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Some identities of special numbers and polynomials arising from p-adic integrals on $\mathbb{Z}_p$

Dae San Kim[1], Han Young Kim[2], Sung-Soo Pyo[3*] and Taekyun Kim[2]

*Correspondence:
ssoopyo@gmail.com
[3]Department of Mathematics Education, Silla University, Busan, Republic of Korea
Full list of author information is available at the end of the article

Abstract

In recent years, studying degenerate versions of various special polynomials and numbers has attracted many mathematicians. Here we introduce degenerate type 2 Bernoulli polynomials, fully degenerate type 2 Bernoulli polynomials, and degenerate type 2 Euler polynomials, and their corresponding numbers, as degenerate and type 2 versions of Bernoulli and Euler numbers. Regarding those polynomials and numbers, we derive some identities, distribution relations, Witt type formulas, and analogues for the Bernoulli interpretation of powers of the first m positive integers in terms of Bernoulli polynomials. The present study was done by using the bosonic and fermionic p-adic integrals on $\mathbb{Z}_p$.

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1 Introduction

Studies on degenerate versions of some special polynomials and numbers began with the papers by Carlitz in [3, 4]. In recent years, studying degenerate versions of various special polynomials and numbers has regained interest of many mathematicians. The research has been carried out by several different methods like generating functions, combinatorial approaches, umbral calculus, p-adic analysis, and differential equations. This idea of studying degenerate versions of some special polynomials and numbers turned out to be very fruitful so as to introduce degenerate Laplace transforms and degenerate gamma functions (see [11]).

In this paper, we introduce degenerate type 2 Bernoulli polynomials, fully degenerate type 2 Bernoulli polynomials, and degenerate type 2 Euler polynomials, and their corresponding numbers, as degenerate and type 2 versions of Bernoulli and Euler numbers. We investigate those polynomials and numbers by means of bosonic and fermionic p-adic integrals and derive some identities, distribution relations, Witt type formulas, and analogues for the Bernoulli interpretation of powers of the first m positive integers in terms of Bernoulli polynomials. In more detail, our main results are as follows.

As to the analogues for the Bernoulli interpretation of power sums, in Theorem 2.6 we express powers of the first m odd integers in terms of type 2 Bernoulli polynomials $b_n(x)$, in Theorem 2.11 alternating sum of powers of the first m odd integers in terms of type 2 Euler polynomials $E_n(x)$, in Theorem 2.9 sum of the values of the generalized falling factorials at the first m odd positive integers in terms of degenerate Carlitz type 2 Bernoulli polynomials $b_{n,\lambda}(x)$, and in Theorem 2.17 alternating sum of the values of the generalized falling factorials at the first m odd positive integers in terms of degenerate type 2 Euler polynomials $E_{n,\lambda}(x)$. Witt type formulas are obtained for $b_n(x), B_{n,\lambda}(x), E_n(x)$, and $E_{n,\lambda}(x)$ respectively in Lemma 2.1, Theorem 2.7, Lemma 2.10, and Theorem 2.16. Distribution relations are derived for $b_n(x)$ and $E_n(x)$ respectively in Theorem 2.3 and Theorem 2.13.

In the rest of this section, we will introduce type 2 Bernoulli and Euler numbers, recall the bosonic and fermionic p-adic integrals, and mention the degenerate exponential function.

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of an algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm $|\cdot|_p$ is normalized by $|p|_p = \frac{1}{p}$.

It is well known that the ordinary Bernoulli polynomials are defined by

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad \text{(see [2, 5, 14, 15, 17])}. \tag{1}$$

When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers.

Also, the type 2 Bernoulli polynomials are given by

$$\frac{t}{2}\operatorname{csch}\frac{t}{2}e^{xt} = \frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}e^{xt} = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}. \tag{2}$$

For $x = 0$, $b_n = b_n(0)$ are called the type 2 Bernoulli numbers so that they are given by

$$\frac{t}{2}\operatorname{csch}\frac{t}{2} = \frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}} = \sum_{n=0}^{\infty} b_n\frac{t^n}{n!}. \tag{3}$$

In fact, the type 2 Bernoulli polynomials and numbers are slightly differently defined in [12].

The ordinary Euler polynomials are defined by

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n^*(x)\frac{t^n}{n!} \quad \text{(see [1, 6, 9, 10])}. \tag{4}$$

When $x = 0$, $E_n^* = E_n^*(0)$ are called the Euler numbers.

