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Impulsive fractional quantum Hahn difference boundary value problems

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Abstract

In this paper, we study the existence and uniqueness of solutions for two classes of boundary value problems for impulsive Caputo type fractional Hahn difference equations, by using the Banach contraction mapping principle and the nonlinear alternative of Leray–Schauder. The obtained results are well illustrated by examples.

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1 Introduction and preliminaries

Our purpose of this paper is to establish the existence and uniqueness results for two impulsive fractional Hanh difference boundary value problems. More precisely, we consider the first boundary value problem of order ν_k, $0 < \nu_k \le 1$,

$$\begin{cases} {}^{c}_{t_k}D^{\nu_k}_{q_k,\omega_k}x(t) = f(t,x(t)), \quad t \in J_k, k = 0,1,2,\dots,m, \\ \Delta x(t_k) = \varphi_k(x(t_k^-)), \quad k = 1,2,\dots,m, \\ \xi_1 x(0) + \xi_2 x(T) = \xi_3, \end{cases} \tag{1}$$

and the second boundary value problem of order ν_k, $1 < \nu_k \le 2$,

$$\begin{cases} {}^{c}_{t_k}D^{\nu_k}_{q_k,\omega_k}x(t) = f(t,x(t)), \quad t \in J_k, k = 0,1,2,\dots,m, \\ \Delta x(t_k) = \varphi_k(x(t_k^-)), \quad k = 1,2,\dots,m, \\ {}_{t_k}D_{q_k,\omega_k}x(t_k^+) - {}_{t_{k-1}}D_{q_{k-1},\omega_{k-1}}x(t_k^-) = \varphi_k^*(x(t_k^-)), \quad k = 1,2,\dots,m, \\ x(0) = \eta_1, \qquad {}_{t_m}D_{q_m,\omega_m}x(T) = \eta_2, \end{cases} \tag{2}$$

where ${}^{c}_{t_k}D^{\nu_k}_{q_k,\omega_k}$ is the fractional quantum Hahn difference operator of Caputo type, $0 < q_k < 1$, $\omega_k \ge 0$, $k = 0,1,2,\dots,m$, $f : J \times \mathbb{R} \to \mathbb{R}$, $J = [0,T]$, $\varphi, \varphi_k^* : \mathbb{R} \to \mathbb{R}$, $k = 1,2,\dots,m$, are given functions, $\Delta x(t_k) = x(t_k) - x(t_k^-)$, ${}_{t_k}D_{q_k,\omega_k}$ is the first order quantum Hahn difference operator on interval J_k, $k = 0,1,2,\dots,m$, and given constants $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2 \in \mathbb{R}$.

The q-calculus appeared as a connection between mathematics and physics, especially, in elementary particle physics, which have used quantum numbers to present the discrete values of energy levels in atoms. In 1910, Jackson [1] introduced the notion of q-derivative as

$$D_q f(t) = \begin{cases} \frac{f(t)-f(qt)}{t(1-q)}, & t \neq 0, \\ f'(0), & t = 0, \end{cases} \tag{3}$$

provided that $f'(0)$ exists. He also was the first to develop q-calculus and q-difference equations in a systematic way. The book by Kac and Cheung [2] covers many of the fundamental aspects of quantum calculus and also q-special functions. The q-calculus has many applications in mathematical areas such as orthogonal polynomials, basic hypergeometric functions, combinatorics, the calculus of variations, quantum mechanics, and the theory of relativity. Some recent results in quantum calculus can be found in [3–9] and the references cited therein.

Hahn [10] introduced his difference operator $D_{q,\omega}$ as

$$D_{q,\omega} f(t) = \begin{cases} \frac{f(qt+\omega)-f(t)}{t(q-1)+\omega}, & t \neq \omega_0, \\ f'(\omega_0), & t = \omega_0, \end{cases} \tag{4}$$

provided that f is differentiable at ω_0, where $q \in (0,1)$ and $\omega \geq 0$ are fixed. Here f is defined on an interval $I \subseteq \mathbb{R}$ containing $\omega_0 := \omega/(1-q)$. The Hahn difference operator unifies (in the limit) the two most well known and used quantum difference operators: the Jackson q-difference derivative D_q, where $q \in (0,1)$, defined by (3), for $\omega = 0$, and the forward difference D_ω for $q \to 1$, defined by

$$D_\omega f(t) = \frac{f(t+\omega)-f(t)}{\omega}, \tag{5}$$

where $\omega > 0$ is a fixed constant. The Hahn difference operator is a successful tool for constructing families of orthogonal polynomials and investigating some approximation problems (cf. [11–14]). For some recent results on the boundary value problems of Hahn difference equations we refer to [15–19] and references therein.

Let us emphasize that the definition (3) does not remain valid for impulse points t_k, $k \in \mathbb{Z}$, such that $t_k \in (qt, t)$. For example, let $[0, T]$, $T > 4$ be a dense interval and $t = 2$ be an impulsive point, i.e., $f(2^+) \neq f(2^-)$. Then we have $D_{1/2} f(4^+) \neq D_{1/2} f(4^-)$, which implies that $D_{1/2} f(4)$ does not exist. On the other hand, this situation does not arise for impulsive equations on q-time scales $\{0, \dots, q^2 t, qt, t\}$, as the domains consist of isolated points covering the case of consecutive points of t and qt with impulsive points $t_k \notin (qt, t)$. Due to this reason, the subject of impulsive quantum difference equations on dense domains could not be studied. In [20], the authors modified the classical quantum calculus on $[a, b]$ by defining

$$_aD_q f(t) = \begin{cases} \frac{f(t)-f(qt+(1-q)a)}{(1-q)(t-a)}, & t \neq a, \\ \lim_{t\to a}\, {}_aD_q f(t), & t = a. \end{cases} \tag{6}$$

Observe that if t_k, $k = 1, 2, \dots$, are impulse points with $f(t_k^+) = f(t_k)$, then, by setting $[a, b) = [t_k, t_{k+1})$, there is no impulse point in $[a, b)$. With the help of definition (6), a series of impulsive quantum initial and boundary value problems were studied. We refer the interested reader to the recent monograph [21] for details.

In [22], the authors defined a quantum shifting operator by

$$ {}_{a}\Phi_q(m) = qm + (1-q)a, \tag{7}$$

$m, a \in \mathbb{R}$ with $m \geq a$. Then the q-derivative of a function f on an interval $[a,b]$ in (6) can be rewritten as

$$ {}_{a}D_q f(t) = \begin{cases} \frac{f(t)-f({}_{a}\Phi_q(t))}{t-{}_{a}\Phi_q(t)}, & t \neq a, \\ \lim_{t\to a} {}_{a}D_q f(t), & t = a. \end{cases} \tag{8}$$

Now, we consider the interval $[a,b] \subseteq \mathbb{R}$, the quantum numbers $0 < q < 1$, $\omega \geq 0$, and

$$ \theta = \frac{\omega}{1-q} + a, \tag{9}$$

with $\theta \in [a,b]$. The Hahn difference operator was generalized recently in [23] to ${}_{a}D_{q,\omega}$ defined by

$$ {}_{a}D_{q,\omega} f(t) = \begin{cases} \frac{f(t)-f(qt+a(1-q)+\omega)}{(t-a)(1-q)-\omega}, & t \neq \theta, \\ f'(\theta), & t = \theta, \end{cases} \tag{10}$$

provided that f is differentiable at θ.

Next, we introduce a new quantum Hahn shifting operator by

$$ {}_{\theta}\Phi_q(m) = qm + (1-q)\theta. \tag{11}$$

As a special case, if $\omega = a = 0$, then (11) is reduced to classical quantum shifting in [1]. If $\omega = 0$, then (11) is reduced to the q-shifting in (7) studied in [20], and if $a = 0$, then (11) is reduced to Hahn shifting as appeared in [10]. In addition, the iterated k-times of quantum shifting is defined by

$$ {}_{\theta}\Phi_q^k(m) = {}_{\theta}\Phi_q^{k-1}\big({}_{\theta}\Phi_q(m)\big) = q^k m + \big(1-q^k\big)\theta, $$

with ${}_{\theta}\Phi_q^0(m) = m$.

