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# Dynamics analysis of a stochastic non-autonomous one-predator–two-prey system with Beddington–DeAngelis functional response and impulsive perturbations

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## Abstract

In this paper, we explore a stochastic non-autonomous one-predator–two-prey system with Beddington–DeAngelis functional response and impulsive perturbations. First, by using Itô's formula, exponential martingale inequality, Chebyshev's inequality and other mathematical skills, we establish some sufficient conditions for extinction, non-persistence in the mean, weak persistence, persistence in the mean and stochastic permanence of the solution of the stochastic system. Then the limit of the average in time of the sample path of the solution is estimated by two constants. Afterwards, the lower-growth rate and the upper-growth rate of the positive solution are estimated. In addition, sufficient conditions for global attractivity of the system are established. Finally, we carry out some simulations to verify our main results and explain the biological implications: the large stochastic interference is disadvantageous for the persistence of the population and the strong impulsive harvesting can lead to extinct of the population.

**Keywords:** Stochastic one-predator–two-prey; Impulsive effect; Beddington–DeAngelis functional response; Stochastic permanence; Global attractivity

## 1 Introduction

Predator–prey systems, competitive systems and cooperative systems, the three major systems in the ecosystem, play a vital role in promoting the stable operation of biological communities. Among them, predation and competition are the most common phenomena in nature, such as, tiger hunting rabbits, wolves catching deer, two trees in the same forest, eagle and snake feeding on the same mouse and so on. Many scholars have studied predation and competition systems (see [1–13]). Among them, the one-predator–two-prey system (see [14–18]) is the most common system in the ecosystem. Therefore, it is important and meaningful to consider dynamical behavior of the one-predator–two-prey system with interspecies competition. When modeling the one-predator–two-prey system, one of the most important factors should be involved is the functional response

mechanism, which changes the prey density per unit time per predator as a function of prey or both prey and predator species. There are many kinds of famous functional response in the predator–prey system reported in the previous references, such as Holling types [19–21], Beddington–DeAngelis type [22–25], Michaelis–Menton type [26], Ivlev type [27], Hassell–Varley type [28], Crowley–Martin type functional response [29], which are suitable for different kinds of predator–prey systems, respectively. In 1975, Beddington [22] and DeAngelis [23] first introduced the Beddington–DeAngelis type predator–prey model taking the form

$$\begin{cases} \frac{dx}{dt} = r_1x - \alpha_1x^2 - \frac{c_1xy}{a_1+a_2x+a_3y}, \\ \frac{dy}{dt} = r_2y - \alpha_2y^2 + \frac{c_2xy}{a_1+a_2x+a_3y}, \end{cases} \tag{1}$$

where  $x$  and  $y$  denote the population densities of prey and predator, respectively. The term  $\frac{c_1x}{a_1+a_2x+a_3y}$  represents the Beddington–DeAngelis functional response, which turns into the Holling-II functional response if  $a_3 = 0$  and linear functional response if both  $a_2 = 0$  and  $a_3 = 0$ . That is to say, the B-D functional response is affected by both predator and prey. Therefore, the effect of mutual interference on the dynamics of population is worth studying.

On the other hand, the population systems in the real world are always inevitably influenced by all kinds of environmental noises which are an important component in an ecosystem. Usually, there are two types of environmental noises: white noise and color noise. White noise arises from a nearly continuous series of small or moderate perturbations that have small effects on the intrinsic growth rates of the species. Therefore, it is essential to reveal how the environmental noise disturbs the population systems. In recent years, many scholars have proposed and investigated stochastic models with white noise perturbations, please refer to [30–46] and the references therein. For example, Ji et al. [30] considered a predator–prey model with modified Leslie–Gower and Holling type II schemes with stochastic perturbation and the condition for persistence and extinction of the system is established. Liu and Wang in [36] discussed a predator–prey system with Beddington–DeAngelis functional response with stochastic perturbation. They demonstrated that if the positive equilibrium of the deterministic system is globally stable, then the stochastic model will preserve this nice property provided the noise is sufficiently small.

However, periodic behavior often arises in implicit ways in various natural phenomena. For example, due to the seasonal variation, hunting, harvesting and so on, the birth rate, the mortality rate and other parameters in the population systems will not remain constant, but exhibit a more-or-less periodicity. Thus, it is natural to model the population by a periodic environment. Therefore, numerous authors have investigated the effect of seasonal variation and stochasticity (see [47–49]).

Furthermore, population growth in ecosystems is also affected by human activities, such as periodic harvesting or stocking for the species, which cannot be considered continuously. Stochastic systems that consider continuous phenomena are not suitable for these phenomena. Therefore, in this case, we should consider the effect of impulse in order to describe these phenomena more accurately. In recent decades, a variety of population dynamical systems with impulsive effects have been proposed and studied extensively (see [50–56]). For example, in [50] Liu and Wang concerned with an

n-species stochastic nonautonomous Lotka-Volterra competitive system with impulsive effects. They obtained the sufficient conditions for stochastic permanence, extinction and global stability and investigated some dynamical properties. Zhang and Meng et al. [52] discussed a stochastic non-autonomous predator–prey system with impulsive effect. They concluded that the large stochastic disturbances can lead to the extinction of the population, and large impulse harvests can also result in the extinction of the population.

Taking all above influences into consideration, we focus on the stochastic non-autonomous one-predator–two-prey system with the Beddington–DeAngelis functional response and impulsive perturbations

$$\left\{ \begin{array}{l} \left. \begin{array}{l} dx_1(t) = x_1(t) \left[ r_1(t) - \alpha_1(t)x_1(t) - \frac{c_1(t)x_3(t)}{a_1(t)+a_2(t)x_1(t)+a_3(t)x_3(t)} \right. \\ \quad \left. - \beta_1(t)x_2(t) \right] dt + \sigma_1(t)x_1(t) dB_1(t), \\ dx_2(t) = x_2(t) \left[ r_2(t) - \alpha_2(t)x_2(t) - \frac{c_2(t)x_3(t)}{b_1(t)+b_2(t)x_2(t)+b_3(t)x_3(t)} \right. \\ \quad \left. - \beta_2(t)x_1(t) \right] dt + \sigma_2(t)x_2(t) dB_2(t), \\ dx_3(t) = x_3(t) \left[ r_3(t) - \alpha_3(t)x_3(t) + \frac{e_1(t)x_1(t)}{a_1(t)+a_2(t)x_1(t)+a_3(t)x_3(t)} \right. \\ \quad \left. + \frac{e_2(t)x_2(t)}{b_1(t)+b_2(t)x_2(t)+b_3(t)x_3(t)} \right] dt + \sigma_3(t)x_3(t) dB_3(t), \end{array} \right\} t \neq t_k, k \in N, \\ \left. \begin{array}{l} x_1(t^+) = (1 + h_{1k})x_1(t), \\ x_2(t^+) = (1 + h_{2k})x_2(t), \\ x_3(t^+) = (1 + h_{3k})x_3(t), \end{array} \right\} t = t_k, k \in N, \end{array} \right. \tag{2}$$

where  $x_i(t)$  is the size of the  $i$ th population at time  $t$ ,  $r_i(t)$  represents the intrinsic growth rate of the  $i$ th population,  $\alpha_i(t)$  stands for the density-dependent coefficients of the  $i$ th population,  $\beta_1(t)$  and  $\beta_2(t)$  are the competitive coefficient of  $x_1(t)$  and  $x_2(t)$ , respectively,  $c_j(t)$  is the capturing rate of predator,  $e_j(t)$  represents the rate of conversion of nutrients into the reproduction of predator,  $B_i(t)$  ( $i = 1, 2, 3$ ) is for independent standard Brownian motions defined on a complete probability space and  $\sigma_i(t)$  is for the intensities of  $B_i(t)$ .  $r_i(t)$ ,  $\alpha_i(t)$ ,  $\beta_j(t)$ ,  $a_i(t)$ ,  $b_i(t)$ ,  $c_j(t)$ ,  $e_j(t)$ ,  $\sigma_i(t)$  are positive, continuous and bounded functions defined on  $\mathbb{R}^+ = (0, \infty)$ ,  $N$  denotes the set of positive integers,  $0 < t_1 < t_2 < \dots$ ,  $\lim_{k \rightarrow +\infty} t_k = +\infty$ ,  $i = 1, 2, 3$ ,  $j = 1, 2$ ,  $k \in N$ .

We impose the following restriction on system (2) which is a reasonable way for giving biological meaning:  $h_{ik} + 1 > 0$ ,  $i = 1, 2, 3$ ,  $k \in N$ . When  $h_{ik} > 0$ , the impulsive effects represent releasing the specie, but if  $h_{ik} < 0$ , the impulsive effects denote harvesting for the  $i$ th population.

The main goals of this paper are to investigate how impulsive perturbations and the white noises affect the permanence, persistence, extinction and global attractivity of system (2). The rest of the paper is organized as follows. In Sect. 2, we give some definition and prove the existence of a unique positive solution of the system. In Sect. 3, we will derive main theoretical results of this paper, such as sufficient conditions for the extinction, non-persistence in the mean, weak persistence, persistence in the mean and stochastic permanence of the system. Meanwhile the limit of the average in time of the sample path of the solution is estimated by two constants. In Sect. 4, the lower-growth rate and the upper-growth rate of the solutions are estimated. In Sect. 5, we investigate the global attractivity of the system. In Sect. 6, we give the con-

clusions and several examples and numerical simulations to illustrate our theoretical results.

## 2 Preliminary

Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$  be a complete probability space with a filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  satisfying the common conditions (i.e. it is increasing and right continuous while  $\mathcal{F}_0$  contains all  $\mathcal{P}$ -null sets). Let  $B(t) = (B_1(t), B_2(t), B_3(t))^T$  be an  $n$ -dimensional Brownian motion defined on this probability space. Let  $\mathbb{R}_+^3 = \{x \in \mathbb{R}^3 : x_i > 0, i \leq i \leq 3\}$ . We define the norm as  $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ .

If  $f(t)$  is a bounded continuous function on  $[0, +\infty)$ , define

$$f^l = \inf_{t \rightarrow \mathbb{R}^+} f(t), \quad f^u = \sup_{t \rightarrow \mathbb{R}^+} f(t), \quad f_* = \liminf_{t \rightarrow +\infty} f(t), \quad f^* = \limsup_{t \rightarrow +\infty} f(t).$$

For the constants  $m_i, M_i, f_i^u, f_i^l$  ( $i = 1, 2, 3$ ), we denote

$$m = \min_{1 \leq i \leq 3} m_i, \quad M = \max_{1 \leq i \leq 3} M_i, \quad \hat{f} = \min_{1 \leq i \leq 3} f_i^l, \quad \check{f} = \max_{1 \leq i \leq 3} f_i^u.$$

### Definition 2.1

1.  $x(t)$  is said to be extinctive if  $\lim_{t \rightarrow +\infty} x(t) = 0$ .
2.  $x(t)$  is said to be non-persistent in the mean if  $\lim_{t \rightarrow +\infty} \frac{\int_0^t x(s) ds}{t} = 0$ .
3.  $x(t)$  is said to be weakly persistent if  $\limsup_{t \rightarrow +\infty} x(t) > 0$ .
4.  $x(t)$  is said to be persistent in the mean if  $\liminf_{t \rightarrow +\infty} \frac{\int_0^t x(s) ds}{t} > 0$ .
5.  $x(t)$  is said to be stochastically permanent if for every  $\varepsilon \in (0, 1)$  there are two constants  $\beta > 0, \delta > 0$  such that

$$\liminf_{t \rightarrow +\infty} \mathbb{P}\{x(t) \geq \beta\} \geq 1 - \varepsilon, \quad \liminf_{t \rightarrow +\infty} \mathbb{P}\{x(t) \leq \delta\} \geq 1 - \varepsilon.$$

Now we give an assumption which will be used in the following proof.

**Assumption 2.1** There exist constants  $m_i$  and  $M_i$  ( $i = 1, 2, 3$ ) such that

$$m_i \leq \prod_{0 < t_k < t} (1 + h_{ik}) \leq M_i.$$

*Remark 1* Assumption 2.1 is easy to satisfy. For example, if  $h_{ik} = e^{\frac{(-1)^{k+1}}{k^2}} - 1, i = 1, 2, 3$ , then  $1 \leq \prod_{0 < t_k < t} (1 + h_{ik}) \leq e$ .

**Theorem 2.1** For any given initial value  $(x_1(0), x_2(0), x_3(0))^T \in \mathbb{R}_+^3$ , system (2) exists a unique positive solution  $x(t) = (x_1(t), x_2(t), x_3(t))^T$  on  $\mathbb{R}^+$  and the positive solution will remain  $\mathbb{R}_+^3$  a.s.

*Proof* Consider the following stochastic differential equations (SDEs) without impulses:

$$\left\{ \begin{aligned}
 dy_1(t) &= y_1(t) \left[ r_1(t) - \alpha_1(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) - \beta_1(t) \prod_{0 < t_k < t} (1 + h_{2k}) y_2(t) \right. \\
 &\quad \left. - \frac{c_1(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)}{a_1(t) + a_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) + a_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \right] dt \\
 &\quad + \sigma_1(t) y_1(t) dB_1(t), \\
 dy_2(t) &= y_2(t) \left[ r_2(t) - \alpha_2(t) \prod_{0 < t_k < t} (1 + h_{2k}) y_2(t) - \beta_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) \right. \\
 &\quad \left. - \frac{c_2(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)}{b_1(t) + b_2(t) \prod_{0 < t_k < t} (1 + h_{2k}) y_2(t) + b_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \right] dt \\
 &\quad + \sigma_2(t) y_2(t) dB_2(t), \\
 dy_3(t) &= y_3(t) \left[ r_3(t) - \alpha_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t) \right. \\
 &\quad + \frac{e_1(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t)}{a_1(t) + a_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) + a_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \\
 &\quad + \frac{e_2(t) \prod_{0 < t_k < t} (1 + h_{2k}) y_2(t)}{b_1(t) + b_2(t) \prod_{0 < t_k < t} (1 + h_{2k}) y_2(t) + b_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \left. \right] dt \\
 &\quad + \sigma_3(t) y_3(t) dB_3(t),
 \end{aligned} \right. \tag{3}$$

with initial value  $(y_1(0), y_2(0), y_3(0))^T = (x_1(0), x_2(0), x_3(0))^T$ . By the classic theory of SDEs without impulses (see [57]), system (3) has a unique global positive solution  $y(t) = (y_1(t), y_2(t), y_3(t))^T$ .

Let  $x_i(t) = \prod_{0 < t_k < t} (1 + h_{ik}) y_i(t)$ , we show that  $x(t) = (x_1(t), x_2(t), x_3(t))^T$  is the solution of system (2) with initial value  $(x_1(0), x_2(0), x_3(0))^T$ .