Now, we define the type 2 Euler polynomials by

$$\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \quad \text{(see [6, 12, 13])}. \tag{5}$$

For $x = 0$, $E_n = E_n(0)$ are called the type 2 Euler numbers so that they are given by

$$\frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}} = \operatorname{sech}\frac{t}{2} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{6}$$

Again, the type 2 Euler polynomials and numbers are slightly differently defined in [12]. From (4) and (6), we note that

$$E_n^*\left(\frac{1}{2}\right) = E_n \quad (n \geq 0), (\text{see } [12]).$$

Let f be a uniformly differentiable function on $\mathbb{Z}_p$. The bosonic (also called Volkenborn) p-adic integral on $\mathbb{Z}_p$ is defined by

$$I_1(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) \quad (\text{see } [8–10]). \tag{7}$$

From (7), we note that

$$I_1(f_1) - I_1(f) = f'(0), \tag{8}$$

where $f_1(x) = f(x+1)$ and $f'(0) = \frac{df}{dx}|_{x=0}$.

The fermionic p-adic integral on $\mathbb{Z}_p$ was introduced by Kim as

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x f(x) \quad (\text{see } [9, 10]). \tag{9}$$

By (9), we easily get

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0). \tag{10}$$

For $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} \quad (\text{see } [3, 4, 6, 11–13]). \tag{11}$$

Note that $\lim_{\lambda\to 0} e_\lambda^x(t) = e^{xt}$. From (11) we have

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{k=0}^{\infty} (x)_{k,\lambda} \frac{t^k}{k!}, \tag{12}$$

where $(x)_{k,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(k-1)\lambda), (k \geq 1)$, and $(x)_{0,\lambda} = 1$.

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From (8), we note that

$$\int_{\mathbb{Z}_p} e^{(x+y+\frac{1}{2})t}\, d\mu_1(y) = \frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} e^{xt}$$

$$
\begin{aligned}
&= \frac{t}{2}\operatorname{csch}\frac{t}{2}e^{xt} \\
&= \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}.
\end{aligned}
\tag{13}
$$

On the other hand, we have

$$
\int_{\mathbb{Z}_p} e^{(x+y+\frac{1}{2})t}\,d\mu_1(x) = \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\left(x+y+\frac{1}{2}\right)^n d\mu_1(x)\frac{t^n}{n!}. \tag{14}
$$

Therefore, by (13) and (14), we obtain the following lemma.

Lemma 2.1 *For $n \geq 0$, we have*

$$
\int_{\mathbb{Z}_p}\left(x+y+\frac{1}{2}\right)^n d\mu_1(x) = b_n(x).
$$

By (7), we get

$$
\begin{aligned}
\int_{\mathbb{Z}_p} f(x)\,d\mu_1(x) &= \lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1} f(x) = \lim_{N\to\infty}\frac{1}{dp^N}\sum_{x=0}^{dp^N-1} f(x) \\
&= \frac{1}{d}\sum_{a=0}^{d-1}\lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1} f(a+xd) = \frac{1}{d}\sum_{a=0}^{d-1}\int_{\mathbb{Z}_p} f(a+xd)\,d\mu_1(x),
\end{aligned}
\tag{15}
$$

where d is a positive integer.

Therefore, by (15), we obtain the following lemma.

Lemma 2.2 *For $d \in \mathbb{N}$, we have*

$$
\int_{\mathbb{Z}_p} f(x)\,d\mu_1(x) = \frac{1}{d}\sum_{a=0}^{d-1}\int_{\mathbb{Z}_p} f(a+xd)\,d\mu_1(x).
$$

Applying Lemma 2.2 to $f(x) = e^{(x+y+1/2)t}$, we have

$$
\begin{aligned}
\int_{\mathbb{Z}_p} e^{(x+y+1/2)t}\,d\mu_1(y) &= \frac{1}{d}\sum_{a=0}^{d-1}\int_{\mathbb{Z}_p} e^{(x+a+dy+1/2)t}\,d\mu_1(y) \\
&= \frac{1}{d}\sum_{a=0}^{d-1}\int_{\mathbb{Z}_p} e^{d(y+\frac{1}{d}(x+a+\frac{1-d}{2})+1/2)t}\,d\mu_1(y).
\end{aligned}
\tag{16}
$$

Thus, by (16), we get

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\left(x+y+\frac{1}{2}\right)^n d\mu_1(y)\frac{t^n}{n!} \\
&\quad = \sum_{n=0}^{\infty} d^{n-1}\sum_{a=0}^{d-1}\int_{\mathbb{Z}_p}\left(y+\frac{1}{d}\left(a+x+\frac{1-d}{2}\right)+1/2\right)^n d\mu_1(y)\frac{t^n}{n!}.
\end{aligned}
\tag{17}
$$