Proposition 1 *The following relations hold*:

(i) $(1-q)(t-a) - \omega = (1-q)(t-\theta) = t - {}_{\theta}\Phi_q(t)$;

(ii) ${}_{\theta}\Phi_q(\theta) = \theta$;

(iii) $(1-q^k)a + \omega[k]_q = (1-q^k)\theta$, *where* $[k_q] = (1-q^k)/(1-q)$, $k = 0, 1, 2, \ldots$.

The next definition modifies the definition (10) (studied in [23]), taking into account Proposition 1(i)–(iii).

Definition 1 Let f be a function defined on $[a,b]$. The quantum Hahn difference operator is defined by

$$ {}_aD_{q,\omega}f(t) = \begin{cases} \frac{f(t)-f({}_\theta\Phi_q(t))}{t-{}_\theta\Phi_q(t)}, & t \neq \theta, \\ f'(\theta), & t = \theta, \end{cases} \tag{12}$$

provided that f is differentiable at θ.

Definition 2 Assume $f : [a,b] \to \mathbb{R}$ is a given function and consider two points $c, d \in [a,b]$. The q, ω-quantum Hahn integral of f from c to d is defined by

$$\int_c^d f(s)_a\, d_{q,\omega}s := \int_\theta^d f(s)_a\, d_{q,\omega}s - \int_\theta^c f(s)_a\, d_{q,\omega}s, \tag{13}$$

where

$$\int_\theta^t f(s)_a\, d_{q,\omega}s = \bigl[t - {}_\theta\Phi_q(t)\bigr] \sum_{i=0}^{\infty} q^i f\bigl({}_\theta\Phi_q^i(t)\bigr), \tag{14}$$

for $t \in [a,b]$, provided that the series converges at $t = c$ and $t = d$.

Let us define the θ-power function as

$$(n-m)_\theta^{(0)} = 1, \qquad (n-m)_\theta^{(k)} = \prod_{i=0}^{k-1}\bigl(n - {}_\theta\Phi_q^i(m)\bigr), \quad k \in \mathbb{N} \cup \{\infty\}. \tag{15}$$

For example, $(n-m)_\theta^{(4)} = (n-m)(n-{}_\theta\Phi_q(m))(n-{}_\theta\Phi_q^2(m))(n-{}_\theta\Phi_q^3(m))$. More generally, if $\gamma \in \mathbb{R}$, then

$$(n-m)_\theta^{(\gamma)} = \prod_{i=0}^{\infty} \frac{(n-{}_\theta\Phi_q^i(m))}{(n-{}_\theta\Phi_q^{\gamma+i}(m))},$$

with ${}_\theta\Phi_q^\gamma(m) = q^\gamma m + (1-q^\gamma)\theta$, $\gamma \in \mathbb{R}$. For example,

$$(n-m)_\theta^{(\frac{3}{2})} = \frac{(n-m)(n-{}_\theta\Phi_q(m))(n-{}_\theta\Phi_q^2(m))\cdots}{(n-{}_\theta\Phi_q^{\frac{3}{2}}(m))(n-{}_\theta\Phi_q^{\frac{5}{2}}(m))(n-{}_\theta\Phi_q^{\frac{7}{2}}(m))\cdots}.$$

Let us state the definitions of Riemann–Liouville type of fractional derivative and integral of quantum Hahn calculus and also Caputo type fractional derivative, which can be found in [24].

Definition 3 The fractional quantum Hahn difference of Riemann–Liouville type of order $\nu \geq 0$ on an interval $[a,b]$ is defined by $({}_aD_{q,\omega}^0 f)(t) = f(t)$ and

$$\bigl({}_aD_{q,\omega}^\nu f\bigr)(t) = \frac{1}{\Gamma_q(l-\nu)}\, {}_aD_{q,\omega}^l \int_a^t \bigl(t - {}_\theta\Phi_q(s)\bigr)_\theta^{(l-\nu-1)} f(s)_a\, d_{q,\omega}s, \quad \nu > 0,$$

where l is the smallest integer greater than or equal to ν.

Definition 4 Let $\nu \geq 0$ and f be a function defined on $[a,b]$. The fractional quantum Hahn integral of Riemann–Liouville type is given by $({}_aI^0_{q,\omega}f)(t) = f(t)$ and

$$\left({}_aI^{\nu}_{q,\omega}f\right)(t) = \frac{1}{\Gamma_q(\nu)}\int_a^t \left(t - {}_{\theta}\Phi_q(s)\right)_{\theta}^{(\nu-1)} f(s)_a\, d_{q,\omega}s, \quad \nu > 0, t \in [a,b].$$

Definition 5 The fractional quantum Hahn difference of Caputo type $\nu \geq 0$ on an interval $[a,b]$ is defined by $({}^c_aD^0_{q,\omega}f)(t) = f(t)$ and

$$\left({}^c_aD^{\nu}_{q,\omega}f\right)(t) = \frac{1}{\Gamma_q(l-\nu)}\int_a^t \left(t - {}_{\theta}\Phi_q(s)\right)_{\theta}^{(l-\nu-1)} {}_aD^l_{q,\omega}f(s)_a\, d_{q,\omega}s, \quad \nu > 0,$$

where l is the smallest integer greater than or equal to ν.

If $\omega = 0$, then $\theta = a$ and the above fractional quantum Hahn calculus is reduced to fractional quantum calculus on the interval $[a,b]$ as appeared in [22].

Theorem 1 ([24]) *Let $\alpha, \beta \in \mathbb{R}^+$, $\lambda \in (-1,\infty)$ and $\theta \in [a,b]$. The following formulas hold*:
(i) $({}_aI^{\alpha}_{q,\omega}(x-a)^{(\lambda)}_{\theta})(t) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)}(t-a)^{(\alpha+\lambda)}_{\theta}$;
(ii) $({}_aD^{\alpha}_{q,\omega}(x-a)^{(\lambda)}_{\theta})(t) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\alpha+1)}(t-a)^{(\lambda-\alpha)}_{\theta}$.

Theorem 2 ([24]) *Let $f(t)$ be a function defined on an interval $[a,b]$, $\beta, \nu \in \mathbb{R}^+$, $\alpha \in (N-1,N)$ and $\theta \in [a,b]$. Then, we have*:
(i) $({}_aI^{\beta}_{q,\omega}{}_aI^{\nu}_{q,\omega}f)(t) = ({}_aI^{\nu}_{q,\omega}{}_aI^{\beta}_{q,\omega}f)(t) = ({}_aI^{\beta+\nu}_{q,\omega}f)(t)$;
(ii) $({}_aD^{\beta}_{q,\omega}{}_aI^{\beta}_{q,\omega}f)(t) = ({}^c_aD^{\beta}_{q,\omega}{}_aI^{\beta}_{q,\omega}f) = f(t)$;
(iii) $({}_aI^{\alpha}_{q,\omega}{}_aD^{\alpha}_{q,\omega}f)(t) = f(t) + c_1(t-a)^{(\alpha-1)}_{\theta} + c_2(t-a)^{(\alpha-2)}_{\theta} + \cdots + c_N(t-a)^{(\alpha-N)}_{\theta}$;
(iv) $({}_aI^{\alpha}_{q,\omega}{}^c_aD^{\alpha}_{q,\omega}f)(t) = f(t) + d_0 + d_1(t-a)^{(1)}_{\theta} + d_2(t-a)^{(2)}_{\theta} + \cdots + d_{N-1}(t-a)^{(N-1)}_{\theta}$, *for some $c_i, d_j \in \mathbb{R}$, $i = 1,2,\ldots,N$, $j = 0,1,\ldots,N-1$.*

The rest of the paper is organized as follows: In Sect. 2.1 we prove the existence and uniqueness results for the the impulsive Hahn difference boundary value problem (1), while the corresponding results for the impulsive Hahn difference boundary value problem (2) are presented in Sect. 2.2. Examples illustrating the obtained results are presented in Sect. 3.