In fact, since  $x_1(t)$  is continuous on each interval  $(t_k, t_{k+1}) \subset \mathbb{R}^+$  and for  $t \neq t_k, k \in N$ , we have

$$\begin{aligned}
 dx_1(t) &= \prod_{0 < t_k < t} (1 + h_{1k}) dy_1(t) \\
 &= \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) \left[ r_1(t) - \alpha_1(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) - \beta_1(t) \prod_{0 < t_k < t} (1 + h_{2k}) y_2(t) \right. \\
 &\quad \left. - \frac{c_1(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)}{a_1(t) + a_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) + a_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \right] dt \\
 &\quad + \sigma_1(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) dB_1(t) \\
 &= x_1(t) \left[ r_1(t) - \alpha_1(t) x_1(t) - \frac{c_1(t) x_3(t)}{a_1(t) + a_2(t) x_1(t) + a_3(t) x_3(t)} - \beta_1(t) x_2(t) \right] dt \\
 &\quad + \sigma_1(t) x_1(t) dB_1(t).
 \end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
 dx_2(t) &= x_2(t) \left[ r_2(t) - \alpha_2(t) x_2(t) - \frac{c_2(t) x_3(t)}{b_1(t) + b_2(t) x_2(t) + b_3(t) x_3(t)} \right. \\
 &\quad \left. - \beta_2(t) x_1(t) \right] dt + \sigma_2(t) x_2(t) dB_2(t), \\
 dx_3(t) &= x_3(t) \left[ r_3(t) - \alpha_3(t) x_3(t) + \frac{e_1(t) x_1(t)}{a_1(t) + a_2(t) x_1(t) + a_3(t) x_3(t)} \right. \\
 &\quad \left. + \frac{e_2(t) x_2(t)}{b_1(t) + b_2(t) x_2(t) + b_3(t) x_3(t)} \right] dt + \sigma_3(t) x_3(t) dB_3(t).
 \end{aligned}$$

And for each  $t_k \in \mathbb{R}^+$ , it is not difficult to show that

$$\begin{aligned} x_i(t_k^+) &= \lim_{t \rightarrow t_k^+} x_i(t) = \prod_{0 < t_j < t_k} (1 + h_{ij}) y_i(t_k^+) \\ &= (1 + h_{ik}) \prod_{0 < t_j < t_k} (1 + h_{ij}) y_i(t_k) = (1 + h_{ik}) x_i(t_k). \end{aligned}$$

Moreover,

$$x_i(t_k^-) = \lim_{t \rightarrow t_k^-} x_i(t) = \prod_{0 < t_j < t_k} (1 + h_{ij}) y_i(t_k^-) = \prod_{0 < t_j < t_k} (1 + h_{ij}) y_i(t_k) = x_i(t_k).$$

Therefore,  $x(t) = (x_1(t), x_2(t), x_3(t))^T$  is the unique global positive solution of system (2). This completes the proof of Theorem 2.1.  $\square$

### 3 Extinction and persistence

In this section we will derive sufficient conditions for the extinction, non-persistence in the mean, weak persistence, persistence in the mean and stochastic permanence of the solutions of system (2).

**Theorem 3.1** *Suppose that  $x(t) = (x_1(t), x_2(t), x_3(t))^T$  is a solution of system (2), then*

$$\limsup_{t \rightarrow +\infty} \frac{\ln x_i(t)}{t} \leq \limsup_{t \rightarrow +\infty} \frac{1}{t} \left[ \sum_{0 < t_k < t} \ln(1 + h_{ik}) + \int_0^t \delta_i(s) ds \right] := \delta_i^*, \quad a.s.,$$

where

$$\begin{cases} \delta_1(t) = r_1(t) - \frac{\sigma_1^2(t)}{2}, \\ \delta_2(t) = r_2(t) - \frac{\sigma_2^2(t)}{2}, \\ \delta_3(t) = r_3(t) + \frac{e_1(t)}{a_2(t)} + \frac{e_2(t)}{b_2(t)} - \frac{\sigma_3^2(t)}{2}. \end{cases}$$

Particularly, if  $\delta_i^* < 0$ , then  $\lim_{t \rightarrow +\infty} x_i(t) = 0$  a.s., namely, the  $i$ th species ( $i = 1, 2, 3$ ) in system (2) is extinct.

*Proof* Applying Itô’s formula to the first equation of system (3), we could find that

$$\begin{aligned} d \ln y_1(t) &= \left[ r_1(t) - \frac{\sigma_1^2(t)}{2} - \alpha_1(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) - \beta_1(t) \prod_{0 < t_k < t} (1 + h_{2k}) y_2(t) \right. \\ &\quad \left. - \frac{c_1(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)}{a_1(t) + a_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) + a_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \right] dt \\ &\quad + \sigma_1(t) dB_1(t) \\ &\leq \left[ r_1(t) - \frac{\sigma_1^2(t)}{2} - \alpha_1(t) x_1(t) \right] dt + \sigma_1(t) dB_1(t) \\ &= [\delta_1(t) - \alpha_1(t) x_1(t)] dt + \sigma_1(t) dB_1(t). \end{aligned} \tag{4}$$

In the same way, combining with the last two equations of system (3) we have

$$\begin{aligned}
 d \ln y_2(t) &= \left[ r_2(t) - \frac{\sigma_2^2(t)}{2} - \alpha_2(t) \prod_{0 < t_k < t} (1 + h_{2k}) y_2(t) - \beta_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) \right. \\
 &\quad \left. - \frac{c_2(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)}{b_1(t) + b_2(t) \prod_{0 < t_k < t} (1 + h_{2k}) y_2(t) + b_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \right] dt \\
 &\quad + \sigma_2(t) dB_2(t) \\
 &\leq \left[ r_2(t) - \frac{\sigma_2^2(t)}{2} - \alpha_2(t)x_2(t) \right] dt + \sigma_2(t) dB_2(t) \\
 &= [\delta_2(t) - \alpha_2(t)x_2(t)] dt + \sigma_2(t) dB_2(t), \tag{5}
 \end{aligned}$$

which leads to

$$\begin{aligned}
 d \ln y_3(t) &= \left[ r_3(t) - \alpha_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t) \right] dt + \sigma_3(t) dB_3(t) \\
 &\quad + \left[ \frac{e_1(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t)}{a_1(t) + a_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) + a_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \right. \\
 &\quad \left. + \frac{e_2(t) \prod_{0 < t_k < t} (1 + h_{2k}) y_2(t)}{b_1(t) + b_2(t) \prod_{0 < t_k < t} (1 + h_{2k}) y_2(t) + b_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \right] dt \\
 &\leq \left[ r_3(t) + \frac{e_1(t)}{a_2(t)} + \frac{e_2(t)}{b_2(t)} - \frac{\sigma_3^2(t)}{2} - \alpha_3(t)x_3(t) \right] dt + \sigma_3(t) dB_3(t) \\
 &= [\delta_3(t) - \alpha_3(t)x_3(t)] dt + \sigma_3(t) dB_3(t). \tag{6}
 \end{aligned}$$

Integrating both sides of inequalities (4), (5) and (6) on the interval  $[0, t]$ , one can easily see that

$$\ln y_i(t) - \ln y_i(0) \leq \int_0^t \delta_i(s) ds - \int_0^t \alpha_i(s)x_i(s) ds + M_i(t), \tag{7}$$

where  $M_i(t) = \int_0^t \sigma_i(s) dB_i(s)$ ,  $i = 1, 2, 3$ . Note that  $M_i(t)$  are local martingales, whose quadratic variations are  $\langle M_i(t), M_i(t) \rangle = \int_0^t \sigma_i^2(s) ds \leq (\sigma_i^u)^2 t$ . Making use of the strong law of large numbers for local martingales (see [58]) results in

$$\lim_{t \rightarrow +\infty} \frac{M_i(t)}{t} = 0 \quad \text{a.s.}$$

On the other hand, it follows from (7) that

$$\begin{aligned}
 &\sum_{0 < t_k < t} \ln(1 + h_{ik}) + \ln y_i(t) - \ln y_i(0) \\
 &\leq \sum_{0 < t_k < t} \ln(1 + h_{ik}) + \int_0^t \delta_i(s) ds - \int_0^t \alpha_i(s)x_i(s) ds + M_i(t).
 \end{aligned}$$

In other words, we can compute that

$$\begin{aligned} \ln x_i(t) &\leq \ln y_i(0) + \sum_{0 < t_k < t} \ln(1 + h_{ik}) + \int_0^t \delta_i(s) ds - \int_0^t \alpha_i(s)x_i(s) ds + M_i(t) \\ &\leq \ln y_i(0) + \sum_{0 < t_k < t} \ln(1 + h_{ik}) + \int_0^t \delta_i(s) ds + M_i(t). \end{aligned} \tag{8}$$

Taking superior limit on both sides of (8) and noting that  $\lim_{t \rightarrow +\infty} \frac{\ln y_i(0)}{t} = 0$ , we obtain

$$\limsup_{t \rightarrow +\infty} \frac{\ln x_i(t)}{t} \leq \limsup_{t \rightarrow +\infty} \frac{1}{t} \left[ \sum_{0 < t_k < t} \ln(1 + h_{ik}) + \int_0^t \delta_i(s) ds \right] := \delta_i^*, \quad \text{a.s.}$$

This completes the proof. □

**Theorem 3.2** *Suppose that  $x(t) = (x_1(t), x_2(t), x_3(t))^T$  is a solution of system (2), then*

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x_i(s) ds \leq \frac{\delta_i^*}{\alpha_i^l} = \bar{x}_i^*.$$

*Particularly, if  $\delta_i^* = 0$ , then  $\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x_i(s) ds = 0$ , that is, the  $i$ th species ( $i = 1, 2, 3$ ) in system (2) is non-persistent in the mean.*

*Proof* According to the definition of the limit, for arbitrary fixed  $\epsilon_i > 0$ , there exists a constant  $T_0 > 0$ , for every  $t > T_0$ , such that

$$\frac{\ln y_i(0)}{t} \leq \frac{\epsilon_i}{3}, \quad \frac{M_i(t)}{t} \leq \frac{\epsilon_i}{3}, \quad \frac{1}{t} \left[ \sum_{0 < t_k < t} \ln(1 + h_{ik}) + \int_0^t \delta_i(s) ds \right] \leq \delta_i^* + \frac{\epsilon_i}{3}.$$

Substituting above inequalities into (8) yields

$$\begin{aligned} \ln x_i(t) &\leq \ln y_i(0) + \sum_{0 < t_k < t} \ln(1 + h_{ik}) + \int_0^t \delta_i(s) ds - \int_0^t \alpha_i(s)x_i(s) ds + M_i(t) \\ &\leq (\delta_i^* + \epsilon_i)t - \int_0^t \alpha_i(s)x_i(s) ds \\ &\leq \lambda_i t - \alpha_i^l \int_0^t x_i(s) ds \quad \text{a.s.,} \end{aligned} \tag{9}$$

for all  $t > T_0$ , where  $\lambda_i = \delta_i^* + \epsilon_i$ .

Denote  $g_i(t) = \int_0^t x_i(s) ds$ , we get  $\frac{dg_i(t)}{dt} = x_i(t)$ . Taking exponent on both sides of (9), we can show that

$$e^{\alpha_i^l g_i(t)} \frac{dg_i(t)}{dt} \leq e^{\lambda_i t}. \tag{10}$$

Integrating inequality (10) from  $T_0$  to  $t$  yields

$$e^{\alpha_i^l g_i(t)} \leq \frac{\alpha_i^l}{\lambda_i} e^{\lambda_i t} + e^{\alpha_i^l g_i(T_0)} - \frac{\alpha_i^l}{\lambda_i} e^{\lambda_i T_0}. \tag{11}$$



Taking logarithm of both sides of inequality (11), we can derive that

$$\int_0^t x_i(s) ds \leq \frac{1}{\alpha_i^l} \ln \left[ \frac{\alpha_i^l}{\lambda_i} e^{\lambda_i t} + e^{\alpha_i^l g_i(T_0)} - \frac{\alpha_i^l}{\lambda_i} e^{\lambda_i T_0} \right]. \tag{12}$$

Taking superior limit on (12) elicits that

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x_i(s) ds \leq \limsup_{t \rightarrow +\infty} \frac{1}{\alpha_i^l t} \ln \left[ \frac{\alpha_i^l}{\lambda_i} e^{\lambda_i t} + e^{\alpha_i^l g_i(T_0)} - \frac{\alpha_i^l}{\lambda_i} e^{\lambda_i T_0} \right].$$

Then it follows from L'Hospital's rule that

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x_i(s) ds \leq \limsup_{t \rightarrow +\infty} \frac{\lambda_i}{\alpha_i^l} = \frac{\delta_i^*}{\alpha_i^l} = \bar{x}_i^*.$$

This completes the proof of this theorem. □

**Theorem 3.3** *Suppose that  $x(t) = (x_1(t), x_2(t), x_3(t))^T$  is a solution of system (2). If  $\delta_i^* > 0$ , then the  $i$ th species ( $i = 1, 2, 3$ ) in system (2) is weakly persistent a.s., i.e.  $\limsup_{t \rightarrow +\infty} x_i(t) > 0$  a.s.*

*Proof* If this assertion is not true, then  $P(S) > 0$ , where  $S$  is the set  $S = \limsup_{t \rightarrow +\infty} x_i(t) = 0$ . It follows from (8) that

$$\begin{aligned} \frac{\ln x_i(t) - \ln x_i(0)}{t} &\leq \frac{1}{t} \left[ \sum_{0 < t_k < t} \ln(1 + h_{ik}) + \int_0^t \delta_i(s) ds \right] \\ &\quad - \frac{1}{t} \int_0^t \alpha_i(s) x_i(s) ds + \frac{M_i(t)}{t}. \end{aligned} \tag{13}$$

On the other hand, for  $\forall \omega \in S$ , we have  $\lim_{t \rightarrow +\infty} x_i(t, \omega) = 0$ . Thus it follows from the boundedness of  $\alpha_i(t)$  that

$$\limsup_{t \rightarrow +\infty} \frac{\ln x_i(t) - \ln x_i(0)}{t} \leq 0, \quad \limsup_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \alpha_i(s) x_i(s) ds = 0.$$

Substituting these inequalities into (13) and making use of  $\lim_{t \rightarrow +\infty} \frac{M_i(t)}{t} = 0$  a.s., one can obtain the contradiction  $0 \geq \limsup_{t \rightarrow +\infty} \ln x_i(t, \omega) = \delta_i^* > 0$ . This completes the proof. □

*Remark 2* Theorems 3.1–3.3 have an interesting biological interpretation. Observe that the extinction and persistence of species  $x_i(t)$  only depend on  $\delta_i^*$ . If  $\delta_i^* > 0$ , the population  $x_i(t)$  is weakly persistent. If  $\delta_i^* < 0$ , the population  $x_i(t)$  goes to extinction.

**Theorem 3.4** *Suppose that  $x(t) = (x_1(t), x_2(t), x_3(t))^T$  is a solution of system (2), then*

$$\liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x_i(s) ds \geq \frac{\theta_i^*}{\alpha_i^u} = \underline{x}_i^*, \quad a.s.$$

where

$$\begin{aligned} \theta_{1*} &= \delta_{1*} - \left( \frac{\beta_1^u \delta_2^*}{\alpha_2^l} + \eta_1 \right), & \theta_{2*} &= \delta_{2*} - \left( \frac{\beta_2^u \delta_1^*}{\alpha_1^l} + \eta_2 \right), & \theta_{3*} &= \delta_{3*}, \\ \delta_{i*} &= \liminf_{t \rightarrow +\infty} \frac{1}{t} \left[ \sum_{0 < t_k < t} \ln(1 + h_{ik}) + \int_0^t \left( r_i(s) - \frac{1}{2} \sigma_i^2(s) \right) ds \right], & i &= 1, 2, 3, \\ \eta_1 &= \min \left\{ \frac{c_1^u}{a_3^l}, \frac{c_1^u \delta_3^*}{a_1^l \alpha_3^l} \right\}, & \eta_2 &= \min \left\{ \frac{c_2^u}{b_3^l}, \frac{c_2^u \delta_3^*}{b_1^l \alpha_3^l} \right\}. \end{aligned}$$

Particularly, if  $\theta_{i*} > 0$ , then the  $i$ th species ( $i = 1, 2, 3$ ) in system (2) is persistent in the mean a.s.

