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On impulsive nonlocal integro-initial value problems involving multi-order Caputo-type generalized fractional derivatives and generalized fractional integrals

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Abstract

In this paper, we present sufficient criteria ensuring the existence and uniqueness of solutions for nonlinear impulsive multi-order Caputo-type generalized fractional differential equations supplemented with nonlocal integro-initial value conditions involving generalized fractional integrals. Extremal solutions for the given problem are also discussed. The main tools of our study include Krasnoselskii's fixed point theorem, Banach contraction mapping principle and monotone iterative technique. Examples are constructed for illustrating the obtained results.

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1 Introduction

Impulsive dynamical systems involve some continuous variable dynamic characteristics, together with certain reset maps generating impulsive switching among them. The dynamical behavior of impulsive systems is much more complex than that associated to non-impulsive dynamical systems. Such systems appear in real-time software verification [1], transportation systems [2, 3], automotive control [4, 5], etc. In consequence, the topic of impulsive differential equations has emerged as an important area of investigation as it accounts for several phenomena which are not addressed by the non-impulsive equations.

Arbitrary (non-integer) order differential and integral operators serve as better modeling tools than their corresponding integer-order counterparts, as these operators are capable to retrieve the historical effects of the systems and processes involved in the phenomena. Fractional-order initial and boundary value problems have been investigated by many authors in recent years; for instance, see [6–17].

Factional differential equations with impulse effects also received considerable attention in view of their applications in modeling the physical problems experiencing instantaneous changes. For some recent works on impulsive fractional differential equations, we refer the reader to the papers [18–29] and the references cited therein. In a recent work [26], the authors discussed the existence of extremal solutions for a nonlinear impulsive differential equations with multi-order fractional derivatives and integral boundary conditions.

In this paper, we introduce a new class of nonlinear nonlocal impulsive multi-order problems involving Caputo-type generalized fractional derivatives and generalized fractional integrals (in the sense of Katugampola). In precise terms, we investigate the following problem:

$$\begin{cases} {}^{\rho}_{c}D^{\alpha_k}_{t_k^+}y(t) = f(t,y(t)), \quad 1 < \alpha_k \le 2, k = 0,1,2,\ldots,p, t \in J', \\ \triangle y(t_k) = S_k(y(t_k)), \qquad \triangle\delta y(t_k) = S_k^*(y(t_k)), \quad k = 1,2,\ldots,p, \\ y(0) = \sum_{k=0}^{p}\lambda_k {}^{\rho}I^{\beta_k}_{t_k^+}y(\xi_k) + \eta, \qquad \delta y(0) = 0, \quad t_k < \xi_k < t_{k+1}, \end{cases} \tag{1.1}$$

where ${}^{\rho}_{c}D^{\alpha_k}_{t_k^+}$ is the Caputo-type generalized fractional derivative of order $\alpha_k, \rho > 0$, ${}^{\rho}I^{\beta_k}_{t_k^+}$ is the generalized fractional integral of order $\beta_k > 0, \rho > 0$ (defined in the next section), $f \in C(J \times \mathbb{R},\mathbb{R}), S_k, S_k^* \in C(\mathbb{R},\mathbb{R})$; λ_k, ξ_k are positive constants; $J = [0,T]$ $(T > 0), \eta \in \mathbb{R}, 0 = t_0 < t_1 < \cdots < t_k < \cdots < t_p < t_{p+1} = T, J' = J \setminus \{t_1,t_2,\ldots,t_m\}, \triangle y(t_k) = y(t_k^+) - y(t_k^-)$, where $y(t_k^+)$ and $y(t_k^-)$ denote the right and the left limits of $y(t)$ at $t = t_k (k = 1,2,\ldots,p)$, respectively; $\triangle\delta y(t_k)$ have a similar meaning for $\delta y(t)$, where $\delta = t^{1-\rho}\frac{d}{dt}$.

In Sect. 2, we present the background material related to our work and prove an important lemma which plays a key role in the sequel. Section 3 contains the existence and uniqueness results for problem (1.1). In Sect. 4, we prove a new comparison result and use it to obtain the extremal solutions for problem (1.1).