By comparing the coefficients on both sides of (17), we get

$$\int_{\mathbb{Z}_p}\left(x+y+\frac{1}{2}\right)^n d\mu_1(y)=d^{n-1}\sum_{a=0}^{d-1}\int_{\mathbb{Z}_p}\left(y+\frac{1}{d}\left(a+x+\frac{1-d}{2}\right)+1/2\right)^n d\mu_1(y). \tag{18}$$

By Lemma 2.1 and (18), we get

$$b_n(x)=d^{n-1}\sum_{a=0}^{d-1}b_n\left(\frac{x+a+\frac{1}{2}(1-d)}{d}\right)\quad (n\geq 0), \tag{19}$$

where d is a positive integer.

Theorem 2.3 *For $d\in\mathbb{N}$ and $n\in\mathbb{N}\cup\{0\}$, we have*

$$b_n(x)=d^{n-1}\sum_{a=0}^{d-1}b_n\left(\frac{x+a+\frac{1}{2}(1-d)}{d}\right).$$

For $r\in\mathbb{N}$, we consider the multivariate p-adic integral on $\mathbb{Z}_p$ as follows:

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{(x_1+x_2+\cdots+x_r+r/2)t}\,d\mu_1(x_1)\,d\mu_1(x_2)\cdots d\mu_1(x_r)\\&\quad=\left(\frac{t}{e^{t/2}-e^{-t/2}}\right)^r=\left(\frac{t}{2}\operatorname{csch}\frac{t}{2}\right)^r.\end{aligned} \tag{20}$$

Now, we define the type 2 Bernoulli numbers of order r by

$$\left(\frac{t}{e^{t/2}-e^{-t/2}}\right)^r=\left(\frac{t}{2}\operatorname{csch}\frac{t}{2}\right)^r=\sum_{n=0}^{\infty}b_n^{(r)}\frac{t^n}{n!}. \tag{21}$$

By (20) and (21), we see that

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(x_1+x_2+\cdots+x_r+\frac{r}{2}\right)^n d\mu_1(x_1)\cdots d\mu_1(x_r)=b_n^{(r)}\quad (n\geq 0). \tag{22}$$

On the other hand,

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(x_1+x_2+\cdots+x_r+\frac{r}{2}\right)^n d\mu_1(x_1)\cdots d\mu_1(x_r)\\&\quad=\sum_{\substack{i_1+i_2+\cdots+i_r=n\\ i_1,i_2,\ldots,i_r\geq 0}}\binom{n}{i_1,i_2,\ldots,i_r}\int_{\mathbb{Z}_p}\left(x_1+\frac{1}{2}\right)^{i_1}d\mu_1(x_1)\cdots\int_{\mathbb{Z}_p}\left(x_r+\frac{1}{2}\right)^{i_r}d\mu_1(x_r)\\&\quad=\sum_{\substack{i_1+i_2+\cdots+i_r=n\\ i_1,i_2,\ldots,i_r\geq 0}}\binom{n}{i_1,i_2,\ldots,i_r}b_{i_1}b_{i_2}\cdots b_{i_r}.\end{aligned} \tag{23}$$

Therefore, by (22) and (23), we obtain the following theorem.

Theorem 2.4 *For $n \geq 0, r \in \mathbb{N}$, we have*

$$b_n^{(r)} = \sum_{\substack{i_1+i_2+\cdots+i_r=n \\ i_1,i_2,\ldots,i_r \geq 0}} \binom{n}{i_1, i_2, \ldots, i_r} b_{i_1} b_{i_2} \cdots b_{i_r}.$$

From (21), we have

$$\begin{aligned} t^r &= \sum_{l=0}^{\infty} b_l^{(r)} \frac{t^l}{l!} \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^r = \sum_{l=0}^{\infty} b_l^{(r)} \frac{t^l}{l!} r! \sum_{m=r}^{\infty} T(m,r) \frac{t^m}{m!} \\ &= \sum_{n=r}^{\infty} r! \sum_{m=r}^{n} \binom{n}{m} T(m,r) b_{n-m}^{(r)} \frac{t^n}{n!}, \end{aligned} \tag{24}$$

where $T(m,r)$ are the central factorial numbers of the second kind.

Therefore, by (24), we obtain the following theorem.

Theorem 2.5 *For $n, r \in \mathbb{N} \cup \{0\}$ with $n \geq r$, we have*

$$\sum_{m=r}^{n} \binom{n}{m} T(m,r) b_{n-m}^{(r)} = \begin{cases} 1 & \text{if } n = r, \\ 0 & \text{if } n > r, \end{cases}$$

where $T(m,r)$ is the central factorial number of the second kind.