*Proof* Applying Itô’s formula to the first equation of system (3), we can observe that

$$\begin{aligned} d \ln y_1(t) &= \left[ r_1(t) - \frac{\sigma_1^2(t)}{2} - \alpha_1(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) - \beta_1(t) \prod_{0 < t_k < t} (1 + h_{2k}) y_2(t) \right. \\ &\quad \left. - \frac{c_1(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)}{a_1(t) + a_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) + a_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \right] dt \\ &\quad + \sigma_1(t) dB_1(t) \\ &\geq \left[ r_1(t) - \frac{\sigma_1^2(t)}{2} - \alpha_1(t) x_1(t) - \beta_1(t) x_2(t) - \frac{c_1(t) x_3(t)}{a_1(t) + a_3(t) x_3(t)} \right] dt \\ &\quad + \sigma_1(t) dB_1(t), \end{aligned} \tag{14}$$

Applying general calculations to (14), it is easy to verify that

$$\begin{aligned} \ln x_1(t) &\geq \ln y_1(0) + \sum_{0 < t_k < t} \ln(1 + h_{1k}) + \int_0^t \left( r_1(s) - \frac{1}{2} \sigma_1^2(s) \right) ds - \int_0^t \alpha_1(s) x_1(s) ds \\ &\quad - \int_0^t \beta_1(s) x_2(s) ds - \int_0^t \frac{c_1(s) x_3(s)}{a_1(s) + a_3(s) x_3(s)} ds + M_1(t) \\ &\geq \ln y_1(0) + \sum_{0 < t_k < t} \ln(1 + h_{1k}) + \int_0^t \left( r_1(s) - \frac{1}{2} \sigma_1^2(s) \right) ds - \alpha_1^u \int_0^t x_1(s) ds \\ &\quad - \beta_1^u \int_0^t x_2(s) ds - \int_0^t \frac{c_1(s) x_3(s)}{a_1(s) + a_3(s) x_3(s)} ds + M_1(t). \end{aligned} \tag{15}$$

It then follows from Theorem 3.2 that

$$\limsup_{t \rightarrow +\infty} \frac{\int_0^t x(s) ds}{t} \leq \frac{\delta_i^*}{\alpha_i^l}, \quad i = 1, 2, 3.$$

Since  $\lim_{t \rightarrow +\infty} \frac{y_i(0)}{t} = 0$ ,  $\lim_{t \rightarrow +\infty} \frac{M_i(t)}{t} = 0$ ,  $i = 1, 2, 3$ , for  $\forall \epsilon_1 > 0$  there exists a  $T_1 > 0$ , such that

$$\int_0^t x_1(s) ds \leq \left( \frac{\delta_1^*}{\alpha_1^l} + \frac{\epsilon_1}{\beta_2^u} \right) t, \quad \int_0^t x_2(s) ds \leq \left( \frac{\delta_2^*}{\alpha_2^l} + \frac{\epsilon_1}{\beta_1^u} \right) t,$$

$$\int_0^t x_3(s) ds \leq \left( \frac{\delta_3^*}{\alpha_3^l} + \frac{a_1^l \epsilon_1}{c_1^u} \right) t,$$

$$\sum_{0 < t_k < t} \ln(1 + h_{1k}) + \int_0^t \left( r_1(s) - \frac{1}{2} \sigma_1^2(s) \right) ds \geq (\delta_{1*} - \epsilon_1)t,$$

$$y_1(0) \geq -\epsilon_1 t, \quad M_1(t) \geq -\epsilon_1 t.$$

Substituting the above inequalities into (15), we get, for  $t > T_1$ ,

$$\ln x_1(t) \geq \theta_1 t - \alpha_1^u \int_0^t x_1(s) ds, \tag{16}$$

where  $\theta_1 = \delta_{1*} - \left( \frac{\beta_1^u \delta_2^*}{\alpha_2^l} + \eta_1 \right) - \epsilon_1$ , and  $\eta_1 = \min \left\{ \frac{c_1^u}{a_3^l}, \frac{c_1^u \delta_3^*}{a_1^l \alpha_3^l} \right\}$ .

In the similar way, we can conclude that, for any  $\epsilon_i$ , there exists some  $T_i > 0$  such that

$$\ln x_i(t) \geq \theta_i t - \alpha_i^u \int_0^t x_i(s) ds, \quad t > T_i, \tag{17}$$

where

$$\theta_2 = \delta_{2*} - \left( \frac{\beta_2^u \delta_1^*}{\alpha_1^l} + \eta_2 \right) - \epsilon_2, \quad \theta_3 = \delta_{3*} - \epsilon_3,$$

and

$$\eta_2 = \min \left\{ \frac{c_2^u}{b_3^l}, \frac{c_2^u \delta_3^*}{b_1^l \alpha_3^l} \right\}, \quad i = 2, 3.$$

Let  $T^* = \min_{1 \leq i \leq 3} T_i > 0$ , then from (16) and (17), we can easily see that

$$\ln x_i(t) \geq \theta_i t - \alpha_i^u \int_0^t x_i(s) ds, \quad t > T^*, i = 1, 2, 3. \tag{18}$$

Denote  $g_i(t) = \int_0^t x_i(s) ds$ , we get  $\frac{dg_i(t)}{dt} = x_i(t)$ . Taking the exponent on both sides of (18), we can obtain

$$e^{\alpha_i^u g_i(t)} \frac{dg_i(t)}{dt} \geq e^{\theta_i t}. \tag{19}$$

Integrating inequality (19) from  $T^*$  to  $t$  yields

$$e^{\alpha_i^u g_i(t)} \geq \frac{\alpha_i^u}{\theta_i} e^{\theta_i t} + e^{\alpha_i^u g_i(T^*)} - \frac{\alpha_i^u}{\theta_i} e^{\theta_i T^*}. \tag{20}$$

Taking logarithm of both sides of inequality (20), it can be verified straightforwardly that

$$\int_0^t x_i(s) ds \geq \frac{1}{\alpha_i^u} \ln \left[ \frac{\alpha_i^u}{\theta_i} e^{\theta_i t} + e^{\alpha_i^u g_i(T^*)} - \frac{\alpha_i^u}{\theta_i} e^{\theta_i T^*} \right]. \tag{21}$$

Taking superior limit on both sides of (21), we obtain

$$\liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x_i(s) ds \geq \liminf_{t \rightarrow +\infty} \frac{1}{\alpha_i^u t} \ln \left[ \frac{\alpha_i^u}{\theta_i} e^{\theta_i t} + e^{\alpha_i^u g_i(T^*)} - \frac{\alpha_i^u}{\theta_i} e^{\theta_i T^*} \right].$$

Then it follows from L'Hospital's rule that

$$\liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x_i(s) ds \geq \liminf_{t \rightarrow +\infty} \frac{\theta_i}{\alpha_i^u} = \frac{\theta_{i*}}{\alpha_i^u} = \underline{x}_i^*.$$

This completes the proof of this theorem. □

**Theorem 3.5** *If Assumption 2.1 holds and  $(\check{\sigma})^2 < 2\hat{r}$ , then system (2) is stochastically permanent.*

*Proof* First, we prove that, for arbitrary  $\varepsilon > 0$ , there exists a constant  $\beta > 0$  such that

$$\liminf_{t \rightarrow +\infty} P\{x(t) \geq \beta\} \geq 1 - \varepsilon.$$

Define

$$V_1(y) = \frac{1}{U^2(y)}, \quad V_2(y) = (1 + V_1(y))^\varrho, \quad V_3(y) = e^{\kappa t} V_2(y),$$

where  $U(y) = \sum_{i=1}^3 y_i(t)$ ,  $\varrho > 0$ ,  $\kappa$  is a positive constant to be determined.

Applying Itô's formula and system (3) once again, we can calculate that

$$\begin{aligned} dV_1(y) = & -\frac{2}{U^3} \left[ \sum_{i=1}^3 y_i(t) \left( r_i(t) - \alpha_i(t) \prod_{0 < t_k < t} (1 + h_{ik}) y_i(t) \right) \right. \\ & - \frac{3}{2U} \sum_{i=1}^3 \sigma_i^2(t) y_i^2(t) - \beta_1(t) \prod_{0 < t_k < t} (1 + h_{2k}) y_1(t) y_2(t) \\ & - \beta_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) y_2(t) \\ & - \frac{c_1(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_1(t) y_3(t)}{a_1(t) + a_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) + a_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \\ & - \frac{c_2(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_2(t) y_3(t)}{b_1(t) + b_2(t) \prod_{0 < t_k < t} (1 + h_{2k}) y_2(t) + b_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \\ & + \frac{e_1(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) y_3(t)}{a_1(t) + a_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) + a_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \\ & \left. + \frac{e_2(t) \prod_{0 < t_k < t} (1 + h_{2k}) y_2(t) y_3(t)}{b_1(t) + b_2(t) \prod_{0 < t_k < t} (1 + h_{2k}) y_2(t) + b_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \right] dt \\ & - \frac{2}{U^3} \sum_{i=1}^3 \sigma_i(t) y_i(t) dB_i(t). \end{aligned}$$

Thus,

$$\begin{aligned} dV_1(y) \leq & \frac{2}{U^3} \left[ -\hat{r} \sum_{i=1}^3 y_i(t) + \left( M\check{\alpha} + \frac{3(\check{\sigma})^2}{2U} \right) \sum_{i=1}^3 y_i^2(t) + (\beta_1^u M_2 + \beta_2^u M_1) y_1(t) y_2(t) \right. \\ & \left. + \frac{c_1^u M_3}{a_1^l} y_1(t) y_3(t) + \frac{c_2^u M_3}{b_1^l} y_2(t) y_3(t) \right] dt - \frac{2}{U^3} \sum_{i=1}^3 \sigma_i(t) y_i(t) dB_i(t). \end{aligned}$$

Substituting inequality  $y_i(t)y_j(t) \leq \frac{y_i^2(t)+y_j^2(t)}{2}$  ( $i, j = 1, 2, 3$ ) into the above inequality and making some estimations yield

$$\begin{aligned} dV_1(y) &\leq \frac{2}{U^3} \left[ -\hat{r}U + \left( M \left( \check{\alpha} + 2\check{\beta} + \frac{c_1^u}{a_1^l} + \frac{c_2^u}{b_1^l} \right) + \frac{3(\check{\sigma})^2}{2U} \right) \sum_{i=1}^3 y_i^2(t) \right] dt \\ &\quad - \frac{2}{U^3} \sum_{i=1}^3 \sigma_i(t)y_i(t) dB_i(t) \\ &= \frac{2}{U^3} \left[ -\hat{r}U + \left( M\phi + \frac{3(\check{\sigma})^2}{2U} \right) \sum_{i=1}^3 y_i^2(t) \right] dt - \frac{2}{U^3} \sum_{i=1}^3 \sigma_i(t)y_i(t) dB_i(t), \end{aligned} \tag{22}$$

where  $\phi = (\check{\alpha} + 2\check{\beta} + \frac{c_1^u}{a_1^l} + \frac{c_2^u}{b_1^l})$ .

Further, when  $y_i > 0$ ,  $\sum_{i=1}^3 y_i^2(t) < (\sum_{i=1}^3 y_i(t))^2 = U^2$ , then from (22), we can derive that

$$dV_1(y) \leq \frac{2}{U^2} \left[ -\hat{r} + \frac{3(\check{\sigma})^2}{2} + M\phi U \right] dt - \frac{2}{U^3} \sum_{i=1}^3 \sigma_i(t)y_i(t) dB_i(t). \tag{23}$$

On the other hand, it follows from Itô's integration by parts formula and applying (23) that

$$\begin{aligned} dV_2(y) &= \varrho(1 + V_1(y))^{e-1} dV_1(y) + \frac{1}{2}(\varrho - 1)(1 + V_1(y))^{e-2} (dV_1(y))^2 \\ &\leq \varrho(1 + V_1(y))^{e-2} \left[ (1 + V_1(y)) \left( -\frac{2\hat{r}}{U^2} + \frac{3(\check{\sigma})^2}{U^2} + \frac{2M\phi}{U} \right) + \frac{2(\varrho - 1)(\check{\sigma})^2}{U^4} \right] dt \\ &\quad - \frac{2\varrho}{U^3} (1 + V_1(y))^{e-1} \sum_{i=1}^3 \sigma_i(t)y_i(t) dB_i(t) \\ &= \varrho(1 + V_1(y))^{e-2} \left[ -(2\hat{r} - (2\varrho + 1)(\check{\sigma})^2) V_1^2(y) + 2M\phi V_1^{\frac{3}{2}}(y) + 2M\phi V_1^{\frac{1}{2}}(y) \right. \\ &\quad \left. + (3(\check{\sigma})^2 - 2\hat{r}) V_1(y) \right] dt - \frac{2\varrho}{U^3} (1 + V_1(y))^{e-1} \sum_{i=1}^3 \sigma_i(t)y_i(t) dB_i(t). \end{aligned} \tag{24}$$

We can choose positive constant  $\kappa$  small enough such that

$$0 < \kappa < \varrho(2\hat{r} - (2\varrho + 1)(\check{\sigma})^2).$$

Then

$$\begin{aligned} dV_3(y) &= \kappa e^{\kappa t} V_2(y) dt + e^{\kappa t} dV_2(y) \\ &\leq \varrho e^{\kappa t} (1 + V_1(y))^{e-2} \left[ -\left( 2\hat{r} - (2\varrho + 1)(\check{\sigma})^2 - \frac{\kappa}{\varrho} \right) V_1^2(y) + 2M\phi V_1^{\frac{3}{2}}(y) \right. \\ &\quad \left. + \left( 3(\check{\sigma})^2 + \frac{2\kappa}{\varrho} - 2\hat{r} \right) V_1(y) + 2M\phi V_1^{\frac{1}{2}}(y) + \frac{\kappa}{\varrho} \right] dt \\ &\quad - \frac{2\varrho}{U^3} (1 + V_1(y))^{e-1} \sum_{i=1}^3 \sigma_i(t)y_i(t) dB_i(t) \end{aligned}$$

$$= e^{\kappa t} H(y) dt - \frac{2\varrho}{U^3} (1 + V_1(y))^{e-1} \sum_{i=1}^3 \sigma_i(t) y_i(t) dB_i(t),$$

where

$$H(y) = \varrho(1 + V_1(y))^{e-2} \left[ -\left( 2\hat{r} - (2\varrho + 1)(\check{\sigma})^2 - \frac{\kappa}{\varrho} \right) V_1^2(y) + 2M\phi V_1^{\frac{3}{2}}(y) + \left( 3(\check{\sigma})^2 + \frac{2\kappa}{\varrho} - 2\hat{r} \right) V_1(y) + 2M\phi V_1^{\frac{1}{2}}(y) + \frac{\kappa}{\varrho} \right].$$

By the definition of  $\kappa$ ,  $H(y)$  is upper bounded in  $\mathbb{R}^+$ , we let  $H = \sup_{y \in \mathbb{R}^+} H(y) < +\infty$ , we could find that

$$dV_3(y) \leq e^{\kappa t} H dt - \frac{2\varrho}{U^3} (1 + V_1(y))^{e-1} \sum_{i=1}^3 \sigma_i(t) y_i(t) dB_i(t). \tag{25}$$

