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A class of nonlocal problems of fractional differential equations with composition of derivative and parameters

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Abstract

In this paper, we study existence and nonexistence of positive solutions for a class of Riemann–Stieltjes integral boundary value problems of fractional differential equations with parameters. By using the fixed point index theory, some new sufficient conditions for the existence of at least one, two and the nonexistence of positive solutions are obtained. The results we obtain show the influence of parameter λ and parameter a on the existence of positive solutions. Finally, some examples are given to illustrate our main results.

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1 Introduction

In this paper, we investigate existence and nonexistence of positive solutions for a class of Riemann–Stieltjes integral boundary value problems of fractional differential equations with parameters

$$
\begin{cases}
D_{0^+}^{\alpha}(p(t)D_{0^+}^{\beta}u(t)) + \lambda f(t, u(t)) = 0, & t \in (0, 1), \\
\lim_{t \to 0^+} t^{2-\beta} u(t) = a, \qquad u(1) = \int_0^1 u(s)\,dA(s), \\
\lim_{t \to 0^+} t^{1-\alpha} p(t) D_{0^+}^{\beta} u(t) = 0,
\end{cases}
\tag{1.1}
$$

where $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ are the Riemann–Liouville fractional derivatives with $0 < \alpha \le 1$, $1 < \beta \le 2$. The parameters $\lambda > 0$, $a \ge 0$, $p \in C([0,1], (0, +\infty))$, $f : [0,1] \times [0, +\infty) \to [0, +\infty)$ are given functions, and f may be discontinuous but satisfies the L^q-Carathéodory conditions. $\int_0^1 u(s)\,dA(s)$ denotes the Riemann–Stieltjes integral with respect to A.

By using the fixed point index theory, some new sufficient conditions for the existence of at least one, two and the nonexistence of positive solutions are obtained. The theorems we obtain show the influence of parameter λ and parameter a on the existence of positive solutions.

In recent decades, with the wide applications of fractional differential equations in physics, engineering, biology, chemistry, and many other fields, researchers have been

paying more and more attention to them, see [1–12] and the references therein. At the same time, many problems of fluid mechanics, bioengineering, chemical engineering, and so on could be attributed to the integral boundary value problems, which are nonlocal problems. Therefore, a lot of meaningful research results have been obtained, see [13–20] and the references therein. The eigenvalue problem is a relatively active part of the differential equation theory, and there have been many results, see [21–30] and the references therein. Nowadays, when solving many practical problems, there will inevitably be errors and those errors will often affect the existence of the solution to a large extent. Therefore, it is meaningful to study the boundary value problem of fractional differential equations with disturbance parameters, see [31–35] and the references therein.

As a generalization of classical Riemann integral, Riemann–Stieltjes integral boundary value problem has a stronger applicability, which not only contains the classical Riemann integral boundary value problem, but also includes two-point boundary value and multi-point boundary value. In this paper, we investigate existence and nonexistence of positive solutions for a class of Riemann–Stieltjes integral boundary value problems of fractional differential equations with parameters (1.1).

The paper is organized as follows. In Sect. 2, we present some necessary definitions and lemmas which will be used to prove our main results. We study the properties of integral kernels and obtain inequalities about the integral kernels. We prove the complete continuity of operators. In Sect. 3, we investigate the existence of at least one positive solution for boundary value problem (1.1). In Sect. 4, sufficient conditions for the existence of at least two positive solution of boundary value problem (1.1) and the nonexistence of positive solution of boundary value problem (1.1) are established. In Sect. 5, we give some examples to illustrate our main result.

Throughout this paper, we assume that $A(t)$ is a monotone increasing function, $\int_0^1 s^{\beta-2} \, dA(s)$ exists, and

$$1 - \int_0^1 s^{\beta-1} \, dA(s) > 0.$$

f satisfies the L^q-Carathéodory conditions, that is,

    (1) $f(\cdot, u)$ is measurable for all $u \in [0, +\infty)$;

    (2) $f(t, \cdot)$ is continuous for a.e. $t \in [0, 1]$;

    (3) for every $r > 0$, there exists $\varphi_r \in L^q[0, 1]$ such that

$$\left| f\left(t, t^{\beta-2}u\right) \right| \leq \varphi_r(t) \quad \text{for all } u \in [0, r] \text{ and a.e. } t \in [0, 1],$$

where $q > \frac{1}{\alpha}$ if $0 < \alpha < 1$ and $q = 1$ if $\alpha = 1$.

For $L^q[0, 1]$, we denote the norm $\|\varphi\|_{L^q} = (\int_0^1 |\varphi(t)|^q \, dt)^{\frac{1}{q}}$.

2 Preliminaries

The definitions of fractional integral and fractional derivative and the related lemmas can be found in [3, 4].

Lemma 2.1 (See [3], Theorem 2.4 and [4], Lemma 2.5) *Let $p > 0$ and $n = \lceil p \rceil = \min\{z \in \mathbb{Z} : z \geq p\}$. If $u \in L^1[0,1]$ and $I_{0^+}^{n-p} u \in AC^n[0,1]$, then the equality*

$$I_{0^+}^p \left(D_{0^+}^p u \right)(t) = u(t) - \sum_{k=1}^n c_k t^{p-k}$$

holds a.e. on $[0,1]$.

Lemma 2.2 *If $0 < \alpha < 1$, $D_{0^+}^\alpha u \in L^1[0,1]$ and $\lim_{t \to 0^+} t^{1-\alpha} u(t) = c$, where c is a constant, then $I_{0^+}^{1-\alpha} u \in AC^1[0,1]$.*

Proof Since $\lim_{t \to 0^+} t^{1-\alpha} u(t) = c$, then for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that $|t^{1-\alpha} u(t) - c| < \frac{\varepsilon}{\Gamma(\alpha)}$ whenever $0 < t < \delta$, and

$$
\begin{aligned}
\left| I_{0^+}^{1-\alpha} u(t) - c\Gamma(\alpha) \right| &= \left| I_{0^+}^{1-\alpha} u(t) - c I_{0^+}^{1-\alpha} t^{\alpha-1} \right| \\
&\leq \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \left| u(s) - cs^{\alpha-1} \right| ds \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} s^{\alpha-1} \left| s^{1-\alpha} u(s) - c \right| ds \\
&< \varepsilon.
\end{aligned}
$$

Hence, we have $\lim_{t \to 0^+} I_{0^+}^{1-\alpha} u(t) = c\Gamma(\alpha)$.

Let $\phi(t) = D_{0^+}^\alpha u(t) = \frac{d}{dt} I_{0^+}^{1-\alpha} u(t)$, then $\phi \in L^1[0,1]$ and

$$I_{0^+}^{1-\alpha} u(t) = c\Gamma(\alpha) + \int_0^t \phi(s)\, ds.$$

Therefore, $I_{0^+}^{1-\alpha} u \in AC^1[0,1]$. □

Let

$$E := C_{2-\beta}[0,1] = \left\{ u \in C(0,1] : t^{2-\beta} u(t) \in C[0,1] \right\},$$

then E is a Banach space with the norm $\|u\| = \sup_{t \in [0,1]} t^{2-\beta} |u(t)|$.

Definition 2.1 A function $u = u(t)$ is called a solution of fractional boundary value problem (1.1) if $u \in E$ and satisfies (1.1). Furthermore, $u = u(t)$ is called a positive solution of fractional boundary value problem (1.1) if $u(t) > 0$, $t \in (0,1)$.

Lemma 2.3 *For any $y \in L^q[0,1]$, the fractional differential initial value problem*

$$
\begin{cases}
D_{0^+}^\alpha v(t) + y(t) = 0, & t \in (0,1), \\
\lim_{t \to 0^+} t^{1-\alpha} v(t) = 0
\end{cases}
\tag{2.1}
$$

has a unique solution

$$v(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds. \tag{2.2}$$

Proof Suppose that $v = v(t)$ is a solution of initial value problem (2.1). Since $y \in L^q[0,1]$, then $D_{0^+}^\alpha v \in L^1[0,1]$. Because $\lim_{t \to 0^+} t^{1-\alpha} v(t) = 0$, it follows $I_{0^+}^{1-\alpha} v \in AC^1[0,1]$ from Lemma 2.2. Thus, by Lemma 2.1, we have

$$v(t) = -I_{0^+}^\alpha y(t) + c_1 t^{\alpha-1}. \tag{2.3}$$

The initial condition $\lim_{t \to 0^+} t^{1-\alpha} v(t) = 0$ implies that $c_1 = 0$. Thus,

$$v(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) \, ds.$$

On the other hand, if $v = v(t)$ satisfies (2.2), we can easily show that v satisfies the equation of initial value problem (2.1).

Next, we show that $\lim_{t \to 0^+} t^{1-\alpha} v(t) = 0$.

