\partial^2 Y^*(H^*)}{\partial h_i \partial h_j})_{1\le i,j\le 3}$. Then condition (61) implies that matrix M is negative definite. We hence obtain that $Y^*(H)$ has a unique maximum value $Y^*(H^*)$. This shows that H^* is an optimal harvesting strategy, and MESY is given in (62).

Proof of conclusion $(\mathcal{A}_2)$. From conclusions (1) and (2) of Theorem 1, we can obtain $\lim_{t\to\infty} x_i(t) = 0$ $(i = 1,2,3)$ if condition $(\mathcal{B}_1)$ holds. Hence, the optimal harvesting does not exist.

Assume that condition $(\mathcal{B}_2)$ or $(\mathcal{B}_3)$ holds. If there is an optimal harvesting strategy $\widetilde{H}^* = (\widetilde{h}_1^*, \widetilde{h}_2^*, \widetilde{h}_3^*)$, then $\widetilde{H}^* \in \mathcal{U}$. That is,

$$\Delta_{33}|_{h_1=\widetilde{h}_1^*, h_2=\widetilde{h}_2^*, h_3=\widetilde{h}_3^*} > 0, \quad \widetilde{h}_1^* \ge 0, \widetilde{h}_2^* \ge 0, \widetilde{h}_3^* \ge 0. \tag{66}$$

On the other hand, if $\widetilde{H}^* = (\widetilde{h}_1^*, \widetilde{h}_2^*, \widetilde{h}_3^*) \in \mathcal{U}$ is the optimal harvesting strategy, then we also have $(\widetilde{h}_1^*, \widetilde{h}_2^*, \widetilde{h}_3^*)$ must be the unique solution of the following system:

$$\frac{\partial Y^*(H)}{\partial h_1} = 0, \qquad \frac{\partial Y^*(H)}{\partial h_2} = 0, \qquad \frac{\partial Y^*(H)}{\partial h_3} = 0.$$

Therefore, we have $(h_1^*, h_2^*, h_3^*) = (\widetilde{h}_1^*, \widetilde{h}_2^*, \widetilde{h}_3^*)$. Thus, condition (66) becomes

$$\Delta_{33}|_{h_1=h_1^*, h_2=h_2^*, h_3=h_3^*} > 0, \quad h_1^* \ge 0, h_2^* \ge 0, h_3^* \ge 0,$$

which contradicts both $(\mathcal{B}_2)$ and $(\mathcal{B}_3)$.

Lastly, we consider condition $(\mathcal{B}_4)$. We can assume that conditions $(\mathcal{B}_2)$ and $(\mathcal{B}_3)$ do not hold. Hence, $h_1^* \ge 0$, $h_2^* \ge 0$, and $h_3^* \ge 0$, and $\Delta_{33}|_{h_1=h_1^*, h_2=h_2^*, h_3=h_3^*} > 0$. Thus, $\mathcal{U}$ is not empty. Condition $(\mathcal{B}_4)$ implies that matrix M is not negative semidefinite. Therefore, there is not any maximum point. This completes the proof. □

5 Numerical examples

In this section, we will provide the numerical examples to illustrate our main results. The numerical approaches are proposed in [13], and also refer to [16]. Firstly, we indicate in the following numerical examples that the initial values always are fixed by $x_1(\theta) = 0.3e^\theta$, $x_2(\theta) = 0.2e^\theta$, and $x_3(\theta) = 0.3e^\theta$ for all $\theta \in [-\ln 2, 0]$, and $\tau_{12} = \tau_{21} = \tau_{23} = \tau_{32} = \ln 2$.

Example 1 In model (1), parameters $r_1 = 2.0$, $r_2 = 1.0$, $r_3 = 0.5$, and $h_1 = h_2 = h_3 = 0$ are fixed. We consider the following cases.

Case 1. Taking parameters $a_{11} = 1$, $a_{22} = 0.5$, $a_{33} = 0.25$, $a_{12} = 1$, $a_{21} = 1$, $a_{23} = 1$, $a_{32} = 1$, $\sigma_1 = 2.5$, $\sigma_2 = 0.1$, and $\sigma_3 = 0.05$, we have $\Delta_{11} = -1.125 < 0$. Hence, the conditions of conclusion (1) in Theorem 1 are satisfied. The numerical simulations given in Fig. 1 illustrate that all species x_i $(i = 1, 2, 3)$ are extinct with probability one.