2 Impulsive fractional Hahn difference equations

To establish our results, we define intervals $J_k = [t_k, t_{k+1})$, $k = 0,1,2,\ldots,m-1$, $J_m = [t_m, T]$ and $J = [0,T]$, with impulsive points $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots < t_m < t_{m+1} = T$. In addition, we define the space $PC(J,\mathbb{R}) = \{x : J \to \mathbb{R} : x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^+) = x(t_k), k = 1,2,\ldots,m\}$. Observe that $PC(J,\mathbb{R})$ is a Banach space equipped with the norm $\|x\| = \sup\{|x(t)| : t \in J\}$. From Sect. 1, we replace all parameters, a, q, ω and ν of fractional quantum Hahn calculus in Definitions 3–5 by t_k, q_k, ω_k and ν_k, $k = 0,1,2,\ldots,m$, respectively. Also we assume that

$$\theta_k = \frac{\omega_k}{1-q_k} + t_k \in J_k, \quad k = 0,1,2,\ldots,m.$$

In the next subsections, the fractional quantum Hahn calculus is used to establish the existence and uniqueness results for the impulsive fractional Hahn difference boundary value problems (1) and (2).

2.1 Impulsive problem of fractional quantum Hahn difference equation of order $0 < \nu_k \leq 1$

In this subsection, we investigate the impulsive Hahn difference boundary value problem (1).

The following lemma deals with the linear variant of problem (1) and gives a representation of the solution.

Lemma 1 *Let $\xi_1 + \xi_2 \neq 0$ and $h \in C(J,\mathbb{R})$ be a given function. Then, the function x is a solution of the impulsive Hahn difference boundary value problem*

$$\begin{cases} {}^{c}_{t_k}D^{\nu_k}_{q_k,\omega_k}x(t) = h(t), & t \in J_k, k = 0,1,2,\dots,m, \\ \Delta x(t_k) = \varphi_k(x(t_k^-)), & k = 1,2,\dots,m, \\ \xi_1 x(0) + \xi_2 x(T) = \xi_3, \end{cases} \tag{16}$$

if and only if

$$\begin{aligned} x(t) = {} & \frac{\xi_3}{\xi_1+\xi_2} - \frac{\xi_2}{\xi_1+\xi_2}\left[\sum_{i=0}^{m}\left({}_{t_i}I^{\nu_i}_{q_i,\omega_i}h\right)\left(t_{i+1}^-\right) + \sum_{j=1}^{m}\varphi_j\left(x\left(t_j^-\right)\right)\right] \\ & + \sum_{i=0}^{k-1}\left({}_{t_i}I^{\nu_i}_{q_i,\omega_i}h\right)\left(t_{i+1}^-\right) + \sum_{j=1}^{k}\varphi_j\left(x\left(t_j^-\right)\right) + \left({}_{t_k}I^{\nu_k}_{q_k,\omega_k}h\right)(t), \end{aligned} \tag{17}$$

for $t \in J_k$, $k = 0,1,2,\dots,m$, with $\sum_a^b(\cdot) = 0$, when $b < a$.

Proof Applying Theorem 2(iv), for $t \in J_0$, we obtain

$${}_{t_0}I^{\nu_0}_{q_0,\omega_0}\left({}^{c}_{t_0}D^{\nu_0}_{q_0,\omega_0}x\right)(t) = x(t) = c_0 + \left({}_{t_0}I^{\nu_0}_{q_0,\omega_0}h\right)(t),$$

for some $c_0 \in \mathbb{R}$. In particular, when $t = t_1^-$, it follows that

$$x\left(t_1^-\right) = c_0 + \left({}_{t_0}I^{\nu_0}_{q_0,\omega_0}h\right)\left(t_1^-\right).$$

For $t \in J_1$, using the same process, we have

$$x(t) = x(t_1) + \left({}_{t_1}I^{\nu_1}_{q_1,\omega_1}h\right)(t).$$

The impulsive condition, $x(t_1) = x(t_1^-) + \varphi_1(x(t_1^-))$, yields

$$\begin{aligned} x(t) &= x\left(t_1^-\right) + \varphi_1\left(x\left(t_1^-\right)\right) + \left({}_{t_1}I^{\nu_1}_{q_1,\omega_1}h\right)(t) \\ &= c_0 + \left({}_{t_0}I^{\nu_0}_{q_0,\omega_0}h\right)\left(t_1^-\right) + \varphi_1\left(x\left(t_1^-\right)\right) + \left({}_{t_1}I^{\nu_1}_{q_1,\omega_1}h\right)(t). \end{aligned}$$

Repeating the above argument, for $t \in J_k$, $k = 0, 1, 2, \ldots, m$, we get

$$x(t) = c_0 + \sum_{i=0}^{k-1} \left({}_{t_i}I^{\nu_i}_{q_i,\omega_i} h \right)\left(t^-_{i+1}\right) + \sum_{j=1}^{k} \varphi_j\left(x\left(t^-_j\right)\right) + \left({}_{t_k}I^{\nu_k}_{q_k,\omega_k} h \right)(t), \tag{18}$$

with $\sum_a^b(\cdot) = 0$, when $b < a$. Since $x(0) = c_0$ and

$$x(T) = c_0 + \sum_{i=0}^{m} \left({}_{t_i}I^{\nu_i}_{q_i,\omega_i} h \right)\left(t^-_{i+1}\right) + \sum_{j=1}^{m} \varphi_j\left(x\left(t^-_j\right)\right),$$

with $t^-_{m+1} = T$, we can compute, with boundary condition in (16), that

$$c_0 = \frac{\xi_3}{\xi_1 + \xi_2} - \frac{\xi_2}{\xi_1 + \xi_2}\left[\sum_{i=0}^{m} \left({}_{t_i}I^{\nu_i}_{q_i,\omega_i} h \right)\left(t^-_{i+1}\right) + \sum_{j=1}^{m} \varphi_j\left(x\left(t^-_j\right)\right) \right].$$

Substituting the constant c_0 in the integral equation (18), we obtain the desired result in (17). The converse follows by direct computation. The proof is completed. □

In the following, for convenience we use the abbreviation

$$\begin{aligned}\left({}_{t_k}I^{\nu_k}_{q_k,\omega_k} f\left(s, x(s)\right)\right)(t) &= \frac{1}{\Gamma_{q_k}(\nu_k)} \int_{t_k}^{t} \left(t - {}_{\theta_k}\Phi_{q_k}(s)\right)^{(\nu_k - 1)}_{\theta_k} f\left(s, x(s)\right)_{t_k} d_{q_k,\omega_k} s \\ &= \left({}_{t_k}I^{\nu_k}_{q_k,\omega_k} f_x \right)(t),\end{aligned}$$

for $k = 0, 1, 2, \ldots, m$, and put

$$\Lambda_1 = \left(1 + \frac{|\xi_2|}{|\xi_1 + \xi_2|}\right)\left(L_1 \sum_{i=0}^{m} \frac{(t_{i+1} - t_i)^{(\nu_i)}_{\theta_i}}{\Gamma_{q_i}(\nu_i + 1)} + \frac{L_2}{2} m(m+1) \right),$$

$$\Lambda_2 = \frac{|\xi_3|}{|\xi_1 + \xi_2|} + \left(1 + \frac{|\xi_2|}{|\xi_1 + \xi_2|}\right)\left(M \sum_{i=0}^{m} \frac{(t_{i+1} - t_i)^{(\nu_i)}_{\theta_i}}{\Gamma_{q_i}(\nu_i + 1)} + \frac{N}{2} m(m+1) \right).$$

Now, we are in the position to establish the existence of a unique solution of problem (1) by using the Banach contraction mapping principle.