Let

$$\phi_1(t) = \begin{cases} t^{\alpha-1}, & 0 < t \le 1, \\ 0, & \text{else}, \end{cases} \quad \text{and} \quad \phi_2(t) = \begin{cases} y(t), & 0 \le t \le 1, \\ 0, & \text{else}. \end{cases}$$

$F(t)$ is given by the convolution form, that is,

$$F(t) = (\phi_1 * \phi_2)(t) = \int_{-\infty}^{+\infty} \phi_1(t-s)\phi_2(s) \, ds.$$

If $0 < \alpha < 1$, since $q > \frac{1}{\alpha}$, we have $\frac{q(\alpha-1)}{q-1} > -1$ and $\phi_1 \in L^{\frac{q}{q-1}}(\mathbb{R})$. Hence,

$$\lim_{\Delta t \to 0^+} \int_{\mathbb{R}} \left| \phi_1(s + \Delta t) - \phi_1(s) \right|^{\frac{q}{q-1}} ds = 0.$$

In view of $\phi_2 \in L^q(\mathbb{R})$, we can get that

$$\begin{aligned}
\left| F(t + \Delta t) - F(t) \right| &= \left| \int_{\mathbb{R}} \phi_1(t + \Delta t - s)\phi_2(s) \, ds - \int_{\mathbb{R}} \phi_1(t-s)\phi_2(s) \, ds \right| \\
&\le \int_{\mathbb{R}} \left| \phi_1(t + \Delta t - s) - \phi_1(t-s) \right| \left| \phi_2(s) \right| ds \\
&\le \left(\int_{\mathbb{R}} \left| \phi_1(s + \Delta t) - \phi_1(s) \right|^{\frac{q}{q-1}} ds \right)^{\frac{q-1}{q}} \left(\int_{\mathbb{R}} \left| \phi_2(s) \right|^q ds \right)^{\frac{1}{q}} \\
&\le \|y\|_{L^q} \left(\int_{\mathbb{R}} \left| \phi_1(s + \Delta t) - \phi_1(s) \right|^{\frac{q}{q-1}} ds \right)^{\frac{q-1}{q}} \\
&\to 0 \quad (\Delta t \to 0).
\end{aligned}$$

If $\alpha = 1$, it is obvious that $|F(t + \Delta t) - F(t)| \to 0$ $(\Delta t \to 0)$.

Hence, $F(t)$ is uniformly continuous on $\mathbb{R}$, then we can get that

$$\lim_{t \to 0} t^{1-\alpha} F(t) = 0.$$

Because

$$F(t) = v(t) = \int_0^t (t-s)^{\alpha-1} h(s)\, ds, \quad 0 \le t \le 1.$$

Then we have $\lim_{t \to 0^+} t^{1-\alpha} v(t) = 0$. $\qquad\square$

Remark 2.1 For any $y \in L^q[0,1]$, $v = v(t)$, which satisfies (2.2), is uniformly continuous on $[0,1]$.

For convenience, we denote

$$g(t) = \left(1 - \int_0^1 s^{\beta-1}\, dA(s)\right)^{-1} \left(\int_0^1 s^{\beta-2}\, dA(s) - 1\right) t^{\beta-1} + t^{\beta-2}. \tag{2.4}$$

Lemma 2.4 *For any $h \in C[0,1]$, the integral boundary value problem of linear fractional differential equation*

$$\begin{cases} D_{0^+}^\beta u(t) = h(t), & t \in (0,1), \\ \lim_{t \to 0^+} t^{2-\beta} u(t) = a, & u(1) = \int_0^1 u(s)\, dA(s) \end{cases} \tag{2.5}$$

has a unique solution

$$u(t) = - \int_0^1 G_1(t,s) h(s)\, ds + a g(t), \tag{2.6}$$

where

$$G_1(t,s) = K(t,s) + \frac{t^{\beta-1}}{1 - \int_0^1 s^{\beta-1}\, dA(s)} \int_0^1 K(\tau,s)\, dA(\tau), \tag{2.7}$$

$$K(t,s) = \frac{1}{\Gamma(\beta)} \begin{cases} t^{\beta-1}(1-s)^{\beta-1} - (t-s)^{\beta-1}, & 0 \le s < t \le 1, \\ t^{\beta-1}(1-s)^{\beta-1}, & 0 \le t \le s \le 1. \end{cases} \tag{2.8}$$

Proof Suppose that $u = u(t)$ is a solution of boundary value problem (2.5). Since $h \in C[0,1]$, then $I_{0^+}^{2-\beta} u \in AC^2[0,1]$. Thus, by Lemma 2.1, we have

$$u(t) = I_{0^+}^\beta h(t) + c_1 t^{\beta-1} + c_2 t^{\beta-2}. \tag{2.9}$$

The boundary condition $\lim_{t \to 0^+} t^{2-\beta} u(t) = a$ implies that $c_2 = a$. Then

$$u(t) = I_{0^+}^\beta h(t) + c_1 t^{\beta-1} + a t^{\beta-2}.$$

Hence

$$u(1) = \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s)\, ds + c_1 + a,$$

$$\int_0^1 u(s)\,dA(s) = \frac{1}{\Gamma(\beta)} \int_0^1 \left(\int_0^s (s-\tau)^{\beta-1} h(\tau)\,d\tau \right) dA(s)$$
$$+ c_1 \int_0^1 s^{\beta-1}\,dA(s) + a \int_0^1 s^{\beta-2}\,dA(s).$$

By the boundary condition $u(1) = \int_0^1 u(s)\,dA(s)$, we obtain that

$$c_1 = \frac{1}{1 - \int_0^1 s^{\beta-1}\,dA(s)} \left(\frac{1}{\Gamma(\beta)} \int_0^1 \left(\int_0^s (s-\tau)^{\beta-1} h(\tau)\,d\tau \right) dA(s) \right.$$
$$\left. - \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s)\,ds + a \int_0^1 s^{\beta-2}\,dA(s) - a \right).$$

Substituting c_1 and c_2 into (2.9), we can get that

$$u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h(s)\,ds + c_1 t^{\beta-1} + c_2 t^{\beta-2}$$

$$= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h(s)\,ds$$
$$+ \frac{t^{\beta-1}}{1 - \int_0^1 s^{\beta-1}\,dA(s)} \left(\frac{1}{\Gamma(\beta)} \int_0^1 \left(\int_0^s (s-\tau)^{\beta-1} h(\tau)\,d\tau \right) dA(s) \right.$$
$$\left. - \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s)\,ds + a \int_0^1 s^{\beta-2}\,dA(s) - a \right) + a t^{\beta-2}$$

$$= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h(s)\,ds$$
$$- \left(\frac{t^{\beta-1}}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s)\,ds + \frac{t^{\beta-1} \int_0^1 \tau^{\beta-1}\,dA(\tau) \int_0^1 (1-s)^{\beta-1} h(s)\,ds}{\Gamma(\beta)(1 - \int_0^1 s^{\beta-1}\,dA(s))} \right)$$
$$+ \frac{t^{\beta-1} \int_0^1 (\int_s^1 (\tau-s)^{\beta-1}\,dA(\tau)) h(s)\,ds}{\Gamma(\beta)(1 - \int_0^1 s^{\beta-1}\,dA(s))} + \frac{a t^{\beta-1} (\int_0^1 s^{\beta-2}\,dA(s) - 1)}{1 - \int_0^1 s^{\beta-1}\,dA(s)} + a t^{\beta-2}$$

$$= - \int_0^1 \left(K(t,s) + \frac{t^{\beta-1}}{1 - \int_0^1 s^{\beta-1}\,dA(s)} \int_0^1 K(\tau,s)\,dA(\tau) \right) h(s)\,ds + a g(t)$$

$$= - \int_0^1 G_1(t,s) h(s)\,ds + a g(t).$$

On the other hand, if u satisfies (2.6), then u satisfies (2.9), too. It follows from (2.9)

$$D_{0^+}^{\beta} u(t) = D_{0^+}^{\beta} I_{0^+}^{\beta} h(t) = D^2 \left(I_{0^+}^{2-\beta} I_{0^+}^{\beta} h \right)(t) = h(t),$$

which implies that the equation of boundary value problem (2.5) is satisfied.

We can easily show that u satisfies the boundary conditions of boundary value problem (2.5). □

Lemma 2.5 *If $u \in E$, then boundary value problem* (1.1) *is equivalent to the following integral equation*:

$$u(t) = \lambda \int_0^1 G(t,s) f(s, u(s))\,ds + a g(t), \tag{2.10}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \int_s^1 \frac{1}{p(\tau)} G_1(t,\tau)(\tau-s)^{\alpha-1} \, d\tau. \tag{2.11}$$

Proof Let $u = u(t)$ be a solution of boundary value problem (1.1) and denote $v(t) = p(t)D_{0^+}^\beta u(t)$, $y(t) = \lambda f(t, u(t))$, $h(t) = \frac{v(t)}{p(t)}$. By Lemma 2.3 and Lemma 2.4, we have

$$u(t) = \lambda \int_0^1 \frac{1}{p(s)} G_1(t,s) \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau, u(\tau)) \, d\tau \, ds + ag(t). \tag{2.12}$$

By exchanging integral order, we can get that

$$\begin{aligned} u(t) &= \lambda \int_0^1 \frac{1}{\Gamma(\alpha)} f(s, u(s)) \int_s^1 \frac{1}{p(\tau)} G_1(t,\tau)(\tau-s)^{\alpha-1} \, d\tau \, ds + ag(t) \\ &= \lambda \int_0^1 G(t,s) f(s, u(s)) \, ds + ag(t), \end{aligned}$$

where $G(t,s)$ is defined by (2.11).