2 Preliminaries

Let us fix $J_0 = [0,t_1], J_k = (t_k,t_{k+1}], k = 1,2,\ldots,p$ with $t_{p+1} = T$, and define $PC(J,\mathbb{R}) = \{y : J \to \mathbb{R} : y \in C(J_k,\mathbb{R}), k = 0,1,\ldots,p$ and $y(t_k^+)$ and $y(t_k^-)$ exist with $y(t_k^-) = y(t_k), k = 1,2,\ldots,p\}$, where $C(J,\mathbb{R})$ denotes the space of all continuous real-valued functions on J and $PC^1_\delta(J,\mathbb{R}) = \{y : J \to \mathbb{R} : \delta y \in PC(J,\mathbb{R}); \delta y(t_k^+), \delta y(t_k^-)$ exist and δy is left continuous at t_k for $k = 1,2,\ldots,p, \delta = t^{1-\rho}\frac{d}{dt}\}$ with the norm $\|y\| = \sup_{t\in J}\{\|y(t)\|_{PC}, \|\delta y(t)\|_{PC^1_\delta}\}$. We further recall that $AC^n(J,\mathbb{R}) = \{h : J \to \mathbb{R} : h, h', \ldots, h^{(n-1)} \in C(J,\mathbb{R})$ and $h^{(n-1)}$ is absolutely continuous$\}$. For $0 \le \epsilon < 1$, we define $C_{\epsilon,\rho}(J,\mathbb{R}) = \{f : J \to \mathbb{R} : (t^\rho - a^\rho)^\epsilon f(t) \in C(J,\mathbb{R})\}$ endowed with the norm $\|f\|_{C_{\epsilon,\rho}} = \|(t^\rho - a^\rho)^\epsilon f(t)\|_C$. Moreover, we define the class of functions f that have absolutely continuous δ^{n-1}-derivative, denoted by $AC^n_\delta(J,\mathbb{R})$, as follows: $AC^n_\delta(J,\mathbb{R}) = \{f : J \to \mathbb{R} : \delta^{n-1}f \in AC(J,\mathbb{R}), \delta = t^{1-\rho}\frac{d}{dt}\}$, which is equipped with the norm $\|f\|_{C^n_\delta} = \sum_{k=0}^{n-1}\|\delta^k f\|_C$. More generally, let $C^n_{\delta,\epsilon}(J,\mathbb{R}) = \{f : J \to \mathbb{R} : \delta^{n-1}f \in C(J,\mathbb{R}), \delta^n f \in C_{\epsilon,\rho}(J,\mathbb{R}), \delta = t^{1-\rho}\frac{d}{dt}\}$ be the space of functions endowed with the norm $\|f\|_{C^n_{\delta,\epsilon}} = \sum_{k=0}^{n-1}\|\delta^k f\|_C + \|\delta^n f\|_{C_{\epsilon,\rho}}$. Here we use the convention $C^n_{\delta,0} = C^n_\delta$.

For $c \in \mathbb{R}, 1 \le q \le \infty$, let $X^q_c(a,b)$ denote the space of all Lebesgue measurable functions ϕ on (a,b) equipped with the norm

$$\|\phi\|_{X^q_c} = \left(\int_a^b \left|x^c\phi(x)\right|^q \frac{dx}{x}\right)^{1/q} < \infty.$$

Definition 2.1 ([30]) The generalized fractional integral of order $\alpha > 0$ and $\rho > 0$ of $f \in X^q_c(a,b)$, for $-\infty < a < t < b < \infty$, is defined by

$$\left({}^{\rho}I^{\alpha}_{a^+}f\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}}f(s)\,ds. \tag{2.1}$$

Note that the integral in (2.1) is called the left-sided fractional integral. Similarly, we can define the right-sided fractional integral ${}^{\rho}I^{\alpha}_{b-}f$ as

$$\left({}^{\rho}I^{\alpha}_{b-}f\right)(t)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_t^b\frac{s^{\rho-1}}{(s^{\rho}-t^{\rho})^{1-\alpha}}f(s)\,ds. \tag{2.2}$$

