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Existence of positive solutions of discrete third-order three-point BVP with sign-changing Green's function

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Abstract

Consider positive solutions and multiple positive solutions for a discrete nonlinear third-order boundary value problem

$$\begin{cases} \Delta^3 u(t-1) = a(t)f(t,u(t)), & t \in [1,T-2]_{\mathbb{Z}}, \\ \Delta u(0) = u(T) = 0, & \Delta^2 u(\eta) - \alpha \Delta u(T-1) = 0, \end{cases}$$

which has the sign-changing Green's function. Here $T > 8$ is a positive integer, $[1,T-1]_{\mathbb{Z}} = \{1,2,\dots,T-2\}$, $\alpha \in [0,\frac{1}{T-1})$, $a : [0,T-2]_{\mathbb{Z}} \to (0,+\infty)$, $f : [1,T-2]_{\mathbb{Z}} \times [0,\infty) \to [0,\infty)$ is continuous.

Keywords: Discrete third-order three-point boundary value problem; Positive solutions; Cone; Fixed point; Sign-changing Green's function

1 Introduction

Multi-point boundary value problems for differential equations have a wide application in computational physics, economics, and modern biological fields [1]. In 1992, Gupta [2] studied solvability of differential equation three-point boundary value problem. Soon afterwards, there arose many results on multi-point nonlinear boundary value problems [3–6]. In 1999, Ma [7] studied the existence of positive solution for a second-order differential equation three-point boundary value problem. Thereafter, many results for the existence of positive solutions on multi-point boundary value problems have been studied [7–19]. With the development of the computing science and the computer simulation, multi-point boundary value problems should be discretized, so we need to study corresponding difference equation [20–26]. In 1998, by using Krasnosel'skii's fixed point theorem, Agarwal and Henderson [24] studied the discrete problem

$$\begin{cases} \Delta^3 u(t-1) = \lambda a(t) f(t,u(t)), & t \in [2,T]_{\mathbb{Z}}, \\ u(0) = u(1) = u(T+3) = 0. \end{cases}$$

They obtained the existence of positive solutions in two cases for $\lambda = 1$ and $\lambda \neq 1$. Later, there were many interesting results on the positive solutions to the discrete boundary value problems, see, for instance, [23–26] and the references therein. It is noted that Green's functions are positive in most of these results. However, when the Green's function is sign-changing, could we also obtain the existence of positive solutions to these kinds of problems?

In 2015, by using the Guo–Krasnosel'skii fixed point theorem, Wang and Gao [25] studied the existence of positive solutions to the discrete third-order three-point boundary value problem

$$\begin{cases} \Delta^3 u(t-1) + a(t)f(t,u(t)) = 0, & t \in [1, T-1]_{\mathbb{Z}}, \\ u(0) = \Delta u(T) = \Delta^2 u(\eta) = 0, \end{cases}$$

where $\eta \in [1, [\frac{3T^2-3T-2}{6T+3}]]_{\mathbb{Z}}$ and the Green's function is sign-changing. In 2016, Geng and Gao [26], by using the Guo–Krasnosel'skii fixed point theorem, studied the discrete third-order three-point boundary value problem

$$\begin{cases} \Delta^3 u(t-1) + \lambda a(t)f(t,u(t)) = 0, & t \in [1, T-1]_{\mathbb{Z}}, \\ u(0) = \Delta u(t) = \Delta^2 u(\eta) = 0, \end{cases}$$

when f satisfies some superlinear and sublinear condition on 0 and ∞. For the continuous case, which has the sign-changing Green's function, one can see [27–29] and the references therein. Inspired by the above works, in this paper, we consider the existence and multiple positive solutions to the following discrete nonlinear third-order three-point BVP:

$$\begin{cases} \Delta^3 u(t-1) = a(t)f(t,u(t)), & t \in [1, T-2]_{\mathbb{Z}}, \\ \Delta u(0) = u(T) = 0, \qquad \Delta^2 u(\eta) - \alpha \Delta u(T-1) = 0, \end{cases} \tag{1.1}$$

where $T > 8$ is a positive integer, $\alpha \in [0, \frac{1}{T-1})$, $a : [1, T-2]_{\mathbb{Z}} \to (0, +\infty)$, $f : [1, T-2]_{\mathbb{Z}} \times [0, \infty) \to [0, \infty)$ is continuous, η satisfies

(H_0) $\eta \in [\lfloor \frac{T-2}{2} \rfloor + 1, T-2]_{\mathbb{Z}}$.

Under assumption (H_0), the Green's function of (1.1) changes its sign. The proof of our main results is based upon the following well-known Guo–Krasnoselskii fixed point theorems [30].

Theorem 1.1 *Let E be a Banach space and $K \subset E$ be a cone. Assume that Ω_1, Ω_2 are open bounded subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. If*

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$$

is a completely continuous operator such that

(i) $\|Au\| \le \|u\|$, $u \in K \cap \partial\overline{\Omega}_1$, *and* $\|Au\| \ge \|u\|$, $u \in K \cap \partial\overline{\Omega}_2$,

or

(ii) $\|Au\| \ge \|u\|$, $u \in K \cap \partial\overline{\Omega}_1$, *and* $\|Au\| \le \|u\|$, $u \in K \cap \partial\overline{\Omega}_2$,

then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Theorem 1.2 *Let E be a Banach space and K be a cone in E. For some $p > 0$, define $K_p = \{x \in K \mid \|x\| \le p\}$. Assume that $A : K_p \to K$ is a compact operator; if $x \in \partial K_p = \{x \in k \mid \|x\| = p\}$, $Ax \ne x$, we have*

(i) *For* $\forall x \in \partial K_p$, *if* $\|Ax\| \geq \|x\|$, *then* $i(A, K_p, K) = 0$,
(ii) *For* $\forall x \in \partial K_p$, *if* $\|Ax\| \leq \|x\|$, *then* $i(A, K_p, K) = 1$.

2 Preliminaries

Lemma 2.1 *Suppose that* (H_0) *holds. Then the linear problem*

$$\begin{cases} \Delta^3 u(t-1) = y(t), & t \in [1, T-2]_{\mathbb{Z}}, \\ \Delta u(0) = u(T) = 0, & \Delta^2 u(\eta) - \alpha \Delta u(T-1) = 0 \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \sum_{s=1}^{T-2} G(t,s) y(s), \tag{2.2}$$

where

$$G(t,s) = g(t,s) + k(t,s) + \begin{cases} \frac{T(T-1)-t(t-1)}{2-2\alpha(T-1)}, & s \leq \eta, \\ 0, & \eta < s, \end{cases}$$

$$g(t,s) = -\frac{\alpha(T-s-1)[T(T-1)-t(t-1)]}{2-2\alpha(T-1)} - \frac{(T-s)(T-s-1)}{2},$$

$$k(t,s) = \begin{cases} \frac{(t-s)(t-s-1)}{2}, & 0 < s \leq t-2 \leq T-2, \\ 0, & 0 \leq t-2 < s \leq T-2, \end{cases}$$

and

$$G(0,s) = G(1,s) = \begin{cases} \frac{-\alpha T(T-1)(T-s-1)+T(T-1)}{2-2\alpha(T-1)} - \frac{(T-1)(T-s-1)}{2}, & s \leq \eta, \\ \frac{-\alpha T(T-1)(T-s-1)}{2-2\alpha(T-1)} - \frac{(T-1)(T-s-1)}{2}, & \eta < s. \end{cases}$$