On the other hand, if u satisfies (2.10), the u will also satisfy (2.12). By Lemma 2.3 and Lemma 2.4, u satisfies boundary value problem (1.1). $\qquad\square$

Denote constants

$$m_i = \int_0^1 s^{\beta-i} \, dA(s), \quad i = 0, 1, 2, \tag{2.13}$$

$$\gamma_0 = \frac{(\beta-1)(m_1 - m_0)}{1 - m_1 + m_2}, \tag{2.14}$$

and the function

$$g_1(s) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)(1-m_1)} \int_s^1 \frac{\tau(1-\tau)^{\beta-1}(\tau-s)^{\alpha-1}}{p(\tau)} \, d\tau, \quad s \in [0,1]. \tag{2.15}$$

Remark 2.2 Since $p \in C[0,1]$ and $p(t) > 0$ for $t \in [0,1]$, we have $g_1(s) > 0$ for $s \in [0,1)$ and

$$\int_0^1 g_1(s) \, ds = \frac{1}{\Gamma(\alpha+1)\Gamma(\beta)(1-m_1)} \int_0^1 \frac{\tau^{\alpha+1}(1-\tau)^{\beta-1}}{p(\tau)} \, d\tau. \tag{2.16}$$

Lemma 2.6 (See [11]) *The function $K(t,s)$, which is defined by (2.8), has the following properties*:

(1) $K(t,s)$ *is continuous for any* $t,s \in [0,1]$ *and* $K(t,s) > 0$ *for any* $t,s \in (0,1)$;

(2)

$$\frac{t^{\beta-1}(1-t)s(1-s)^{\beta-1}}{\Gamma(\beta-1)} \le K(t,s) \le \frac{t^{\beta-1}(1-t)(1-s)^{\beta-2}}{\Gamma(\beta)}, \quad t,s \in (0,1);$$

(3)

$$K(t,s) \le \frac{1}{\Gamma(\beta)} t^{\beta-2} s(1-s)^{\beta-1} < \frac{1}{\Gamma(\beta)} t^{\beta-2}, \quad t,s \in (0,1).$$

Lemma 2.7 *The function $G_1(t,s)$, which is defined by* (2.7), *has the following properties*:

(1) $G_1(t,s)$ *is continuous for $t,s \in [0,1]$ and $G_1(t,s) > 0$ for $t,s \in (0,1)$;*

(2)

$$\frac{(m_1 - m_0)s(1-s)^{\beta-1}t^{\beta-1}}{\Gamma(\beta-1)(1-m_1)} < G_1(t,s) < \frac{(1-m_1+m_2)s(1-s)^{\beta-1}t^{\beta-2}}{\Gamma(\beta)(1-m_1)}, \quad t,s \in (0,1),$$

where m_i ($i = 0,1,2$) are defined by (2.13).

Proof (1) By the expression of $G_1(t,s)$ and Lemma 2.6, it is easy to check that (1) holds.

(2) For any $t,s \in (0,1)$, from Lemma 2.6, we have

$$
\begin{aligned}
G_1(t,s) &= K(t,s) + \frac{t^{\beta-1}}{1 - \int_0^1 s^{\beta-1}\, dA(s)} \int_0^1 K(\tau,s)\, dA(\tau) \\
&> \frac{t^{\beta-1}}{1 - \int_0^1 s^{\beta-1}\, dA(s)} \int_0^1 \frac{\tau^{\beta-1}(1-\tau)s(1-s)^{\beta-1}}{\Gamma(\beta-1)}\, dA(\tau) \\
&= \frac{(m_1 - m_0)s(1-s)^{\beta-1}t^{\beta-1}}{\Gamma(\beta-1)(1-m_1)}.
\end{aligned}
$$

On the other hand, for any $t,s \in (0,1)$, $1 < \beta \le 2$, implies that $t^{\beta-1} < t^{\beta-2}$, thus, we have

$$
\begin{aligned}
G_1(t,s) &\le \frac{t^{\beta-2}s(1-s)^{\beta-1}}{\Gamma(\beta)} + \frac{t^{\beta-1}}{1 - \int_0^1 s^{\beta-1}\, dA(s)} \int_0^1 \frac{\tau^{\beta-2}s(1-s)^{\beta-1}}{\Gamma(\beta)}\, dA(\tau) \\
&< \frac{t^{\beta-2}s(1-s)^{\beta-1}}{\Gamma(\beta)} + \frac{t^{\beta-2}s(1-s)^{\beta-1}}{\Gamma(\beta)(1-m_1)} \int_0^1 \tau^{\beta-2}\, dA(\tau) \\
&= \frac{(1-m_1+m_2)s(1-s)^{\beta-1}t^{\beta-2}}{\Gamma(\beta)(1-m_1)}. \qquad\qquad \square
\end{aligned}
$$

Lemma 2.8 *The function $G(t,s)$, which is defined by* (2.11), *has the following properties*:

(1) $G(t,s)$ *is continuous for $t,s \in [0,1]$ and $G(t,s) > 0$ for $t,s \in (0,1)$;*

(2)

$$(\beta-1)(m_1-m_0)g_1(s)t < t^{2-\beta}G(t,s) < (1-m_1+m_2)g_1(s), \quad t,s \in (0,1),$$

where m_i, $g_1(s)$ are defined by (2.13) *and* (2.15), *respectively.*

Proof (1) By the expression of $G(t,s)$, we can easily get the results.

(2) According to the definition of $G(t,s)$ and Lemma 2.7, for any $t,s \in (0,1)$, we can obtain that

$$
\begin{aligned}
t^{2-\beta}G(t,s) &= \frac{t^{2-\beta}}{\Gamma(\alpha)} \int_s^1 \frac{1}{p(\tau)} G_1(t,\tau)(\tau-s)^{\alpha-1}\, d\tau \\
&> \frac{t^{2-\beta}}{\Gamma(\alpha)} \int_s^1 \frac{(m_1-m_0)\tau(1-\tau)^{\beta-1}t^{\beta-1}(\tau-s)^{\alpha-1}}{\Gamma(\beta-1)(1-m_1)p(\tau)}\, d\tau \\
&= (\beta-1)(m_1-m_0)g_1(s)t.
\end{aligned}
$$

On the other hand, for any $t, s \in (0, 1)$, by Lemma 2.7, we can show that

$$
\begin{aligned}
t^{2-\beta} G(t, s) &= \frac{t^{2-\beta}}{\Gamma(\alpha)} \int_s^1 \frac{1}{p(\tau)} G_1(t, \tau)(\tau - s)^{\alpha-1} \, d\tau \\
&< \frac{t^{2-\beta}}{\Gamma(\alpha)} \int_s^1 \frac{(1 - m_1 + m_2)\tau(1 - \tau)^{\beta-1} t^{\beta-2}(\tau - s)^{\alpha-1}}{\Gamma(\beta)(1 - m_1)p(\tau)} \, d\tau \\
&= (1 - m_1 + m_2)g_1(s). \qquad \qquad \qquad \qquad \qquad \qquad \quad \square
\end{aligned}
$$

Let

$$
P = \left\{ u \in E : t^{2-\beta} u(t) \geq \gamma_0 t \|u\|, t \in [0, 1] \right\}.
$$

Then P is a cone in E.

Lemma 2.9 *If u is a positive solution of boundary value problem* (1.1), *then $u \in P$.*

Proof If u is a positive solution of boundary value problem (1.1), then from Definition 2.1, we can get that $u(t) > 0$ for $t \in (0, 1)$ and u satisfies (2.10). It is easy to see $u \in E$.

For any $t \in [0, 1]$, by Lemma 2.8, we have

$$
\begin{aligned}
t^{2-\beta} u(t) &= t^{2-\beta} \left(\lambda \int_0^1 G(t, s) f\left(s, u(s)\right) ds + ag(t) \right) \\
&\geq \lambda(\beta - 1)(m_1 - m_0) t \int_0^1 g_1(s) f\left(s, u(s)\right) ds + \frac{at(m_2 - m_1)}{1 - m_1} \\
&\geq \lambda(\beta - 1)(m_1 - m_0) t \int_0^1 g_1(s) f\left(s, u(s)\right) ds + \frac{at(m_1 - m_0)}{1 - m_1} \\
&\geq \frac{t(\beta - 1)(m_1 - m_0)(\lambda(1 - m_1) \int_0^1 g_1(s) f(s, u(s)) ds + a)}{1 - m_1}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\|u\| &= \sup_{t \in [0,1]} t^{2-\beta} \left| \lambda \int_0^1 G(t, s) f\left(s, u(s)\right) ds + ag(t) \right| \\
&\leq \lambda(1 - m_1 + m_2) \int_0^1 g_1(s) f\left(s, u(s)\right) ds + \frac{a \max\{1 - m_1, m_2 - m_1\}}{1 - m_1} \\
&\leq \lambda(1 - m_1 + m_2) \int_0^1 g_1(s) f\left(s, u(s)\right) ds + \frac{a(1 - m_1 + m_2)}{1 - m_1} \\
&= \frac{(1 - m_1 + m_2)(\lambda(1 - m_1) \int_0^1 g_1(s) f(s, u(s)) ds + a)}{1 - m_1}.
\end{aligned}
$$

Then $t^{2-\beta} u(t) \geq \gamma_0 t \|u\|$, which implies $u \in P$. $\qquad \qquad \qquad \qquad \qquad \square$

We define $T : P \to E$ by

$$
(Tu)(t) = \lambda \int_0^1 G(t, s) f\left(s, u(s)\right) ds + ag(t).
$$

Lemma 2.10 *The operator $T : P \to P$ is completely continuous.*

Proof By Lemma 2.9, we have $Tu \in P$ for $u \in P$, then $T : P \to P$.

(1) T is a continuous operator.