Definition 2.2 ([31]) The generalized fractional derivatives of $f\in X^q_c(a,b)$ of order $\alpha\in(n-1,n], n\in\mathbb{N}$, associated with the generalized fractional integrals (2.1) and (2.2), are defined for $0\leq a<x<b<\infty$ by

$$\begin{aligned}\left({}^{\rho}D^{\alpha}_{a^+}f\right)(t)&=\left(t^{1-\rho}\frac{d}{dt}\right)^n\left({}^{\rho}I^{n-\alpha}_{a^+}f\right)(t)\\&=\frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n\int_a^t\frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{\alpha-n+1}}f(s)\,ds\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\left({}^{\rho}D^{\alpha}_{b-}f\right)(t)&=\left(-t^{1-\rho}\frac{d}{dt}\right)^n\left({}^{\rho}I^{n-\alpha}_{b-}f\right)(t)\\&=\frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(-t^{1-\rho}\frac{d}{dt}\right)^n\int_t^b\frac{s^{\rho-1}}{(s^{\rho}-t^{\rho})^{\alpha-n+1}}f(s)\,ds.\end{aligned} \tag{2.4}$$

Definition 2.3 ([32]) For $\alpha\geq 0$ and $f\in AC^n_{\delta}[a,b]$, the Caputo-type generalized fractional derivatives ${}^{\rho}_{c}D^{\alpha}_{a^+}$ and ${}^{\rho}_{c}D^{\alpha}_{b-}$ are defined in terms of (2.3) and (2.4) as follows:

$${}^{\rho}_{c}D^{\alpha}_{a^+}f(x)={}^{\rho}D^{\alpha}_{a^+}\left[f(t)-\sum_{k=0}^{n-1}\frac{\delta^k f(a)}{k!}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^k\right](x),\qquad \delta=x^{1-\rho}\frac{d}{dx}, \tag{2.5}$$

$${}^{\rho}_{c}D^{\alpha}_{b-}f(x)={}^{\rho}D^{\alpha}_{b-}\left[f(t)-\sum_{k=0}^{n-1}\frac{(-1)^k\delta^k f(b)}{k!}\left(\frac{b^{\rho}-t^{\rho}}{\rho}\right)^k\right](x),\qquad \delta=x^{1-\rho}\frac{d}{dx}. \tag{2.6}$$

Lemma 2.4 ([32]) *Let $\alpha\geq 0, n=[\alpha]+1$, where $[\alpha]$ denotes the integer part of α, and $f\in AC^n_{\delta}[a,b]$ with $0<a<b<\infty$.*

1. *If $\alpha\notin\mathbb{N}$, then*

$${}^{\rho}_{c}D^{\alpha}_{a^+}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{n-\alpha-1}\frac{(\delta^n f)(s)\,ds}{s^{1-\rho}}={}^{\rho}I^{n-\alpha}_{a^+}\left(\delta^n f\right)(t), \tag{2.7}$$

$${}^{\rho}_{c}D^{\alpha}_{b-}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_t^b\left(\frac{s^{\rho}-t^{\rho}}{\rho}\right)^{n-\alpha-1}\frac{(-1)^n(\delta^n f)(s)\,ds}{s^{1-\rho}}={}^{\rho}I^{n-\alpha}_{b-}\left(\delta^n f\right)(t). \tag{2.8}$$

2. *If $\alpha\in\mathbb{N}$, then*

$${}^{\rho}_{c}D^{\alpha}_{a^+}f=\delta^n f,\qquad {}^{\rho}_{c}D^{\alpha}_{b-}f=(-1)^n\delta^n f. \tag{2.9}$$

Lemma 2.5 ([32]) *If $f\in AC^n_{\delta}[a,b]$ or $C^n_{\delta}[a,b]$ and $\alpha\in\mathbb{R}$, then*

$${}^{\rho}I^{\alpha}_{a^+}\,{}^{\rho}_{c}D^{\alpha}_{a^+}f(x)=f(x)-\sum_{k=0}^{n-1}\frac{(\delta^k f)(a)}{k!}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^k,$$

$$ {}^{\rho}I_{b^-}^{\alpha}\,{}^{\rho}_{c}D_{b^-}^{\alpha}f(x) = f(x) - \sum_{k=0}^{n-1}\frac{(-1)^k(\delta^k f)(a)}{k!}\left(\frac{b^\rho - x^\rho}{\rho}\right)^k. $$

In particular, for $0 < \alpha \leq 1$, *we have*

$$ {}^{\rho}I_{a^+}^{\alpha}\,{}^{\rho}_{c}D_{a^+}^{\alpha}f(x) = f(x) - f(a), $$
$$ {}^{\rho}I_{b^-}^{\alpha}\,{}^{\rho}_{c}D_{b^-}^{\alpha}f(x) = f(x) - f(b). $$

Definition 2.6 A function $y \in PC_\delta^1(J,\mathbb{R}) \cap AC_\delta^2(J_k)$ with its Caputo generalized derivative of order α_k, $k = 0,1,\ldots,p$, is a solution of (1.1) if it satisfies (1.1).