Proof Summing both sides of equation (2.1) from $s = 1$ to $s = t-1$, we get

$$\Delta^2 u(t-1) = \Delta^2 u(0) + \sum_{s=1}^{t-1} y(s),$$

and

$$\Delta u(t-1) = (t-1)\Delta^2 u(0) + \sum_{s=1}^{t-2} (t-s-1) y(s).$$

Then summing both sides from $\tau = 1$ to $\tau = t$, we get

$$u(t) = u(0) + \frac{t(t-1)}{2}\Delta^2 u(0) + \sum_{s=1}^{t-2} \frac{(t-s)(t-s-1)}{2} y(s).$$

From boundary conditions $\Delta u(0) = u(T) = 0$, $\Delta^2 u(\eta) - \alpha \Delta u(T-1) = 0$, we have

$$\begin{cases} u(0) + \frac{T(T-1)}{2}\Delta^2 u(0) + \sum_{s=1}^{T-2} \frac{(T-s)(T-s-1)}{2} y(s) = 0; \\ \Delta^2 u(0) + \sum_{s=1}^{\eta} y(s) - \alpha(T-1)\Delta^2 u(0) - \alpha \sum_{s=1}^{T-2}(T-s-1)y(s) = 0. \end{cases}$$

Furthermore, we get

$$\begin{cases} u(0) = \sum_{s=1}^{T-2} \frac{-\alpha T(T-1)(T-s-1)}{2-2\alpha(T-1)} y(s) + \sum_{s=1}^{\eta} \frac{T(T-1)}{2-2\alpha(T-1)} y(s) \\ \qquad - \sum_{s=1}^{T-2} \frac{(T-s)(T-s-1)}{2} y(s), \\ \Delta^2 u(0) = \sum_{s=1}^{T-2} \frac{\alpha(T-s-1)}{1-\alpha(T-1)} y(s) - \sum_{s=1}^{\eta} \frac{1}{1-\alpha(T-1)} y(s). \end{cases}$$

Then we have

$$\begin{aligned} u(t) = {} & \sum_{s=1}^{T-2} \left(-\frac{\alpha(T-s-1)[T(T-1)-t(t-1)]}{2-2\alpha(T-1)} - \frac{(T-s)(T-s-1)}{2} \right) y(s) \\ & + \sum_{s=1}^{t-2} \frac{(t-s)(t-s-1)}{2} y(s) \\ & + \sum_{s=1}^{\eta} \frac{T(T-1)-t(t-1)}{2-2\alpha(T-1)} y(s). \end{aligned}$$

□

Lemma 2.2 *Suppose that* (H_0) *holds. Then the Green's function* $G(t,s)$ *changes its sign on* $[0,T]_{\mathbb{Z}} \times [0,T]_{\mathbb{Z}}$. *More precisely,*

(i) *If* $s \in [1,\eta]_{\mathbb{Z}}$, *then* $\Delta_t G(t,s) \le 0$, $G(t,s) \ge 0$ *for* $\forall t \in [0,T]_{\mathbb{Z}}$. *Furthermore,*

$$\min_{t\in[0,T]_{\mathbb{Z}}} G(t,s) = G(T,s) = 0,$$
$$\max_{t\in[0,T]_{\mathbb{Z}}} G(t,s) = G(0,s) \le \frac{T(T-1)(1+\alpha\eta)}{1-\alpha(T-1)}.$$

(ii) *If* $s \in [\eta+1, T-2]_{\mathbb{Z}}$, *then* $\Delta_t G(t,s) \ge 0$, $G(t,s) \le 0$ *for* $\forall t \in [0,T]_{\mathbb{Z}}$. *Furthermore,*

$$\max_{t\in[0,T]_{\mathbb{Z}}} G(t,s) = G(T,s) = 0,$$
$$\min_{t\in[0,T]_{\mathbb{Z}}} G(t,s) = G(0,s) \ge \frac{-T(T-1)(1+\alpha\eta)}{1-\alpha(T-1)}.$$

Proof (i) If $s \in [1,\eta]_{\mathbb{Z}}$, we have

$$\Delta_t G(t,s) = \begin{cases} \frac{\alpha t(T-s-1)-2t}{1-\alpha(T-1)}, & t-2 \le s, \\ \frac{\alpha t(T-s-1)-2t}{1-\alpha(T-1)} + t - s, & s \le t-2. \end{cases}$$

If $s \le t-2$, since $\alpha(T-1)-1 < 0$, we get

$$\begin{aligned} \Delta_t G(t,s) &= \frac{2\alpha t(T-s-1)-2t}{2-2\alpha(T-1)} + t - s \\ &= \frac{2\alpha t(T-s-1)-2t+(t-s)(2-2\alpha(T-1))}{2-2\alpha(T-1)} \end{aligned}$$

$$= \frac{-2s\alpha + 2s[\alpha(T-1)-1]}{2-2\alpha(T-1)}$$
$$< 0.$$

If $t-2 \le s$, since $\alpha < \frac{1}{T-1}$, so $2-2\alpha(T-1) > 0$, $\alpha(T-s-1)-1 < 0$, we have

$$\Delta_t G(t,s) = \frac{2\alpha t(T-s-1)-2t}{2-2\alpha(T-1)} < 0.$$

For $\forall t \in [0,T-1]_{\mathbb{Z}}$, $\Delta_t G(t,s) \le 0$. If $s \in [1,\eta]_{\mathbb{Z}}$, $G(t,s) \ge 0$. So

$$\min_{t\in[0,T]_{\mathbb{Z}}} G(t,s) = G(T,s) = 0,$$

$$\begin{aligned}\max_{t\in[0,T]_{\mathbb{Z}}} G(t,s) = G(0,s) & \\ &= \frac{T(T-1)[1-\alpha(T-s-1)]}{2-2\alpha(T-1)} - \frac{(T-s)(T-s-1)}{2} \\ &\le \frac{T(T-1)[1-\alpha(T-\eta-1)]}{2-2\alpha(T-1)} - \frac{(T-\eta)(T-\eta-1)}{2} \\ &\le \frac{T(T-1)(1+\alpha\eta)}{1-\alpha(T-1)}.\end{aligned}$$

(ii) If $s \in [\eta+1,T]_{\mathbb{Z}}$, we have

$$\Delta_t G(t,s) = \begin{cases} \frac{2\alpha t(T-s-1)}{2-2\alpha(T-1)}, & s \ge t-2, \\ \frac{2\alpha t(T-s-1)}{2-2\alpha(T-1)} + t - s, & s \le t-2. \end{cases}$$