If $\{u_n\} \subset P$, $u \in P$, and $\|u_n - u\| \to 0$ as $n \to \infty$, there exists a constant $\gamma > 0$ such that $\|u_n\| \le \gamma$ and $\|u\| \le \gamma$, that is, $\sup_{t \in [0,1]} |t^{2-\beta} u_n(t)| \le \gamma$ and $\sup_{t \in [0,1]} |t^{2-\beta} u(t)| \le \gamma$. Then there exists $\varphi_\gamma \in L^q[0,1]$, we have

$$
t^{2-\beta} G(t,s) \big| f\big(s, u_n(s)\big) - f\big(s, u(s)\big) \big|
$$
$$
\le 2(1 - m_1 + m_2) g_1(s) \varphi_\gamma(s) \quad \text{for } t \in [0,1] \text{ and a.e. } s \in [0,1].
$$

Since f satisfies the L^q-Carathéodory conditions, for a.e. $t \in [0,1]$, we have

$$
\lim_{n \to \infty} f\big(t, u_n(t)\big) = \lim_{n \to \infty} f\big(t, t^{\beta-2} t^{2-\beta} u_n(t)\big) = f\big(t, t^{\beta-2} t^{2-\beta} u(t)\big) = f\big(t, u(t)\big).
$$

By the Lebesgue dominated convergence theorem, we can get

$$
\lim_{n \to \infty} \|Tu_n - Tu\| \le \lim_{n \to \infty} \sup_{t \in [0,1]} \lambda \int_0^1 t^{2-\beta} G(t,s) \big| f\big(s, u_n(s)\big) - f\big(s, u(s)\big) \big| \, ds
$$
$$
\le \lim_{n \to \infty} \lambda(1 - m_1 + m_2) \int_0^1 g_1(s) \big| f\big(s, u_n(s)\big) - f\big(s, u(s)\big) \big| \, ds
$$
$$
= \lambda(1 - m_1 + m_2) \int_0^1 \lim_{n \to \infty} g_1(s) \big| f\big(s, u_n(s)\big) - f\big(s, u(s)\big) \big| \, ds
$$
$$
= 0, \quad n \to \infty.
$$

Hence, $T : P \to P$ is continuous.

(2) T is relatively compact.

Let $\Omega \subset P$ be any bounded set, then there exists a constant $r > 0$ such that $\|u\| \le r$ for each $u \in \Omega$, that is, $\sup_{t \in [0,1]} |t^{2-\beta} u(t)| \le r$. There exists $\varphi_r \in L^q[0,1]$, for any $u \in \Omega$, we have

$$
\big| f\big(t, u(t)\big) \big| = \big| f\big(t, t^{\beta-2} t^{2-\beta} u(t)\big) \big| \le \varphi_r(t), \quad \text{a.e. } t \in [0,1].
$$

Therefore, by Lemma 2.8, we have

$$
\|Tu\| = \sup_{t \in [0,1]} t^{2-\beta} \left| \lambda \int_0^1 G(t,s) f\big(s, u(s)\big) \, ds + ag(t) \right|
$$
$$
\le \lambda(1 - m_1 + m_2) \|\varphi_r\|_{L^q} \int_0^1 g_1(s) \, ds + \frac{a \max\{1 - m_1, m_2 - m_1\}}{1 - m_1},
$$

which implies that $T(\Omega)$ is uniformly bounded.

In addition, because $G(t,s)$ is continuous on $[0,1] \times [0,1]$, then it must be uniformly continuous on $[0,1] \times [0,1]$. Thus, for any $\varepsilon > 0$, there exists a constant $\delta \in (0, \frac{\varepsilon(1-m_1)}{2a|1-m_2|+1})$

such that

$$\left| t_1^{2-\beta} G(t_1, s_1) - t_2^{2-\beta} G(t_2, s_2) \right| < \frac{\varepsilon}{2\lambda \|\varphi_r\|_{L^q} + 1}$$

whenever $|t_1 - t_2| < \delta$ and $|s_1 - s_2| < \delta$, where $t_1, t_2, s_1, s_2 \in [0, 1]$.

Then, for any $u \in \Omega$ and $t_1, t_2 \in [0, 1]$ with $|t_1 - t_2| < \delta$, we have

$$\left| t_1^{2-\beta} Tu(t_1) - t_2^{2-\beta} Tu(t_2) \right|$$
$$= \lambda \int_0^1 \left| t_1^{2-\beta} G(t_1, s) - t_2^{2-\beta} G(t_2, s) \right| f\bigl(s, u(s)\bigr) \, \mathrm{d}s + \frac{a|1 - m_2|}{1 - m_1} |t_1 - t_2|$$
$$\leq \frac{\lambda \varepsilon \|\varphi_r\|_{L^q}}{2\lambda \|\varphi_r\|_{L^q} + 1} + \frac{a|1 - m_2|\delta}{1 - m_1}$$
$$< \varepsilon.$$

Thus, we prove that $T(\Omega)$ is equicontinuous.

According to the Arzela–Ascoli theorem, T is relatively compact.

Therefore, $T : P \to P$ is completely continuous. $\qquad\qquad\square$

Lemma 2.11 (See [36], Lemma 2.3.1) *Let E be a Banach space and $P \subseteq E$ be a cone. Assume that Ω is a bounded open subset of E and $\theta \in \Omega$ and that $T : P \cap \bar{\Omega} \to P$ is completely continuous. If*

$$Tu \neq \tau u \quad \text{for all } u \in P \cap \partial\Omega \text{ and } \tau \geq 1, \tag{2.17}$$

then the fixed point index $i(T, P \cap \Omega, P) = 1$.

Lemma 2.12 (See [36], Corollary 2.3.1) *Let E be a Banach space and $P \subseteq E$ be a cone. Assume that Ω is a bounded open subset of E and that $T : P \cap \bar{\Omega} \to P$ is completely continuous. If there exists $u_0 \in P\backslash\{\theta\}$ such that*

$$u - Tu \neq \tau u_0 \quad \text{for all } u \in P \cap \partial\Omega \text{ and } \tau \geq 0, \tag{2.18}$$

then the fixed point index $i(T, P \cap \Omega, P) = 0$.

Corollary 2.1 *Let E be a Banach space and $P \subseteq E$ be a cone. Assume that Ω is a bounded open subset of E and $\theta \in \Omega$ and that $T : P \cap \bar{\Omega} \to P$ is completely continuous.*

  (1) *If $\|u\| > \|Tu\|$ for $u \in P \cap \partial\Omega$, then $i(T, P \cap \Omega, P) = 1$;*
  (2) *If $\|u\| < \|Tu\|$ for $u \in P \cap \partial\Omega$, then $i(T, P \cap \Omega, P) = 0$.*

Proof (1) If $\|u\| > \|Tu\|$ for $u \in P \cap \partial\Omega$, then we can show that (2.17) holds.

Otherwise, there exist $u^* \in P \cap \partial\Omega$ and $\tau^* \geq 1$ such that $Tu^* = \tau^* u^*$, then

$$\left\| Tu^* \right\| = \left\| \tau^* u^* \right\| \geq \left\| u^* \right\|,$$

which contradicts $\|u\| > \|Tu\|$. In view of Lemma 2.11, we can get $i(T, P \cap \Omega, P) = 1$.

  (2) If $\|u\| < \|Tu\|$ for $u \in P \cap \partial\Omega$, we can prove that (2.18) holds.

In fact, if for any $u \in P \setminus \{\theta\}$ there exist $u^* \in P \cap \partial\Omega$ and $\tau^* \geq 0$ such that $u^* - Tu^* = \tau^* u$, then

$$u^* = Tu^* + \tau^* u \geq Tu^*.$$

Thus, $\|u^*\| \geq \|Tu^*\|$, in contradiction with $\|u\| < \|Tu\|$.

From Lemma 2.12, we can get $i(T, P \cap \Omega, P) = 0$. □

3 The existence of at least one positive solution

For convenience, we denote

$$f^\infty = \limsup_{u \to +\infty} \sup_{t \in [0,1]} \frac{f(t, t^{\beta-2} u)}{u}; \qquad f_\infty = \liminf_{u \to +\infty} \inf_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{f(t, t^{\beta-2} u)}{u};$$

$$f^0 = \limsup_{u \to 0^+} \sup_{t \in [0,1]} \frac{f(t, t^{\beta-2} u)}{u}; \qquad f_0 = \liminf_{u \to 0^+} \inf_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{f(t, t^{\beta-2} u)}{u}.$$

Let $B_r = \{u \in E : \|u\| < r\}$, $\partial B_r = \{u \in E : \|u\| = r\}$, $P_r = P \cap B_r$, $\partial P_r = P \cap \partial B_r$.

Theorem 3.1 *Suppose that there exist constants $\xi, \eta > 0$ such that $f^0 < \xi$ and $f_\infty > \eta$. If*
$$\xi < \frac{\gamma_0^2 \eta \int_{\frac{1}{4}}^{\frac{3}{4}} g_1(s)\,ds}{4 \int_0^1 g_1(s)\,ds} \text{ and } \lambda \text{ satisfies}$$

$$4\left((\beta-1)(m_1 - m_0)\gamma_0 \eta \int_{\frac{1}{4}}^{\frac{3}{4}} g_1(s)\,ds\right)^{-1} \leq \lambda \leq \left((1 - m_1 + m_2)\xi \int_0^1 g_1(s)\,ds\right)^{-1}, \quad (3.1)$$

then there exists a constant $a_\lambda > 0$ such that boundary value problem (1.1) with $0 \leq a \leq a_\lambda$ has at least one positive solution.

Proof Since $f^0 < \xi$, there exists a constant $r_1 > 0$ such that

$$f\left(t, t^{\beta-2} u\right) < \xi u, \quad t \in (0, 1], \text{ and } u \in [0, r_1].$$

Let

$$a_\lambda = \frac{(1 - m_1)(1 - \lambda(1 - m_1 + m_2)\xi \int_0^1 g_1(s)\,ds)r_1}{\max\{1 - m_1, m_2 - m_1\}}.$$

Because λ satisfies (3.1) and $0 \leq a \leq a_\lambda$, by Lemma 2.8, for any $u \in \partial P_{r_1}$, we have $0 < t^{2-\beta} u(t) \leq r_1$ for $t \in (0, 1]$ and

$$\|Tu\| = \sup_{t \in [0,1]} t^{2-\beta} \left| \lambda \int_0^1 G(t, s) f\left(s, u(s)\right)\,ds + ag(t) \right|$$

$$< \lambda(1 - m_1 + m_2)\xi \int_0^1 g_1(s) s^{2-\beta} u(s)\,ds + \frac{a_\lambda \max\{1 - m_1, m_2 - m_1\}}{1 - m_1}$$

$$\leq \lambda(1 - m_1 + m_2)\xi r_1 \int_0^1 g_1(s)\,ds + \frac{a_\lambda \max\{1 - m_1, m_2 - m_1\}}{1 - m_1}$$

$$= r_1.$$

Hence,

$$\|Tu\| < \|u\|, \quad u \in \partial P_{r_1}.$$

It follows from Corollary 2.1(1) $i(T, P_{r_1}, P) = 1$.