Lemma 2.7 *For any* $h \in C([0,T],\mathbb{R}), y \in PC_\delta^1(J,\mathbb{R}) \cap AC_\delta^2(J_k)$, *the constants* S_k, S_k^* ($k = 1,2,\ldots,p$) *and*

$$ \Omega = 1 - \sum_{k=0}^{p}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k + 1)} \neq 0, \tag{2.10} $$

the integral representation of the solution for the following impulsive nonlocal integro-initial value problem

$$ \begin{cases} {}^{\rho}_{c}D_{t_k^+}^{\alpha_k}y(t) = h(t), \quad 0 < \alpha_k \leq 2, k = 0,1,2,\ldots,p, t \in J', \\ \triangle y(t_k) = S_k, \qquad \triangle\delta y(t_k) = S_k^*, \quad k = 1,2,\ldots,p, \\ y(0) = \sum_{k=0}^{p}\lambda_k {}^{\rho}I_{t_k^+}^{\beta_k}y(\xi_k) + \eta, \qquad \delta y(0) = 0, \quad t_k < \xi_k < t_{k+1}, \end{cases} \tag{2.11} $$

is given by

$$ y(t) = \begin{cases} {}^{\rho}I_{0^+}^{\alpha_0}h(t) + A, \quad t \in J_0, \\ {}^{\rho}I_{t_k^+}^{\alpha_k}h(t) + \sum_{i=1}^{k}[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}}h(t_i) + S_i] + \sum_{i=1}^{k-1}(\frac{t_k^\rho - t_i^\rho}{\rho})[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1}h(t_i) + S_i^*] \\ \quad + \sum_{i=1}^{k}(\frac{t^\rho - t_k^\rho}{\rho})[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1}h(t_i) + S_i^*] + A, \\ \quad t \in J_k, k = 1,2,\ldots,p, \end{cases} \tag{2.12} $$

where

$$ \begin{aligned} A = \frac{1}{\Omega}\Bigg\{ & \sum_{k=0}^{p}\lambda_k {}^{\rho}I_{t_k^+}^{\alpha_k+\beta_k}h(\xi_k) + \sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)}\big[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}}h(t_i) + S_i\big] \\ & + \sum_{k=2}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}(t_k^\rho - t_i^\rho)}{\rho^{\beta_k+1}\Gamma(\beta_k+1)}\big[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1}h(t_i) + S_i^*\big] \\ & + \sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k+1}}{\rho^{\beta_k+1}\Gamma(\beta_k+2)}\big[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1}h(t_i) + S_i^*\big] + \eta\Bigg\}. \end{aligned} \tag{2.13} $$

Proof Applying the operator ${}^{\rho}I_{t_k^+}^{\alpha_k}$ to the fractional differential equation in (2.11) and using Lemma 2.5, we obtain

$$ y(t) = {}^{\rho}I_{t_k^+}^{\alpha_k}h(t) + c_{1,k} + c_{2,k}\left(\frac{t^\rho - t_k^\rho}{\rho}\right) $$

$$= \frac{\rho^{1-\alpha_k}}{\Gamma(\alpha_k)} \int_{t_k}^{t} s^{\rho-1}\left(t^\rho - s^\rho\right)^{\alpha_k-1} h(s)\,ds + c_{1,k} + c_{2,k}\left(\frac{t^\rho - t_k^\rho}{\rho}\right), \quad t \in J_k, \tag{2.14}$$

where $c_{1,k}, c_{2,k} \in \mathbb{R}, k = 0, 1, \ldots, p$. Taking δ-derivative of (2.14), we get

$$\delta y(t) = {}^\rho I_{t_k^+}^{\alpha_k-1} h(t) + c_{2,k} = \frac{\rho^{2-\alpha_k}}{\Gamma(\alpha_k - 1)} \int_{t_k}^{t} s^{\rho-1}\left(t^\rho - s^\rho\right)^{\alpha_k-2} h(s)\,ds + c_{2,k}. \tag{2.15}$$