If $t-2 \le s$, since $\alpha < \frac{1}{T-1}$, $2-2\alpha(T-1) > 0$, we have

$$\Delta_t G(t,s) = \frac{2\alpha t(T-s-1)}{2-2\alpha(T-1)} > 0.$$

If $s \le t-2$, then $t-s > 0$, so

$$\Delta_t G(t,s) = \frac{2\alpha t(T-s-1)}{2-2\alpha(T-1)} + t - s > 0.$$

If $s \in [\eta+1,T]_{\mathbb{Z}}$, for $\forall t \in [0,T]_{\mathbb{Z}}$, we have

$$\max_{t\in[0,T]_{\mathbb{Z}}} G(t,s) = G(T,s) = 0,$$

$$\begin{aligned}\min_{t\in[0,T]_{\mathbb{Z}}} G(t,s) = G(0,s) & \\ &= \frac{-\alpha T(T-1)(T-s-1)}{2-2\alpha(T-1)} - \frac{(T-s)(T-s-1)}{2} \\ &\ge \frac{-\alpha T(T-1)(T-\eta-1)}{2-2\alpha(T-1)} - \frac{(T-\eta)(T-\eta-1)}{2} \\ &\ge \frac{-T(T-1)(1+\alpha\eta)}{1-\alpha(T-1)}.\end{aligned}$$

In conclusion, if $s \in [1,\eta]_{\mathbb{Z}}$, $G(t,s)$ is decreasing, then $G(t,s) \geq 0$, since $\min_{t\in[0,T]_{\mathbb{Z}}} G(t,s) = 0$; if $s \in [\eta+1,T]_{\mathbb{Z}}$, $G(t,s)$ is increasing, then $G(t,s) \leq 0$ since $\max_{t\in[0,T]_{\mathbb{Z}}} G(t,s) = 0$. So, the Green's function $G(t,s)$ is a sign-changing function on $[0,T]_{\mathbb{Z}} \times [0,T]_{\mathbb{Z}}$. □

Remark Now, we give a brief explanation for the reason why we choose

$$\eta \in \left[\left\lfloor \frac{T-2}{2} \right\rfloor + 1, T-2\right]_{\mathbb{Z}}. \tag{2.3}$$

Consider the problem

$$\begin{cases} \Delta^3 u(t-1) = 1, \quad t \in [1,T-2]_{\mathbb{Z}}, \\ \Delta u(0) = u(T) = 0, \qquad \Delta^2 u(\eta) - \alpha \Delta u(T-1) = 0. \end{cases} \tag{2.4}$$

From Lemma 2.1, we get

$$\begin{aligned} u(t) = {} & \frac{1}{12-12\alpha(T-1)} \bigl\{3\alpha\bigl[t(t-1) - T(T-1)\bigr](T-1)(T-2) \\ & + 6\eta\bigl[T(T-1) - t(t-1)\bigr] - 2T(T-1)(T-2)\bigl[1-\alpha(T-1)\bigr] \\ & + 2t(t-1)(t-2)\bigl[1-\alpha(T-1)\bigr]\bigr\} \\ = {} & \frac{\phi(t)}{12-12\alpha(T-1)}, \end{aligned}$$

where

$$\begin{aligned} \phi(t) = {} & 3\alpha\bigl[t(t-1) - T(T-1)\bigr](T-1)(T-2) + 6\eta\bigl[T(T-1) - t(t-1)\bigr] \\ & - 2T(T-1)(T-2)\bigl[1-\alpha(T-1)\bigr] + 2t(t-1)(t-2)\bigl[1-\alpha(T-1)\bigr]. \end{aligned}$$

Clearly, $u(t) \geq 0$ is equivalent to $\phi(t) \geq 0$, and

$$\Delta\phi(t) = 6t\bigl\{(t-1)\bigl[1-\alpha(T-1)\bigr] - 2\eta + \alpha(T-1)(T-2)\bigr\}.$$

Let $\Delta\phi(t) = 0$. Then $t = 0$ or $t = 1 + \frac{2\eta-\alpha(T-1)(T-2)}{1-\alpha(T-1)}$. Therefore,

$$\Delta\phi(t) > 0 \quad \Leftrightarrow \quad t > \frac{2\eta+1-\alpha(T-1)^2}{1-\alpha(T-1)}.$$

Now, we claim that if (2.3) holds, then $\phi(t)$ is a positive solution of (2.4). In fact, if (2.3) holds, then $\frac{2\eta+1-\alpha(T-1)^2}{1-\alpha(T-1)} \geq T-1$ in this case, this implies that

$$\Delta\phi(t) \leq 0, \quad t \in [0,T-1]_{\mathbb{Z}}.$$

More precisely, $\Delta\phi(0) = 0$, $\Delta\phi(t) < 0$ for $t \in [1,T-2]_{\mathbb{Z}}$ and $\Delta\phi(T-1) < 0$ for $\frac{2\eta+1-\alpha(T-1)^2}{1-\alpha(T-1)} > T-1$ and $\Delta\phi(T-1) = 0$ for $\frac{2\eta+1-\alpha(T-1)^2}{1-\alpha(T-1)} = T-1$. Therefore, $\phi(t)$ is a positive solution of the linear problem (2.4) since $\phi(T) = 0$.

Let

$$E = \{u : [0,T]_{\mathbb{Z}} \to \mathbb{R} | \Delta u(0) = u(T), \Delta^2 u(\eta) - \alpha \Delta u(T-1) = 0\}.$$

Then E is a Banach space with norm $\|u\| = \max_{t\in[0,T]_{\mathbb{Z}}} |u(t)|$.

Let

$$K_0 = \{y \in E : y(t) \geq 0, t \in [0,T]_{\mathbb{Z}}; \Delta y(t) \leq 0, t \in [0,T-1]_{\mathbb{Z}}\}.$$

Then K_0 is a cone in E.

Lemma 2.3 *Let* (H_0) *hold. If* $y \in K_0$, *then the solution* $u(t)$ *of problem* (2.1) *belongs to* K_0, *i.e.,* $u \in K_0$. *Furthermore,* $u(t)$ *is concave on* $[0,\eta]_{\mathbb{Z}}$.

Proof Firstly, if $0 \leq t-2 \leq \eta$, then

$$\begin{aligned} u(t) = {} & \sum_{s=\eta+1}^{T-2} \left\{ \frac{-\alpha(T-s-1)[T(T-1)-t(t-1)]}{2-2\alpha(T-1)} - \frac{(T-s)(T-s-1)}{2} \right\} y(s) \\ & + \sum_{s=1}^{\eta} \left\{ \frac{[1-\alpha(T-s-1)][T(T-1)-t(t-1)]}{2-2\alpha(T-1)} - \frac{(T-s)(T-s-1)}{2} \right\} y(s) \\ & + \sum_{s=1}^{t-2} \frac{(t-s)(t-s-1)}{2} y(s). \end{aligned}$$

Then we obtain that

$$\begin{aligned} \Delta u(t) &= u(t+1) - u(t) \\ &= -\sum_{s=1}^{\eta} \frac{t[1-\alpha(T-s-1)]}{1-\alpha(T-1)} y(s) + \sum_{s=1}^{t-2} (t-s)y(s) + y(t-1) + \sum_{s=\eta+1}^{T-2} \frac{\alpha t(T-s-1)}{1-\alpha(T-1)} y(s) \\ &= \sum_{s=1}^{t-2} \frac{-\alpha ts + s(\alpha(T-1)-1)}{1-\alpha(T-1)} y(s) - \sum_{s=t-1}^{\eta} \frac{t[1-\alpha(T-s-1)]}{1-\alpha(T-1)} y(s) + y(t-1) \\ &\quad + \sum_{s=\eta+1}^{T-2} \frac{\alpha t(T-s-1)}{1-\alpha(T-1)} y(s). \end{aligned} \tag{2.5}$$