By $f_\infty > \eta$, there exists a constant $r_2 > r_1$ such that

$$f\big(t, t^{\beta-2} u\big) > \eta u, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right] \text{ and } u \in \left[\frac{1}{4}\gamma_0 r_2, +\infty\right).$$

For any $u \in \partial P_{r_2}$, we have

$$t^{2-\beta} u(t) \geq \gamma_0 t \|u\| \geq \frac{1}{4}\gamma_0 r_2, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right],$$

and by Lemma 2.8,

$$
\begin{aligned}
\|Tu\| &= \sup_{t \in [0,1]} t^{2-\beta} \left| \lambda \int_0^1 G(t,s) f\big(s, u(s)\big)\, ds + a g(t) \right| \\
&> \sup_{t \in [0,1]} \lambda(\beta-1)(m_1 - m_0) t \int_{\frac{1}{4}}^{\frac{3}{4}} g_1(s) f\big(s, s^{\beta-2} s^{2-\beta} u(s)\big)\, ds \\
&> \lambda(\beta-1)(m_1 - m_0) \int_{\frac{1}{4}}^{\frac{3}{4}} \eta s^{2-\beta} u(s) g_1(s)\, ds \\
&\geq \frac{1}{4}\lambda(\beta-1)(m_1 - m_0)\gamma_0 r_2 \eta \int_{\frac{1}{4}}^{\frac{3}{4}} g_1(s)\, ds \\
&\geq r_2.
\end{aligned}
$$

Therefore,

$$\|Tu\| > \|u\|, \quad u \in \partial P_{r_2}. \tag{3.2}$$

It follows from Corollary 2.1(2) $i(T, P_{r_2}, P) = 0$.

According to the additivity property of the fixed point index, we obtain

$$i(T, P_{r_2} \backslash \bar{P}_{r_1}, P) = i(T, P_{r_2}, P) - i(T, P_{r_1}, P) = -1.$$

Then T has at least one fixed point $u \in P \cap (P_{r_2} \backslash \bar{P}_{r_1})$ with $r_1 < \|u\| < r_2$. Because $u \in P$, we have $t^{2-\beta} u(t) \geq \gamma_0 t \|u\| > 0$ for $t \in (0,1]$, that is, $u(t) > 0$ for $t \in (0,1)$, $u = u(t)$ is a positive solution for boundary value problem (1.1) with $0 \leq a \leq a_\lambda$. $\qquad\square$

Theorem 3.2 *If $f^0 = 0$, $f_\infty = +\infty$, and $\lambda > 0$, then there exists a constant $a_\lambda > 0$ such that boundary value problem (1.1) with $0 \leq a \leq a_\lambda$ has at least one positive solution.*

Proof Let $\lambda > 0$, $0 < \xi < (\lambda(1 - m_1 + m_2) \int_0^1 g_1(s)\, ds)^{-1}$ and $\eta \geq 4(\lambda(\beta-1)(m_1 - m_0)\gamma_0 \times \int_{\frac{1}{4}}^{\frac{3}{4}} g_1(s)\, ds)^{-1}$.

By $f^0 = 0$, there exists a constant $r_1 > 0$ such that

$$f\left(t, t^{\beta-2}u\right) < \xi u, \quad t \in (0, 1], \text{and } u \in [0, r_1],$$

and by $f_\infty = +\infty$, there exists a constant $r_2 > r_1$ such that

$$f\left(t, t^{\beta-2}u\right) > \eta u, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right] \text{ and } u \in \left[\frac{1}{4}\gamma_0 r_2, +\infty\right).$$

Let

$$a_\lambda = \frac{(1 - m_1)(1 - \lambda(1 - m_1 + m_2)\xi \int_0^1 g_1(s)\,ds)r_1}{\max\{1 - m_1, m_2 - m_1\}}.$$

For $0 \le a \le a_\lambda$, similar to the proof of Theorem 3.1, we have

$$i(T, P_{r_1}, P) = 1$$

and

$$i(T, P_{r_2}, P) = 0.$$

According to the additivity property of the fixed point index,

$$i(T, P_{r_2} \backslash \bar{P}_{r_1}, P) = i(T, P_{r_2}, P) - i(T, P_{r_1}, P) = -1.$$

Then T has at least one fixed point $u \in P \cap (P_{r_2} \backslash \bar{P}_{r_1})$ with $r_1 < \|u\| < r_2$, that is, u is a positive solution for boundary value problem (1.1) with $0 \le a \le a_\lambda$. $\qquad\square$

Theorem 3.3 *Suppose that there exist constants ξ, $\eta > 0$ such that $f^\infty < \xi$ and $f_0 > \eta$. If $\xi < \dfrac{\gamma_0^2 \eta \int_{\frac{1}{4}}^{\frac{3}{4}} g_1(s)\,ds}{12 \int_0^1 g_1(s)\,ds}$ and λ satisfies*

$$4\left((\beta - 1)(m_1 - m_0)\gamma_0 \eta \int_{\frac{1}{4}}^{\frac{3}{4}} g_1(s)\,ds\right)^{-1} \le \lambda \le \left(3(1 - m_1 + m_2)\xi \int_0^1 g_1(s)\,ds\right)^{-1}, \quad (3.3)$$

then boundary value problem (1.1) with $a \ge 0$ has at least one positive solution.

Proof By $f_0 > \eta$, there exists a constant $R_1 > 0$ such that

$$f\left(t, t^{\beta-2}u\right) > \eta u, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right], \text{and } u \in [0, R_1].$$

When λ satisfies (3.3) and $a \ge 0$, similar to the proof of Theorem 3.1, we can obtain

$$\|Tu\| > \|u\|, \quad u \in \partial P_{R_1},$$

and

$$i(T, P_{R_1}, P) = 0.$$

On the other hand, by $f^\infty < \xi$, there exists a constant $M > 0$ such that

$$f\left(t, t^{\beta-2}u\right) < \xi u, \quad t \in [0,1], \text{and } u \in [M, +\infty).$$

Since f satisfies the L^q-Carathéodory conditions, for the above $M > 0$, there exists $\varphi_M \in L^q[0,1]$ such that

$$\left|f\left(t, t^{\beta-2}u\right)\right| \le \varphi_M(t), \quad \text{a.e. } t \in [0,1] \text{ and } u \in [0, M].$$

Let

$$R_2 > \max\left\{M, R_1, 3\lambda(1 - m_1 + m_2)\int_0^1 g_1(s)\varphi_M(s)\,ds, \frac{3a \max\{1 - m_1, m_2 - m_1\}}{1 - m_1}\right\}.$$

For any $u \in \partial P_{R_2}$, by Lemma 2.8, we have

$$
\begin{aligned}
\|Tu\| &= \sup_{t\in[0,1]} t^{2-\beta}\left|\lambda\int_0^1 G(t,s)f\left(s, u(s)\right)ds + ag(t)\right| \\
&\le \lambda(1 - m_1 + m_2)\left(\int_{0\le s^{2-\beta}u(s)\le M} g_1(s)f\left(s, u(s)\right)ds + \int_{s^{2-\beta}u(s)\ge M} g_1(s)f\left(s, u(s)\right)ds\right) \\
&\quad + \frac{a \max\{1 - m_1, m_2 - m_1\}}{1 - m_1} \\
&< \lambda(1 - m_1 + m_2)\left(\int_0^1 g_1(s)\varphi_M(s)\,ds + \xi R_2\int_0^1 g_1(s)\,ds\right) \\
&\quad + \frac{a \max\{1 - m_1, m_2 - m_1\}}{1 - m_1} \\
&< \frac{R_2}{3} + \frac{R_2}{3} + \frac{R_2}{3} \\
&= R_2.
\end{aligned}
$$

That is,

$$\|Tu\| < \|u\|, \quad u \in \partial P_{R_2}.$$

From Corollary 2.1(1), we can get $i(T, P_{R_2}, P) = 1$.