Case 2. Taking parameters $a_{11} = 1$, $a_{22} = 0.5$, $a_{33} = 0.25$, $a_{12} = 1$, $a_{21} = 1$, $a_{23} = 1$, $a_{32} = 1$, $\sigma_1 = 2.0$, $\sigma_2 = 0.1$, and $\sigma_3 = 0.05$, we have $\Delta_{11} = 0$. Hence, the conditions of conclusion (2) in Theorem 1 are satisfied. The numerical simulations given in Fig. 2 illustrate that all species x_i $(i = 1, 2, 3)$ also are extinct with probability one.

Case 3. Taking parameters $a_{11} = 1$, $a_{22} = 0.5$, $a_{33} = 0.25$, $a_{12} = 1$, $a_{21} = 0.7$, $a_{23} = 1$, $a_{32} = 1$, $\sigma_1 = 1.0$, $\sigma_2 = 0.6$, and $\sigma_3 = 0.05$, we have $\Delta_{11} = 1.5 > 0$ and $\Delta_{22} = -0.13 < 0$. Hence, the conditions of conclusion (3) in Theorem 1 are satisfied. The numerical simulations given in Fig. 3 illustrate that species $x_1(t)$ is persistent in the mean while species $x_i(t)$ $(i = 2, 3)$ go to extinction.

Case 4. Taking parameters $a_{11} = 1$, $a_{22} = 0.5$, $a_{33} = 0.25$, $a_{12} = 1$, $a_{21} = 0.78667$, $a_{23} = 1$, $a_{32} = 1$, $\sigma_1 = 1.0$, $\sigma_2 = 0.5$, and $\sigma_3 = 0.3$, we have $\Delta_{22} = 0$. Hence, the conditions of conclusion (4) in Theorem 1 are satisfied. The numerical simulations given in Fig. 4 illustrate

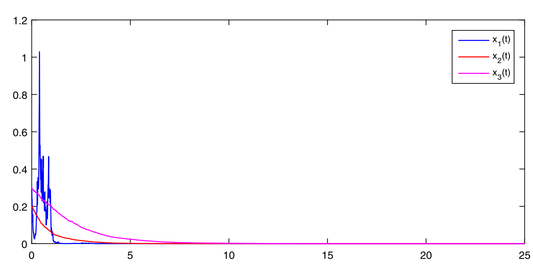

Figure 1 Species x_i $(i = 1, 2, 3)$ are extinct with probabilty one

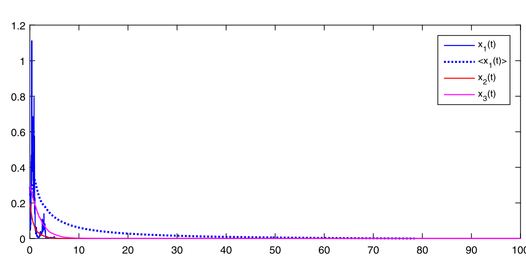

Figure 2 Species x_i $(i = 1, 2, 3)$ are also extinct with probability one

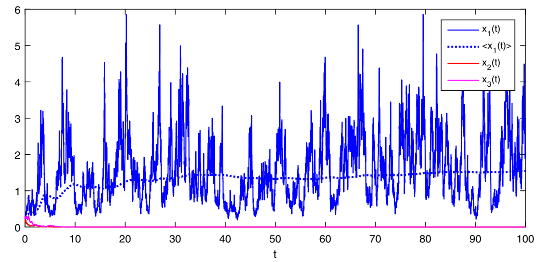

Figure 3 Species $x_1(t)$ is persistent in the mean while species $x_i(t)$ ($i = 2, 3$) go to extinction

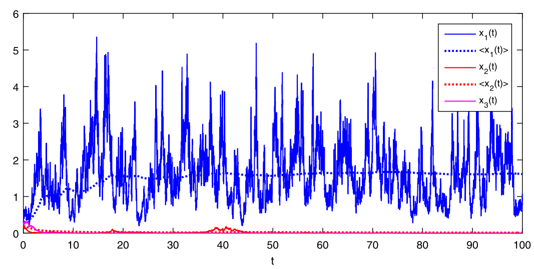

Figure 4 Species $x_1(t)$ is persistent in the mean, species $x_2(t)$ is extinct in the mean and $x_3(t)$ is extinct