Theorem 3 *Let $f : J \times \mathbb{R} \to \mathbb{R}$ and $\varphi_k : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \ldots, m$, be given functions satisfying*

$$\left| f(t,x) - f(t,y) \right| \le L_1 |x - y|, \quad L_1 > 0, \forall t \in J, x, y \in \mathbb{R}, \tag{19}$$

and

$$\left| \varphi_k(x) - \varphi_k(y) \right| \le L_2 |x - y|, \quad L_2 > 0, \forall x, y \in \mathbb{R}. \tag{20}$$

If

$$\Lambda_1 < 1, \tag{21}$$

then problem (1) *has a unique solution on J.*

Proof In view of Lemma 1, we transform the boundary value problem (1), into an operator equation $x(t) = Ax(t)$, where $A : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is defined by

$$\begin{aligned} Ax(t) := {}& \frac{\xi_3}{\xi_1+\xi_2} - \frac{\xi_2}{\xi_1+\xi_2}\left[\sum_{i=0}^{m}\left({}_{t_i}I^{\nu_i}_{q_i,\omega_i}f_x\right)\left(t_{i+1}^-\right) + \sum_{j=1}^{m}\varphi_j\left(x\left(t_j^-\right)\right)\right] \\ & + \sum_{i=0}^{k-1}\left({}_{t_i}I^{\nu_i}_{q_i,\omega_i}f_x\right)\left(t_{i+1}^-\right) + \sum_{j=1}^{k}\varphi_j\left(x\left(t_j^-\right)\right) + \left({}_{t_k}I^{\nu_k}_{q_k,\omega_k}f_x\right)(t), \quad t \in J. \end{aligned}$$

Also we define a set B_r by $B_r = \{x \in PC(J,\mathbb{R}) : \|x\| \le r\}$ where $r > \Lambda_2/(1-\Lambda_1)$. It should be shown that $AB_r \subset B_r$. Setting $\sup_{t\in J}|f_0| = M$, $\max_j|\varphi_j(0)| = N$, where $f_0 = f(t,0)$, and using $|f_x| \le |f_x - f_0| + |f_0|$ and $|\varphi_j(x)| \le |\varphi_j(x) - \varphi_j(0)| + |\varphi_j(0)|$, $j = 1,2,3,\dots,m$, for any $x \in B_r$, we have

$$\begin{aligned} &\left|Ax(t)\right| \\ &\quad\le \frac{|\xi_3|}{|\xi_1+\xi_2|} + \frac{|\xi_2|}{|\xi_1+\xi_2|}\left[\sum_{i=0}^{m}\left({}_{t_i}I^{\nu_i}_{q_i,\omega_i}|f_x|\right)\left(t_{i+1}^-\right) + \sum_{j=1}^{m}\left|\varphi_j\left(x\left(t_j^-\right)\right)\right|\right] \\ &\qquad + \sum_{i=0}^{k-1}\left({}_{t_i}I^{\nu_i}_{q_i,\omega_i}|f_x|\right)\left(t_{i+1}^-\right) + \sum_{j=1}^{k}\left|\varphi_j\left(x\left(t_j^-\right)\right)\right| + \left({}_{t_k}I^{\nu_k}_{q_k,\omega_k}|f_x|\right)(t) \\ &\quad\le \frac{|\xi_3|}{|\xi_1+\xi_2|} + \frac{|\xi_2|}{|\xi_1+\xi_2|}\left[\sum_{i=0}^{m}\left({}_{t_i}I^{\nu_i}_{q_i,\omega_i}\left(|f_x-f_0| + |f_0|\right)\right)\left(t_{i+1}^-\right)\right. \\ &\qquad \left. + \sum_{j=1}^{m}\left(\left|\varphi_j\left(x(t_j)\right) - \varphi_j(0)\right| + \left|\varphi_j(0)\right|\right)\right] + \sum_{i=0}^{k-1}\left({}_{t_i}I^{\nu_i}_{q_i,\omega_i}\left(|f_x-f_0| + |f_0|\right)\right)\left(t_{i+1}^-\right) \\ &\qquad + \sum_{j=1}^{k}\left(\left|\varphi_j\left(x(t_j)\right) - \varphi_j(0)\right| + \left|\varphi_j(0)\right|\right) + \left({}_{t_k}I^{\nu_k}_{q_k,\omega_k}\left(|f_x-f_0| + |f_0|\right)\right)(t) \\ &\quad\le \frac{|\xi_3|}{|\xi_1+\xi_2|} + \frac{|\xi_2|}{|\xi_1+\xi_2|}\left[(L_1r+M)\sum_{i=0}^{m}\left({}_{t_i}I^{\nu_i}_{q_i,\omega_i}(1)\right)\left(t_{i+1}^-\right) + (L_2r+N)\sum_{j=1}^{m}(1)\right] \\ &\qquad + (L_1r+M)\sum_{i=0}^{m}\left({}_{t_i}I^{\nu_i}_{q_i,\omega_i}(1)\right)\left(t_{i+1}^-\right) + (L_2r+N)\sum_{j=1}^{m}(1) \\ &\quad= \Lambda_1 r + \Lambda_2 < r. \end{aligned}$$

This show that $\|Ax\| \le r$, which leads to $AB_r \subset B_r$. Now, we will prove that the operator A is a contraction by using (21). For any $x, y \in B_r$, we have

$$\begin{aligned} &\left|Ax(t) - Ay(t)\right| \\ &\quad\le \frac{|\xi_2|}{|\xi_1+\xi_2|}\left[\sum_{i=0}^{m}\left({}_{t_i}I^{\nu_i}_{q_i,\omega_i}|f_x-f_y|\right)\left(t_{i+1}^-\right) + \sum_{j=1}^{m}\left|\varphi_j\left(x\left(t_j^-\right)\right) - \varphi_j\left(y\left(t_j^-\right)\right)\right|\right] \\ &\qquad + \sum_{i=0}^{k-1}\left({}_{t_i}I^{\nu_i}_{q_i,\omega_i}|f_x-f_y|\right)\left(t_{i+1}^-\right) + \sum_{j=1}^{k}\left|\varphi_j\left(x\left(t_j^-\right)\right) - \varphi_j\left(y\left(t_j^-\right)\right)\right| \end{aligned}$$

$$\begin{aligned}
&+ \left({}_{t_k}I^{\nu_k}_{q_k,\omega_k}|f_x - f_y|\right)(t) \\
&\leq L_1\|x-y\|\left(\frac{|\xi_2|}{|\xi_1+\xi_2|}\sum_{i=0}^{m}\left({}_{t_i}I^{\nu_i}_{q_i,\omega_i}(1)\right)\left(t_{i+1}^-\right) + \sum_{i=0}^{m}\left({}_{t_i}I^{\nu_i}_{q_i,\omega_i}(1)\right)\left(t_{i+1}^-\right)\right) \\
&\quad + L_2\|x-y\|\left(\frac{|\xi_2|}{|\xi_1+\xi_2|}+1\right)\sum_{j=1}^{m}(1) \\
&= \Lambda_1\|x-y\|,
\end{aligned}$$

which yields $\|Ax - Ay\| \leq \Lambda_1\|x-y\|$. From (21), we conclude that the operator A is a contraction on B_r. By the Banach contraction mapping principle, therefore, the impulsive boundary value problem of fractional quantum Hahn difference equation (1) has a unique solution x on J such that $\|x\| \leq r$. The proof is complete. □

Corollary 1 *Let constants $\xi_1 \neq 0$ and $\xi_2 = 0$ in (1), then we have the impulsive initial value problem*

$$\begin{cases}
{}^{c}_{t_k}D^{\nu_k}_{q_k,\omega_k}x(t) = f(t,x(t)), & t \in J_k, k = 0,1,2,\ldots,m, \\
\Delta x(t_k) = \varphi_k(x(t_k^-)), & k = 1,2,\ldots,m, \\
x(0) = \frac{\xi_3}{\xi_1}.
\end{cases} \tag{22}$$

If the functions f and φ_i, $i = 1,2,\ldots,m$, satisfy (19) and (20), respectively, and if

$$L_1\sum_{i=0}^{m}\frac{(t_{i+1}-t_i)^{(\nu_i)}_{\theta_i}}{\Gamma_{q_i}(\nu_i+1)} + \frac{L_2}{2}m(m+1) < 1,$$

then the impulsive initial value problem of fractional quantum Hahn difference equation (22) has a unique solution on J.

2.2 Impulsive problem of fractional quantum Hahn difference equation of order $1 < \nu_k \leq 2$

Consider now the impulsive fractional Hahn difference boundary value problem (2).