According to the additivity property of the fixed point index, we obtain

$$i(T, P_{R_2}\backslash\bar{P}_{R_1}, P) = i(T, P_{R_2}, P) - i(T, P_{R_1}, P) = 1.$$

Then T has at least one fixed point $u \in P \cap (P_{R_2}\backslash\bar{P}_{R_1})$ with $R_1 < \|u\| < R_2$. That is, u is a positive solution for boundary value problem (1.1) with $a \ge 0$. $\square$

Theorem 3.4 *If $f_0 = +\infty, f^\infty = 0, \lambda > 0$, and $a \ge 0$, then boundary value problem (1.1) has at least one positive solution.*

Proof Denote

$$\eta = 4\left(\lambda(\beta - 1)(m_1 - m_0)\gamma_0 \int_{\frac{1}{4}}^{\frac{3}{4}} g_1(s)\,\mathrm{d}s\right)^{-1},$$

$$\xi = \left(3\lambda(1 - m_1 + m_2) \int_0^1 g_1(s)\,\mathrm{d}s\right)^{-1}.$$

By $f_0 = +\infty$, there exists a constant $R_1 > 0$ such that

$$f\left(t, t^{\beta-2}u\right) \geq \eta u, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right], \text{and } u \in [0, R_1].$$

On the other hand, by $f^\infty = 0$, there exists a constant $M > 0$ such that

$$f\left(t, t^{\beta-2}u\right) \leq \xi u, \quad t \in (0, 1], \text{and } u \in [M, +\infty).$$

Let

$$R_2 > \max\left\{M, R_1, 3\lambda(1 - m_1 + m_2) \int_0^1 g_1(s)\varphi_M(s)\,\mathrm{d}s, \frac{3a\max\{1 - m_1, m_2 - m_1\}}{1 - m_1}\right\}.$$

Similar to the proof of Theorem 3.3, we can obtain T has at least one fixed point $u \in P \cap (P_{R_2} \setminus \bar{P}_{R_1})$ with $R_1 < \|u\| < R_2$. That is, $u = u(t)$ is a positive solution for boundary value problem (1.1). □

Theorem 3.5 *If* $f_\infty = +\infty$, $r > 0$ *is a constant and* λ *satisfies*

$$0 < \lambda \leq r\left((1 - m_1 + m_2) \int_0^1 g_1(s)\varphi_r(s)\,\mathrm{d}s\right)^{-1}, \tag{3.4}$$

then there exists a constant $a_\lambda > 0$ *such that boundary value problem* (1.1) *with* $0 \leq a \leq a_\lambda$ *has at least one positive solution* u *with* $\|u\| > r$.

Proof For any given $r > 0$, when λ satisfies (3.4), let

$$a_\lambda = \frac{(1 - m_1)(r - \lambda(1 - m_1 + m_2) \int_0^1 g_1(s)\varphi_r(s)\,\mathrm{d}s)}{\max\{1 - m_1, m_2 - m_1\}}.$$

For $0 \leq a \leq a_\lambda$ and any $u \in \partial P_r$, by Lemma 2.8, we have

$$\|Tu\| = \sup_{t \in [0,1]} t^{2-\beta}\left|\lambda \int_0^1 G(t, s)f\left(s, u(s)\right)\,\mathrm{d}s + ag(t)\right|$$

$$< \lambda(1 - m_1 + m_2) \int_0^1 g_1(s)\varphi_r(s)\,\mathrm{d}s + \frac{a_\lambda \max\{1 - m_1, m_2 - m_1\}}{1 - m_1}$$

$$= r.$$

That is,

$$\|Tu\| < \|u\|, \quad u \in \partial P_r.$$

By Corollary 2.1(1), we can get $i(T, P_r, P) = 1$.

Let $\eta = 4(\lambda(\beta - 1)(m_1 - m_0)\gamma_0 \int_{\frac{1}{4}}^{\frac{3}{4}} g_1(s)\,ds)^{-1}$. Since $f_\infty = +\infty$, there exists a constant $r_1 > r$ such that

$$f(t, t^{\beta-2}u) \geq \eta u, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right], \text{and } u \in \left[\frac{1}{4}\gamma_0 r_1, +\infty\right).$$

Similar to the proof of Theorem 3.2, we obtain

$$\|Tu\| > \|u\|, \quad u \in \partial P_{r_1}.$$

By Corollary 2.1(2)

$$i(T, P_{r_1}, P) = 0.$$

According to the additivity property of the fixed point index,

$$i(T, P_{r_1} \backslash \bar{P}_r, P) = i(T, P_{r_1}, P) - i(T, P_r, P) = -1.$$

Then T has at least one fixed point $u \in P \cap (P_{r_1} \backslash \bar{P}_r)$ with $r < \|u\| < r_1$. That is, $u = u(t)$ is a positive solution for boundary value problem (1.1) with $0 \leq a \leq a_\lambda$. □

4 The multiplicity and nonexistence of positive solutions

In this section, we present the existence of at least two positive solutions and nonexistence positive solutions.

Theorem 4.1 *Suppose that there exist constants $\eta_1, \eta_2 > 0$ such that $f_0 > \eta_1$ and $f_\infty > \eta_2$. Let a constant*

$$r > 4 \int_0^1 g_1(s)\varphi_r(s)\,ds \left(\min\{\eta_1, \eta_2\}\gamma_0^2 \int_{\frac{1}{4}}^{\frac{3}{4}} g_1(s)\,ds\right)^{-1}. \tag{4.1}$$

If λ satisfies

$$4\left(\min\{\eta_1, \eta_2\}(\beta - 1)(m_1 - m_0)\gamma_0 \int_{\frac{1}{4}}^{\frac{3}{4}} g_1(s)\,ds\right)^{-1}$$

$$\leq \lambda \leq r\left((1 - m_1 + m_2) \int_0^1 g_1(s)\varphi_r(s)\,ds\right)^{-1}, \tag{4.2}$$

then there exists a constant $a_\lambda \geq 0$ such that boundary value problem (1.1) with $0 \leq a \leq a_\lambda$ has at least two positive solutions u_1 and u_2.

Proof Let

$$a_\lambda = \frac{(1 - m_1)(r - \lambda(1 - m_1 + m_2) \int_0^1 g_1(s)\varphi_r(s)\,ds)}{\max\{1 - m_1, m_2 - m_1\}},$$

by (4.1) and (4.2), we have $a_\lambda \geq 0$.

For $0 \le a \le a_\lambda$ and any $u \in \partial P_r$, similar to the proof of Theorem 3.5, we have

$$\|Tu\| < \|u\|, \quad u \in \partial P_r,$$

and

$$i(T, P_r, P) = 1.$$

Since $f_0 > \eta_1$, there exists a constant $0 < \bar{r}_1 < r$ such that

$$f\left(t, t^{\beta-2}u\right) > \eta_1 u, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right], \text{and } u \in [0, \bar{r}_1].$$

By $f_\infty > \eta_2$, there exists a constant $\bar{r}_2 > r$ such that

$$f\left(t, t^{\beta-2}u\right) > \eta_2 u, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right], \text{and } u \in \left[\frac{1}{4}\gamma_0\bar{r}_2, +\infty\right).$$

Similar to the proof of Theorem 3.3 and Theorem 3.1, we can obtain $\|Tu\| > \|u\|$, $u \in \partial P_{\bar{r}_1}$, and $\|Tu\| > \|u\|$, $u \in \partial P_{\bar{r}_2}$. Hence, by Corollary 2.1(2), we can get $i(T, P_{\bar{r}_1}, P) = 0$ and $i(T, P_{\bar{r}_2}, P) = 0$.

According to the additivity property of the fixed point index, we can show

$$i(T, P_r \backslash \bar{P}_{\bar{r}_1}, P) = i(T, P_r, P) - i(T, P_{\bar{r}_1}, P) = 1$$

and

$$i(T, P_{\bar{r}_2} \backslash \bar{P}_r, P) = i(T, P_{\bar{r}_2}, P) - i(T, P_r, P) = -1.$$

Then T has at least two fixed points $u_1 \in P \cap (P_r \backslash \bar{P}_{\bar{r}_1})$ with $\bar{r}_1 < \|u_1\| < r$ and $u_2 \in P \cap (\bar{P}_{\bar{r}_2} \backslash P_r)$ with $r < \|u_2\| < \bar{r}_2$. That is, u_1 and u_2 are positive solutions of boundary value problem (1.1) with $0 \le a \le a_\lambda$. □

Theorem 4.2 *Suppose that $f_0 = +\infty, f_\infty = +\infty$, and a constant $r > 0$ hold. If λ satisfies*

$$0 < \lambda \le r\left((1 - m_1 + m_2)\int_0^1 g_1(s)\varphi_r(s)\,ds\right)^{-1}, \tag{4.3}$$

then there exists a constant $a_\lambda \ge 0$ such that boundary value problem (1.1) with $0 \le a \le a_\lambda$ has at least two positive solutions u_1 and u_2.

Proof Let

$$a_\lambda = \frac{(1 - m_1)(r - \lambda(1 - m_1 + m_2)\int_0^1 g_1(s)\varphi_r(s)\,ds)}{\max\{1 - m_1, m_2 - m_1\}}.$$

Since λ satisfies (4.3), we have $a_\lambda \ge 0$.