Since $y \in K_0$, we know that y is decreasing on $[0,T]_{\mathbb{Z}}$. That is to say, $y(t) \geq y(\eta)$ for $t \leq \eta$ and $y(t) \leq y(\eta)$ for $t \geq \eta$. Therefore, if $t-1 \leq \eta$, then

$$\begin{aligned} & \Delta u(t) \\ & = \sum_{s=1}^{t-1} \frac{-\alpha ts + s(\alpha(T-1)-1)}{1-\alpha(T-1)} y(s) - \sum_{s=t-1}^{\eta} \frac{t[1-\alpha(T-s-1)]}{1-\alpha(T-1)} y(s) \\ & \quad + \sum_{s=\eta+1}^{T-2} \frac{\alpha t(T-s-1)}{1-\alpha(T-1)} y(s) \end{aligned}$$

$$\begin{aligned}
&\le y(\eta)\left\{\sum_{s=1}^{t-1}\frac{-\alpha ts+s(\alpha(T-1)-1)}{1-\alpha(T-1)}-\sum_{s=t-1}^{\eta}\frac{t[1-\alpha(T-s-1)]}{1-\alpha(T-1)}+\sum_{s=\eta+1}^{T-2}\frac{\alpha t(T-s-1)}{1-\alpha(T-1)}\right\}\\
&= y(\eta)\left\{-\sum_{s=1}^{\eta}\frac{t[1-\alpha(T-s-1)]}{1-\alpha(T-1)}+\sum_{s=1}^{t-1}(t-s)+\sum_{s=\eta+1}^{T-2}\frac{\alpha t(T-s-1)}{1-\alpha(T-1)}\right\}.
\end{aligned}$$

If $t-1>\eta$, then $y(t-1)\le y(\eta)$. Therefore, by (2.5), no matter $t-1\le\eta$ or $t-1>\eta$, we always have

$$\begin{aligned}
\Delta u(t) &\le y(\eta)\left\{-\sum_{s=1}^{\eta}\frac{t[1-\alpha(T-s-1)]}{1-\alpha(T-1)}+\sum_{s=1}^{t-1}(t-s)+\sum_{s=\eta+1}^{T-2}\frac{\alpha t(T-s-1)}{1-\alpha(T-1)}\right\}\\
&= \frac{ty(\eta)}{1-\alpha(T-1)}\left\{-\eta+\frac{(t-1)[1-\alpha(T-1)]}{2}+\frac{\alpha(T-1)(T-2)}{2}\right\}\\
&= \frac{1}{1-\alpha(T-1)}\bigl[-2\eta+(t-1)+\alpha(T-1)(T-t-1)\bigr].
\end{aligned}$$

Combining this with $\eta\ge\frac{T-2}{2}$, we have

$$\Delta u(t)\le (T-t-1)\bigl(\alpha(T-1)-1\bigr)<0.$$

Furthermore, we have

$$\begin{aligned}
&\Delta^2 u(t-1)\\
&\quad= -\sum_{s=1}^{\eta}\frac{1-\alpha(T-s-1)}{1-\alpha(T-1)}y(s)+\sum_{s=1}^{t}y(s)+\sum_{s=\eta+1}^{T-2}\frac{\alpha(T-s-1)}{1-\alpha(T-1)}y(s)\\
&\quad= -\sum_{s=1}^{\eta}\frac{1-\alpha(T-s-1)}{1-\alpha(T-1)}y(s)+\sum_{s=1}^{t-2}y(s)+y(t-1)+y(t)+\sum_{s=\eta+1}^{T-2}\frac{\alpha(T-s-1)}{1-\alpha(T-1)}y(s).
\end{aligned}$$

Now, if $t-1\le\eta$ and $t\le\eta$, then

$$\begin{aligned}
&\Delta^2 u(t-1)\\
&\quad\le -\sum_{s=1}^{t-2}\frac{s\alpha}{1-\alpha(T-1)}y(s)-\sum_{s=t-1}^{\eta}\frac{1-\alpha(T-s-1)}{1-\alpha(T-1)}y(s)+\sum_{s=\eta+1}^{T-2}\frac{\alpha(T-s-1)}{1-\alpha(T-1)}y(s)+2y(\eta).
\end{aligned}$$

Combining this with the monotonicity of y, we get that

$$\begin{aligned}
&\Delta^2 u(t-1)\\
&\quad\le y(\eta)\left\{-\sum_{s=1}^{t-2}\frac{s\alpha}{1-\alpha(T-1)}-\sum_{s=t-1}^{\eta}\frac{1-\alpha(T-s-1)}{1-\alpha(T-1)}+\sum_{s=\eta+1}^{T-2}\frac{\alpha(T-s-1)}{1-\alpha(T-1)}+2\right\}\\
&\quad= y(\eta)\left\{-\sum_{s=1}^{\eta}\frac{1-\alpha(T-s-1)}{1-\alpha(T-1)}+t+\sum_{s=\eta+1}^{T-2}\frac{\alpha(T-s-1)}{1-\alpha(T-1)}\right\}\\
&\quad= \frac{y(\eta)}{1-\alpha(T-1)}\left\{-\eta+\frac{t[1-\alpha(T-1)]}{2}+\frac{\alpha(T-1)}{2}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{y(\eta)}{1-\alpha(T-1)} \frac{-2\eta+2t+\alpha(T-1)(T-2)}{2} \\
&\leq 0.
\end{aligned}
$$

For other cases, $t-1 \leq \eta < t$ and $t-1 > \eta$, we could also obtain $\Delta^2 u(t-1) \leq 0$ for $0 \leq t-2 \leq \eta$.

Secondly, if $\eta < t-2 \leq T-2$, then

$$
\begin{aligned}
u(t) ={}& \sum_{s=t-1}^{T-2} \left\{ -\frac{\alpha(T-s-1)[T(T-1)-t(t-1)]}{2-2\alpha(T-1)} - \frac{(T-s)(T-s-1)}{2} \right\} y(s) \\
&+ \sum_{s=1}^{t-2} \left\{ \frac{-\alpha(T-s-1)[T(T-1)-t(t-1)]}{2-2\alpha(T-1)} - \frac{(T-s)(T-s-1)}{2} \right. \\
&\left. + \frac{(t-s)(t-s-1)}{2} \right\} y(s) + \sum_{s=1}^{\eta} \frac{T(T-1)-t(t-1)}{2-2\alpha(T-1)} y(s)
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta u(t) &= \sum_{s=1}^{T-2} \frac{\alpha t(T-s-1)}{1-\alpha(T-1)} y(s) + \sum_{s=1}^{t-1} (t-s) y(s) - \sum_{s=1}^{\eta} \frac{t}{1-\alpha(T-1)} y(s) \\
&= -\sum_{s=1}^{\eta} \frac{\alpha t s + s(1-\alpha(T-1))}{1-\alpha(T-1)} y(s) + \sum_{s=\eta+1}^{T-2} \frac{\alpha t(T-s-1)}{(1-\alpha(T-1))} y(s) + \sum_{s=\eta+1}^{t-1} (t-s) y(s).
\end{aligned}
$$