From Lemma 2.1, we note that

$$\begin{aligned} b_n(x) &= \int_{\mathbb{Z}_p} \left(y + x + \frac{1}{2}\right)^n d\mu_1(y) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} \int_{\mathbb{Z}_p} \left(y + \frac{1}{2}\right)^l d\mu_1(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} x^{n-l} b_l. \end{aligned} \tag{25}$$

By (25), we get

$$b_n(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} b_l. \tag{26}$$

Now, we observe that

$$\begin{aligned} \sum_{k=0}^{n-1} e^{(k+\frac{1}{2})t} = e^{\frac{1}{2}t} \sum_{k=0}^{n-1} e^{kt} &= \frac{1}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} \left(e^{nt} - 1\right) \\ &= \left(\frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} e^{nt} - \frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}\right) \frac{1}{t} \\ &= \frac{1}{t} \sum_{m=0}^{\infty} \left(b_m(n) - b_m\right) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \frac{(b_{m+1}(n) - b_{m+1})}{m+1} \frac{t^m}{m!}. \end{aligned} \tag{27}$$

On the other hand,

$$\sum_{k=0}^{n-1} e^{(k+\frac{1}{2})t} = \sum_{m=0}^{\infty}\sum_{k=0}^{n-1}\left(k+\frac{1}{2}\right)^m \frac{t^m}{m!}. \tag{28}$$

By (27) and (28), we get

$$\sum_{k=0}^{n-1}(2k+1)^m = 2^m\left(\frac{b_{m+1}(n)-b_{m+1}}{m+1}\right). \tag{29}$$

Therefore, by (29), and interchanging m and n, we obtain the following theorem.

Theorem 2.6 *For $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, we have*

$$1^n + 3^n + \cdots + (2m-1)^n = 2^n\left(\frac{b_{n+1}(m)-b_{n+1}}{n+1}\right).$$

We define the fully degenerate type 2 Bernoulli polynomials by

$$\frac{1}{\lambda}\left(\frac{\log(1+\lambda t)}{e_\lambda^{1/2}(t)-e_\lambda^{-1/2}(t)}\right)e_\lambda^x(t) = \sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!}. \tag{30}$$

When $x = 0, B_{n,\lambda} = B_{n,\lambda}(0)$ are called the fully degenerate type 2 Bernoulli numbers.

We note that

$$\begin{aligned}\int_{\mathbb{Z}_p} e_\lambda^{x+y+1/2}(t)\,d\mu_1(y) &= \frac{\log(1+\lambda t)}{\lambda}\cdot\frac{1}{e_\lambda^{1/2}(t)-e_\lambda^{-1/2}(t)}e_\lambda^x(t) \\ &= \sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!}.\end{aligned} \tag{31}$$

Thus, by (31) and (12) we obtain

$$\int_{\mathbb{Z}_p}\left(x+y+\frac{1}{2}\right)_{n,\lambda} d\mu_1(y) = B_{n,\lambda}(x). \tag{32}$$

As is known, the degenerate Stirling numbers of the first kind are defined by

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{1,\lambda}(n,l)x^l \quad (n \geq 0). \tag{33}$$

By (32), (33), and Lemma 2.1, we have

$$B_{n,\lambda}(x) = \sum_{l=0}^{n} S_{1,\lambda}(n,l)b_l(x). \tag{34}$$

Also, from (12) and (31) we observe that

$$\int_{\mathbb{Z}_p} e_\lambda^{x+y+1/2}(t)\,d\mu_1(y) = e_\lambda^x(t)\int_{\mathbb{Z}_p} e_\lambda^{y+1/2}(t)\,d\mu_1(y)$$

$$= \sum_{l=0}^{\infty} (x)_{l,\lambda} \frac{t^l}{l!} \sum_{m=0}^{\infty} B_{m,\lambda} \frac{t^m}{m!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} B_{m,\lambda}(x)_{n-m,\lambda} \frac{t^n}{n!}. \tag{35}$$

Therefore, from (32), (34), and (35), we have the following theorem.