For $0 \le a \le a_\lambda$ and any $u \in \partial P_r$, similar to the proof of Theorem 3.5, we have

$$\|Tu\| < \|u\| \quad \text{for } u \in \partial P_r.$$

By Corollary 2.1(1), we can get $i(T, P_r, P) = 1$.
Denote

$$\eta_1 = \eta_2 = 4\left(\lambda(\beta-1)(m_1-m_0)\gamma_0 \int_{\frac{1}{4}}^{\frac{3}{4}} g_1(s)\,\mathrm{d}s\right)^{-1}.$$

By $f_0 = +\infty$, there exists a constant $0 < \bar{r}_1 < r$ such that

$$f\left(t, t^{\beta-2}u\right) \ge \eta_1 u, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right], \text{and } u \in [0, \bar{r}_1].$$

And by $f_\infty = +\infty$, there exists a constant $\bar{r}_2 > r$ such that

$$f\left(t, t^{\beta-2}u\right) \ge \eta_2 u, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right], \text{and } u \in \left[\frac{1}{4}\gamma_0\bar{r}_2, +\infty\right).$$

Similar to the proof of Theorem 3.3 and Theorem 3.1, we obtain

$$\|Tu\| > \|u\|, \quad u \in \partial P_{\bar{r}_1}, \quad \text{and} \quad \|Tu\| > \|u\|, \quad u \in \partial P_{\bar{r}_2}.$$

By Corollary 2.1(2), we can get $i(T, P_{\bar{r}_1}, P) = 0$ and $i(T, P_{\bar{r}_2}, P) = 0$.
According to the additivity property of the fixed point index, we obtain

$$i(T, P_r \backslash \bar{P}_{\bar{r}_1}, P) = i(T, P_r, P) - i(T, P_{\bar{r}_1}, P) = 1$$

and

$$i(T, P_{\bar{r}_2} \backslash \bar{P}_r, P) = i(T, P_{\bar{r}_2}, P) - i(T, P_r, P) = -1.$$

Then T has at least two fixed points $u_1 \in P \cap (P_r \backslash \bar{P}_{\bar{r}_1})$ with $\bar{r}_1 < \|u_1\| < r$, and $u_2 \in P \cap (\bar{P}_{\bar{r}_2} \backslash P_r)$ with $r < \|u_2\| < \bar{r}_2$. That is, u_1 and u_2 are positive solutions for boundary value problem (1.1) with $0 \le a \le a_\lambda$. □

Theorem 4.3 *Suppose that* $0 < \liminf_{u \to +\infty} \inf_{t \in [\frac{1}{4}, \frac{3}{4}]} f(t, t^{\beta-2}u) < +\infty, f^0 = 0, \text{ and } f^\infty = 0$ *hold. Then there exist constants* $\lambda^* > 0$ *and* $a_0 > 0$ *such that boundary value problem* (1.1) *has at least two positive solutions with* $\lambda \ge \lambda^*$ *and* $0 \le a \le a_0$.

Proof By $0 < \liminf_{u \to +\infty} \inf_{t \in [\frac{1}{4}, \frac{3}{4}]} f(t, t^{\beta-2}u) < +\infty$, there exist constants $L > 0$ and $R_2 > 0$ such that

$$f\left(t, t^{\beta-2}u\right) > L, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right], \text{and } u \in [R_2, +\infty),$$

Let

$$\lambda^* = \frac{4R_2}{(m_1 - m_0)(\beta-1)L \int_{\frac{1}{4}}^{\frac{3}{4}} g_1(s)\,\mathrm{d}s}.$$

We denote $\xi = (3\lambda(1 - m_1 + m_2) \int_0^1 g_1(s)\,ds))^{-1}$ for $\lambda \geq \lambda^*$.

By $f^\infty = 0$, there exists a constant $M > R_2$ such that

$$f(t, t^{\beta-2}u) \leq \xi u, \quad t \in (0,1], \text{ and } u \in [M, +\infty).$$

By $f^0 = 0$, there exists a constant $0 < R_1 < R_2$ such that

$$f(t, t^{\beta-2}u) \leq \xi u, \quad t \in (0,1], \text{ and } u \in [0, R_1].$$

Let

$$R_3 > \left\{ 4^{2-\beta}R_2, 3\lambda(1 - m_1 + m_2) \int_0^1 g_1(s)\varphi_M(s)\,ds \right\},$$

for $0 < R_1 < R_2 < R_3$, we define

$$\Omega_1 = \left\{ u : u \in E, \|u\| < R_1 \right\},$$

$$\Omega_2 = \left\{ u : u \in E, \|u\| < R_3, \min_{t \in [\frac{1}{4}, \frac{3}{4}]} t^{2-\beta} u(t) > R_2 \right\},$$

$$\Omega_3 = \left\{ u : u \in E, \|u\| < R_3 \right\}.$$

It is easy to see that Ω_1, Ω_2, and Ω_3 are nonempty bounded convex open sets in E, and $\Omega_1 \subset \Omega_3$, $\Omega_2 \subset \Omega_3$, and $\Omega_1 \cap \Omega_2 = \varnothing$. Let

$$a_0 = \frac{R_1(1 - m_1)}{3\max\{1 - m_1, m_2 - m_1\}}.$$

Then, when $0 \leq a \leq a_0$, for any $u \in P \cap \partial\Omega_1$, similar to Theorem 3.1, we obtain

$$\|Tu\| < \|u\|, \quad u \in P \cap \partial\Omega_1$$

and by Corollary 2.1(1), we can get that

$$i(T, P \cap \Omega_1, P) = 1. \tag{4.4}$$

For any $u \in P \cap \partial\Omega_3$, by Lemma 2.8, we have

$$
\begin{aligned}
\|Tu\| &= \sup_{t \in [0,1]} t^{2-\beta} \left| \lambda \int_0^1 G(t,s)f(s,u(s))\,ds + ag(t) \right| \\
&\leq \lambda(1 - m_1 + m_2)\left(\int_{0 \leq s^{2-\beta}u(s) \leq M} g_1(s)f(s,u(s))\,ds + \int_{s^{2-\beta}u(s) \geq M} g_1(s)f(s,u(s))\,ds \right) \\
&\quad + \frac{a\max\{1 - m_1, m_2 - m_1\}}{1 - m_1} \\
&< \lambda(1 - m_1 + m_2)\left(\int_0^1 g_1(s)\varphi_M(s)\,ds + \xi R_3 \int_0^1 g_1(s)\,ds \right)
\end{aligned}
$$

$$+ \frac{a \max\{1 - m_1, m_2 - m_1\}}{1 - m_1}$$

$$< \frac{R_3}{3} + \frac{R_3}{3} + \frac{R_3}{3} = R_3,$$

that is,

$$\|Tu\| < \|u\| \quad \text{for } u \in P \cap \partial \Omega_3,$$

it follows $i(T, P \cap \Omega_3, P) = 1$ from Corollary 2.1(1).

Similarly, for any $u \in P \cap \partial \Omega_2$, we have

$$\|Tu\| \leq \lambda(1 - m_1 + m_2)\left(\int_{0 \leq s^{2-\beta} u(s) \leq M} g_1(s)f(s, u(s)) \, ds + \int_{s^{2-\beta} u(s) \geq M} g_1(s)f(s, u(s)) \, ds \right)$$

$$+ \frac{a \max\{1 - m_1, m_2 - m_1\}}{1 - m_1}$$

$$< \lambda(1 - m_1 + m_2)\left(\int_0^1 g_1(s)\varphi_M(s) \, ds + \xi R_3 \int_0^1 g_1(s) \, ds \right)$$

$$+ \frac{a \max\{1 - m_1, m_2 - m_1\}}{1 - m_1}$$

$$< \frac{R_3}{3} + \frac{R_3}{3} + \frac{R_3}{3}$$

$$= R_3,$$

then $\|Tu\| < R_3$, and

$$\min_{t \in [\frac{1}{4}, \frac{3}{4}]} t^{2-\beta} Tu(t) \geq \min_{t \in [\frac{1}{4}, \frac{3}{4}]} t^{2-\beta} \lambda \int_0^1 G(t,s)f(s, u(s)) \, ds$$

$$\geq \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \lambda(\beta - 1)(m_1 - m_0)t \int_{\frac{1}{4}}^{\frac{3}{4}} g_1(s)f(s, u(s)) \, ds$$

$$> \frac{1}{4}\lambda^*(\beta - 1)(m_1 - m_0)L \int_{\frac{1}{4}}^{\frac{3}{4}} g_1(s) \, ds$$

$$= R_2,$$

so, $Tu \in P \cap \Omega_2$.

Let $u_0 \equiv \frac{1}{2}(4^{2-\beta} R_2 + R_3)$, and

$$H(\tau, u) = (1 - \tau)Tu + \tau u_0, \quad (\tau, u) \in [0, 1] \times (P \cap \bar{\Omega}_2).$$

Because $u_0 \in E$, $\|u_0\| = \frac{1}{2}(4^{2-\beta} R_2 + R_3)$, $\gamma_0 < 1$, then $t^{2-\beta} u_0 \geq \gamma_0 t \|u_0\|$, that is, $u_0 \in P$. Since $R_3 > 4^{2-\beta} R_2$, we can see that $\|u_0\| < R_3$, $\min_{t \in [\frac{1}{4}, \frac{3}{4}]} t^{2-\beta} u_0 > R_2$, which implies $u_0 \in P \cap \Omega_2$. So, we have

$$H(\tau, u) \in P \cap \Omega_2, \quad (\tau, u) \in [0, 1] \times (P \cap \bar{\Omega}_2).$$

Therefore, we have

$$H(\tau, u) \neq u, \quad (\tau, u) \in [0,1] \times (P \cap \partial\Omega_2).$$

By the complete continuity of the operator T and the definition of H, we can know that $H : [0,1] \times (P \cap \Omega_2) \to P$ is completely continuous.

According to the homotopy invariance and normality of fixed point index,

$$i(T, P \cap \Omega_2, P) = i(u_0, P \cap \Omega_2, P) = 1.$$

Thus, T has one fixed point u_1 in $P \cap \Omega_2$.

By the additivity of the fixed point index, we obtain

$$i\big(T, P \cap \big(\Omega_3 \backslash (\bar\Omega_1 \cup \bar\Omega_2)\big), P\big)$$
$$= i(T, P \cap \Omega_3, P) - i(T, P \cap \Omega_2, P) - i(T, P \cap \Omega_1, P)$$
$$= 1 - 1 - 1 = -1.$$

Thus, T has one fixed point u_2 in $P \cap (\Omega_3 \backslash (\bar\Omega_1 \cup \bar\Omega_2))$.

Consequently, u_1 and u_2 are positive solutions of boundary value problem (1.1) with $\lambda \geq \lambda^*$ and $0 \leq a \leq a_0$. □

Remark 4.1 If $f(t,u) \neq 0$, by (4.4), we can get that boundary value problem (1.1) has at least one positive solution in $P \cap \Omega_1$. Then boundary value problem (1.1) with $\lambda \geq \lambda^*$ and $0 \leq a \leq a_0$ has at least three positive solutions in $P \cap \Omega_3$.