Lemma 2 *Let $g \in C(J,\mathbb{R})$. Then, the function x is a solution of the impulsive Hahn difference boundary value problem*

$$\begin{cases}
{}^{c}_{t_k}D^{\nu_k}_{q_k,\omega_k}x(t) = g(t), & t \in J_k, k = 0,1,2,\ldots,m, \\
\Delta x(t_k) = \varphi_k(x(t_k^-)), & k = 1,2,\ldots,m, \\
{}_{t_k}D_{q_k,\omega_k}x(t_k^+) - {}_{t_{k-1}}D_{q_{k-1},\omega_{k-1}}x(t_k^-) = \varphi_k^*(x(t_k^-)), & k = 1,2,\ldots,m, \\
x(0) = \eta_1, \qquad {}_{t_m}D_{q_m,\omega_m}x(T) = \eta_2,
\end{cases} \tag{23}$$

if and only if

$$\begin{aligned} x(t) &= \eta_1 + \left(\eta_2 - \sum_{j=0}^{m} \left({}_{t_j}I_{q_j,\omega_j}^{\nu_j-1} g\right)\left(t_{j+1}^-\right) - \sum_{i=1}^{m} \varphi_i^*\left(x\left(t_i^-\right)\right)\right) \sum_{i=0}^{k}\left([t_{i+1}] - t_i\right) \\ &\quad + \sum_{i=0}^{k-1}\left[\left({}_{t_i}I_{q_i,\omega_i}^{\nu_i} g\right)\left(t_{i+1}^-\right) + \varphi_{i+1}\left(x\left(t_{i+1}^-\right)\right)\right] \\ &\quad + \sum_{i=1}^{k}\left\{\left([t_{i+1}] - t_i\right) \sum_{j=0}^{i-1}\left[\left({}_{t_j}I_{q_j,\omega_j}^{\nu_j-1} g\right)\left(t_{j+1}^-\right) + \varphi_{j+1}^*\left(x\left(t_{j+1}^-\right)\right)\right]\right\} \\ &\quad + \left({}_{t_k}I_{q_k,\omega_k}^{\nu_k} g\right)(t), \end{aligned} \tag{24}$$

for $t \in J_k$, $k = 0, 1, 2, \ldots, m$, *with* $\sum_a^b(\cdot) = 0$, *when* $b < a$ *and*

$$[t_{i+1}] = \begin{cases} t_{i+1}, & t_{i+1} \le t_k, \\ t, & t_{i+1} > t_k. \end{cases}$$

Proof Taking the Riemann–Liouville fractional quantum Hahn integral of order ν_0 to the first equation in (23) and applying Theorem 2(iv), for $t \in J_0$, we get

$$ {}_{t_0}I_{q_0,\omega_0}^{\nu_0}\left({}_{t_0}^{c}D_{q_0,\omega_0}^{\nu_0} x\right)(t) = x(t) = c_0 + c_1(t - t_0)_{\theta_0}^{(1)} + \left({}_{t_0}I_{q_0,\omega_0}^{\nu_0} g\right)(t). \tag{25}$$

From the first condition, $x(0) = \eta_1$, we have $c_0 = \eta_1$ and from (15) with $k = 1$, for $t = t_1^-$, we obtain

$$x\left(t_1^-\right) = \eta_1 + c_1(t_1 - t_0) + \left({}_{t_0}I_{q_0,\omega_0}^{\nu_0} g\right)\left(t_1^-\right), \tag{26}$$

with $(a - b)_\theta^{(1)} = (a - b)$, $a, b \in \mathbb{R}$. In addition, we can formulate from (25) that

$$ {}_{t_0}D_{q_0,\omega_0} x(t) = c_1 + \left({}_{t_0}I_{q_0,\omega_0}^{\nu_0-1} g\right)(t),$$

and then ${}_{t_0}D_{q_0,\omega_0} x(t_1^-) = c_1 + ({}_{t_0}I_{q_0,\omega_0}^{\nu_0-1} g)(t_1^-)$.

For $t \in [t_1, t_2) = J_1$, we have

$$x(t) = x\left(t_1^+\right) + {}_{t_1}D_{q_1,\omega_1} x\left(t_1^+\right)(t - t_1) + \left({}_{t_1}I_{q_1,\omega_1}^{\nu_1} g\right)(t). \tag{27}$$

Since

$$\begin{aligned} x\left(t_1^+\right) &= x\left(t_1^-\right) + \varphi_1\left(x\left(t_1^-\right)\right) \\ &= \eta_1 + c_1(t_1 - t_0) + \left({}_{t_0}I_{q_0,\omega_0}^{\nu_0} g\right)\left(t_1^-\right) + \varphi_1\left(x\left(t_1^-\right)\right), \end{aligned}$$

and

$$\begin{aligned} {}_{t_1}D_{q_1,\omega_1} x\left(t_1^+\right) &= {}_{t_0}D_{q_0,\omega_0} x\left(t_1^-\right) + \varphi_1^*\left(x\left(t_1^-\right)\right) \\ &= c_1 + \left({}_{t_0}I_{q_0,\omega_0}^{\nu_0-1} g\right)\left(t_1^-\right) + \varphi_1^*\left(x\left(t_1^-\right)\right), \end{aligned}$$

then (27) can be written as

$$
\begin{aligned}
x(t) &= \eta_1 + c_1(t_1 - t_0) + \left({}_{t_0}I^{\nu_0}_{q_0,\omega_0}g\right)\left(t_1^-\right) + \varphi_1\left(x\left(t_1^-\right)\right) \\
&\quad + (t - t_1)\left[c_1 + \left({}_{t_0}I^{\nu_0-1}_{q_0,\omega_0}g\right)\left(t_1^-\right) + \varphi_1^*\left(x\left(t_1^-\right)\right)\right] + \left({}_{t_1}I^{\nu_1}_{q_1,\omega_1}g\right)(t).
\end{aligned}
$$

Repeating the above process, for $t \in J_k$, we get

$$
\begin{aligned}
x(t) &= \eta_1 + c_1\sum_{i=0}^{k}\left([t_{i+1}] - t_i\right) + \sum_{i=0}^{k-1}\left[\left({}_{t_i}I^{\nu_i}_{q_i,\omega_i}g\right)\left(t_{i+1}^-\right) + \varphi_{i+1}\left(x\left(t_{i+1}^-\right)\right)\right] \\
&\quad + \sum_{i=1}^{k}\left\{\left([t_{i+1}] - t_i\right)\sum_{j=0}^{i-1}\left[\left({}_{t_j}I^{\nu_j-1}_{q_j,\omega_j}g\right)\left(t_{j+1}^-\right) + \varphi_{j+1}^*\left(x\left(t_{j+1}^-\right)\right)\right]\right\} \\
&\quad + \left({}_{t_k}I^{\nu_k}_{q_k,\omega_k}g\right)(t).
\end{aligned} \tag{28}
$$

To compute c_1, we have

$$
\eta_2 = {}_{t_m}D_{q_m,\omega_m}x(T) = c_1 + \sum_{j=0}^{m}\left({}_{t_j}I^{\nu_j-1}_{q_j,\omega_j}g\right)\left(t_{j+1}^-\right) + \sum_{i=1}^{m}\varphi_i^*\left(x\left(t_i^-\right)\right),
$$

which leads to

$$
c_1 = \eta_2 - \sum_{j=0}^{m}\left({}_{t_j}I^{\nu_j-1}_{q_j,\omega_j}g\right)\left(t_{j+1}^-\right) - \sum_{i=1}^{m}\varphi_i^*\left(x\left(t_i^-\right)\right).
$$

Therefore, the result in (24) holds when substituting the constant c_1 in (28). The converse follows by direct computation. This competes the proof. □

To accomplish our goal, we define the operator $G : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ by

$$
\begin{aligned}
Gx(t) &= \eta_1 + \left(\eta_2 - \sum_{j=0}^{m}\left({}_{t_j}I^{\nu_j-1}_{q_j,\omega_j}f_x\right)\left(t_{j+1}^-\right) - \sum_{i=1}^{m}\varphi_i^*\left(x\left(t_i^-\right)\right)\right)\sum_{i=0}^{k}\left([t_{i+1}] - t_i\right) \\
&\quad + \sum_{i=0}^{k-1}\left[\left({}_{t_i}I^{\nu_i}_{q_i,\omega_i}f_x\right)\left(t_{i+1}^-\right) + \varphi_{i+1}\left(x\left(t_{i+1}^-\right)\right)\right] \\
&\quad + \sum_{i=1}^{k}\left\{\left([t_{i+1}] - t_i\right)\sum_{j=0}^{i-1}\left[\left({}_{t_j}I^{\nu_j-1}_{q_j,\omega_j}f_x\right)\left(t_{j+1}^-\right) + \varphi_{j+1}^*\left(x\left(t_{j+1}^-\right)\right)\right]\right\} \\
&\quad + \left({}_{t_k}I^{\nu_k}_{q_k,\omega_k}f_x\right)(t).
\end{aligned} \tag{29}
$$