For $t \in J_0$, we have

$$y(t) = {}^\rho I_{0^+}^{\alpha_0} h(t) + c_{1,0} + c_{2,0}\frac{t^\rho}{\rho} \tag{2.16}$$

and

$$\delta y(t) = {}^\rho I_{0^+}^{\alpha_0-1} h(t) + c_{2,0}. \tag{2.17}$$

Using the condition $\delta y(0) = 0$ in (2.17), we get $c_{2,0} = 0$. In consequence, (2.16) and (2.17) take the form

$$y(t) = {}^\rho I_{0^+}^{\alpha_0} h(t) + c_{1,0}, \quad t \in J_0, \tag{2.18}$$

and

$$\delta y(t) = {}^\rho I_{0^+}^{\alpha_0-1} h(t), \quad t \in J_0. \tag{2.19}$$

Next, for $t \in J_1$, we have

$$y(t) = {}^\rho I_{t_1^+}^{\alpha_1} h(t) + c_{1,1} + c_{2,1}\left(\frac{t^\rho - t_1^\rho}{\rho}\right), \tag{2.20}$$

$$\delta y(t) = {}^\rho I_{t_1^+}^{\alpha_1-1} h(t) + c_{2,1}, \tag{2.21}$$

which imply that

$$y\left(t_1^-\right) = {}^\rho I_{0^+}^{\alpha_0} h(t_1) + c_{1,0}, \qquad y\left(t_1^+\right) = c_{1,1}, \tag{2.22}$$

$$\delta y\left(t_1^-\right) = {}^\rho I_{0^+}^{\alpha_0-1} h(t_1), \qquad \delta y\left(t_1^+\right) = c_{2,1}. \tag{2.23}$$

Using the impulse conditions $\triangle y(t_1) = y(t_1^+) - y(t_1^-) = S_1, \triangle \delta y(t_k) = \delta y(t_1^+) - \delta y(t_1^-) = S_1^*$ in (2.22) and (2.23), we find that

$$c_{1,1} = {}^\rho I_{0^+}^{\alpha_0} h(t_1) + c_{1,0} + S_1, \qquad c_{2,1} = {}^\rho I_{0^+}^{\alpha_0-1} h(t_1) + S_1^*.$$

Substituting the values of $c_{1,1}$ and $c_{2,1}$ in (2.20), we obtain

$$y(t) = {}^\rho I_{t_1^+}^{\alpha_1} h(t) + {}^\rho I_{0^+}^{\alpha_0} h(t_1) + S_1 + \left(\frac{t^\rho - t_1^\rho}{\rho}\right)\left[{}^\rho I_{0^+}^{\alpha_0-1} h(t_1) + S_1^*\right] + c_{1,0}, \quad t \in J_1.$$

By a similar process, for $t \in J_k$, we get

$$y(t) = {}^{\rho}I_{t_k^+}^{\alpha_k} h(t) + \sum_{i=1}^{k} \left[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}} h(t_i) + S_i \right] + \sum_{i=1}^{k-1} \left(\frac{t_k^{\rho} - t_i^{\rho}}{\rho} \right) \left[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1} h(t_i) + S_i^* \right]$$
$$+ \sum_{i=1}^{k} \left(\frac{t^{\rho} - t_k^{\rho}}{\rho} \right) \left[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1} h(t_i) + S_i^* \right] + c_{1,0}, \quad k = 1, 2, \dots, p. \tag{2.24}$$