Similar to the proof of Theorem 4.3, we can prove the following theorem.

Theorem 4.4 *Suppose that $\lambda > 0$, $\liminf_{u \to +\infty} \inf_{t \in [\frac{1}{4}, \frac{3}{4}]} f(t, t^{\beta-2}u) = +\infty$, $f^0 = 0$, and $f^\infty = 0$ hold. Then there exists a constant $a_0 > 0$ such that boundary value problem (1.1) has at least two positive solutions with $0 \leq a \leq a_0$.*

Theorem 4.5 *If $f_\infty > 0$, then there exist constants $\lambda^* > 0$ and $a_0 > 0$ such that boundary value problem with $\lambda \geq \lambda^*$ and $a \geq a_0$ (1.1) has no positive solution.*

Proof Since $f_\infty > 0$, there exist constants $\eta > 0$ and $r_1 > 0$ such that

$$f\big(t, t^{\beta-2}u\big) > \eta u, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right], \text{and } u \in \left[\frac{1}{4}\gamma_0 r_1, +\infty\right). \tag{4.5}$$

Let

$$\lambda^* = 8\left((m_1 - m_0)(\beta - 1)\eta\gamma_0 \int_{\frac{1}{4}}^{\frac{3}{4}} g_1(s)\, ds\right)^{-1},$$

$$a_0 = 2(1 - m_1)r_1(m_2 + 1 - 2m_1)^{-1}.$$

If u is a positive solution of boundary value problem (1.1) with $\lambda \geq \lambda^*$ and $a \geq a_0$, we will show that this leads to a contradiction.

In fact, since $Tu = u$, we have

$$\left(\frac{1}{2}\right)^{2-\beta} u\left(\frac{1}{2}\right) = \lambda \left(\frac{1}{2}\right)^{2-\beta} \int_0^1 G\left(\frac{1}{2}, s\right) f\left(s, u(s)\right) \, ds + \frac{a(\frac{1}{2}m_2 + \frac{1}{2} - m_1)}{1 - m_1}$$

$$> \frac{a(\frac{1}{2}m_2 + \frac{1}{2} - m_1)}{1 - m_1}$$

$$\geq \frac{a_0(\frac{1}{2}m_2 + \frac{1}{2} - m_1)}{1 - m_1}$$

$$= r_1.$$

Hence, $\|u\| > r_1$.

Because $u \in P$, by (4.5), we can get that

$$t^{2-\beta} u(t) \geq \gamma_0 t \|u\| > \frac{1}{4} \gamma_0 r_1, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right],$$

and

$$\left(\frac{1}{2}\right)^{2-\beta} u\left(\frac{1}{2}\right) = \lambda \left(\frac{1}{2}\right)^{2-\beta} \int_0^1 G\left(\frac{1}{2}, s\right) f\left(s, u(s)\right) \, ds + \frac{a(\frac{1}{2}m_2 + \frac{1}{2} - m_1)}{1 - m_1}$$

$$> \frac{1}{8} \lambda^* (\beta - 1)(m_1 - m_0) \gamma_0 \eta \|u\| \int_{\frac{1}{4}}^{\frac{3}{4}} g_1(s) \, ds + r_1$$

$$= \|u\| + r_1.$$

That is, $\|u\| > \|u\| + r_1$, which is a contradiction. Therefore, boundary value problem (1.1) with $\lambda \geq \lambda^*$ and $a \geq a_0$ has no positive solution. $\qquad\square$

5 Illustration

To illustrate our main results, we present the following examples.

Example 5.1 Consider the boundary value problem

$$\begin{cases} D_{0^+}^{\frac{1}{4}} \left(\frac{1}{t+1} D_{0^+}^{\frac{7}{4}} u(t)\right) + \lambda (t^2 + 1)\left(160 t^{\frac{1}{4}} u(t) - \frac{51{,}199}{320} \sin(t^{\frac{1}{4}} u(t))\right) = 0, \quad t \in (0, 1), \\ \lim_{t \to 0^+} t^{\frac{1}{4}} u(t) = a, \qquad u(1) = \frac{1}{2} u(\frac{1}{4}) + \frac{1}{2} u(\frac{3}{4}), \\ \lim_{t \to 0^+} \frac{1}{(t+1)t^{\frac{3}{4}}} D_{0^+}^{\frac{7}{4}} u(t) = 0, \end{cases} \tag{5.1}$$

where $\alpha = \frac{1}{4}, \beta = \frac{7}{4}, f(t, u) = (t^2 + 1)(160 t^{\frac{1}{4}} u - \frac{51{,}199}{320} \sin(t^{\frac{1}{4}} u)), p(t) = \frac{1}{t+1} > 0$, and

$$A(t) = \begin{cases} 0, & 0 \leq t < \frac{1}{4}, \\ \frac{1}{2}, & \frac{1}{4} \leq t < \frac{3}{4}, \\ 1, & \frac{3}{4} \leq t < 1. \end{cases}$$

Hence,

$$0 < \int_0^1 s^{\beta-1} \, dA(s) \approx 0.57974 < 1, \qquad \gamma_0 \approx 0.105123 < 1,$$

$$m_0 \approx 0.346417, \qquad m_1 \approx 0.57974, \qquad m_2 \approx 1.24439,$$

$$f^0 = \frac{1}{160} < \xi = \frac{1}{80}, \qquad f_\infty = 170 > \eta = 160.$$

We can obtain the following results.

(1) It is easy to check that all the conditions of Theorem 3.1 are satisfied. By Theorem 3.1, for each λ with $2.75654 \le \lambda \le 62.0449$, let a constant $r_1 = 0.01$, then for each a satisfying $0 \le a \le 0.00632303 - 0.00010191\lambda$, boundary value problem (5.1) has at least one positive solution.

(2) It is easy to see that all the conditions of Theorem 4.5 are satisfied. By Theorem 4.5, let $r_1 = 381$, for all $\lambda \ge 5.51308$ and $a \ge 295.175$, boundary value problem (5.1) has no positive solution.

Example 5.2 Consider the boundary value problem

$$\begin{cases} D_{0+}^{\frac{1}{2}}((t+1)D_{0+}^{\frac{3}{2}}u(t)) + \lambda(e^{tu^2(t)} + t\sin(t^{\frac{1}{2}}u(t))) = 0, & t \in (0,1), \\ \lim_{t \to 0^+} t^{\frac{1}{2}}u(t) = a, \qquad u(1) = \frac{1}{4}u(\frac{1}{4}) + \frac{1}{8}u(\frac{3}{4}), \\ \lim_{t \to 0^+} t^{\frac{1}{2}}(t+1)D_{0+}^{\frac{3}{2}}u(t) = 0, \end{cases} \tag{5.2}$$

where $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$, $f(t,u) = (e^{tu^2} + t\sin(t^{\frac{1}{2}}u))$, $p(t) = t+1 > 0$ for any $t \in [0,1]$, and

$$A(t) = \begin{cases} 0, & 0 \le t < \frac{1}{4}, \\ \frac{1}{4}, & \frac{1}{4} \le t < \frac{3}{4}, \\ \frac{1}{4} + \frac{1}{8}, & \frac{3}{4} \le t < 1. \end{cases}$$

Then

$$\int_0^1 s^{\beta-1}\, dA(s) \approx 0.233253 < 1, \qquad \gamma_0 \approx 0.0428085 < 1,$$

$$m_0 \approx 0.11244, \qquad m_1 \approx 0.233253, \qquad m_2 \approx 0.644338,$$

$$f_\infty = +\infty, \qquad f_0 = +\infty.$$

All the conditions of Theorem 4.2 are satisfied. By Theorem 4.2, for given $r > 1$, $0 < \lambda \le 1.1124$ and each a satisfying $0 \le a \le 1 - 0.402948\lambda$, boundary value problem (5.2) has at least two positive solutions u_1, u_2.

Example 5.3 Consider the boundary value problem

$$\begin{cases} D_{0+}^{\frac{1}{2}}((t+1)D_{0+}^{\frac{3}{2}}u(t)) + \lambda f(t,u(t)) = 0, & t \in (0,1), \\ \lim_{t \to 0^+} t^{\frac{1}{2}}u(t) = a, \qquad u(1) = \int_0^1 su(s)\, ds, \\ \lim_{t \to 0^+} t^{\frac{1}{2}}(t+1)D_{0+}^{\frac{3}{2}}u(t) = 0, \end{cases} \tag{5.3}$$

where $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$, $p(t) = t + 1$, $A(t) = \frac{1}{2}t^2$, and

$$f(t, u) = \begin{cases} t^2 u^2, & 0 \le t \le 1 \text{ and } 0 \le u < 1, \\ t^2 \sqrt{u}, & 0 \le t \le 1 \text{ and } u \ge 1. \end{cases}$$

Then $\liminf_{u \to +\infty} \inf_{t \in [\frac{1}{4}, \frac{3}{4}]} f(t, t^{\beta-2}u) = +\infty$, $f^0 = 0$, and $f^\infty = 0$,

$$0 < \int_0^1 s^{\beta-1} \, dA(s) = \int_0^1 s^{\frac{3}{2}} \, ds = \frac{2}{5} < 1,$$

$$m_0 = \frac{2}{7}, \qquad m_1 = \frac{2}{5}, \qquad m_2 = \frac{2}{3}, \qquad \gamma_0 = \frac{6}{133} < 1.$$

All the conditions of Theorem 4.4 are satisfied. By Theorem 4.4, there exists a constant $a_0 > 0$ such that boundary value problem (5.3) with $0 \le a \le a_0$ has at least two positive solutions for $\lambda > 0$.

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