The Banach fixed point theorem and Leray–Schauder's nonlinear alternative will be used to study the existence and uniqueness results for the impulsive Hahn difference boundary value problem (2). Now, we set

$$\Lambda_3 = T\sum_{i=0}^{m}\frac{(t_{i+1}-t_i)_{\theta_i}^{(\nu_i-1)}}{\Gamma_{q_i}(\nu_i)} + \sum_{i=0}^{m}\frac{(t_{i+1}-t_i)_{\theta_i}^{(\nu_i)}}{\Gamma_{q_i}(\nu_i+1)} + \sum_{i=1}^{m}(t_{i+1}-t_i)\sum_{j=0}^{i-1}\frac{(t_{j+1}-t_j)_{\theta_j}^{(\nu_j-1)}}{\Gamma_{q_j}(\nu_j)},$$

$$\Lambda_4 = mT + \sum_{i=1}^{m}(t_{i+1}-t_i)i, \qquad \Lambda_5 = L_1\Lambda_3 + L_2 m + L_3\Lambda_4, \qquad \Lambda_6 = |\eta_1| + |\eta_2|T.$$

Theorem 4 *Let f and φ_k be given functions satisfying (19) and (20), respectively, for all $k = 1, 2, \ldots, m$. Assume the $\varphi_k^* : \mathbb{R} \to \mathbb{R}$ such that*

$$\left|\varphi_k^*(x) - \varphi_k^*(y)\right| \le L_3|x-y|, \quad L_3 > 0, \forall x, y \in \mathbb{R}. \tag{30}$$

If $\Lambda_5 < 1$, then the impulsive fractional quantum Hahn difference boundary value problem (2) has a unique solution on J.

Proof The existence of a unique solution for the problem (2) will be proved by considering an operator equation $x = Gx$, where G is defined by (29). Consider the set $B_R = \{x \in PC(J,\mathbb{R}) : \|x\| \le R\}$, where a positive constant R satisfies

$$R > \frac{\Lambda_6 + \Lambda_3 M + mN + \Lambda_4 K}{1 - \Lambda_5},$$

and $K = \max_j |\varphi_j^*(0)|$ and constants M, N are defined in the proof of Theorem 3. We claim that $GB_R \subset B_R$. Since

$$\begin{aligned}
&\left|Gx(t)\right| \\
&\quad \le |\eta_1| + \left(|\eta_2| + \sum_{j=0}^{m}\left({}_{t_j}I_{q_j,\omega_j}^{\nu_j-1}|f_x|\right)\left(t_{j+1}^-\right) + \sum_{i=1}^{m}\left|\varphi_i^*\left(x\left(t_i^-\right)\right)\right|\right)\sum_{i=0}^{m}(t_{i+1}-t_i) \\
&\qquad + \sum_{i=0}^{m-1}\left[\left({}_{t_i}I_{q_i,\omega_i}^{\nu_i}|f_x|\right)\left(t_{i+1}^-\right) + \left|\varphi_{i+1}\left(x\left(t_{i+1}^-\right)\right)\right|\right] \\
&\qquad + \sum_{i=1}^{m}\left\{(t_{i+1}-t_i)\sum_{j=0}^{i-1}\left[\left({}_{t_j}I_{q_j,\omega_j}^{\nu_j-1}|f_x|\right)\left(t_{j+1}^-\right) + \left|\varphi_{j+1}^*\left(x\left(t_{j+1}^-\right)\right)\right|\right]\right\} + \left({}_{t_m}I_{q_m,\omega_m}^{\nu_m}|f_x|\right)(T),
\end{aligned}$$

and $|f_x| \le |f_x - f_0| + |f_0| \le L_1 R + M$, $|\varphi_j(x)| \le |\varphi_j(x) - \varphi_j(0)| + |\varphi_j(0)| \le L_2 R + N$, $|\varphi_j^*(x)| \le |\varphi_j^*(x) - \varphi_j^*(0)| + |\varphi_j^*(0)| \le L_3 R + K$, $j = 1, 2, 3, \ldots, m$, for any $x \in B_R$, we have

$$\left|Gx(t)\right| \le \Lambda_6 + R\Lambda_5\Lambda_3 M + mN + \Lambda_4 K < R,$$

which yields $\|Gx\| \le R$. To prove the contraction property of operator G, for any $x, y \in B_R$, we consider the inequalities

$$\begin{aligned}
&\left|Gx(t) - Gy(t)\right| \\
&\quad \le T\sum_{j=0}^{m}\left({}_{t_j}I_{q_j,\omega_j}^{\nu_j-1}|f_x - f_y|\right)\left(t_{j+1}^-\right) + T\sum_{i=1}^{m}\left|\varphi_i^*\left(x\left(t_i^-\right)\right) - \varphi_i^*\left(y\left(t_i^-\right)\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=0}^{m-1}\Big[\big({}_{t_i}I_{q_i,\omega_i}^{\nu_i}|f_x-f_y|\big)\big(t_{i+1}^-\big)+\big|\varphi_{i+1}\big(x\big(t_{i+1}^-\big)\big)-\varphi_{i+1}\big(y\big(t_{i+1}^-\big)\big)\big|\Big]\\
&+\sum_{i=1}^{m}\Bigg\{(t_{i+1}-t_i)\sum_{j=0}^{i-1}\Big[\big({}_{t_j}I_{q_j,\omega_j}^{\nu_j-1}|f_x-f_y|\big)\big(t_{j+1}^-\big)\\
&+\big|\varphi_{j+1}^*\big(x\big(t_{j+1}^-\big)\big)-\varphi_{j+1}^*\big(y\big(t_{j+1}^-\big)\big)\big|\Big]\Bigg\}+\big({}_{t_m}I_{q_m,\omega_m}^{\nu_m}|f_x-f_y|\big)(T)\\
\le{}& L_1\|x-y\|T\sum_{j=0}^{m}\big({}_{t_j}I_{q_j,\omega_j}^{\nu_j-1}(1)\big)\big(t_{j+1}^-\big)+L_3\|x-y\|T\sum_{i=1}^{m}(1)\\
&+L_1\|x-y\|\sum_{i=0}^{m-1}\big({}_{t_i}I_{q_i,\omega_i}^{\nu_i}(1)\big)\big(t_{i+1}^-\big)+L_2\|x-y\|\sum_{i=0}^{m-1}(1)\\
&+L_1\|x-y\|\sum_{i=1}^{m}(t_{i+1}-t_i)\sum_{j=0}^{i-1}\big({}_{t_j}I_{q_j,\omega_j}^{\nu_j-1}(1)\big)\big(t_{j+1}^-\big)\\
&+L_3\|x-y\|\sum_{i=1}^{m}(t_{i+1}-t_i)\sum_{j=0}^{i-1}(1)+L_1\|x-y\|\big({}_{t_m}I_{q_m,\omega_m}^{\nu_m}(1)\big)(T)\\
={}& L_1\Lambda_3\|x-y\|+L_2m\|x-y\|+L_3\Lambda_4\|x-y\|\\
={}& \Lambda_5\|x-y\|.
\end{aligned}
$$

Then we get $\|Gx-Gy\|\le\Lambda_5\|x-y\|$ which implies that G is a contraction operator as $\Lambda_5<1$. Therefore problem (2) has a unique solution x on J. □

Lemma 3 (Nonlinear alternative for single valued maps, [25]) *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0\in U$. Suppose that $F:\overline{U}\to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *There is a $u\in\partial U$ (the boundary of U in C) and $\lambda\in(0,1)$ with $u=\lambda F(u)$.*

Theorem 5 *Assume that the functions $f:J\times\mathbb{R}\to\mathbb{R}$, φ_k, $\varphi_k^*:\mathbb{R}\to\mathbb{R}$, $k=1,2,\dots,m$, are continuous. In addition, we suppose that:*

(H_1) *There exist a continuous nondecreasing function $\psi_1:[0,\infty)\to(0,\infty)$ and a continuous function $p:J\to\mathbb{R}^+$ such that*

$$\big|f(t,x)\big|\le p(t)\psi_1\big(|x|\big)\quad\textit{for each }(t,x)\in J\times\mathbb{R}.$$

(H_2) *There exist continuous nondecreasing functions $\psi_2,\psi_3:[0,\infty)\to(0,\infty)$ such that*

$$\big|\varphi_k(x)\big|\le\psi_2\big(|x|\big)\quad\textit{and}\quad\big|\varphi_k^*(x)\big|\le\psi_3\big(|x|\big),$$

for all $x\in\mathbb{R}$, $k=1,2,\dots,m$.