Integrating inequality (25) on the interval  $[0, t]$ , then multiplying  $e^{-\kappa t}$  and taking expectations on both sides, it is not difficult to show that

$$\mathbb{E}[(1 + V_1(y))^e] \leq V_2(y_0) \mathbb{E}[e^{-\kappa t}] + \frac{H}{\kappa} \mathbb{E}[1 - e^{-\kappa t}],$$

where  $y_0 = \sum_{i=1}^3 y_i(0)$ . Thus,

$$\begin{aligned} \limsup_{t \rightarrow +\infty} \mathbb{E} \left[ \frac{1}{U^{2e}}(y) \right] &\leq \limsup_{t \rightarrow +\infty} \mathbb{E}[(1 + V_1(y))^e] \\ &\leq \limsup_{t \rightarrow +\infty} \left[ \frac{V_2(y_0)}{e^{\kappa t}} + \frac{H(1 - e^{-\kappa t})}{\kappa} \right] = \frac{H}{\kappa}. \end{aligned} \tag{26}$$

On the other hand, since  $m \leq m_i \leq \prod_{0 < t_k < t} (1 + h_{ik}) \leq M_i \leq M$  and by the previous transformation  $x_i(t) = \prod_{0 < t_k < t} (1 + h_{ik}) y_i(t)$ , we have

$$M^{-2} \left( \sum_{i=1}^3 x_i \right)^2 \leq U^2(y) \leq \left( \sum_{i=1}^3 y_i \right)^2 \leq m^{-2} \left( \sum_{i=1}^3 x_i \right)^2 \leq 4m^{-2} \sum_{i=1}^3 x_i^2,$$

which yields

$$\left( \frac{m^2}{4} \right)^e \left( \sum_{i=1}^3 x_i^2 \right)^{-e} \leq U^{-2e}(y).$$

Consequently,

$$\limsup_{t \rightarrow +\infty} \mathbb{E} \left[ \left( \frac{m^2}{4} \right)^e \left( \sum_{i=1}^3 x_i^2 \right)^{-e} \right] \leq \limsup_{t \rightarrow +\infty} \mathbb{E}[U^{-2e}(y)] \leq \frac{H}{\kappa},$$

which leads to

$$\limsup_{t \rightarrow +\infty} \mathbb{E}[|x(t)|^{-e}] \leq \left( \frac{m^2}{4} \right)^{-e} \limsup_{t \rightarrow +\infty} \mathbb{E}[U^{-2e}(y)] \leq \frac{4^e H}{\kappa m^{2e}}.$$

Then, for any  $\varepsilon > 0$ , set  $\beta = (\frac{\kappa m^{2q} \varepsilon}{4^q H})^{\frac{1}{q}}$ , it follows from Chebyshev’s inequality (see [57]) that

$$\limsup_{t \rightarrow +\infty} \mathbb{P}\{|x(t)| < \beta\} = \limsup_{t \rightarrow +\infty} \mathbb{P}\{|x(t)|^{-q} > \beta^{-q}\} \leq \lim_{t \rightarrow +\infty} \frac{\mathbb{E}[|x(t)|^{-q}]}{\beta^{-q}} \leq \varepsilon.$$

In other words,

$$\liminf_{t \rightarrow +\infty} \mathbb{P}\{|x(t)| \geq \beta\} \geq 1 - \varepsilon. \tag{27}$$

Next we show that, for arbitrary  $\varepsilon > 0$ , there exists a constant  $\delta > 0$  such that

$$\liminf_{t \rightarrow +\infty} \mathbb{P}\{x(t) \leq \delta\} \geq 1 - \varepsilon.$$

Let  $q > 2$ , applying Itô’s formula to the non-impulsive system (3),

$$\begin{aligned} d(e^t y_1^q(t)) &= e^t y_1^q(t) dt + q e^t y_1^{q-1}(t) dy_1(t) + \frac{1}{2} q(q-1) e^t y_1^{q-2}(t) (dy_1(t))^2 \\ &= e^t y_1^q(t) \left[ 1 + q \left( r_1(t) + \frac{(q-1)\sigma_1^2(t)}{2} - \alpha_1(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) \right. \right. \\ &\quad \left. \left. - \frac{c_1(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)}{a_1(t) + a_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) + a_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \right. \right. \\ &\quad \left. \left. - \beta_1(t) \prod_{0 < t_k < t} (1 + h_{2k}) y_2(t) \right) \right] dt + q e^t y_1^q(t) \sigma_1(t) dB_1(t) \\ &\leq e^t y_1^q(t) \left[ 1 + q r_1^u + \frac{q(q-1)(\sigma_1^u)^2}{2} - q \alpha_1^l m_1 y_1(t) \right] dt \\ &\quad + q e^t \sigma_1(t) y_1^q(t) dB_1(t). \end{aligned} \tag{28}$$

Integrating (28) on the interval  $[0, t]$  yields

$$\begin{aligned} e^t y_1^q(t) - y_1^q(0) &\leq \int_0^t e^s y_1^q(s) \left[ 1 + q r_1^u + \frac{q(q-1)(\sigma_1^u)^2}{2} - q \alpha_1^l m_1 y_1(s) \right] ds \\ &\quad + q \int_0^t e^s \sigma_1(s) y_1^q(s) dB_1(s). \end{aligned} \tag{29}$$

Taking expectations on both sides of (29) we obtain

$$\mathbb{E}[e^t y_1^q(t)] \leq y_1^q(0) + \mathbb{E} \left[ \int_0^t e^s y_1^q(s) \left[ 1 + q r_1^u + \frac{q(q-1)(\sigma_1^u)^2}{2} - q \alpha_1^l m_1 y_1(s) \right] ds \right].$$

Denote

$$g(y_1) = y_1^q \left[ 1 + q r_1^u + \frac{q(q-1)(\sigma_1^u)^2}{2} - q \alpha_1^l m_1 y_1 \right],$$

then we have

$$g'(y_1) = q \left[ 1 + q r_1^u + \frac{q(q-1)(\sigma_1^u)^2}{2} - (q+1) \alpha_1^l m_1 y_1 \right] y_1^{q-1}$$

and

$$g''(y_1) = q \left( (q-1) \left[ 1 + qr_1^\mu + \frac{q(q-1)(\sigma_1^\mu)^2}{2} \right] - q(q+1)\alpha_1^l m_1 y_1 \right) y_1^{q-2}.$$

It is easy to see that  $g(y_1)$  has a unique maximum  $y_1^* = \frac{1+qr_1^\mu + \frac{q(q-1)(\sigma_1^\mu)^2}{2}}{(q+1)\alpha_1^l m_1}$  since

$$g''(y_1^*) = -q \left[ 1 + qr_1^\mu + \frac{q(q-1)(\sigma_1^\mu)^2}{2} \right] (y_1^*)^{q-2} < 0.$$

Therefore,

$$g(y_1) \leq g(y_1^*) = \frac{[1 + qr_1^\mu + \frac{q(q-1)(\sigma_1^\mu)^2}{2}]^{q+1}}{(q+1)^{q+1}(\alpha_1^l m_1)^q} := \Theta_1(q),$$

which yields

$$\mathbb{E}[e^t y_1^q(t)] \leq y_1^q(0) + \Theta_1(q) \mathbb{E} \left[ \int_0^t e^s y_1^q(s) ds \right] = y_1^q(0) + \Theta_1(q)(e^t - 1).$$

On the other hand, by applying Itô's formula and the last two equations of system (3) then making some estimations, we can easily see that

$$\begin{aligned} d(e^t y_2^q(t)) &\leq e^t y_2^q(t) \left[ 1 + qr_2^\mu + \frac{q(q-1)(\sigma_2^\mu)^2}{2} - q\alpha_2^l m_2 y_2(t) \right] dt + qe^t \sigma_2(t) y_2^q(t) dB_2(t), \\ d(e^t y_3^q(t)) &\leq e^t y_3^q(t) \left[ 1 + q \left( r_3^\mu + \frac{e_1^\mu}{a_2^l} + \frac{e_2^\mu}{b_2^l} \right) + \frac{q(q-1)(\sigma_3^\mu)^2}{2} - q\alpha_3^l m_3 y_3(t) \right] dt \\ &\quad + qe^t \sigma_3(t) y_3^q(t) dB_3(t). \end{aligned}$$

Then, similar to the above discussions, we can also derive that

$$\mathbb{E}[e^t y_i^q(t)] \leq y_i^q(0) + \Theta_i(q) \mathbb{E} \left[ \int_0^t e^s y_i^q(s) ds \right] = y_i^q(0) + \Theta_i(q)(e^t - 1), \quad i = 2, 3, \tag{30}$$

where

$$\begin{aligned} \Theta_2(q) &= \frac{[1 + qr_2^\mu + \frac{q(q-1)(\sigma_2^\mu)^2}{2}]^{q+1}}{(q+1)^{q+1}(\alpha_2^l m_2)^q}, \\ \Theta_3(q) &= \frac{[1 + q(r_3^\mu + \frac{e_1^\mu}{a_2^l} + \frac{e_2^\mu}{b_2^l}) + \frac{q(q-1)(\sigma_3^\mu)^2}{2}]^{q+1}}{(q+1)^{q+1}(\alpha_3^l m_3)^q}. \end{aligned}$$

Combining (29) and (30), we can conclude that

$$\mathbb{E}[e^t y_i^q(t)] \leq y_i^q(0) + \Theta_i(q)(e^t - 1), \quad i = 1, 2, 3. \tag{31}$$

Multiplying  $e^{-t}$  on both sides of (31) and taking the superior limit yield

$$\limsup_{t \rightarrow +\infty} \mathbb{E}[y_i^q(t)] \leq \limsup_{t \rightarrow +\infty} \frac{y_i^q(0) + \Theta_i(q)(e^t - 1)}{e^t} = \Theta_i(q), \quad i = 1, 2, 3. \tag{32}$$



This leads to

$$\limsup_{t \rightarrow +\infty} \mathbb{E}[x_i^q(t)] \leq \limsup_{t \rightarrow +\infty} \mathbb{E} \left[ \prod_{0 < t_k < t} (1 + h_{ik})^q y_i^q(t) \right] \leq \Theta_i(q)(M_i)^q, \quad i = 1, 2, 3.$$

Then, for any  $\varepsilon > 0$ , let  $\delta = \sqrt{\frac{\Theta}{\varepsilon}}$ , it follows from Chebyshev’s inequality that

$$\limsup_{t \rightarrow +\infty} \mathbb{P}\{|x(t)| > \delta\} = \limsup_{t \rightarrow +\infty} \mathbb{P}\{|x(t)|^2 > \delta^2\} \leq \lim_{t \rightarrow +\infty} \frac{\mathbb{E}[|x(t)|^2]}{\delta^2} = \varepsilon,$$

where  $\Theta = \sum_{i=1}^3 \Theta_i(q)(M_i)^2$ . As a consequence,

$$\limsup_{t \rightarrow +\infty} \mathbb{P}\{|x(t)| \leq \delta\} \geq 1 - \varepsilon. \tag{33}$$

According to Definition 2.1, it follows from (27) and (33) that system (2) is stochastically permanent.  $\square$

*Remark 3* From inequality (32), we can get

$$\begin{aligned} \limsup_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \mathbb{E}[y_i^q(t)] ds &\leq [y_i^q(0) - \Theta_i(q)] \limsup_{t \rightarrow +\infty} \frac{1}{t} \int_0^t e^{-s} ds + \Theta_i(q) \\ &= \Theta_i(q), \quad i = 1, 2, 3. \end{aligned}$$

Therefore, system (2) has the property

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \mathbb{E}[x_i^q(t)] ds \leq \Theta_i(q)(M_i)^q, \quad i = 1, 2, 3.$$

#### 4 Asymptotic properties

In this section we will discuss the asymptotic properties of the solution of system (2).

**Theorem 4.1** *If Assumption 2.1 holds and any solution  $x(t) = (x_1(t), x_2(t), x_3(t))^T$  of system (2) has the property that*

$$\limsup_{t \rightarrow +\infty} \frac{\ln x_i(t)}{\ln t} \leq 1 \quad a.s.$$

and, moreover,  $2\hat{r} - (\check{\sigma})^2 > 0$ , then

$$\liminf_{t \rightarrow +\infty} \frac{\ln |x_i(t)|}{\ln t} \geq -\frac{\check{\sigma}^2}{2\hat{r} - (\check{\sigma})^2} \quad a.s.$$

*Proof* It follows from Itô’s formula and combining with inequality (4), (5) and (6) that

$$\begin{aligned} d(e^t \ln y_i(t)) &= e^t \ln y_i(t) dt + e^t d(\ln y_i(t)) \\ &\leq e^t [\ln y_i(t) + \delta_i(t) - m_i \alpha_i^l y_i(t)] dt + e^t \sigma_i(t) dB_i(t). \end{aligned} \tag{34}$$

Integrating above inequality (34) on the interval  $[0, t]$  yields

$$e^t \ln y_i(t) - \ln y_i(0) \leq \int_0^t e^s [\ln y_i(s) + \delta_i(s) - m_i \alpha_i^l y_i(s)] ds + N_i(t), \tag{35}$$

where  $N_i(t) = \int_0^t e^t \sigma_i(s) dB_i(s)$  is the exponential martingale, whose quadratic variation is

$$\langle N_i(t), N_i(t) \rangle = \int_0^t e^{2s} \sigma_i^2(s) ds, \quad i = 1, 2, 3.$$

Thus, it follows from the exponential martingale inequality (see [57]) that

$$\mathbb{P} \left\{ \sup_{0 \leq t \leq k\gamma} \left[ N_i(t) - \frac{1}{2} e^{-k\gamma} \langle N_i(t), N_i(t) \rangle \right] > \rho e^{k\gamma} \ln k \right\} \leq k^{-\rho}, \quad \rho > 1, \gamma > 0.$$

By virtue of the Borel–Cantelli lemma, for almost all  $\omega \in \Omega$ , there exists  $k_0(\omega)$  such that, for every  $k \geq k_0(\omega)$ ,

$$N_i(t) \leq \frac{1}{2} e^{-k\gamma} \langle N_i(t), N_i(t) \rangle + \rho e^{k\gamma} \ln k = \frac{1}{2} e^{-k\gamma} \int_0^t e^{2s} \sigma_i^2(s) ds + \rho e^{k\gamma} \ln k, \tag{36}$$

for  $0 \leq t \leq k\gamma$ . Substituting inequality (36) into (35) and making some estimations yield

$$e^t \ln y_i(t) - \ln y_i(0) \leq \int_0^t e^s \left[ \ln y_i(s) + \delta_i^u + \frac{(\sigma_i^u)^2}{2} - m_i \alpha_i^l y_i(s) \right] ds + \rho e^{k\gamma} \ln k.$$

If we denote

$$f(y_i) = \ln y_i + \delta_i^u + \frac{(\sigma_i^u)^2}{2}.$$

Then  $f'(y_i) = \frac{1}{y_i} - m_i \alpha_i^l, f''(y_i) = -\frac{1}{y_i^2} < 0$ , this means  $y_i^* = \frac{1}{m_i \alpha_i^l}$  is the unique maximum of the function  $f(y_i)$ , i.e.  $f(y_i) \leq f(y_i^*)$ .

Thus,

$$\begin{aligned} e^t \ln y_i(t) &\leq \ln y_i(0) + \frac{1}{m_i \alpha_i^l} \int_0^t e^s ds + \rho e^{k\gamma} \ln k \\ &= \ln y_i(0) + \frac{1}{m_i \alpha_i^l} (e^t - 1) + \rho e^{k\gamma} \ln k. \end{aligned} \tag{37}$$

Multiplying  $e^{-t}$  on both sides of (37) yields

$$\ln y_i(t) \leq \ln e^{-t} y_i(0) + \frac{1}{m_i \alpha_i^l} (1 - e^{-t}) + \rho e^{k\gamma - t} \ln k.$$

For  $(k - 1)\gamma \leq t \leq k\gamma$  and  $k \geq k_0(\omega)$ , if  $t \rightarrow \infty$ , then  $k \rightarrow \infty$ .