Theorem 2.7 *For $n \geq 0$, we have*

$$B_{n,\lambda}(x) = \int_{\mathbb{Z}_p} \left(x + y + \frac{1}{2}\right)_{n,\lambda} d\mu_1(y) = \sum_{l=0}^{n} S_{1,\lambda}(n,l) b_l(x) = \sum_{m=0}^{n} \binom{n}{m} B_{m,\lambda}(x)_{n-m,\lambda}.$$

As is known, the degenerate Carlitz type 2 Bernoulli polynomials are defined by

$$\frac{t}{e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t)} e_\lambda^x(t) = \sum_{n=0}^{\infty} b_{n,\lambda}(x) \frac{t^n}{n!}. \tag{36}$$

When $x = 0, b_{n,\lambda} = b_{n,\lambda}(0), (n \geq 0)$ are called the degenerate Carlitz type 2 Bernoulli numbers.

It is well known that the Daehee numbers, denoted by d_n, are defined by

$$\frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} d_n \frac{t^n}{n!} \quad \text{(see [7, 16])}. \tag{37}$$

Now, from (31), (36), and (37), we observe that

$$\sum_{n=0}^{\infty} B_{n,\lambda} \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e_\lambda^{x+1/2}(t)\, d\mu_1(x) = \frac{\log(1+\lambda t)}{\lambda t} \frac{t}{e_\lambda^{1/2}(t) - e_\lambda^{-1/2}(t)}$$

$$= \sum_{l=0}^{\infty} \lambda^l d_l \frac{t^l}{l!} \sum_{m=0}^{\infty} b_{m,\lambda} \frac{t^m}{m!}$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} \lambda^l d_l b_{n-l,\lambda} \frac{t^n}{n!}. \tag{38}$$

Therefore, by (38) and (12), we obtain the following theorem.

Theorem 2.8 *For $n \geq 0$, we have*

$$B_{n,\lambda} = \int_{\mathbb{Z}_p} \left(x + \frac{1}{2}\right)_{n,\lambda} d\mu_1(x) = \sum_{l=0}^{n} \binom{n}{l} \lambda^l d_l b_{n-l,\lambda}.$$

For $n \in \mathbb{N}$, by (8), we easily get

$$\int_{\mathbb{Z}_p} f(x+m)\, d\mu_1(x) = \sum_{l=0}^{m-1} f'(x) + \int_{\mathbb{Z}_p} f(x)\, d\mu_1(x). \tag{39}$$

By applying (39) to $f(x) = e_\lambda^{x+\frac{1}{2}}(t)$, we get

$$\frac{1}{e_\lambda^{1/2}(t) - e_\lambda^{-1/2}(t)} e_\lambda^m(t) - \frac{1}{e_\lambda^{1/2}(t) - e_\lambda^{-1/2}(t)} = e_\lambda^{1/2}(t) \sum_{l=0}^{m-1} e_\lambda^l(t). \tag{40}$$

From (40), we derive the following equation:

$$\frac{1}{t} \sum_{n=0}^{\infty} \bigl(b_{n,\lambda}(m) - b_{n,\lambda}\bigr) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{l=0}^{m-1} \left(l + \frac{1}{2} \right)_{n,\lambda} \right) \frac{t^n}{n!}. \tag{41}$$

By (41), we get

$$\sum_{n=0}^{\infty} \left(\frac{b_{n+1,\lambda}(m) - b_{n+1,\lambda}}{n+1} \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{1}{2^n} \sum_{l=0}^{m-1} (2l+1)_{n,2\lambda} \right) \frac{t^n}{n!}. \tag{42}$$

Therefore, by (42), we obtain the following theorem.

Theorem 2.9 *For $n \geq 0, m \in \mathbb{N}$, we have*

$$\frac{2^n}{n+1} \bigl(b_{n+1,\lambda}(m) - b_{n+1,\lambda}\bigr) = \sum_{l=0}^{m-1} (2l+1)_{n,2\lambda}.$$

From (10), we observe that

$$\int_{\mathbb{Z}_p} e^{t(x+y+\frac{1}{2})} \, d\mu_{-1}(y) = \frac{2}{e^{t/2} + e^{-t/2}} e^{xt} = \operatorname{sech} \frac{t}{2} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{43}$$

Thus from (43) and (12), we have the following lemma.