(H_3) *There exists a constant $Q > 0$ such that*

$$\frac{Q}{\Lambda_6 + p^*\psi_1(Q)\Lambda_3 + \psi_2(Q)m + \psi_3(Q)\Lambda_4} > 1,$$

where $p^* = \sup\{p(t) : t \in J\}$.

Then the impulsive fractional quantum Hahn difference boundary value problem (2) *has at least one solution on J.*

Proof Let us prove the theorem by applying Lemma 3. For a positive number ρ, we define the set $B_\rho = \{x \in PC(J,\mathbb{R}) : \|x\| \leq \rho\}$. Clearly, B_ρ is a closed, convex subset of $PC(J,\mathbb{R})$. Let $\{x_n\}$ be a sequence converging to x. Then for $t \in J$, we obtain

$$\begin{aligned}
&\bigl|Gx_n(t) - Gx(t)\bigr| \\
&\quad\leq \left(\sum_{j=0}^{m}\bigl({}_{t_j}I_{q_j,\omega_j}^{\nu_j-1}|f_{x_n} - f_x|\bigr)\bigl(t_{j+1}^-\bigr) - \sum_{i=1}^{m}\bigl|\varphi_i^*\bigl(x_n\bigl(t_i^-\bigr)\bigr) - \varphi_i^*\bigl(x\bigl(t_i^-\bigr)\bigr)\bigr|\right) \\
&\qquad\times \sum_{i=0}^{k}\bigl([t_{i+1}] - t_i\bigr) + \sum_{i=0}^{k-1}\bigl[\bigl({}_{t_i}I_{q_i,\omega_i}^{\nu_i}|f_{x_n} - f_x|\bigr)\bigl(t_{i+1}^-\bigr) \\
&\qquad + \bigl|\varphi_{i+1}\bigl(x_n\bigl(t_{i+1}^-\bigr)\bigr) - \varphi_{i+1}\bigl(x\bigl(t_{i+1}^-\bigr)\bigr)\bigr|\bigr] + \sum_{i=1}^{k}\Biggl\{\bigl([t_{i+1}] - t_i\bigr) \\
&\qquad\times \sum_{j=0}^{i-1}\bigl[\bigl({}_{t_j}I_{q_j,\omega_j}^{\nu_j-1}|f_{x_n} - f_x|\bigr)\bigl(t_{j+1}^-\bigr) + \bigl|\varphi_{j+1}^*\bigl(x_n\bigl(t_{j+1}^-\bigr)\bigr) - \varphi_{j+1}^*\bigl(x\bigl(t_{j+1}^-\bigr)\bigr)\bigr|\bigr]\Biggr\} \\
&\qquad + \bigl({}_{t_k}I_{q_k,\omega_k}^{\nu_k}|f_{x_n} - f_x|\bigr)(t) \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Hence the operator G is continuous which is one of assumptions in Lemma 3. In the next step, we will prove the compactness of operator G.

For $t \in J$ and $x \in B_\rho$, we have

$$\begin{aligned}
&\bigl|Gx(t)\bigr| \\
&\quad\leq |\eta_1| + \left(|\eta_2| + \sum_{j=0}^{m}\bigl({}_{t_j}I_{q_j,\omega_j}^{\nu_j-1}|f_x|\bigr)\bigl(t_{j+1}^-\bigr) + \sum_{i=1}^{m}\bigl|\varphi_i^*\bigl(x\bigl(t_i^-\bigr)\bigr)\bigr|\right)\sum_{i=0}^{k}\bigl([t_{i+1}] - t_i\bigr) \\
&\qquad + \sum_{i=0}^{k-1}\bigl[\bigl({}_{t_i}I_{q_i,\omega_i}^{\nu_i}|f_x|\bigr)\bigl(t_{i+1}^-\bigr) + \bigl|\varphi_{i+1}\bigl(x\bigl(t_{i+1}^-\bigr)\bigr)\bigr|\bigr] \\
&\qquad + \sum_{i=1}^{k}\Biggl\{\bigl([t_{i+1}] - t_i\bigr)\sum_{j=0}^{i-1}\bigl[\bigl({}_{t_j}I_{q_j,\omega_j}^{\nu_j-1}|f_x|\bigr)\bigl(t_{j+1}^-\bigr) + \bigl|\varphi_{j+1}^*\bigl(x\bigl(t_{j+1}^-\bigr)\bigr)\bigr|\bigr]\Biggr\} \\
&\qquad + \bigl({}_{t_k}I_{q_k,\omega_k}^{\nu_k}|f_x|\bigr)(t) \\
&\quad\leq |\eta_1| + \left(|\eta_2| + p^*\psi_1(\rho)\sum_{j=0}^{m}\bigl({}_{t_j}I_{q_j,\omega_j}^{\nu_j-1}(1)\bigr)\bigl(t_{j+1}^-\bigr) + \psi_3(\rho)\sum_{i=1}^{m}(1)\right)\sum_{i=0}^{m}(t_{i+1} - t_i) \\
&\qquad + p^*\psi_1(\rho)\sum_{i=0}^{m-1}\bigl({}_{t_i}I_{q_i,\omega_i}^{\nu_i}(1)\bigr)\bigl(t_{i+1}^-\bigr) + \psi_2(\rho)\sum_{i=1}^{m}(1)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\left\{(t_{i+1}-t_i)\sum_{j=0}^{i-1}\bigl[p^*\psi_1(\rho)\bigl({}_{t_j}I_{q_j,\omega_j}^{\nu_j-1}(1)\bigr)\bigl(t_{j+1}^-\bigr)+\psi_3(\rho)\bigr]\right\}\\
&+p^*\psi_1(\rho)\bigl({}_{t_m}I_{q_m,\omega_m}^{\nu_m}(1)\bigr)(T)\\
\le{}&|\eta_1|+|\eta_2|T+p^*\psi_1(\rho)\left(T\sum_{j=0}^{m}\frac{(t_{j+1}-t_j)_{\theta_j}^{(\nu_j-1)}}{\Gamma_{q_j}(\nu_j)}+\sum_{i=0}^{m}\frac{(t_{i+1}-t_i)_{\theta_i}^{(\nu_i)}}{\Gamma_{q_i}(\nu_i+1)}\right.\\
&\left.+\sum_{i=1}^{m}(t_{i+1}-t_i)\sum_{j=0}^{i-1}\frac{(t_{j+1}-t_j)_{\theta_j}^{(\nu_j-1)}}{\Gamma_{q_j}(\nu_j)}\right)+\psi_2(\rho)m\\
&+\psi_3(\rho)\left(mT+\sum_{i=1}^{m}(t_{i+1}-t_i)i\right)\\
:={}&K.
\end{aligned}
$$

Hence, we obtain $\|Gx\|\le K$, which means that the set GB_ρ is a uniformly bounded set. Next we let $\tau_1,\tau_2\in J_k$ for some $k\in\{0,1,2,\dots,m\}$ with $\tau_1<\tau_2$, and let $x\in B_\rho$. Then we see that

$$
\begin{aligned}
\bigl|Gx(\tau_2)-Gx(\tau_1)\bigr|\le{}&\left(|\eta_2|+p^*\psi_1(\rho)\sum_{j=0}^{m}\bigl({}_{t_j}I_{q_j,\omega_j}^{\nu_j-1}(1)\bigr)\bigl(t_{j+1}^-\bigr)+\psi_3(\rho)m\right)|\tau_2-\tau_1|\\
&+|\tau_2-\tau_1|\sum_{j=0}^{k-1}\bigl[p^*\psi_1(\rho)\bigl({}_{t_j}I_{q_j,\omega_j}^{\nu_j-1}(1)\bigr)\bigl(t_{j+1}^-\bigr)+\psi_3(\rho)\bigr]\\
&+p^*\psi_1(\rho)\bigl[\bigl({}_{t_k}I_{q_k,\omega_k}^{\nu_k}(1)\bigr)(\tau_2)-\bigl({}_{t_k}I_{q_k,\omega_k}^{\nu_k}(1)\bigr)(\tau_1)\bigr]\\
={}&\left(|\eta_2|+p^*\psi_1(\rho)\sum_{j=0}^{m}\frac{(t_{j+1}-t_j)_{\theta_j}^{(\nu_j-1)}}{\Gamma_{q_j}(\nu_j)}+\psi_3(\rho)m\right)|\tau_2-\tau_1|\\
&+|\tau_2-\tau_1|\sum_{j=0}^{k-1}\left[p^*\psi_1(\rho)\frac{(t_{j+1}-t_j)_{\theta_j}^{(\nu_j-1)}}{\Gamma_{q_j}(\nu_j)}+\psi_3(\rho)\right]\\
&+p^*\psi_1(\rho)\left[\left(\frac{(\tau_2-t_k)_{\theta_k}^{(\nu_k)}}{\Gamma_{q_k}(\nu_k+1)}\right)-\left(\frac{(\tau_1-t_k)_{\theta_k}^{(\nu_k)}}{\Gamma_{q_k}(\nu_k+1)}\right)\right].
\end{aligned}
$$