Therefore,

$$\limsup_{t \rightarrow +\infty} \frac{\ln y_i(t)}{\ln t} \leq \limsup_{t \rightarrow +\infty} \frac{\ln e^{-t} y_i(0) + \frac{1}{m_i \alpha_i^l} (1 - e^{-t}) + \rho e^{k\gamma - t} \ln k}{\ln t} = \rho e^\gamma.$$

Let  $\rho \rightarrow 1$  and  $\gamma \rightarrow 0$ , then  $\limsup_{t \rightarrow +\infty} \frac{\ln y_i(t)}{\ln t} \leq 1$ . Since Assumption 2.1 holds,

$$\limsup_{t \rightarrow +\infty} \frac{\ln x_i(t)}{\ln t} = \limsup_{t \rightarrow +\infty} \frac{\sum_{0 < t_k < t} \ln(1 + h_{ik}) + \ln y_i(t)}{\ln t} \leq 1.$$

Now, we prove the next part. By (26), there exists a constant  $C_1 > 0$  such that

$$\limsup_{t \rightarrow +\infty} \mathbb{E}[(1 + V_1(y))^e] \leq C_1, \quad t \geq 0. \tag{38}$$

At the same time, it follows from (24) that

$$\begin{aligned} dV_2(y) &\leq \varrho(1 + V_1(y))^{e-2} [-(2\hat{r} - (2\varrho + 1)(\check{\sigma})^2)V_1^2(y) + 2M\phi V_1^{\frac{3}{2}}(y) + 2M\phi V_1^{\frac{1}{2}}(y) \\ &\quad + (3(\check{\sigma})^2 - 2\hat{r})V_1(y)] dt - \frac{2\varrho}{U^3}(1 + V_1(y))^{e-1} \sum_{i=1}^3 \sigma_i(t)y_i(t) dB_i(t) \\ &\leq \varrho C_2(1 + V_1(y))^e dt - \frac{2\varrho}{U^3}(1 + V_1(y))^{e-1} \sum_{i=1}^3 \sigma_i(t)y_i(t) dB_i(t), \end{aligned} \tag{39}$$

where  $C_2 = \max\{|2\hat{r} - (2\varrho + 1)(\check{\sigma})^2|, M\phi, |3(\check{\sigma})^2 - 2\hat{r}|\}$ . Let  $\mu > 0$  be sufficiently small for

$$\varrho C_2 \mu + 24\varrho \mu^{\frac{1}{2}} \sqrt{\check{\sigma}^2} < \frac{1}{2}. \tag{40}$$

Let  $k = 1, 2, \dots$ , making use of (39) shows that

$$\begin{aligned} &\mathbb{E} \left[ \limsup_{(k-1)\mu \leq t \leq k\mu} (1 + V_1(y(t)))^e \right] \\ &\leq \mathbb{E}[(1 + V_1(y((k-1)\mu)))^e] + \mathbb{E} \left[ \limsup_{(k-1)\mu \leq t \leq k\mu} \left| \int_{(k-1)\mu}^t \varrho C_2(1 + V_1(y(s)))^e ds \right| \right] \\ &\quad + \mathbb{E} \left[ \limsup_{(k-1)\mu \leq t \leq k\mu} \left| \int_{(k-1)\mu}^t \frac{2\varrho}{U^3(y(s))} (1 + V_1(y(s)))^{e-1} \sum_{i=1}^3 \sigma_i(s)y_i(s) dB_i(s) \right| \right]. \end{aligned} \tag{41}$$

We compute that

$$\begin{aligned} &\mathbb{E} \left[ \limsup_{(k-1)\mu \leq t \leq k\mu} \left| \int_{(k-1)\mu}^t \varrho C_2(1 + V_1(y(s)))^e ds \right| \right] \\ &\leq \mathbb{E} \left[ \int_{(k-1)\mu}^t |\varrho C_2(1 + V_1(y(s)))^e| ds \right] \\ &\leq \varrho C_2 \mu \mathbb{E} \left[ \limsup_{(k-1)\mu \leq t \leq k\mu} (1 + V_1(y(t)))^e \right]. \end{aligned} \tag{42}$$

On the other hand, by the famous Burkholder–Davis–Gundy inequality (see [57]), it is easy to derive that

$$\begin{aligned}
 & \mathbb{E} \left[ \limsup_{(k-1)\mu \leq t \leq k\mu} \left| \int_{(k-1)\mu}^t \frac{2\varrho}{U^3(y(s))} (1 + V_1(y(s)))^{e-1} \sum_{i=1}^3 \sigma_i(s) y_i(s) dB_i(s) \right| \right] \\
 & \leq \sum_{i=1}^3 \mathbb{E} \left[ \limsup_{(k-1)\mu \leq t \leq k\mu} \left| \int_{(k-1)\mu}^t \frac{2\varrho}{U^3(y(s))} (1 + V_1(y(s)))^{e-1} (\sigma_i(s) y_i(s) dB_i(s)) \right| \right] \\
 & \leq 4 \sum_{i=1}^3 \mathbb{E} \left[ \left( \int_{(k-1)\mu}^{k\mu} 4\varrho^2 (1 + V_1(y(s)))^{2e-2} V_1^3(y(s)) \sigma_i^2(s) y_i^2(s) ds \right)^{\frac{1}{2}} \right] \\
 & = 4 \sum_{i=1}^3 \mathbb{E} \left[ \left( \int_{(k-1)\mu}^{k\mu} 4\varrho^2 (1 + V_1(y(s)))^{2e} \frac{V_1^2(y(s))}{(1 + V_1(y(s)))^2} \frac{\sigma_i^2(s) y_i^2(s)}{(\sum_{i=1}^3 y_i(s))^2} ds \right)^{\frac{1}{2}} \right] \\
 & \leq 8\varrho \sqrt{\check{\sigma}^2} \sum_{i=1}^3 \mathbb{E} \left[ \left( \int_{(k-1)\mu}^{k\mu} (1 + V_1(y(s)))^{2e} ds \right)^{\frac{1}{2}} \right] \\
 & \leq 24\varrho \mu^{\frac{1}{2}} \sqrt{\check{\sigma}^2} \mathbb{E} \left[ \limsup_{(k-1)\mu \leq t \leq k\mu} (1 + V_1(y(t)))^e \right]. \tag{43}
 \end{aligned}$$

Substituting (43) and (42) into (41) results in

$$\begin{aligned}
 & \mathbb{E} \left[ \limsup_{(k-1)\mu \leq t \leq k\mu} (1 + V_1(y(t)))^e \right] \\
 & \leq \mathbb{E} \left[ (1 + V_1(y((k-1)\mu)))^e \right] \\
 & \quad + (\varrho C_2 \mu + 24\varrho \mu^{\frac{1}{2}} \sqrt{\check{\sigma}^2}) \mathbb{E} \left[ \limsup_{(k-1)\mu \leq t \leq k\mu} (1 + V_1(y(t)))^e \right].
 \end{aligned}$$

Applying (38) and (40), we can show that

$$\mathbb{E} \left[ \limsup_{(k-1)\mu \leq t \leq k\mu} (1 + V_1(y(t)))^e \right] \leq 2C_1.$$

Let  $\epsilon > 0$  be arbitrary. Then, by the Chebyshev inequality, we obtain

$$\mathbb{P} \left\{ \omega : \sup_{(k-1)\mu \leq t \leq k\mu} (1 + V_1(y(t)))^e > (k\mu)^{1+\epsilon} \right\} \leq \frac{2C_1}{(k\mu)^{1+\epsilon}}, \quad k = 1, 2, \dots$$

By the Borel–Cantelli lemma [59], for almost all  $\omega \in \Omega$ , there exists an integer  $k_0 = k_0(\omega)$  such that

$$\frac{\ln(1 + V_1(y(t)))^e}{\ln t} \leq \frac{(1 + \epsilon) \ln(k\mu)}{\ln((k-1)\mu)}$$

for  $k \geq k_0$  and  $(k-1)\mu \leq t \leq k\mu$ . That is to say

$$\limsup_{t \rightarrow +\infty} \frac{\ln(1 + V_1(y(t)))^e}{\ln t} \leq 1 + \epsilon.$$

Letting  $\epsilon \rightarrow 0$  gives

$$\limsup_{t \rightarrow +\infty} \frac{\ln(|y(t)|^{-2\varrho})}{\ln t} \leq 1.$$

Consequently,

$$\liminf_{t \rightarrow +\infty} \frac{\ln(|y(t)|)}{\ln t} \geq -\frac{1}{2\varrho}.$$

But this holds for any  $\varrho$  that satisfies  $2\hat{r} > (2\varrho + 1)(\check{\sigma})^2$ , we therefore have

$$\liminf_{t \rightarrow +\infty} \frac{\ln(|y(t)|)}{\ln t} \geq -\frac{\check{\sigma}^2}{2\hat{r} - (\check{\sigma})^2}.$$

It then follows that

$$\liminf_{t \rightarrow +\infty} \frac{\ln(|x(t)|)}{\ln t} \geq \liminf_{t \rightarrow +\infty} \frac{\ln(m|y(t)|)}{\ln t} \geq -\frac{\check{\sigma}^2}{2\hat{r} - (\check{\sigma})^2}.$$

This completes the proof of this theorem. □

*Remark 4* Theorem 4.1 shows that, for any  $\epsilon > 0$ , there exists a random variable  $T_\epsilon > 0$  such that  $t^{-\frac{1}{2\hat{r} - (\check{\sigma})^2} + \epsilon} \leq |x(t)| \leq t^{1 + \epsilon}$  for  $t \geq T_\epsilon$  almost surely. That is to say, the solution will not decay faster than  $t^{-\frac{1}{2\hat{r} - (\check{\sigma})^2} + \epsilon}$  and will not grow faster than  $t^{1 + \epsilon}$  with probability one. We are now in the position to estimate the limit of the average in time of the sample paths of solutions.

### 5 Global attractivity

In this section we give the definition of global attractivity and some useful lemmas to study the global attractivity of system (2).

**Definition 5.1** Let  $x(t) = (x_1(t), x_2(t), x_3(t))^T, z(t) = (z_1(t), z_2(t), z_3(t))^T$  be two arbitrary solutions of system (2) with initial values  $x(0), z(0) \in \mathbb{R}^+$ , respectively. If  $\lim_{t \rightarrow +\infty} |x(t) - z(t)| = 0$  a.s., then we say system (2) is globally attractive.

**Lemma 5.1** (see [60]) Let  $X(t)$  be an  $n$ -dimensional stochastic process on  $t \geq 0$ . Suppose that there exist positive constants  $\alpha, \beta, c$  such that

$$\mathbb{E}|X(t) - X(s)|^\alpha \leq c|t - s|^{1 + \beta}, \quad 0 \leq s, t < \infty.$$

Then there exists a continuous modification  $\bar{X}(t)$  of  $X(t)$  which has the property that for every  $\vartheta \in (0, \frac{\beta}{\alpha})$  there is a positive random variable  $h(\omega)$  such that

$$\mathbb{P} \left\{ \omega : \sup_{0 < |t - s| < h(\omega), 0 \leq s, t < \infty} \frac{|\bar{X}(t, \omega) - X(t, \omega)|}{|t - s|^\vartheta} \leq \frac{2}{1 - 2^{-\vartheta}} \right\} = 1.$$

In other words, almost every sample path of  $\bar{X}(t)$  is locally but uniformly Hölder continuous with exponent  $\vartheta$ .

**Lemma 5.2** (see [60]) *Let Assumption 2.1 hold. If  $y(t) = (y_1(t), y_2(t), y_3(t))^T$  is a solution of (3) with initial values  $y(0) \in \mathbb{R}^+$ , then almost every sample path of  $y_i(t)$  ( $1 \leq i \leq 3$ ) is uniformly continuous for  $t \geq 0$ .*

*Proof* By (32), there exists  $T > 0$ , such that  $\mathbb{E}[y_i^q(t)] \leq \frac{3}{2}\Theta_1(q)$  for all  $t \geq T$ . Moreover, it follows from the continuity of  $\mathbb{E}[y_i^q(t)]$  that there is a  $\Theta_2(q) > 0$  such that  $\mathbb{E}[y_i^q(t)] \leq \Theta_2(q)$  for  $t \geq T$ . Let  $\Theta(q) = \max\{\frac{3}{2}\Theta_1(q), \Theta_2(q)\}$ , then, for all  $t \leq 0$ ,

$$\mathbb{E}[y_i^q(t)] \leq \Theta(q).$$

Clearly, the first equation of system (3) is equivalent to the following equation:

$$\begin{aligned} y_1(t) = & \int_0^t y_1(s) \left[ r_1(s) - \alpha_1(s) \prod_{0 < t_k < s} (1 + h_{1k}) y_1(s) - \beta_1(s) \prod_{0 < t_k < s} (1 + h_{2k}) y_2(s) \right. \\ & \left. - \frac{c_1(s) \prod_{0 < t_k < s} (1 + h_{3k}) y_3(s)}{a_1(s) + a_2(s) \prod_{0 < t_k < s} (1 + h_{1k}) y_1(s) + a_3(s) \prod_{0 < t_k < s} (1 + h_{3k}) y_3(s)} \right] ds \\ & + y_1(0) + \int_0^t \sigma_1(s) y_1(s) dB_1(s). \end{aligned}$$

Therefore,

$$\begin{aligned} & \mathbb{E} \left| y_1(t) \left[ r_1(t) - \alpha_1(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) - \beta_1(t) \prod_{0 < t_k < t} (1 + h_{2k}) y_2(t) \right. \right. \\ & \quad \left. \left. - \frac{c_1(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)}{a_1(t) + a_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) + a_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \right] \right|^q \\ & = \mathbb{E} \left[ |y_1(t)|^q \left| r_1(t) - \alpha_1(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) - \beta_1(t) \prod_{0 < t_k < t} (1 + h_{2k}) y_2(t) \right. \right. \\ & \quad \left. \left. - \frac{c_1(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)}{a_1(t) + a_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) + a_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \right|^q \right] \\ & \leq \frac{1}{2} \mathbb{E}[|y_1(t)|^{2q}] + \frac{1}{2} \mathbb{E} \left[ \left| r_1(t) - \alpha_1(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) - \beta_1(t) \prod_{0 < t_k < t} (1 + h_{2k}) y_2(t) \right. \right. \\ & \quad \left. \left. - \frac{c_1(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)}{a_1(t) + a_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) + a_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \right|^{2q} \right] \\ & \leq \frac{1}{2} \Theta_1(2q) + 4^{2q-1} \left[ (r_1^u)^{2q} + \alpha_1^u M_1 \mathbb{E}[|y_1(t)|^{2q}] + \beta_1^u M_2 \mathbb{E}[|y_2(t)|^{2q}] + \left( \frac{c_1^u}{a_3} \right)^{2q} \right] \\ & \leq \frac{1}{2} \Theta(2q) + 4^{2q-1} \left[ (r_1^u)^{2q} + \left( \frac{c_1^u}{a_3} \right)^{2q} + (\alpha_1^u M_1 + \beta_1^u M_2) \Theta(2q) \right] \\ & := G_1(q). \end{aligned}$$