Combining this with the fact that $y \in K_0$ and the monotonicity of y, we get that

$$
\begin{aligned}
\Delta u(t) &\leq y(\eta) \left\{ -\sum_{s=1}^{\eta} \frac{\alpha t s + s(1-\alpha(T-1))}{1-\alpha(T-1)} + \sum_{s=\eta+1}^{T-2} \frac{\alpha t(T-s-1)}{1-\alpha(T-1)} + \sum_{s=\eta+1}^{t-1} (t-s) \right\} \\
&= \frac{y(\eta)}{2-2\alpha(T-1)} \bigl[\alpha t(T-1)(T-2) + t(t-1)\bigl(1-\alpha(T-1)\bigr) - 2t\eta \bigr].
\end{aligned}
$$

Since $\eta \geq \frac{T-2}{2}$, then we get that

$$
\Delta u(t) \leq y(\eta)\bigl(1-\alpha(T-2)\bigr) t(t-T) \leq 0.
$$

So, for $\forall t \in [0, T-1]_{\mathbb{Z}}$, $\Delta u(t) \leq 0$, which implies that $u(t)$ is decreasing. Since $u(T) = 0$, for $\forall t \in [0, T]_{\mathbb{Z}}$, we have $u(t) \geq 0$ and $u \in K_0$. For $\forall t \in [1, \eta+1]_{\mathbb{Z}}$, $\Delta^2 u(t-1) \leq 0$, we get that $u(t)$ is concave on $[0, \eta+2]_{\mathbb{Z}}$. □

Lemma 2.4 *Let* (H_0) *hold. If* $y \in K_0$, *then the solution* $u(t)$ *of* (2.1) *satisfies*

$$
\min_{t \in [T-\theta, \theta]_{\mathbb{Z}}} u(t) = u(\theta) \geq \frac{\eta+2-\theta}{\eta+2} \|u\| = \theta^* \|u\|,
$$

where $\theta^* = \frac{\eta+2-\theta}{\eta+2}$, $\theta \in [\lfloor \frac{T}{2} \rfloor + 1, \eta+2]_{\mathbb{Z}}$.

Proof From Lemma 2.3, $u(t)$ is concave on $[0,\eta+2]_{\mathbb{Z}}$. So, $u(t)$ satisfies

$$\frac{u(t)-u(0)}{t} \geq \frac{u(\eta+2)-u(0)}{\eta+2}, \quad t \in [0,\eta+2]_{\mathbb{Z}}.$$

Meanwhile, from Lemma 2.3, $u(t)$ is non-increasing on $[0,T]_{\mathbb{Z}}$, which implies that $u(0) = \|u\|$. Therefore,

$$u(t) \geq \frac{\eta+2-t}{\eta+2}u(0) = \frac{\eta+2-t}{\eta+2}\|u\|,$$
$$\min_{t\in[T-\theta,\theta]_{\mathbb{Z}}} u(t) = u(\theta) \geq \frac{\eta+2-\theta}{\eta+2}\|u\| = \theta^*\|u\|. \qquad \square$$

3 Main results

(H_1) $f : [1,T-2]_{\mathbb{Z}} \times [0,\infty) \to [0,\infty)$ is continuous. For $u \in [0,\infty)$, $f(t,u)$ is a decreasing function with respect to t, and for $t \in [1,T-2]_{\mathbb{Z}}$, $f(t,u)$ is an increasing function with respect to u.

(H_2) $a : [1,T-2]_{\mathbb{Z}} \to [0,\infty)$ is a decreasing function.

Let

$$K = \Big\{u \in K_0 \big| \min_{t\in[T-\theta,\theta]_{\mathbb{Z}}} u(t) \geq \theta^*\|u\|\Big\}.$$

Then K is a cone in E. Define the operator $S : K \to E$ as

$$Su(t) = \sum_{s=1}^{T-2} G(t,s)a(s)f\big(s,u(s)\big). \tag{3.1}$$

Lemma 3.1 *$S : K \to K$ is completely continuous.*

Proof It is obvious that $S : K \to E$ is completely continuous since the Banach space E is finite dimensional. Now, let us prove that $S : K \to K$, that is to say, for any $u \in K$, $Su \in K$.

Let $u \in K$. Then $u \in K_0$, which implies that $\Delta u(t) \leq 0$ and u is decreasing on t. Therefore, by (H_1), $f(t,u(t))$ is a decreasing function of t. Let $y(t) := a(t)f(t,u(t))$. Then, from (H_1) and (H_2), we obtain that $y(t) \geq 0$ and y is also a decreasing function of t. Thus, $y \in K_0$. Furthermore, by (3.1), we know that

$$\Delta^3(Su)(t-1) = y(t), \quad t \in [1,T-2]_{\mathbb{Z}}, \tag{3.2}$$

and

$$\Delta(Su)(0) = (Su)(T) = 0, \qquad \Delta^2(Su)(\eta) - \alpha\Delta(Su)(T-1) = 0. \tag{3.3}$$

Therefore, Su satisfies problem (2.1). Now, similar to the proof of Lemma 2.3, and using the fact $y \in K_0$, we obtain that $Su \in K_0$ and Su is concave on $[0,\eta]_{\mathbb{Z}}$. Furthermore, by

Lemma 2.4 and the fact $Su \in K_0$, we know that

$$\min_{t\in[T-\theta,\theta]_{\mathbb{Z}}}(Su)(t) \geq \theta^*\|Su\|.$$

Therefore, $Su \in K$ and $S : K \to K$ is completely continuous. □

From (3.1) and Lemma 3.1, we know that if u is a fixed point of S in K, then u is a positive solution of (1.1). In the rest of this paper, we try to prove S has at least one or two fixed point(s) in K by using Theorem 1.1 and Theorem 1.2.

Let

$$A = \sum_{s=1}^{T-2}\frac{T(T-1)(1+\alpha\eta)}{1-\alpha(T-1)}a(s), \qquad B = \sum_{s=T-\theta}^{\theta}\frac{\theta(T-\theta)[2-\alpha(\theta-1)]}{2-2\alpha(T-1)}a(s).$$

Theorem 3.1 *Suppose that* (H_0), (H_1), *and* (H_2) *hold. If there exist two constants* r *and* R $(r \neq R)$ *such that*

(A_1) $f(t,u) \leq \frac{r}{A}$, $(t,u) \in [1,T-2]_{\mathbb{Z}} \times [0,r]$;

(A_2) $f(t,u) \geq \frac{R}{B}$, $(t,u) \in [1,T-2]_{\mathbb{Z}} \times [\theta^*R,R]$,

then problem (1.1) *has at least one positive solution* $u \in K$ *with* $\min\{r,R\} \leq \|u\| \leq \max\{r,R\}$.