Lemma 2.10 *For $n \geq 0$, we have*

$$\int_{\mathbb{Z}_p} \left(x + y + \frac{1}{2} \right)^n d\mu_{-1}(y) = E_n(x).$$

From Lemma 2.10, we have

$$\begin{aligned} E_n(x) &= \int_{\mathbb{Z}_p} \left(x + y + \frac{1}{2} \right)^n d\mu_{-1}(y) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} \int_{\mathbb{Z}_p} \left(y + \frac{1}{2} \right)^l d\mu_{-1}(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_l \quad (n \geq 0). \end{aligned} \tag{44}$$

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Then, by (10), we get

$$\int_{\mathbb{Z}_p} f(x+d) \, d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x) \, d\mu_{-1}(x) = 2 \sum_{l=0}^{d-1} (-1)^l f(l). \tag{45}$$

Let us take $f(x)=e^{(x+1/2)t}$. Then, by (45), we get

$$e^{mt}\int_{\mathbb{Z}_p} e^{(x+1/2)t}\,d\mu_{-1}(x)+\int_{\mathbb{Z}_p} e^{(x+1/2)t}\,d\mu_{-1}(x)=2\sum_{l=0}^{m-1}(-1)^l e^{(l+1/2)t}. \tag{46}$$

From (46), we have

$$\frac{2}{e^{t/2}+e^{-t/2}}e^{mt}+\frac{2}{e^{t/2}+e^{-t/2}}=2\sum_{l=0}^{m-1}(-1)^l e^{(l+1/2)t}. \tag{47}$$

By (5) and (47), we get

$$\sum_{n=0}^{\infty}\bigl(E_n(m)+E_n\bigr)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(2\sum_{l=0}^{m-1}(-1)^l\left(l+\frac{1}{2}\right)^n\right)\frac{t^n}{n!}. \tag{48}$$

Therefore, by (48), we obtain the following theorem.

Theorem 2.11 *For $m\in\mathbb{N}$ with $m\equiv 1 \pmod 2$, $n\in\mathbb{N}\cup\{0\}$, we have*

$$2^{n-1}\bigl(E_n(m)+E_n\bigr)=\sum_{l=0}^{m-1}(-1)^l(2l+1)^n.$$

The following lemma can be easily shown.

Lemma 2.12

$$\int_{\mathbb{Z}_p} f(x)\,d\mu_{-1}(x)=\sum_{a=0}^{d-1}(-1)^a\int_{\mathbb{Z}_p} f(a+dx)\,d\mu_{-1}(x),$$

where $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$.
Let us apply Lemma 2.12 to $f(y)=(x+y+1/2)^n$. Then we have

$$\begin{aligned}\int_{\mathbb{Z}_p}\left(x+y+\frac{1}{2}\right)^n d\mu_{-1}(y)&=\sum_{a=0}^{d-1}(-1)^a\int_{\mathbb{Z}_p}\left(x+a+dy+\frac{1}{2}\right)^n d\mu_{-1}(y)\\&=d^n\sum_{a=0}^{d-1}(-1)^a\int_{\mathbb{Z}_p}\left(\frac{x+a+\frac{1}{2}(1-d)}{d}+y+\frac{1}{2}\right)^n d\mu_{-1}(y).\end{aligned} \tag{49}$$

Therefore, by (49), we have the following theorem.

Theorem 2.13 *For $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$, $n\in\mathbb{N}\cup\{0\}$, we have*

$$E_n(x)=d^n\sum_{a=0}^{d-1}(-1)^a E_n\left(\frac{x+a+\frac{1}{2}(1-d)}{d}\right).$$

For $r\in\mathbb{N}$, let us consider the following fermionic p-adic integral on $\mathbb{Z}_p$:

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+x_2+\cdots+x_r+\frac{r}{2})t}\,d\mu_{-1}(x_1)\,d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_r)$$

$$= \left(\frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}} \right)^r = \left(\operatorname{sech} \frac{t}{2} \right)^r. \tag{50}$$

Let us define the type 2 Euler numbers of order r by

$$\left(\frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}} \right)^r = \left(\operatorname{sech} \frac{t}{2} \right)^r = \sum_{n=0}^{\infty} E_n^{(r)} \frac{t^n}{n!}. \tag{51}$$

From (50) and (51), we have

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(x_1 + x_2 + \cdots + x_r + \frac{r}{2} \right)^n d\mu_{-1}(x_1)\, d\mu_{-1}(x_2) \cdots d\mu_{-1}(x_r) = E_n^{(r)} \quad (n \geq 0). \tag{52}$$

On the other hand,

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(x_1 + x_2 + \cdots + x_r + \frac{r}{2} \right)^n d\mu_{-1}(x_1)\, d\mu_{-1}(x_2) \cdots d\mu_{-1}(x_r) \\
&= \sum_{\substack{i_1+i_2+\cdots+i_r=n \\ i_1,i_2,\ldots,i_r \geq 0}} \binom{n}{i_1, \ldots, i_r} \int_{\mathbb{Z}_p} \left(x_1 + \frac{1}{2} \right)^{i_1} d\mu_{-1}(x_1) \cdots \int_{\mathbb{Z}_p} \left(x_r + \frac{1}{2} \right)^{i_r} d\mu_1(x_r) \\
&= \sum_{\substack{i_1+i_2+\cdots+i_r=n \\ i_1,i_2,\ldots,i_r \geq 0}} \binom{n}{i_1, \ldots, i_r} E_{i_1} E_{i_2} \cdots E_{i_r}.
\end{aligned} \tag{53}$$

Therefore, by (52) and (53), we obtain the following theorem.