By the famous moment inequality for stochastic integrals (see [58]), we obtain, for  $0 \leq t_1 \leq t_2$  and  $q > 2$ ,

$$\mathbb{E} \left[ \left| \int_{t_1}^{t_2} \sigma_1(s) y_1(s) dB_1(s) \right|^q \right] \leq [(\sigma_1^2)^u]^q \left[ \frac{q(q-1)}{2} \right]^{\frac{q}{2}} (t_2 - t_1)^{\frac{q-2}{2}} \int_{t_1}^{t_2} \mathbb{E}[|y_1(s)|^q] ds$$

$$\leq [(\sigma_1^2)^u]^q \left[ \frac{q(q-1)}{2} \right]^{\frac{q}{2}} (t_2 - t_1)^{\frac{q}{2}} \Theta(q).$$

Then, for  $0 < t_1 < t_2 < \infty$ ,  $t_2 - t_1 \leq 1$ ,  $\frac{1}{q} + \frac{1}{p} = 1$ , we can derive that

$$\begin{aligned} & \mathbb{E} \left[ |y_1(t_2) - y_1(t_1)|^q \right] \\ &= \mathbb{E} \left[ \left| \int_{t_1}^{t_2} y_1(s) \left[ r_1(s) - \alpha_1(s) \prod_{0 < t_k < s} (1 + h_{1k}) y_1(s) - \beta_1(s) \prod_{0 < t_k < s} (1 + h_{2k}) y_2(s) \right. \right. \right. \\ & \quad \left. \left. - \frac{c_1(s) \prod_{0 < t_k < s} (1 + h_{3k}) y_3(s)}{a_1(s) + a_2(s) \prod_{0 < t_k < s} (1 + h_{1k}) y_1(s) + a_3(s) \prod_{0 < t_k < s} (1 + h_{3k}) y_3(s)} \right] ds \right. \\ & \quad \left. + \int_{t_1}^{t_2} \sigma_1(s) y_1(s) dB_1(s) \right|^q \Big] \\ &\leq 2^{q-1} \mathbb{E} \left[ \left| \int_{t_1}^{t_2} y_1(s) \left[ r_1(s) - \alpha_1(s) \prod_{0 < t_k < s} (1 + h_{1k}) y_1(s) \right. \right. \right. \\ & \quad \left. \left. - \beta_1(s) \prod_{0 < t_k < s} (1 + h_{2k}) y_2(s) \right. \right. \\ & \quad \left. \left. - \frac{c_1(s) \prod_{0 < t_k < s} (1 + h_{3k}) y_3(s)}{a_1(s) + a_2(s) \prod_{0 < t_k < s} (1 + h_{1k}) y_1(s) + a_3(s) \prod_{0 < t_k < s} (1 + h_{3k}) y_3(s)} \right] ds \right|^q \Big] \\ & \quad + 2^{q-1} \mathbb{E} \left[ \left| \int_{t_1}^{t_2} \sigma_1(s) y_1(s) dB_1(s) \right|^q \right] \\ &\leq 2^{q-1} (t_2 - t_1)^{\frac{q}{p}} \int_{t_1}^{t_2} \mathbb{E} \left[ |y_1(s) \left[ r_1(s) - \alpha_1(s) \prod_{0 < t_k < s} (1 + h_{1k}) y_1(s) \right. \right. \right. \\ & \quad \left. \left. - \beta_1(s) \prod_{0 < t_k < s} (1 + h_{2k}) y_2(s) \right. \right. \\ & \quad \left. \left. - \frac{c_1(s) \prod_{0 < t_k < s} (1 + h_{3k}) y_3(s)}{a_1(s) + a_2(s) \prod_{0 < t_k < s} (1 + h_{1k}) y_1(s) + a_3(s) \prod_{0 < t_k < s} (1 + h_{3k}) y_3(s)} \right]|^q \right] ds \\ & \quad + 2^{q-1} [(\sigma_1^2)^u]^q \left[ \frac{q(q-1)}{2} \right]^{\frac{q}{2}} (t_2 - t_1)^{\frac{q}{2}} \Theta(q) \\ &\leq 2^{q-1} (t_2 - t_1)^{\frac{q}{p}+1} G_1(q) + 2^{q-1} [(\sigma_1^2)^u]^q \left[ \frac{q(q-1)}{2} \right]^{\frac{q}{2}} (t_2 - t_1)^{\frac{q}{2}} \Theta(q) \\ &\leq 2^{q-1} (t_2 - t_1)^{\frac{q}{2}} \left[ (t_2 - t_1)^{\frac{q}{2}} + \left[ \frac{q(q-1)}{2} \right]^{\frac{q}{2}} \right] G_2(q) \\ &\leq 2^{q-1} (t_2 - t_1)^{\frac{q}{2}} \left[ 1 + \left[ \frac{q(q-1)}{2} \right]^{\frac{q}{2}} \right] G_2(q), \end{aligned}$$

where  $G_2(q) = \max\{G_1(q), [(\sigma_1^2)^u]^q \Theta(q)\}$ . Then it follows from Lemma 5.1 that almost every sample path of  $y_1(t)$  is locally but uniformly Hölder continuous with exponent  $\vartheta$  for every  $\vartheta \in (0, \frac{q-2}{2q})$  and therefore almost every sample path of  $y_1(t)$  is uniformly continuous on  $t \geq 0$ . Similarly, we can show that almost every sample path of  $y_2(t)$  and  $y_3(t)$  are uniformly continuous on  $t \geq 0$ . □

**Lemma 5.3** (see [61]) *Let  $f$  be a non-negative function defined on  $t \geq 0$  such that  $f$  is integrable on  $t \geq 0$  and is uniformly continuous on  $t \geq 0$ . Then  $\lim_{t \rightarrow +\infty} f(t) = 0$ .*

**Theorem 5.1** *If Assumption 2.1 holds and*

$$\begin{cases} A = \alpha_1^l + \beta_2^l + \frac{c_1^l a_2^l}{a_3^l} + \frac{c_2^l b_2^l}{b_3^l} - e_1^u > 0, \\ B = \alpha_2^l + \beta_1^l - e_2^u > 0, \\ C = \alpha_3^l + c_1^l + c_2^l - \frac{e_1^u a_3^u}{a_2^l} - \frac{e_2^u b_3^u}{b_2^l} > 0, \end{cases} \tag{44}$$

then system (2) is globally attractive.

*Proof* Let  $x(t) = (x_1(t), x_2(t), x_3(t))^T$ ,  $z(t) = (z_1(t), z_2(t), z_3(t))^T$  be two arbitrary solutions of system (2) with initial values  $x(0), z(0) \in \mathbb{R}^+$ , respectively. Let  $y(t) = (y_1(t), y_2(t), y_3(t))^T$ ,  $\bar{y}(t) = (\bar{y}_1(t), \bar{y}_2(t), \bar{y}_3(t))^T$  be two arbitrary solution of system (3) with initial values  $y(0), \bar{y}(0) \in \mathbb{R}^+$ , respectively.

Then

$$x_i(t) = \prod_{0 < t_k < t} (1 + h_{ik}) y_i(t), \quad z_i(t) = \prod_{0 < t_k < t} (1 + h_{ik}) \bar{y}_i(t).$$

Define

$$\bar{V}(t) = \sum_{i=1}^3 |\ln y_i(t) - \ln \bar{y}_i(t)|.$$

By Itô's formula

$$\begin{aligned} d^+ \bar{V}(t) &= \sum_{i=1}^3 \operatorname{sgn}(y_i(t) - \bar{y}_i(t)) d(\ln y_i(t) - \ln \bar{y}_i(t)) \\ &= \operatorname{sgn}(y_1(t) - \bar{y}_1(t)) \left[ -\alpha_1(t) \prod_{0 < t_k < t} (1 + h_{1k}) (y_1(t) - \bar{y}_1(t)) \right. \\ &\quad - \left( \frac{c_1(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)}{a_1(t) + a_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) + a_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \right. \\ &\quad \left. \left. - \frac{c_1(t) \prod_{0 < t_k < t} (1 + h_{3k}) \bar{y}_3(t)}{a_1(t) + a_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) \bar{y}_1(t) + a_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) \bar{y}_3(t)} \right) \right. \\ &\quad \left. - \beta_1(t) \prod_{0 < t_k < t} (1 + h_{2k}) (y_2(t) - \bar{y}_2(t)) \right] dt \\ &\quad + \operatorname{sgn}(y_2(t) - \bar{y}_2(t)) \left[ -\alpha_2(t) \prod_{0 < t_k < t} (1 + h_{2k}) (y_2(t) - \bar{y}_2(t)) \right. \\ &\quad \left. - \left( \frac{c_2(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)}{b_1(t) + b_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) + b_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \right) \right. \end{aligned}$$



$$\begin{aligned}
 & - \frac{c_2(t) \prod_{0 < t_k < t} (1 + h_{3k}) \bar{y}_3(t)}{b_1(t) + b_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) \bar{y}_1(t) + b_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) \bar{y}_3(t)} \\
 & - \beta_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) (y_1(t) - \bar{y}_1(t)) \Big] dt \\
 & + \operatorname{sgn}(y_3(t) - \bar{y}_3(t)) \left[ -\alpha_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) (y_3(t) - \bar{y}_3(t)) \right. \\
 & + \left( \frac{e_1(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t)}{a_1(t) + a_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_1(t) + a_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \right. \\
 & - \frac{e_1(t) \prod_{0 < t_k < t} (1 + h_{1k}) \bar{y}_1(t)}{a_1(t) + a_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) \bar{y}_1(t) + a_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) \bar{y}_3(t)} \Big) \\
 & + \left( \frac{e_2(t) \prod_{0 < t_k < t} (1 + h_{2k}) y_2(t)}{b_1(t) + b_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) y_2(t) + b_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) y_3(t)} \right. \\
 & \left. - \frac{e_2(t) \prod_{0 < t_k < t} (1 + h_{2k}) \bar{y}_2(t)}{b_1(t) + b_2(t) \prod_{0 < t_k < t} (1 + h_{1k}) \bar{y}_2(t) + b_3(t) \prod_{0 < t_k < t} (1 + h_{3k}) \bar{y}_3(t)} \right) \Big] dt \\
 \leq & -\alpha_1(t) |x_1(t) - z_1(t)| dt - \alpha_2(t) |x_2(t) - z_2(t)| dt - \alpha_3(t) |x_3(t) - z_3(t)| dt \\
 & - c_1(t) \left| \frac{a_2(t)x_3(t)(z_1(t) - x_1(t)) + (a_1(t) + a_2(t)x_1(t))(x_3(t) - z_3(t))}{(a_1(t) + a_2(t)x_1(t) + a_3(t)x_3(t))(a_1(t) + a_2(t)z_1(t) + a_3(t)z_3(t))} \right| dt \\
 & - \beta_1(t) |x_2(t) - z_2(t)| dt - \beta_2(t) |x_1(t) - z_1(t)| dt \\
 & - c_2(t) \left| \frac{b_2(t)x_3(t)(z_1(t) - x_1(t)) + (b_1(t) + b_2(t)x_1(t))(x_3(t) - z_3(t))}{(b_1(t) + b_2(t)x_1(t) + b_3(t)x_3(t))(b_1(t) + b_2(t)z_1(t) + b_3(t)z_3(t))} \right| dt \\
 & + e_1(t) \left| \frac{a_3(t)x_1(t)(z_3(t) - x_3(t)) + (a_1(t) + a_3(t)x_3(t))(x_1(t) - z_1(t))}{(a_1(t) + a_2(t)x_1(t) + a_3(t)x_3(t))(a_1(t) + a_2(t)z_1(t) + a_3(t)z_3(t))} \right| dt \\
 & + e_2(t) \left| \frac{b_3(t)x_2(t)(z_3(t) - x_3(t)) + (b_1(t) + b_3(t)x_3(t))(x_2(t) - z_2(t))}{(b_1(t) + b_2(t)x_2(t) + b_3(t)x_3(t))(b_1(t) + b_2(t)z_2(t) + b_3(t)z_3(t))} \right| dt \\
 \leq & -\alpha_1^l |x_1(t) - z_1(t)| dt - \alpha_2^l |x_2(t) - z_2(t)| dt - \alpha_3^l |x_3(t) - z_3(t)| dt \\
 & - \frac{c_1^l a_2^l}{a_3^u} |x_1(t) - z_1(t)| dt - c_1^l |x_3(t) - z_3(t)| dt - \beta_1^l |x_2(t) - z_2(t)| dt \\
 & - \beta_2^l |x_1(t) - z_1(t)| dt - \frac{c_2^l b_2^l}{b_3^u} |x_1(t) - z_1(t)| dt - c_2^l |x_3(t) - z_3(t)| dt \\
 & + \frac{e_1^u a_3^u}{a_2^l} |x_3(t) - z_3(t)| dt + e_1^u |x_1(t) - z_1(t)| dt + \frac{e_2^u b_3^u}{b_2^l} |x_3(t) - z_3(t)| dt \\
 & + e_2^u |x_2(t) - z_2(t)| dt \\
 = & -[A|x_1(t) - z_1(t)| + B|x_2(t) - z_2(t)| + C|x_3(t) - z_3(t)|] dt \\
 = & - \left[ A \prod_{0 < t_k < t} (1 + h_{1k}) |y_1(t) - \bar{y}_1(t)| + B \prod_{0 < t_k < t} (1 + h_{2k}) |y_2(t) - \bar{y}_2(t)| \right. \\
 & \left. + C \prod_{0 < t_k < t} (1 + h_{3k}) |y_3(t) - \bar{y}_3(t)| \right] dt \\
 = & -[Am_1 |y_1(t) - \bar{y}_1(t)| + Bm_2 |y_2(t) - \bar{y}_2(t)| + Cm_3 |y_3(t) - \bar{y}_3(t)|] dt.
 \end{aligned}$$

Integrating both sides gives

$$\bar{V}(t) \leq \bar{V}(0) - \int_0^t [Am_1|y_1(t) - \bar{y}_1(t)| + Bm_2|y_2(t) - \bar{y}_2(t)| + Cm_3|y_3(t) - \bar{y}_3(t)|] ds.$$

Therefore

$$\begin{aligned} \bar{V}(t) + \int_0^t [Am_1|y_1(t) - \bar{y}_1(t)| + Bm_2|y_2(t) - \bar{y}_2(t)| + Cm_3|y_3(t) - \bar{y}_3(t)|] ds \\ \leq \bar{V}(0) < \infty. \end{aligned}$$

Making use of  $\bar{V}(t) \geq 0$  and (44) results in

$$|y_i(t) - \bar{y}_i(t)| \in L^1[0, \infty).$$

Consequently, by Lemmas 5.2 and 5.3, one can observe that

$$\lim_{t \rightarrow +\infty} |y_i(t) - \bar{y}_i(t)| = 0 \quad \text{a.s.}$$

Then

$$\lim_{t \rightarrow +\infty} |x_i(t) - z_i(t)| = \lim_{t \rightarrow +\infty} \prod_{0 < t_k < t} (1 + h_{ik}) |y_i(t) - \bar{y}_i(t)| \leq M \lim_{t \rightarrow +\infty} |y_i(t) - \bar{y}_i(t)| = 0, \quad \text{a.s.}$$

This completes the proof. □

### 6 Conclusion and numerical simulations

In this paper, a stochastic non-autonomous one-predator–two-prey system with Beddington–DeAngelis functional response and impulsive perturbations is proposed and investigated. First, we obtain some sufficient conditions for extinction, non-persistence in the mean, weak persistence, persistence in the mean and stochastic permanence of the solution, and we verify some asymptotic behaviors of the solutions of system (2), such as the limit of the average in time, the lower-growth rate, the upper-growth rate and global attractivity. Now we summarize the key results as follows:

(I):

- (1) If  $\delta_i^* = \limsup_{t \rightarrow +\infty} \frac{1}{t} [\sum_{0 < t_k < t} \ln(1 + h_{ik}) + \int_0^t \delta_i(s) ds] < 0$ , then the  $i$ th species ( $i = 1, 2, 3$ ) in system (2) is extinct.
- (2) If  $\delta_i^* = 0$ , then the  $i$ th species ( $i = 1, 2, 3$ ) in system (2) is non-persistent in the mean.
- (3) If  $\delta_i^* > 0$ , then the  $i$ th species ( $i = 1, 2, 3$ ) in system (2) is weakly persistent.
- (4) If  $\theta_{i*} > 0$ , then the  $i$ th species ( $i = 1, 2, 3$ ) in system (2) is persistent in the mean.
- (5) If  $(\check{\sigma})^2 < 2\hat{r}$  and Assumption 2.1 holds, then system (2) is stochastically permanent.