Proof Without loss of generality, suppose that $r < R$, the other case could be treated similarly. Let $\Omega_1 = \{u \in E : \|u\| < r\}$. From Lemma 2.2, $G(t,s) \leq 0$ for $s \in [\eta+1,T-2]_{\mathbb{Z}}$; $G(t,s) \geq 0$ for $s \in [1,\eta]_{\mathbb{Z}}$. Since (A_1), we get, for $\forall u \in K \cap \partial\Omega_1$,

$$\begin{aligned}
\|Su\| &= \max_{t\in[0,T]_{\mathbb{Z}}}\left|\sum_{s=1}^{T-2}G(t,s)a(s)f\bigl(s,u(s)\bigr)\right| \\
&\leq \max_{t\in[0,T]_{\mathbb{Z}}}\sum_{s=1}^{T-2}\bigl|G(t,s)\bigr|a(s)f\bigl(s,u(s)\bigr) \\
&\leq \sum_{s=1}^{T-2}\frac{T(T-1)(1+\alpha\eta)}{1-\alpha(T-1)}a(s)f\bigl(s,u(s)\bigr) \\
&\leq \sum_{s=1}^{T-2}\frac{T(T-1)(1+\alpha\eta)}{1-\alpha(T-1)}a(s)\frac{r}{A} \\
&= r.
\end{aligned}$$

So, for $u \in K \cap \partial\Omega_1$,

$$\|Su\| \leq \|u\|. \tag{3.4}$$

Let $\Omega_2 = \{u \in E : \|u\| < R\}$. Then, for $u \in K \cap \partial\Omega_2$,

$$Su(T-\theta) = \sum_{s=1}^{T-2}G(T-\theta,s)a(s)f\bigl(s,u(s)\bigr) \geq \sum_{s=\theta}^{T-\theta}G(T-\theta,s)a(s)f\bigl(s,u(s)\bigr). \tag{3.5}$$

In fact, by Lemma 2.2,

$$\begin{aligned}
&\sum_{s=1}^{T-\theta-1} G(T-\theta,s)a(s)f\bigl(s,u(s)\bigr) + \sum_{s=\theta+1}^{T-2} G(T-\theta,s)a(s)f\bigl(s,u(s)\bigr) \\
&\quad\geq \sum_{s=1}^{T-\theta-1} G(T-\theta,s)a(s)f\bigl(s,u(s)\bigr) + \sum_{s=\eta+1}^{T-2} G(T-\theta,s)a(s)f\bigl(s,u(s)\bigr) \\
&\quad\geq a(\eta)f\bigl(\eta,u(\eta)\bigr)\left[\sum_{s=1}^{T-\theta-1} G(T-\theta,s) + \sum_{s=\eta+1}^{T-2} G(T-\theta,s)\right].
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\sum_{s=1}^{T-\theta-1} G(T-\theta,s) + \sum_{s=\eta+1}^{T-2} G(T-\theta,s) \\
&\quad= -\sum_{s=1}^{T-\theta-1} \frac{\alpha T(T-1)(T-s-1)}{2-2\alpha(T-1)} - \sum_{s=1}^{T-\theta-1} \frac{(T-s)(T-s-1)}{2} \\
&\qquad- \sum_{s=\eta+1}^{T-2} \frac{\alpha T(T-1)(T-s-1)}{2-2\alpha(T-1)} - \sum_{s=\eta+1}^{T-2} \frac{(T-s)(T-s-1)}{2} + \sum_{s=1}^{T-\theta-1} \frac{T(T-1)}{2-2\alpha(T-1)} \\
&\qquad+ \sum_{s=1}^{T-\theta-1} \frac{\alpha(T-s-1)(T-\theta)(T-\theta-1)}{2-2\alpha(T-1)} + \sum_{s=1}^{T-\theta-1} \frac{(T-\theta-s)(T-\theta-s-1)}{2-2\alpha(T-1)} \\
&\qquad- \sum_{s=1}^{T-\theta-1} \frac{(T-\theta)(T-\theta-1)}{2-2\alpha(T-1)}.
\end{aligned}$$

Let

$$\begin{aligned}
I_1 :={}& -\sum_{s=1}^{T-\theta-1} \frac{\alpha T(T-1)(T-s-1)}{2-2\alpha(T-1)} - \sum_{s=1}^{T-\theta-1} \frac{(T-s)(T-s-1)}{2} \\
&- \sum_{s=\eta+1}^{T-2} \frac{\alpha T(T-1)(T-s-1)}{2-2\alpha(T-1)} - \sum_{s=\eta+1}^{T-2} \frac{(T-s)(T-s-1)}{2} \\
&+ \sum_{s=1}^{T-\theta-1} \frac{T(T-1)}{2-2\alpha(T-1)}
\end{aligned}$$

and

$$\begin{aligned}
I_2 :={}& \sum_{s=1}^{T-\theta-1} \frac{\alpha(T-s-1)(T-\theta)(T-\theta-1)}{2-2\alpha(T-1)} + \sum_{s=1}^{T-\theta-1} \frac{(T-\theta-s)(T-\theta-s-1)}{2-2\alpha(T-1)} \\
&- \sum_{s=1}^{T-\theta-1} \frac{(T-\theta)(T-\theta-1)}{2-2\alpha(T-1)}.
\end{aligned}$$

Then

$$\begin{aligned}
I_1 &\geq -\sum_{s=1}^{T-\theta-1}(T-s-1)\frac{(T-1)(1+\alpha)}{2-2\alpha(T-1)} - \sum_{s=\theta}^{T-2}(T-s-1)\frac{(T-1)(1+\alpha)}{2-2\alpha(T-1)} \\
&\quad + \sum_{s=1}^{T-\theta-1}\frac{T(T-1)}{2-2\alpha(T-1)} \\
&= -\frac{(T-1)(1+\alpha)(T-1-\theta)(T-2+\theta)}{4-4\alpha(T-1)} - \frac{(T-1)(1+\alpha)(T-1-\theta)(T-\theta)}{4-4\alpha(T-1)} \\
&\quad + \frac{T(T-1)(T-\theta-1)}{2-2\alpha(T-1)} \\
&= \frac{(T-1)(T-\theta-1)}{2-2\alpha(T-1)}\bigl(T-(1+\alpha)(T-1)\bigr) \\
&\geq \frac{(T-1)(T-\theta-1)}{2-2\alpha(T-1)}\left(T-\frac{T}{T-1}(T-1)\right) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \frac{(T-\theta)(T-\theta-1)(T-\theta-1)}{2-2\alpha(T-1)}\bigl(\alpha(T-2+\theta)-1\bigr) + \sum_{s=1}^{T-\theta-1}\frac{(T-\theta-s)(T-\theta-s-1)}{2-2\alpha(T-1)} \\
&= \frac{(T-\theta)(T-\theta-1)(T-\theta-1)}{2-2\alpha(T-1)}\bigl(\alpha(T-2+\theta)-1\bigr) + \frac{(T-\theta)(T-\theta-1)(T-\theta-2)}{3} \\
&= \frac{(T-\theta)(T-\theta-1)(T-\theta-2)}{6-6\alpha(T-1)}\bigl[\alpha(T-8)+3(\theta-1)\bigr] \\
&\geq 0.
\end{aligned}$$

Therefore, (3.5) holds. This implies that

$$\begin{aligned}
Su(T-\theta) &\geq \sum_{s=T-\theta}^{\theta} G(T-\theta,s)a(s)f\bigl(s,u(s)\bigr) \\
&= \sum_{s=T-\theta}^{\theta}\left\{\frac{1-\alpha(T-s-s)[T(T-1)-(T-\theta)(T-\theta-1)]}{2-2\alpha(T-1)}\right. \\
&\quad \left. -\frac{(T-s)(T-s-1)}{2} + \frac{(T-\theta-s)(T-\theta-s-1)}{2}\right\}a(s)f\bigl(s,u(s)\bigr) \\
&\geq \sum_{s=T-\theta}^{\theta}\frac{\theta(T-\theta)[2-\alpha(\theta-1)]}{2-2\alpha(T-1)}a(s)f\bigl(s,u(s)\bigr) \\
&\geq \sum_{s=T-\theta}^{\theta}\frac{\theta(T-\theta)[2-\alpha(\theta-1)]}{2-2\alpha(T-1)}a(s)\frac{R}{B} \\
&\geq R.
\end{aligned}$$