For $t \in J_k, k = 0, 1, 2, \dots, p$, we have

$${}^{\rho}I_{t_k^+}^{\beta_k} y(t) = {}^{\rho}I_{t_k^+}^{\alpha_k+\beta_k} h(t) + \sum_{i=1}^{k} \frac{(t^{\rho} - t_k^{\rho})^{\beta_k}}{\rho^{\beta_k} \Gamma(\beta_k + 1)} \left[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}} h(t_i) + S_i \right]$$
$$+ \sum_{i=1}^{k-1} \frac{(t^{\rho} - t_k^{\rho})^{\beta_k} (t_k^{\rho} - t_i^{\rho})}{\rho^{\beta_k+1} \Gamma(\beta_k + 1)} \left[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1} h(t_i) + S_i^* \right]$$
$$+ \sum_{i=1}^{k} \frac{(t^{\rho} - t_k^{\rho})^{\beta_k+1}}{\rho^{\beta_k+1} \Gamma(\beta_k + 2)} \left[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1} h(t_i) + S_i^* \right] + c_{1,0} \frac{(t^{\rho} - t_k^{\rho})^{\beta_k}}{\rho^{\beta_k} \Gamma(\beta_k + 1)}. \tag{2.25}$$

The condition $y(0) = \sum_{k=0}^{p} \lambda_k I_{t_k^+}^{\beta_k} y(\xi_k) + \eta$, together with (2.18) and (2.25), implies that

$$c_{1,0} = \left(1 - \sum_{k=0}^{p} \frac{\lambda_k (\xi_k^{\rho} - t_k^{\rho})^{\beta_k}}{\rho^{\beta_k} \Gamma(\beta_k + 1)} \right)^{-1} \left\{ \sum_{k=0}^{p} \lambda_k {}^{\rho}I_{t_k^+}^{\alpha_k+\beta_k} h(\xi_k) \right.$$
$$+ \sum_{k=1}^{p} \sum_{i=1}^{k} \frac{\lambda_k (\xi_k^{\rho} - t_k^{\rho})^{\beta_k}}{\rho^{\beta_k} \Gamma(\beta_k + 1)} \left[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}} h(t_i) + S_i \right]$$
$$+ \sum_{k=2}^{p} \sum_{i=1}^{k-1} \frac{\lambda_k (\xi_k^{\rho} - t_k^{\rho})^{\beta_k} (t_k^{\rho} - t_i^{\rho})}{\rho^{\beta_k+1} \Gamma(\beta_k + 1)} \left[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1} h(t_i) + S_i^* \right]$$
$$\left. + \sum_{k=1}^{p} \sum_{i=1}^{k} \frac{\lambda_k (\xi_k^{\rho} - t_k^{\rho})^{\beta_k+1}}{\rho^{\beta_k+1} \Gamma(\beta_k + 2)} \left[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1} h(t_i) + S_i^* \right] + \eta \right\}, \tag{2.26}$$

which, on inserting in (2.18) and (2.24), yields the solution (2.12). The converse follows by direct computation. This completes the proof. □

3 Existence and uniqueness results

In this section, we present the existence and uniqueness results for problem (1.1). Let $\mathcal{G} = PC_{\delta}^{1}(J, \mathbb{R}) \cap AC_{\delta}^{2}(J_k)$. By Lemma 2.7, we transform problem (1.1) into a fixed point problem by defining an operator $F : \mathcal{G} \to \mathcal{G}$ as

$$(Fy)(t) = \begin{cases} {}^{\rho}I_{0^+}^{\alpha_0} f(t, y(t)) + A, \quad t \in J_0, \\ {}^{\rho}I_{t_k^+}^{\alpha_k} f(t, y(t)) + \sum_{i=1}^{k} [{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}} f(t_i, y(t_i)) + S_i(y(t_i))] \\ \quad + \sum_{i=1}^{k-1} (\frac{t_k^{\rho} - t_i^{\rho}}{\rho}) [{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1} f(t_i, y(t_i)) + S_i^*(y(t_i))] \\ \quad + \sum_{i=1}^{k} (\frac{t^{\rho} - t_k^{\rho}}{\rho}) [{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1} f(t_i, y(t_i)) + S_i^*(y(t_i))] + A, \\ \quad t \in J_k, k = 1, 2, \dots, p, \end{cases} \tag{3.1}$$

where A is defined by (2.13).