(II): The solution  $x_i(t)$  ( $i = 1, 2, 3$ ) obeys

$$\underline{x}_i^* = \frac{\theta_{i*}}{\alpha_i^u} \leq \liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x_i(s) ds \leq \limsup_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x_i(s) ds \leq \frac{\delta_i^*}{\alpha_i^l} = \bar{x}_i^* \quad \text{a.s.}$$

(III): Under Assumption 2.1, the solution of system (2) satisfies

$$\limsup_{t \rightarrow +\infty} \frac{\ln x_i(t)}{\ln t} \leq 1 \quad \text{a.s.}$$

In addition, if  $2\hat{r} - (\check{\sigma})^2 > 0$ , then

$$\liminf_{t \rightarrow +\infty} \frac{\ln |x_i(t)|}{\ln t} \geq -\frac{1}{2\hat{r} - (\check{\sigma})^2} \quad \text{a.s.}$$

(IV): If  $A, B, C > 0$  and Assumption 2.1 holds, then system (2) is globally attractive. By our results, we can analyze that the smaller stochastic perturbations cannot affect the stochastic permanence and extinction of the population. However, if the stochastic perturbations are larger, the stochastic permanence of the populations will be extinct. Similarly, the small impulsive perturbations have a little influence on the stochastic permanence and extinction of the populations. However, if the impulsive perturbations are large, the stochastic permanence and extinction of the populations could be changed.

We will give some numerical experiments to verify our analytical results by using the Milstein method (see [62]) by supplementing impulsive perturbations into it. We choose the same initial value  $(x_1(0), x_2(0), x_3(0)) = (0.5, 0.5, 0.5)$  and the same parameters in the following numerical examples.

The parameters are as follows:

$$\begin{aligned} r_1(t) &= 1.2 + 0.02 \sin t, & r_2(t) &= 1.12 + 0.02 \sin t, & r_3(t) &= 0.38 + 0.02 \sin t, \\ \alpha_1(t) &= 0.24 + 0.01 \sin t, & \alpha_2(t) &= 0.3 + 0.01 \sin t, & \alpha_3(t) &= 0.45 + 0.01 \sin t, \\ a_1(t) &= 0.9 + 0.01 \sin t, & a_2(t) &= 1.12 + 0.01 \sin t, & a_3(t) &= 0.86 + 0.01 \sin t, \\ b_1(t) &= 1.2 + 0.01 \sin t, & b_2(t) &= 0.76 + 0.01 \sin t, & b_3(t) &= 0.84 + 0.01 \sin t, \\ c_1(t) &= 0.42 + 0.01 \sin t, & e_1(t) &= 0.3 + 0.01 \sin t, & \beta_1(t) &= 0.14 + 0.01 \sin t, \\ c_2(t) &= 0.35 + 0.01 \sin t, & e_2(t) &= 0.28 + 0.01 \sin t, & \beta_2(t) &= 0.1 + 0.01 \sin t. \end{aligned}$$

At first, we will discuss the effects of different stochastic perturbations to system (2) under the same impulse interference in following Examples 1–6.

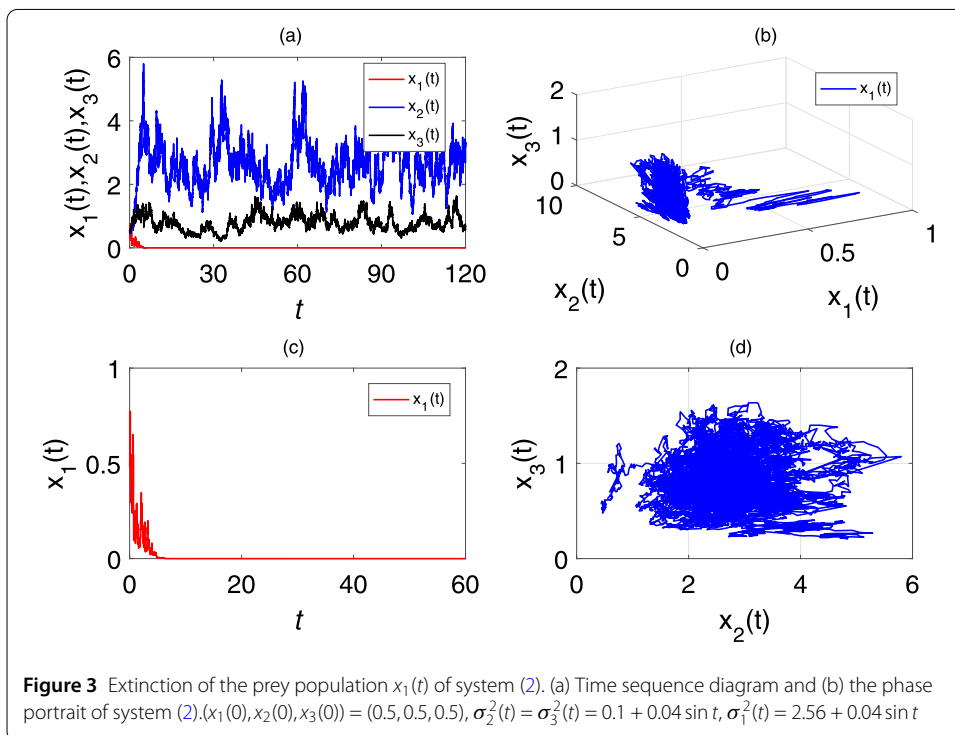
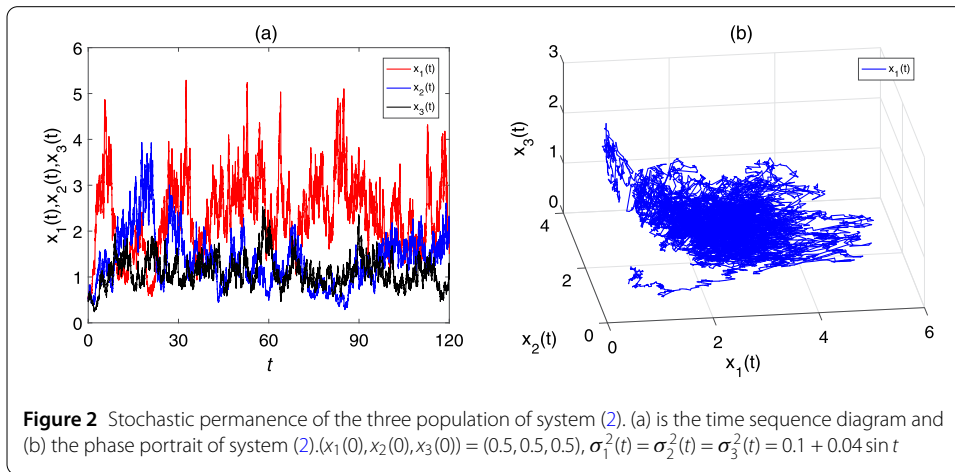
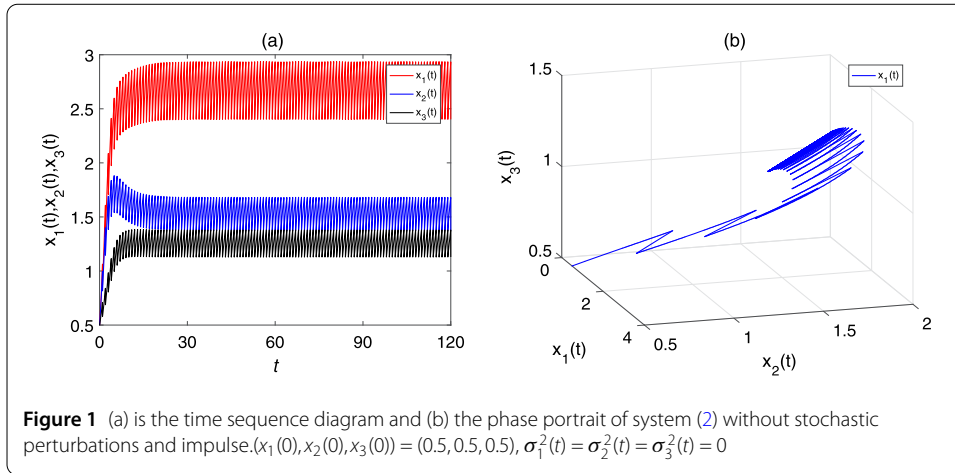
Let  $h_{1k} = h_{2k} = h_{3k} = e^{-0.2} - 1$ , it is easy to verify that

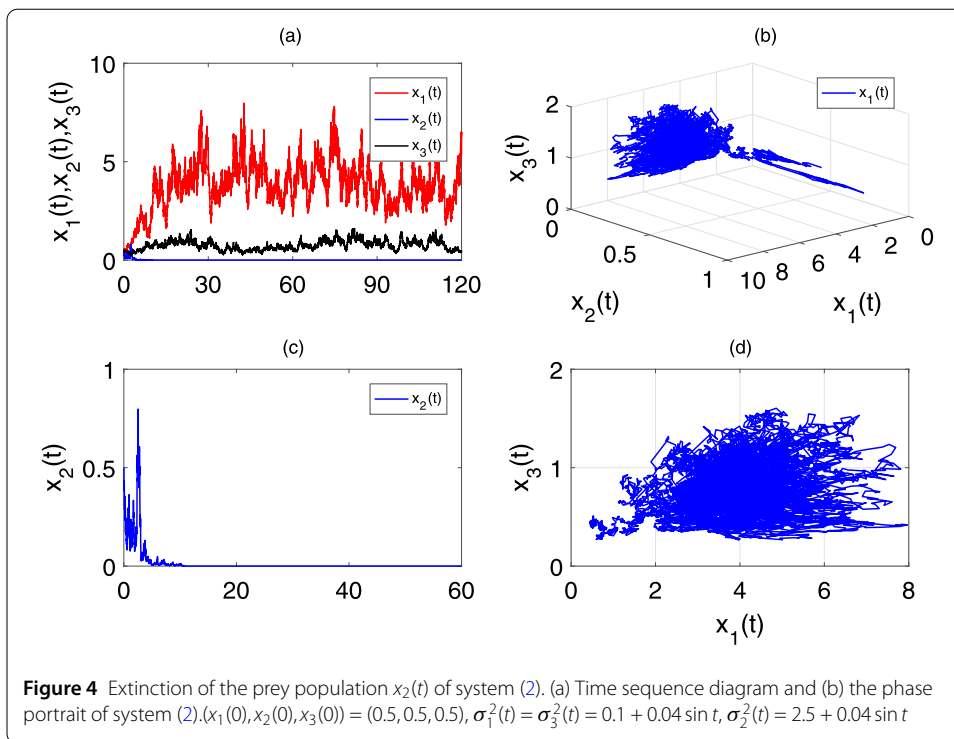
$$e^{-0.4} \leq \prod_{0 < t_k < t} (1 + h_{ik}) \leq e^{-0.1},$$

which means the Assumption 2.1 holds. In system (2) without stochastic perturbations, we can see that the prey and predator populations are all permanent (see Fig. 1).

*Example 1* Let  $\sigma_1^2(t) = \sigma_2^2(t) = \sigma_3^2(t) = 0.1 + 0.04 \sin t$ . Then we get  $(\check{\sigma})^2 = 0.14 < 2\hat{r} = 0.72$ , and the Assumption 2.1 holds. According to Theorem 3.5, we can see that the prey population  $x_1(t)$ ,  $x_2(t)$  and the predator population  $x_3(t)$  are all stochastically permanent (see Fig. 2).

*Example 2* Let  $\sigma_2^2(t) = \sigma_3^2(t) = 0.1 + 0.04 \sin t$ ,  $\sigma_1^2(t) = 2.56 + 0.04 \sin t$ . Then we get  $\delta_1^* = -0.08 < 0$ . By Theorem 3.1, we can see that the prey population  $x_1(t)$  will be extinct (see Fig. 3(a),(c)) and the population  $x_2(t)$ ,  $x_3(t)$  are all stochastically permanent (see Fig. 3(a),(d)).





*Example 3* Let  $\sigma_1^2(t) = \sigma_3^2(t) = 0.1 + 0.04 \sin t$ ,  $\sigma_2^2(t) = 2.5 + 0.04 \sin t$ . Then we get  $\delta_2^* = -0.13 < 0$ . By Theorem 3.1, we can see that the prey population  $x_2(t)$  will be extinct (see Fig. 4(a),(c)) and the population  $x_1(t)$ ,  $x_3(t)$  are all stochastically permanent (see Fig. 4(a),(d)).

*Example 4* Let  $\sigma_1^2(t) = \sigma_2^2(t) = 0.1 + 0.04 \sin t$ ,  $\sigma_3^2(t) = 2.2 + 0.04 \sin t$ . Then we get  $\delta_3^* = -0.0376 < 0$ . By Theorem 3.1, we can see that the predator population  $x_3(t)$  will be extinct (see Fig. 5(a),(c)) and the prey population  $x_1(t)$ ,  $x_2(t)$  are all stochastically permanent (see Fig. 5(a),(d)).