So, for $\forall u \in K \cap \partial\Omega_2$,

$$\|Su\| \geq \|u\|. \tag{3.6}$$

Then, by Theorem 1.1, S has at least one fixed point $u \in K$ and u will be a positive solution of problem (1.1). □

Theorem 3.2 *Suppose that* (H_0), (H_1), *and* (H_2) *hold. If one of the following conditions holds*:

$$(B_1) \qquad f^0 := \lim_{u\to 0^+} \max_{t\in[1,T-2]_{\mathbb{Z}}} \frac{f(t,u)}{u} = 0, \qquad f_\infty := \lim_{u\to \infty} \max_{t\in[1,T-2]_{\mathbb{Z}}} \frac{f(t,u)}{u} = \infty,$$

or

$$(B_2) \qquad f_0 := \lim_{u\to 0^+} \max_{t\in[1,T-2]_{\mathbb{Z}}} \frac{f(t,u)}{u} = \infty, \qquad f^\infty := \lim_{u\to \infty} \max_{t\in[1,T-2]_{\mathbb{Z}}} \frac{f(t,u)}{u} = 0,$$

then (1.1) *has at least one positive solution.*

Proof Firstly, we prove the case that (B_1) holds. Since $f^0 = 0$, there exists a constant $R_1 > 0$ such that

$$\frac{f(t,u)}{u} \le \frac{R_1}{A}, \quad (t,u) \in [1, T-2]_{\mathbb{Z}} \times [0, R_1].$$

Since $f_\infty = \infty$, there exists a constant $R_2 > R_1$ such that

$$f(t,u) \ge \frac{u}{\theta^* B} \ge \frac{\theta^* R_2}{\theta^* B} = \frac{R_2}{B}, \quad (t,u) \in [1, T-2]_{\mathbb{Z}} \times \left[\theta^* R_2, R_2\right].$$

From Theorem 3.1, problem (1.1) has at least one positive solution $u \in K$.

Secondly, suppose that (B_2) holds. Since $f_0 = \infty$, there exists a constant $r_1 > 0$ such that

$$f(t,u) \ge \frac{u}{\theta^* B}, \quad (t,u) \in [1, T-2]_{\mathbb{Z}} \times [0, r_1].$$

Let $\Omega_1 = \{u \in E : \|u\| < r_1\}$. If $u \in K \cap \partial\Omega_1$, we have

$$\min_{s\in[T-\theta,\theta]_{\mathbb{Z}}} u(s) \ge \theta^* \|u\| = \theta^* r_1.$$

Therefore, similar to the proof of (3.6), we have

$$\|Su\| \ge \|u\| \quad \text{for } u \in K \cap \partial\Omega_1.$$

On the other hand, since $f^\infty = 0$, there exists a constant $r_2 > 0$ such that

$$f(t,u) \le \frac{u}{A}, \quad (t,u) \in [1, T-2]_{\mathbb{Z}} \times [r_2, \infty).$$

If f is bounded, then there exists a constant $N > 0$ such that $f \le N$. So, we choose $R' = \max\{2r_1, NA\}$. If f is unbounded, then let $R' > \max\{2r_1, r_2\}$ such that $f(t,u) \le f(t,R_2)$. Let $\Omega_2 = \{u \in K : \|u\| < R'\}$ for $\forall (t,u) \in [1, T-2]_{\mathbb{Z}} \times [0, R_2]$. Similar to the proof of Theorem 3.1, we get, for $\forall u \in k \cap \partial\Omega_2$,

$$\|Su\| \le \|u\|.$$

Thus, by Theorem 1.1, S has at least one fixed point $u \in K \cap \overline{\Omega_2} \setminus \Omega_1$, which is a positive solution of problem (1.1). □

Theorem 3.3 *Assume that* (H_0), (H_1), *and* (H_2) *hold. If*

(C_1) $f_0 := \lim_{u\to 0^+} \min_{t\in[1,T-2]_{\mathbb{Z}}} = \frac{f(t,u)}{u} = +\infty, f_\infty := \lim_{u\to\infty} \min_{t\in[1,T-2]_{\mathbb{Z}}} = \frac{f(t,u)}{u} = +\infty$,

and

(C_2) *There exists a constant* $p > 0$ *such that* $f(t,u) < \gamma p$ *for* $0 \le u \le p$ *and* $t \in [1, T-2]_{\mathbb{Z}}$, *where*

$$\gamma = \left(\frac{T(T-1)(1+\alpha\eta)}{1-\alpha(T-1)}\sum_{s=1}^{T-2} a(s)\right)^{-1},$$

then problem (1.1) *has at least two positive solutions* u_1 *and* u_2 *with* $0 \le \|u_1\| \le p \le \|u_2\|$.

Proof Choose $M > 0$ such that

$$M\theta^* \frac{\theta(T-\theta)[2-\alpha(\theta-1)]}{2-2\alpha(T-1)} \sum_{s=T-\theta}^{\theta} a(s) \ge 1.$$

Since $f_0 = +\infty$, there exists a constant r with $0 < r < p$ such that $f(t,u) \ge Mu$ for $0 \le u \le r$. Then, for $\forall u \in \partial K_r$, we have

$$\begin{aligned} Su(T-\theta) &= \sum_{s=1}^{T-2} G(T-\theta,s)a(s)f\bigl(s,u(s)\bigr) \\ &\ge \sum_{s=T-\theta}^{\theta} G(T-\theta,s)a(s)f\bigl(s,u(s)\bigr) \\ &\ge M\theta^* \sum_{s=T-\theta}^{\theta} G(T-\theta,s)a(s)\|u\| \\ &\ge M\theta^* \frac{\theta(T-\theta)[2-\alpha(\theta-1)]}{2-2\alpha(T-1)} \sum_{s=T-\theta}^{\theta} a(s)\|u\| \\ &\ge \|u\|. \end{aligned}$$

From Theorem 1.2, we get

$$i(S,K_r,K) = 0.$$

Since $f_\infty = +\infty$, there exists a constant $R_1 > 0$ such that $f(t,u) \ge Mu$ for $\forall u \ge R_1$. Choose $R > \max\{p, \frac{R_1}{\theta^*}\}$, then for $\forall u \in \partial K_R$, $\min_{t\in[T-\theta,\theta]_{\mathbb{Z}}} u(t) \ge \theta^*\|u\| > R_1$. Similar to the above proof, we have

$$\|Su\| \ge \|u\| \quad \text{for } u \in \partial K_R.$$

Therefore,

$$i(S,K_R,K) = 0.$$

From (C_2), for $\forall u \in \partial K_p$,

$$\begin{aligned}\|Su\| &= \max_{t\in[0,T]_{\mathbb{Z}}}\left|\sum_{s=1}^{T-2} G(t,s)a(s)f\bigl(s,u(s)\bigr)\right| \\ &\le \max_{t\in[0,T]_{\mathbb{Z}}}\sum_{s=1}^{T-2}\bigl|G(t,s)\bigr|a(s)f\bigl(s,u(s)\bigr) \\ &\le \sum_{s=1}^{T-2}\frac{T(T-1)(1+\alpha\eta)}{1-\alpha(T-1)}a(s)f\bigl(s,u(s)\bigr) \\ &= \|u\|.\end{aligned}$$