Theorem 2.14 *For $n \geq 0$, we have*

$$E_n^{(r)} = \sum_{\substack{i_1+i_2+\cdots+i_r=n \\ i_1,i_2,\ldots,i_r \geq 0}} \binom{n}{i_1, \ldots, i_r} E_{i_1} E_{i_2} \cdots E_{i_r}.$$

From (51), we have

$$\begin{aligned}
2^r &= \sum_{l=0}^{\infty} E_l^{(r)} \frac{t^l}{l!} \left(e^{\frac{t}{2}} + e^{-\frac{t}{2}} \right)^r \\
&= \sum_{l=0}^{\infty} E_l^{(r)} \frac{t^l}{l!} \sum_{j=0}^{r} \binom{r}{j} e^{(j-\frac{r}{2})t} \\
&= \sum_{l=0}^{\infty} E_l^{(r)} \frac{t^l}{l!} \sum_{m=0}^{\infty} \sum_{j=0}^{r} \binom{r}{j} \left(j - \frac{r}{2} \right)^m \frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{j=0}^{r} \binom{r}{j} \binom{n}{m} \left(j - \frac{r}{2} \right)^m E_{n-m}^{(r)} \frac{t^n}{n!}.
\end{aligned} \tag{54}$$

Comparing the coefficients on both sides of (54), we obtain the following theorem.

Theorem 2.15 *For $n \geq 0$, we have*

$$\sum_{m=0}^{n}\sum_{j=0}^{r}\binom{r}{j}\binom{n}{m}\left(j-\frac{r}{2}\right)^{m}E_{n-m}^{(r)}=\begin{cases}2^{r} & \text{if } n=0,\\ 0 & \text{if } n>0.\end{cases}$$

We define the degenerate type 2 Euler polynomials by

$$\frac{2}{e_{\lambda}^{1/2}(t)+e_{\lambda}^{-1/2}(t)}e_{\lambda}^{x}(t)=\sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^{n}}{n!}. \tag{55}$$

When $x=0$, $E_{n,\lambda}=E_{n,\lambda}(0)$ are called the degenerate type 2 Euler numbers.

From (10), we can derive the following equation:

$$\begin{aligned}\int_{\mathbb{Z}_p}e_{\lambda}^{x+y+\frac{1}{2}}(t)\,d\mu_{-1}(y)&=\frac{2}{e_{\lambda}^{1/2}(t)+e_{\lambda}^{-1/2}(t)}e_{\lambda}^{x}(t)\\&=\sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^{n}}{n!}.\end{aligned} \tag{56}$$

By (56) and (12), we get

$$E_{n,\lambda}(x)=\int_{\mathbb{Z}_p}\left(x+y+\frac{1}{2}\right)_{n,\lambda}d\mu_{-1}(y)\quad(n\geq 0). \tag{57}$$

By (57), (33), and Lemma 2.10, we get

$$E_{n,\lambda}(x)=\sum_{l=0}^{n}S_{1,\lambda}(n,l)E_{l}(x). \tag{58}$$

Also, from (12) and (56), we observe that

$$\begin{aligned}\int_{\mathbb{Z}_p}e_{\lambda}^{x+y+1/2}(t)\,d\mu_{-1}(y)&=e_{\lambda}^{x}(t)\int_{\mathbb{Z}_p}e_{\lambda}^{y+1/2}(t)\,d\mu_{-1}(y)\\&=\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^{l}}{l!}\sum_{m=0}^{\infty}E_{m,\lambda}\frac{t^{m}}{m!}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}E_{m,\lambda}(x)_{n-m,\lambda}\frac{t^{n}}{n!}.\end{aligned} \tag{59}$$

Therefore, by (57)–(59), we obtain the following theorem.