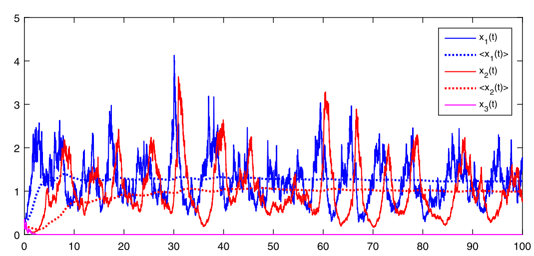

Figure 5 Species $x_1(t)$ and $x_2(t)$ are persistent in the mean while species $x_3(t)$ goes to extinction

that species $x_1(t)$ is persistent in the mean, species $x_2(t)$ is extinct in the mean and $x_3(t)$ is extinct.

Case 5. Taking parameters $a_{11} = 1$, $a_{22} = 0.5$, $a_{33} = 2.5$, $a_{12} = 1$, $a_{21} = 2$, $a_{23} = 1$, $a_{32} = 1$, $\sigma_1 = 0.5$, $\sigma_2 = 0.3$, and $\sigma_3 = 1.5$, we have $\Delta_{22} = 2.505 > 0$ and $\Delta_{33} = -1.5575 < 0$. Hence, the conditions of conclusion (5) in Theorem 1 are satisfied. The numerical simulations given in Fig. 5 illustrate that species $x_1(t)$ and $x_2(t)$ are persistent in the mean while species $x_3(t)$ goes to extinction.

Case 6. Taking parameters $a_{11} = 1$, $a_{22} = 0.5$, $a_{33} = 2.5$, $a_{12} = 1$, $a_{21} = 2$, $a_{23} = 1$, $a_{32} = 1$, $\sigma_1 = 0.5$, $\sigma_2 = 0.3$, and $\sigma_3 = \sqrt{1.004}$, we have $\Delta_{33} = 0$. Hence, the conditions of conclusion (6) in Theorem 1 are satisfied. The numerical simulations given in Fig. 6 illustrate that species $x_1(t)$ and $x_2(t)$ are persistent in the mean while species $x_3(t)$ is extinct in the mean.

Case 7. Taking parameters $a_{11} = 1$, $a_{22} = 0.5$, $a_{33} = 1$, $a_{12} = 1$, $a_{21} = 2$, $a_{23} = 1$, $a_{32} = 2$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, and $\sigma_3 = 0.9$, we have $\Delta_{33} = 0.2425 > 0$. Hence, the conditions of conclusion

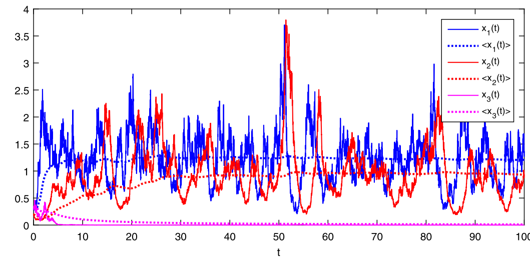

Figure 6 Species $x_1(t)$ and $x_2(t)$ are persistent in the mean while species $x_3(t)$ is extinct in the mean

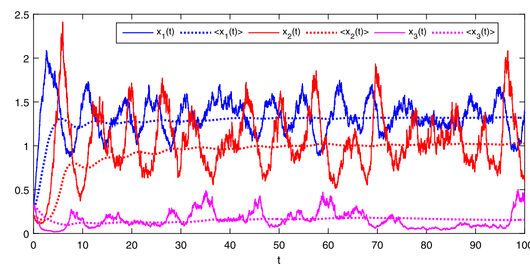

Figure 7 Species $x_i(t)$ $(i = 1, 2, 3)$ are persistent in the mean

(7) in Theorem 1 are satisfied. The numerical simulations given in Fig. 7 illustrate that all species $x_i(t)$ $(i = 1, 2, 3)$ are persistent in the mean.

Example 2 In model (1) we take parameters $r_1 = 1$, $r_2 = 0.3$, $r_3 = 0.1$, $m_1 = 1$, $m_2 = 0.3$, $m_3 = 0.1$, $a_{11} = 0.4$, $a_{12} = 0.1$, $a_{22} = 0.5$, $a_{21} = 0.75$, $a_{23} = 0.1$, $a_{32} = 0.45$, $a_{33} = 0.6$, $\sigma_1 = 0.2$, $\sigma_2 = 0.1$, and $\sigma_3 = \sqrt{0.012}$.