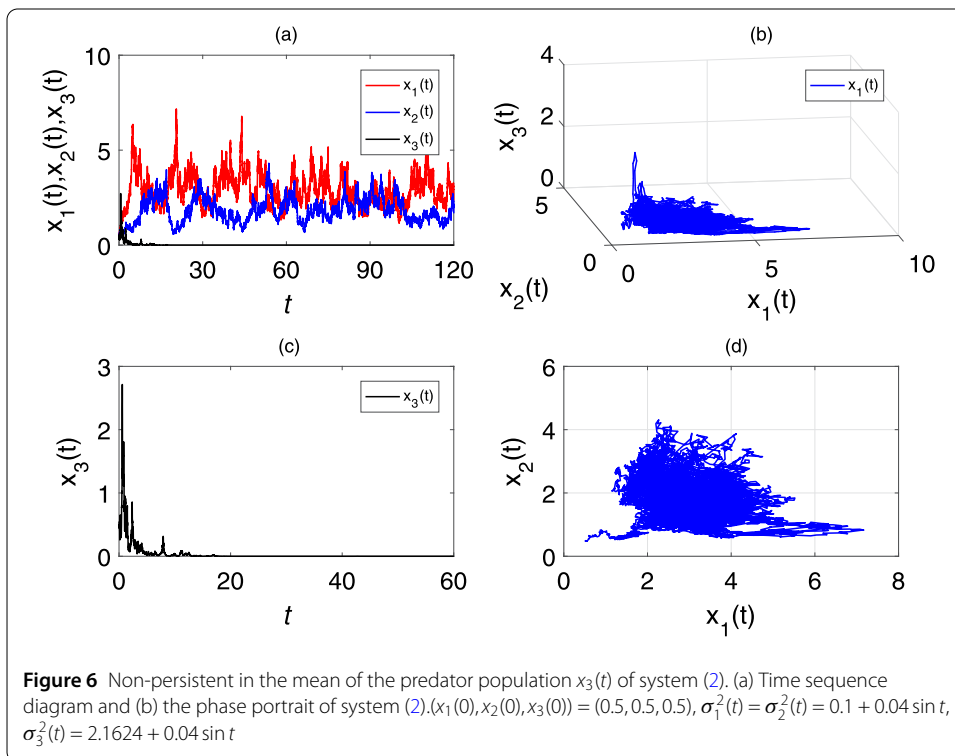
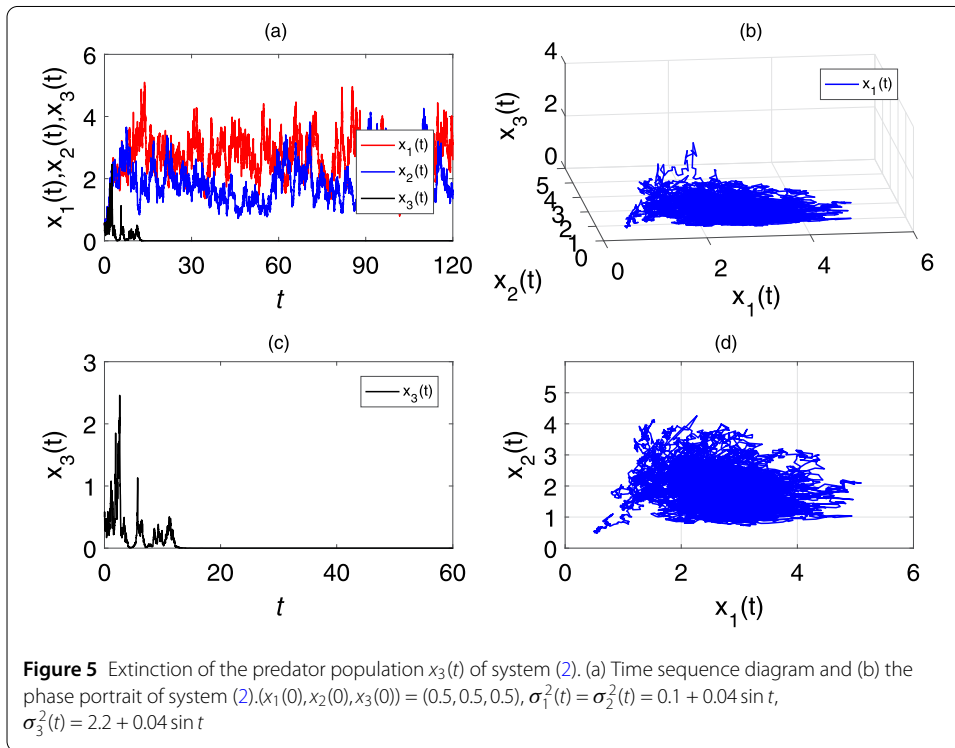
*Example 5* Let  $\sigma_1^2(t) = \sigma_2^2(t) = 0.1 + 0.04 \sin t$ ,  $\sigma_3^2(t) = 2.1624 + 0.04 \sin t$ . Then we get  $\delta_3^* = 0$ . According to Theorem 3.2, we can see that the population  $x_3(t)$  is non-persistent in the mean (see Fig. 6(a),(c)).

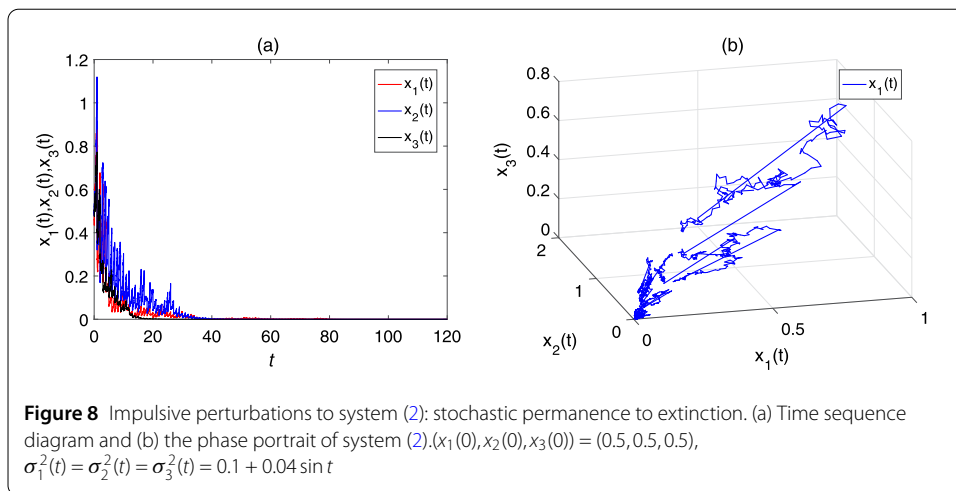
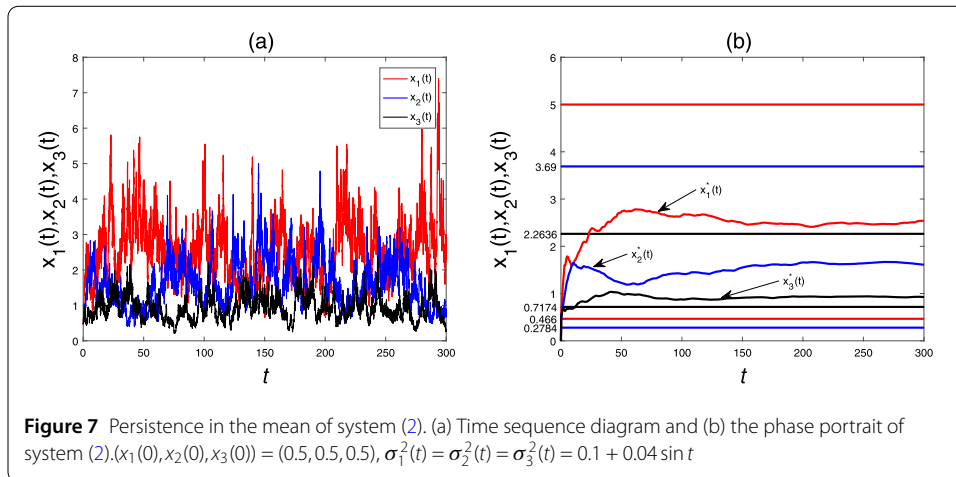
*Example 6* Let  $\sigma_1^2(t) = \sigma_2^2(t) = \sigma_3^2(t) = 0.1 + 0.04 \sin t$ . By Theorem 3.1, 3.2 and 3.4, we can calculate that  $\delta_1^* = 1.15$ ,  $\delta_2^* = 1.07$ ,  $\delta_3^* = 0.996$ ,  $\delta_{1*} = 1.15$ ,  $\delta_{2*} = 1.07$ ,  $\delta_{3*} = 0.33$ ,  $\underline{x}_1^* = 0.466$ ,  $\underline{x}_2^* = 0.2784$ ,  $\underline{x}_3^* = 0.7174$ ,  $\bar{x}_1^* = 5$ ,  $\bar{x}_2^* = 3.6897$ ,  $\bar{x}_3^* = 2.2636$ .

Denote  $x_i^*(t) = \frac{1}{t} \int_0^t x_i(s) ds$  ( $i = 1, 2, 3$ ). Since

$$\liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x_i(s) ds \leq x_i^*(t) \leq \limsup_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x_i(s) ds \quad \text{a.s. } i = 1, 2, 3,$$

then we have  $0.466 \leq x_1^*(t) \leq 5$ ,  $0.2784 \leq x_2^*(t) \leq 3.6897$ ,  $0.7174 \leq x_3^*(t) \leq 2.2636$ . In Fig. 7(a), we can see that the persistence in the mean of system (2). In Fig. 7(b), it is clear to see that the curve of  $x_i^*(t)$  gradually transcend the line  $\underline{x}_i^*(t)$  and stays between the  $\underline{x}_i^*(t)$  and  $\bar{x}_i^*(t)$  lines of the same color, which verify the conclusion.





Finally, we give Example 7 to discuss the effect of the impulsive perturbations on system (2), according to the choice of parameters in Example 1.

*Example 7* Let  $h_{1k} = h_{2k} = e^{-1.2} - 1$ ,  $h_{3k} = e^{-0.6} - 1$ . In Fig. 8, one can easily see that all of the species in system (2) become extinct gradually. This means suitable impulsive control strategy might be useful for the permanence of the system while arbitrary impulsive perturbations might lead to the extinction of system (2).

Therefore, through the numerical simulations given in Examples 1–6, we can see that the large stochastic disturbance is disadvantageous for the persistence of the population. However, the small stochastic perturbation is little effects on the permanence and extinction of the population. By Fig. 8, we can see that the small impulsive perturbations cannot affect the stochastic permanence and extinction of the prey and predator populations. But the large impulsive perturbations can lead to population extinction.

On the other hand, if  $a_3(t) = b_3(t) = 0$ , the Beddington–DeAngelis functional response converts to the Holling II functional response in system (2), then the system (2) be-

comes

$$\left\{ \begin{aligned} dx_1(t) &= x_1(t) \left[ r_1(t) - \alpha_1(t)x_1(t) - \frac{c_1(t)x_3(t)}{a_1(t)+a_2(t)x_1(t)} \right. \\ &\quad \left. - \beta_1(t)x_2(t) \right] dt + \sigma_1(t)x_1(t) dB_1(t), \\ dx_2(t) &= x_2(t) \left[ r_2(t) - \alpha_2(t)x_2(t) - \frac{c_2(t)x_3(t)}{b_1(t)+b_2(t)x_2(t)} \right. \\ &\quad \left. - \beta_2(t)x_1(t) \right] dt + \sigma_2(t)x_2(t) dB_2(t), \\ dx_3(t) &= x_3(t) \left[ r_3(t) - \alpha_3(t)x_3(t) + \frac{e_1(t)x_1(t)}{a_1(t)+a_2(t)x_1(t)} \right. \\ &\quad \left. + \frac{e_2(t)x_2(t)}{b_1(t)+b_2(t)x_2(t)} \right] dt + \sigma_3(t)x_3(t) dB_3(t), \\ x_1(t^+) &= (1 + h_{1k})x_1(t), \\ x_2(t^+) &= (1 + h_{2k})x_2(t), \\ x_3(t^+) &= (1 + h_{3k})x_3(t), \end{aligned} \right. \quad t \neq t_k, k \in N, \tag{45}$$

Therefore, we can obtain the following results.

(I):

- (1) If  $\delta_i^* = \limsup_{t \rightarrow +\infty} \frac{1}{t} [\sum_{0 < t_k < t} \ln(1 + h_{ik}) + \int_0^t \delta_i(s) ds] < 0$ , then the  $i$ th species ( $i = 1, 2, 3$ ) in system (45) is extinct.
- (2) If  $\delta_i^* = 0$ , then the  $i$ th species ( $i = 1, 2, 3$ ) in system (45) is non-persistent in the mean.
- (3) If  $\delta_i^* > 0$ , then the  $i$ th species ( $i = 1, 2, 3$ ) in system (45) is weakly persistent.
- (4) If  $\theta_{i*} > 0$ , then the  $i$ th species ( $i = 1, 2, 3$ ) in system (45) is persistent in the mean, where

$$\theta_{1*} = \delta_{1*} - \left( \frac{\beta_1^u \delta_2^*}{\alpha_2^l} + \frac{c_1^u \delta_3^*}{a_1^l \alpha_3^l} \right), \quad \theta_{2*} = \delta_{2*} - \left( \frac{\beta_2^u \delta_1^*}{\alpha_1^l} + \frac{c_2^u \delta_3^*}{b_1^l \alpha_3^l} \right), \quad \theta_{3*} = \delta_{3*},$$

$$\delta_{i*} = \liminf_{t \rightarrow +\infty} \frac{1}{t} \left[ \sum_{0 < t_k < t} \ln(1 + h_{ik}) + \int_0^t \left( r_i(s) - \frac{1}{2} \sigma_i^2(s) \right) ds \right], \quad i = 1, 2, 3.$$

- (5) If  $(\check{\sigma})^2 < 2\hat{r}$  and Assumption 2.1 holds, then system (45) is stochastically permanent.

(II) The solution  $x_i(t)$  ( $i = 1, 2, 3$ ) obeys

$$\underline{x}_i^* = \frac{\theta_{i*}}{\alpha_i^u} \leq \liminf_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x_i(s) ds \leq \limsup_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x_i(s) ds \leq \frac{\delta_i^*}{\alpha_i^l} = \bar{x}_i^* \quad \text{a.s.}$$

(III) Under Assumption 2.1, the solution of system (45) satisfies

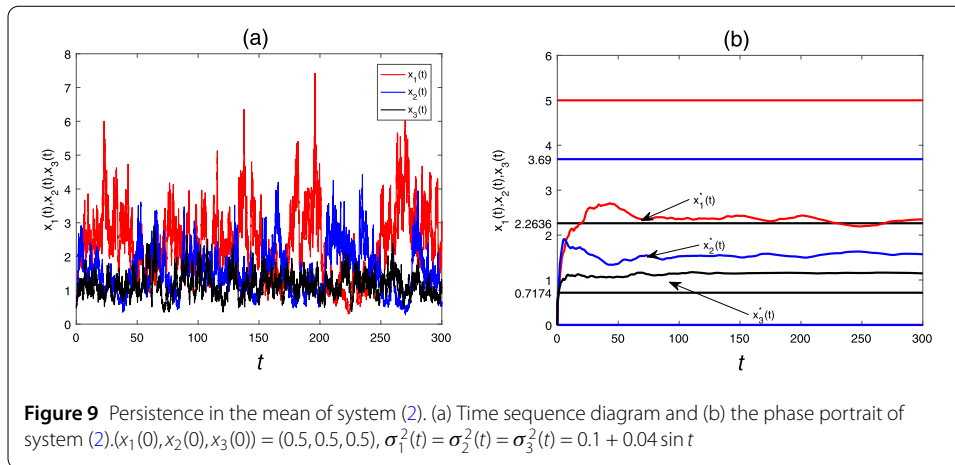
$$\limsup_{t \rightarrow +\infty} \frac{\ln x_i(t)}{\ln t} \leq 1 \quad \text{a.s.}$$

In addition, if  $2\hat{r} - (\check{\sigma})^2 > 0$ , then

$$\liminf_{t \rightarrow +\infty} \frac{\ln |x_i(t)|}{\ln t} \geq -\frac{1}{2\hat{r} - (\check{\sigma})^2} \quad \text{a.s.}$$

By comparison, we see that the results of the B-D functional response are more accurate than those of the Holling II functional response. Now we show some simulations to verify our main results.





**Example 8** The parameter values are the same as those given in Example 6. By the results of the Holling II functional response in system (45), we can calculate that  $\delta_1^* = 1.15$ ,  $\delta_2^* = 1.07$ ,  $\delta_3^* = 0.996$ ,  $\delta_{1*} = 1.15$ ,  $\delta_{2*} = 1.07$ ,  $\delta_{3*} = 0.33$ ,  $\underline{x}_1^* = -0.4713 < 0$ ,  $\underline{x}_2^* = -0.164 < 0$ ,  $\underline{x}_3^* = 0.7174$ ,  $\bar{x}_1^* = 5$ ,  $\bar{x}_2^* = 3.6897$ ,  $\bar{x}_3^* = 2.2636$ . We can see that the values of  $\underline{x}_1^*$  and  $\underline{x}_2^*$  of system (45) are smaller than those of system (2). Therefore, our results can be verified in Fig. 9.

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**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors read and approved the final manuscript.

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