Therefore, for $\forall u \in \partial K_p$, $\|Tu\| \le \|u\|$. By Theorem 1.2,

$$i(S,K_p,K) = 1.$$

Thus,

$$i(S,K_R \setminus \mathring{K}_p,K) = -1, \qquad i(S,K_p \setminus \mathring{K}_r,K) = 1.$$

So, S has a fixed point u_1 in $K_p \setminus \mathring{K}_r$ and another fixed point u_2 in $K_R \setminus \mathring{K}_p$. So, problem (1.1) has at least two positive solutions u_1 and u_2 with $0 \le \|u_1\| \le p \le \|u_2\|$. □

Theorem 3.4 *Assume that (H_0), (H_1), and (H_2) hold. If*

(D_1) $f^0 := \lim_{u\to 0^+}\max_{t\in[1,T-2]_{\mathbb{Z}}}\frac{f(t,u)}{u} = 0, f^\infty := \lim_{u\to\infty}\max_{t\in[1,T-2]_{\mathbb{Z}}}\frac{f(t,u)}{u} = 0,$

and

(D_2) *there exists a constant $p > 0$ such that $f(t,u) > \beta p$ for $\theta^* p \le u \le p$ and $t \in [1,T-2]_{\mathbb{Z}}$, where*

$$\beta = \left(\theta^*\frac{\theta(T-\theta)[2-\alpha(\theta-1)]}{2-2\alpha(T-1)}\sum_{s=T-\theta}^{\theta} a(s)\right)^{-1},$$

then (1.1) has at least two positive solutions u_1 and u_2 with $0 \le \|u_1\| \le p \le \|u_2\|$.

Proof From (D_1), for $\forall \epsilon > 0$, there exists $M_1 > 0$, if $u > 0$, $t \in [0,T]_{\mathbb{Z}}$, we have $f(t,u) \le M_1 + \epsilon u$. Then, for $\forall u \in K$,

$$\begin{aligned}\|Su\| &= \max_{t\in[0,T]_{\mathbb{Z}}}\left|\sum_{s=1}^{T-2} G(t,s)a(s)f\bigl(s,u(s)\bigr)\right| \\ &\le \frac{T(T-1)(1+\alpha\eta)}{1-\alpha(T-1)}\sum_{s=1}^{T-2} a(s)(M_1+\epsilon u).\end{aligned}$$

Choose $\epsilon > 0$ sufficiently small and $R > p$ sufficiently large, then for $\forall u \in \partial K_R$, $\|Su\| \le \|u\|$, from Theorem 1.2, we have

$$i(S,K_R,K) = 1.$$

In a similar way, if $0 < r < p$,

$$i(S, K_R, K) = 1.$$

From (D_2), for $\forall u \in \partial K_p$,

$$\begin{aligned}
Su(T-\theta) &= \sum_{s=1}^{T-2} G(T-\theta, s)a(s)f\bigl(s, u(s)\bigr) \\
&\geq \sum_{s=T-\theta}^{\theta} G(T-\theta, s)a(s)f\bigl(s, u(s)\bigr) \\
&> \beta\theta^* p \sum_{s=T-\theta}^{\theta} G(T-\theta, s)a(s) \\
&\geq \beta\theta^* p \frac{\theta(T-\theta)[2-\alpha(\theta-1)]}{2-2\alpha(T-1)} \sum_{s=T-\theta}^{\theta} a(s) \\
&= p.
\end{aligned}$$

Then, for $\forall u \in \partial K_p$, $\|Su\| \geq \|u\|$. From Theorem 1.2, we have

$$i(S, K_p, K) = 0.$$

Then, problem (1.1) has at least two solutions u_1 and u_2 with $0 \leq \|u_1\| \leq p \leq \|u_2\|$. □

4 Example

Example 4.1 Consider the discrete three-point boundary problem

$$\begin{cases} \Delta^3 u(t-1) - a(t)f(t, u(t)) = 0, & t \in [1, 7]_{\mathbb{Z}}, \\ \Delta u(0) = u(9) = 0, & \Delta^2 u(7) - \frac{1}{9}\Delta u(8) = 0, \end{cases} \tag{4.1}$$

where $a(t) = \frac{9-t}{10}$, and

$$f(t, u) = \begin{cases} 15 - t + \frac{u}{1000}, & (t, u) \in [1, 7]_{\mathbb{Z}} \times [0, 1000], \\ \sqrt[3]{u} + 6 - t, & (t, u) \in [1, 7]_{\mathbb{Z}} \times [1000, \infty]. \end{cases}$$

Since $T = 9$ and $\alpha = \frac{1}{9}$, $\eta \in [4, 7]_{\mathbb{Z}}$. Without loss of generality, let $\eta = 4$. Then, by the direct calculation, we get $\theta \in [5, 6]_{\mathbb{Z}}$. Choose $\theta = 5$, then $\theta^* = \frac{\eta+2-\theta}{\eta+2} = \frac{1}{6}$. So,

$$\begin{aligned}
A &= \sum_{s=1}^{T-2} \frac{T(T-1)(1+\alpha\eta)}{1-\alpha(T-1)} a(s) = 936 \sum_{s=1}^{7} \frac{9-s}{10} = 3276, \\
B &= \sum_{s=T-\theta}^{\theta} \frac{\theta(T-\theta)[2-\alpha(\theta-1)]}{2-2\alpha(T-1)} a(s) = 140 \sum_{s=4}^{5} \frac{9-s}{10} = 136.
\end{aligned}$$

If we choose $R = 330$, $r = 1{,}000{,}000$, from Theorem 3.1, problem (4.1) has at least one positive solution.

Example 4.2 In this example, we continue to consider problem (4.1) with

$$f(t,u)=\begin{cases}\frac{u^2(10-t)}{10}, & (t,u)\in[1,7]_{\mathbb{Z}}\times[0,1],\\ \frac{\sqrt[3]{u}+9-t}{10}, & (t,u)\in[1,7]_{\mathbb{Z}}\times[1,3],\\ \frac{\sqrt[3]{u}+1890-t}{1000}, & (t,u)\in[1,7]_{\mathbb{Z}}\times[3,\infty).\end{cases}$$

Continue to take $\alpha=\frac{1}{9}$, $\eta=4$, $\theta=5$, and $\theta^*=\frac{1}{6}$. Then

$$\beta=\left(\theta^*\frac{\theta(T-\theta)[2-\alpha(\theta-1)]}{2-2\alpha(T-1)}\sum_{s=T-\theta}^{\theta}a(s)\right)^{-1}=\frac{3}{68}.$$

Furthermore, if we choose $p=1$, then for $\theta^*p\le u\le p$, $f(t,u)\ge\beta p=\frac{3}{68}$. From Theorem 3.4, problem (4.1) has at least two positive solutions u_1 and u_2 with $0\le\|u_1\|\le p\le\|u_2\|$.

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Availability of data and materials
The datasets used and analysed during the current study are available from the corresponding author on reasonable request.

Competing interests
The authors declare that they do not have any competing interests in this manuscript.

Consent for publication
The authors confirm that the work described has not been published before (except in the form of an abstract or as part of a published lecture, review, or thesis), that its publication has been approved by all co-authors.

Authors' contributions
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