The right-hand side of the above inequality tends to zero as $\tau_1\to\tau_2$ (independently of x). This shows that the set GB_ρ is an equicontinuous set. Therefore the set GB_ρ is relatively compact. From the above and Arzelá–Ascoli theorem, the operator G is completely continuous or compact. Hence one more of assumptions of Lemma 3 holds.

The result will follow from Lemma 3 if we can prove the boundedness of the set of all solutions to equations $x=\lambda Gx$ for $\lambda\in(0,1)$. Let x be a solution of problem (2). Then, for $t\in J$, we recall the computations in proving that G is bounded. For $\lambda\in(0,1)$, let $x=\lambda Gx$. Then we have

$$
\bigl|x(t)\bigr|\le\Lambda_6+p^*\psi_1\bigl(\|x\|\bigr)\Lambda_3+\psi_2\bigl(\|x\|\bigr)m+\psi_3\bigl(\|x\|\bigr)\Lambda_4,
$$

which yields

$$\frac{\|x\|}{\Lambda_6 + p^*\psi_1(\|x\|)\Lambda_3 + \psi_2(\|x\|)m + \psi_3(\|x\|)\Lambda_4} \leq 1.$$

By assumption (H_3), there exists a positive constant Q such that $\|x\| \neq Q$. Let us define $U = \{x \in B_\rho : \|x\| < Q\}$. It is obvious that $G : \overline{U} \to PC(J, \mathbb{R})$ is continuous and completely continuous. Therefore, there is no $x \in \partial U$ such that $x = \lambda Gx$ for some $\lambda \in (0, 1)$. By Lemma 3, thus, we get the result that G has a fixed point $x \in \overline{U}$ which is a solution of problem (2) on J. The proof is completed. □

3 Examples

In this section we give examples to illustrate the usefulness of our main results.

Example 1 Consider the following impulsive fractional quantum Hahn difference boundary value problem:

$$\begin{cases} {}^{c}_{t_k}D^{\frac{k+2}{k+3}}_{\frac{k+1}{k+2},\frac{1}{k^2+5k+6}}x(t) = \frac{1}{2}(\frac{e^{-t}}{t+17})(\frac{x^2(t)+2|x(t)|}{1+|x(t)|}) + \frac{2}{7}, \quad t \in J_k, J = [0,4], \\ \Delta x(t_k) = \frac{1}{k+18}|\sin x(t_k^-)|, \quad t_k = k, k = 1,2,3, \\ \frac{1}{2}x(0) + \frac{2}{3}x(4) = \frac{4}{5}. \end{cases} \tag{31}$$

Here $\nu_k = (k+2)/(k+3) < 1$, $q_k = (k+1)/(k+2)$, $\omega_k = 1/(k^2+5k+6)$, $k = 0,1,2,3$, $m = 3$, $T = 4$, $\xi_1 = 1/2$, $\xi_2 = 2/3$, $\xi_3 = 4/5$.

In addition, we observe that $\theta_k = (1/(k+3)) + k \in J_k$, $k = 0,1,2,3$. By using a mathematical program, we can find that

$$\left(1 + \frac{|\xi_2|}{|\xi_1 + \xi_2|}\right)\sum_{i=0}^{m}\frac{(t_{i+1} - t_i)^{(\nu_i)}_{\theta_i}}{\Gamma_{q_i}(\nu_i + 1)} \approx 8.357863592111553,$$
$$\frac{1}{2}\left(1 + \frac{|\xi_2|}{|\xi_1 + \xi_2|}\right)m(m+1) \approx 9.428571428571431.$$

Setting

$$f(t,x) = \frac{1}{2}\left(\frac{e^{-t}}{t+17}\right)\left(\frac{x^2 + 2|x|}{1+|x|}\right) + \frac{2}{7} \quad \text{and} \quad \varphi_k(x) = \frac{1}{k+18}|\sin x|,$$

we compute that

$$\left|f(t,x) - f(t,y)\right| \leq (1/17)|x-y| \quad \text{and} \quad \left|\varphi_k(x) - \varphi_k(y)\right| \leq (1/19)|x-y|.$$

Then we get $\Lambda_1 \approx 0.987879636333851 < 1$, by using $L_1 = 1/17$ and $L_2 = 1/19$. Hence, by Theorem 3, the impulsive fractional quantum Hahn difference boundary value problem (31) has a unique solution on $[0,4]$.

Example 2 Consider the impulsive fractional quantum Hahn difference boundary value problem

$$\begin{cases} {}_{t_k}^{c}D_{\frac{k+3}{k+5},\frac{1}{k+5}}^{\frac{2k+3}{k+2}}x(t) = \frac{1}{(t+10)^2}\left(\frac{x^8(t)}{|x^7(t)|+1}+1\right), \quad t \in J_k, J = [0,4], \\ \Delta x(t_k) = \frac{k^2}{10(k^2+1)}|x(t_k^-)| + 2, \quad k = 1,2,3, t_k = k, \\ {}_{t_k}D_{\frac{k+3}{k+5},\frac{1}{k+5}}x(t_k^+) - {}_{t_{k-1}}D_{\frac{k+2}{k+4},\frac{1}{k+4}}x(t_k^-) = \frac{\sin^2 k}{100}|x(t_k^-)| + 3, \quad k = 1,2,3, t_k = k, \\ x(0) = \frac{4}{7}, \qquad {}_{t_3}D_{\frac{3}{4},\frac{1}{8}}x(4) = \frac{5}{9}. \end{cases} \tag{32}$$

Here $\nu_k = (2k+3)/(k+2)$, $1 < \nu_k \leq 2$, $q_k = (k+3)/(k+5)$, $\omega_k = 1/(k+5)$, $k = 0,1,2,3$, $m = 3$, $T = 4$, $\eta_1 = 4/7$, $\eta_2 = 5/9$. We find that $\theta_k = (1/2) + k \in J_k$, $k = 0,1,2,3$. By using a mathematical program, we obtain constants as

$$\Lambda_3 \approx 40.771217238380451, \qquad \Lambda_4 = 18, \qquad \Lambda_6 \approx 2.793650793650793.$$

Setting

$$f(t,x) = \frac{1}{(t+10)^2}\left(\frac{x^8}{|x^7|+1}+1\right),$$
$$\varphi_k(x) = \frac{k^2}{10(k^2+1)}|x| + 2 \quad \text{and} \quad \varphi_k^*(x) = \frac{\sin^2 k}{100}|x| + 3,$$

and choosing $p(t) = (1/(t+10)^2)$, $\psi_1(u) = u+1$, $\psi_2(u) = (1/10)u+2$ and $\psi_3(u) = (1/100)u+3$, it follows that

$$\left|f(t,u)\right| \leq p(t)\psi_1(|u|), \qquad \left|\varphi_k(u)\right| \leq \psi_2(|u|), \qquad \left|\varphi_k^*(u)\right| \leq \psi_3(|u|),$$

for all $u \in \mathbb{R}$, $k = 1,2,3$, which imply that conditions (H_1)–(H_2) hold. For $p^* = (1/100)$, by direct computation, there exists a constant $Q > 562.8514177160808$ satisfying the inequality in (H_3). Applying Theorem 5, we deduce that the impulsive fractional quantum Hahn difference boundary value problem (32) has at least one solution on $[0,4]$.

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