Theorem 2.16 *For $n \geq 0$, we have*

$$E_{n,\lambda}(x)=\int_{\mathbb{Z}_p}\left(x+y+\frac{1}{2}\right)_{n,\lambda}d\mu_{-1}(y)=\sum_{l=0}^{n}S_{1,\lambda}(n,l)E_{l}(x)=\sum_{m=0}^{n}\binom{n}{m}E_{m,\lambda}(x)_{n-m,\lambda}.$$

For $m\in\mathbb{N}$ with $m\equiv 1 \pmod 2$, from (45) we have

$$\int_{\mathbb{Z}_p}e_{\lambda}^{m+x+1/2}(t)\,d\mu_{-1}(x)+\int_{\mathbb{Z}_p}e_{\lambda}^{x+1/2}(t)\,d\mu_{-1}(x)=2\sum_{l=0}^{m-1}(-1)^{l}e_{\lambda}^{l+1/2}(t). \tag{60}$$

From (60), we have

$$\sum_{n=0}^{\infty}\left(E_{n,\lambda}(m)+E_{n,\lambda}\right)\frac{t^n}{n!}=2\sum_{n=0}^{\infty}\sum_{l=0}^{m-1}(-1)^l\left(l+\frac{1}{2}\right)_{n,\lambda}\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n-1}\sum_{l=0}^{m-1}(-1)^l(2l+1)_{n,2\lambda}\frac{t^n}{n!}. \tag{61}$$

Therefore, by (61), we obtain the following theorem.

Theorem 2.17 *For $n\geq 0, m\in\mathbb{N}$ with $m\equiv 1 \pmod 2$, we have*

$$2^{n-1}\left(E_{n,\lambda}(m)+E_{n,\lambda}\right)=\sum_{l=0}^{m-1}(-1)^l(2l+1)_{n,2\lambda}.$$

For $r\in\mathbb{N}$, we have

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e_{\lambda}^{x_1+\cdots+x_r+r/2}(t)\,d\mu_{-1}(x_1)\,d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_r)$$
$$=\left(\frac{2}{e_{\lambda}^{1/2}(t)+e_{\lambda}^{-1/2}(t)}\right)^r. \tag{62}$$

Now, we define the degenerate type 2 Euler numbers of order r which are given by

$$\left(\frac{2}{e_{\lambda}^{1/2}(t)+e_{\lambda}^{-1/2}(t)}\right)^r=\sum_{n=0}^{\infty}E_{n,\lambda}^{(r)}\frac{t^n}{n!}. \tag{63}$$

By (62), (63), and (12), we get

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(x_1+x_2+\cdots+x_r+\frac{r}{2}\right)_{n,\lambda}d\mu_{-1}(x_1)\,d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_r)=E_{n,\lambda}^{(r)}\quad(n\geq 0).$$

3 Conclusion

In recent years, studying degenerate versions of various special polynomials and numbers has attracted many mathematicians and has been carried out by several different methods like generating functions, combinatorial approaches, umbral calculus, p-adic analysis, and differential equations. In this paper, we introduced degenerate type 2 Bernoulli polynomials, fully degenerate type 2 Bernoulli polynomials, and degenerate type 2 Euler polynomials, and their corresponding numbers, as degenerate and type 2 versions of Bernoulli and Euler numbers. We investigated those polynomials and numbers by means of bosonic and fermionic p-adic integrals and derived some identities, distribution relations, Witt type formulas, and analogues for the Bernoulli interpretation of powers of the first m positive integers in terms of Bernoulli polynomials. In more detail, our main results are as follows.

As to the analogues for the Bernoulli interpretation of power sums, in Theorem 2.6 we expressed powers of the first m odd integers in terms of type 2 Bernoulli polynomials $b_n(x)$, in Theorem 2.11 alternating sum of powers of the first m odd integers in terms of type 2 Euler polynomials $E_n(x)$, in Theorem 2.9 sum of the values of the generalized falling factorials at the first m odd positive integers in terms of degenerate Carlitz type 2 Bernoulli polynomials $b_{n,\lambda}(x)$, and in Theorem 2.17 alternating sum of the values of the generalized falling factorials at the first m odd positive integers in terms of degenerate type 2 Euler polynomials $E_{n,\lambda}(x)$. Witt type formulas were obtained for $b_n(x), B_{n,\lambda}(x), E_n(x)$, and $E_{n,\lambda}(x)$ respectively in Lemma 2.1, Theorem 2.7, Lemma 2.10, and Theorem 2.16. Distribution relations were derived for $b_n(x)$ and $E_n(x)$ respectively in Theorem 2.3 and Theorem 2.13.

As one of our future projects, we would like to continue to do research on degenerate versions of various special numbers and polynomials and to find many applications of them in mathematics, science, and engineering.

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Author details
[1]Department of Mathematics, Sogang University, Seoul, Republic of Korea. [2]Department of Mathematics, Kwangwoon University, Seoul, Republic of Korea. [3]Department of Mathematics Education, Silla University, Busan, Republic of Korea.

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