We have $m_1a_{11} - m_2a_{21} = 0.1075 > 0$, $m_2a_{22} - m_1a_{12} - m_3a_{32} = 0.005 > 0$, and $m_3a_{33} - m_2a_{23} = 0.03 > 0$. Calculating h_i^* $(i = 1, 2, 3)$ in Theorem 2, we have $h_1^* = 0.0023 > 0$, $h_2^* = 0.1414 > 0$, and $h_3^* = 0.0958 > 0$. Furthermore, we also have $4a_{11}a_{22}a_{33} - a_{33}(a_{12} - a_{21})^2 - a_{11}(a_{23} - a_{32})^2 = 0.1175 > 0$, $\Delta_{33}|_{h_1=h_1^*,h_2=h_2^*,h_3=h_3^*} = 0.4769 > 0$, and $H_3 = 0.183$. Hence, all conditions of conclusion $(\mathcal{A}_1)$ in Theorem 2 are satisfied. Hence, there is an optimal harvesting strategy $H^* = (0.0023, 0.1414, 0.0958)^T$, and the maximum of expectation of sustainable yield (MESY) is $\frac{Y^*(H^*)}{H_3} = 0.3464$. The numerical simulations are given in Fig. 8.

6 Conclusion

Ecological and mathematical improvements have provided that three species are more advantageous than two-species models (Pimm [27], Hastings and Powell [28]). Besides, considering the influence of distributed delays and environmental noise, we analyze a stochastic three species food-chain model with harvesting in this paper. By using the stochastic integral inequalities, Lyapunov function method, and the inequality estimation technique, some criteria on the existence of global positive solutions, stochastic boundedness, extinction, global asymptotic stability in the mean and the probability distribution, and the effect of harvesting are established. Our results show some meaningful facts:

(i) Theorem 1 shows the sufficient and necessary conditions for the extinction and global asymptotic stability in the mean with probability one. In addition, Theorem 1

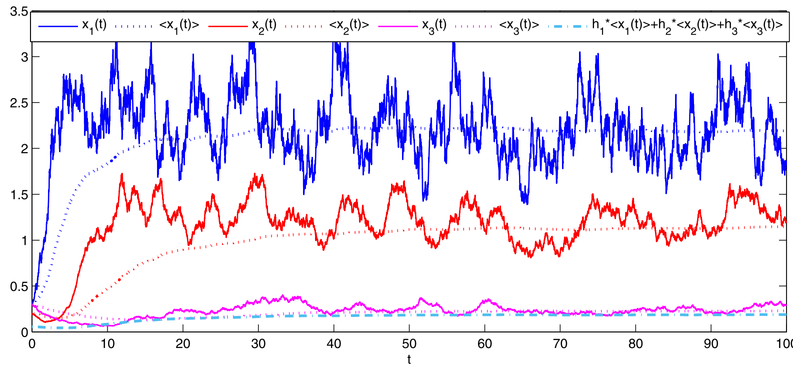

Figure 8 Species $x_i(t)$ ($i = 1, 2, 3$) are also persistent in the mean. There is an optimal harvesting strategy $H^* = (h_1^*, h_2^*, h_3^*)^T = (0.0023, 0.1414, 0.0958)^T$, and the maximum of expectation of sustainable yield (MESY) is $\frac{Y^*(H^*)}{H_3} = 0.3464$

also reveals the effects of harvesting for the extinction and permanence in the mean of prey, middle predator, and top predator.

(ii) Theorem 2 and Theorem 3 guarantee the global attractivity in the expectation and the global asymptotic stability in distribution, respectively.

(iii) Theorem 4 reveals the existence of optimal harvesting strategy and MESY are affected by environmental fluctuations.

There are still some problems waiting for further investigation. Firstly, it is meaningful to study more complex systems, for example, stochastic systems with Lévy jumps (see, for example, [22, 29]), Markovian switching (see, for example, [30]) and nonlinear functional responses (see, for example, [31]), and general stochastic many species food-chain systems. Furthermore, the optimal harvesting problem for other stochastic population systems with distributed delays, for instance, competitive systems and cooperative systems, still are rarely investigated at present. We will leave to investigate these problems in the future.

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