For convenience, for $p \geq 1$, we set

$$\Lambda_1 = (1+p)\frac{\max_{0\leq i\leq p} T^{\alpha_i \rho}}{\min_{0\leq i\leq p}\{\rho^{\alpha_i}\Gamma(\alpha_i+1)\}} + (2p-1)\frac{\max_{0\leq i\leq p} T^{\alpha_i \rho}}{\min_{0\leq i\leq p}\{\rho^{\alpha_i}\Gamma(\alpha_i)\}}, \tag{3.2}$$

$$\begin{aligned}\Lambda_2 = \frac{1}{|\Omega|}\Bigg\{&\sum_{k=0}^{p}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\alpha_k+\beta_k}}{\rho^{\alpha_k+\beta_k}\Gamma(\alpha_k+\beta_k+1)} \\ &+ \sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}(t_i^\rho - t_{i-1}^\rho)^{\alpha_{i-1}}}{\rho^{\beta_k+\alpha_{i-1}}\Gamma(\beta_k+1)\Gamma(\alpha_{i-1}+1)} \\ &+ \sum_{k=2}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}(t_k^\rho - t_i^\rho)(t_i^\rho - t_{i-1}^\rho)^{\alpha_{i-1}-1}}{\rho^{\beta_k+\alpha_{i-1}}\Gamma(\beta_k+1)\Gamma(\alpha_{i-1})} \\ &+ \sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k+1}(t_i^\rho - t_{i-1}^\rho)^{\alpha_{i-1}-1}}{\rho^{\beta_k+\alpha_{i-1}}\Gamma(\beta_k+2)\Gamma(\alpha_{i-1})}\Bigg\},\end{aligned} \tag{3.3}$$

$$\Lambda_3 = p + \frac{1}{|\Omega|}\sum_{k=1}^{p} k\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)}, \tag{3.4}$$

and

$$\begin{aligned}\Lambda_4 = (2p-1)\frac{T^\rho}{\rho} + \frac{1}{|\Omega|}\Bigg\{&\sum_{k=2}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}(t_k^\rho - t_i^\rho)}{\rho^{\beta_k+1}\Gamma(\beta_k+1)} \\ &+ \sum_{k=1}^{p} k\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k+1}}{\rho^{\beta_k+1}\Gamma(\beta_k+2)}\Bigg\}.\end{aligned} \tag{3.5}$$

Our first existence result for problem (1.1) relies on Krasnoselskii's fixed point theorem [33], which is stated below.

Lemma 3.1 (Krasnoselskii's fixed point theorem) *Suppose $\mathcal{S}$ is a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that* (i) *$Ax + By \in \mathcal{S}$ whenever $x, y \in \mathcal{S}$*; (ii) *A is compact and continuous*; *and* (iii) *B is a contraction mapping. Then there exists $w \in \mathcal{S}$ such that $w = Aw + Bw$.*

Theorem 3.2 *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function, and $S_k, S_k^* \in C(\mathbb{R},\mathbb{R})$. Assume there exist positive constants L_2, L_3, M_2, M_3 such that the following conditions hold*:

(H_1) $|S_k(x) - S_k(y)| \leq L_2|x-y|$, $|S_k^*(x) - S_k^*(y)| \leq L_3|x-y|$ *with* $\|S_k(x)\| \leq M_2$, $\|S_k^*(x)\| \leq M_3$, $\forall x, y \in \mathbb{R}$, $k = 1, 2, \ldots, p$;

(H_2) $|f(t,y)| \leq \phi(t)$, $\forall (t,y) \in [0,T] \times \mathbb{R}$, *and* $\phi \in C([0,T],\mathbb{R}^+)$.

Then problem (1.1) *has at least one solution on J, provided that*

$$L_2\Lambda_3 + L_3\Lambda_4 < 1. \tag{3.6}$$

Proof Consider $B_r = \{y \in \mathcal{G} : \|y\| \le r\}$ with $r > \|\phi\|(\Lambda_1 + \Lambda_2) + M_2\Lambda_3 + M_3\Lambda_4 + \frac{|\eta|}{|\Omega|}$, $\|\phi\| = \sup_{t\in[0,T]} |\phi(t)|$ and define operators $\mathcal{P}$ and $\mathcal{Q}$ on B_r as follows:

$$(\mathcal{P}y)(t) = \begin{cases} {}^\rho I_{0^+}^{\alpha_0} f(t,y(t)) + \frac{1}{\Omega}\{\sum_{k=0}^{p} \lambda_k {}^\rho I_{t_k^+}^{\alpha_k+\beta_k} f(\xi_k, y(\xi_k)) \\ \quad + \sum_{k=1}^{p}\sum_{i=1}^{k} \frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)} {}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}} f(t_i, y(t_i))) \\ \quad + \sum_{k=2}^{p}\sum_{i=1}^{k-1} \frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}(t_k^\rho - t_i^\rho)}{\rho^{\beta_k+1}\Gamma(\beta_k+1)} {}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}-1} f(t_i, y(t_i))) \\ \quad + \sum_{k=1}^{p}\sum_{i=1}^{k} \frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k+1}}{\rho^{\beta_k+1}\Gamma(\beta_k+2)} {}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}-1} f(t_i, y(t_i))\}, \quad t \in J_0, \\ {}^\rho I_{t_k^+}^{\alpha_k} f(t,y(t)) + \sum_{i=1}^{k} {}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}} f(t_i, y(t_i)) + \sum_{i=1}^{k-1}(\frac{t_k^\rho - t_i^\rho}{\rho}) {}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}-1} f(t_i, y(t_i)) \\ \quad + \sum_{i=1}^{k}(\frac{t^\rho - t_k^\rho}{\rho}) {}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}-1} f(t_i, y(t_i)) + \frac{1}{\Omega}\{\sum_{k=0}^{p} \lambda_k {}^\rho I_{t_k^+}^{\alpha_k+\beta_k} f(\xi_k, y(\xi_k)) \\ \quad + \sum_{k=1}^{p}\sum_{i=1}^{k} \frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)} {}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}} f(t_i, y(t_i))) \\ \quad + \sum_{k=2}^{p}\sum_{i=1}^{k-1} \frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}(t_k^\rho - t_i^\rho)}{\rho^{\beta_k+1}\Gamma(\beta_k+1)} {}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}-1} f(t_i, y(t_i))) \\ \quad + \sum_{k=1}^{p}\sum_{i=1}^{k} \frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k+1}}{\rho^{\beta_k+1}\Gamma(\beta_k+2)} {}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}-1} f(t_i, y(t_i))\}, \quad t \in J_k, k = 1,2,\ldots,p, \end{cases}$$

and

$$(\mathcal{Q}y)(t) = \begin{cases} \frac{1}{\Omega}\{\sum_{k=1}^{p}\sum_{i=1}^{k} \frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)} S_i(y(t_i)) + \sum_{k=2}^{p}\sum_{i=1}^{k-1} \frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}(t_k^\rho - t_i^\rho)}{\rho^{\beta_k+1}\Gamma(\beta_k+1)} S_i^*(y(t_i)) \\ \quad + \sum_{k=1}^{p}\sum_{i=1}^{k} \frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k+1}}{\rho^{\beta_k+1}\Gamma(\beta_k+2)} S_i^*(y(t_i)) + \eta\}, \quad t \in J_0, \\ \sum_{i=1}^{k} S_i(y(t_i)) + \sum_{i=1}^{k-1}(\frac{t_k^\rho - t_i^\rho}{\rho}) S_i^*(y(t_i)) + \sum_{i=1}^{k}(\frac{t^\rho - t_k^\rho}{\rho}) S_i^*(y(t_i)) \\ \quad + \frac{1}{\Omega}\{\sum_{k=1}^{p}\sum_{i=1}^{k} \frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)} S_i(y(t_i)) + \sum_{k=2}^{p}\sum_{i=1}^{k-1} \frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}(t_k^\rho - t_i^\rho)}{\rho^{\beta_k+1}\Gamma(\beta_k+1)} S_i^*(y(t_i)) \\ \quad + \sum_{k=1}^{p}\sum_{i=1}^{k} \frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k+1}}{\rho^{\beta_k+1}\Gamma(\beta_k+2)} S_i^*(y(t_i)) + \eta\}, \quad t \in J_k, k = 1,2,\ldots,p. \end{cases}$$

Observe that $\mathcal{P} + \mathcal{Q} = F$, where the operator $F : \mathcal{G} \to \mathcal{G}$ is defined by (3.1). For $x, y \in B_r$ and $t \in J_0$, we have

$$\begin{aligned} &\|\mathcal{P}x + \mathcal{Q}y\| \\ &\quad \le \sup_{t\in J}\Bigg\{ {}^\rho I_{0^+}^{\alpha_0} |f(t,x(t))| + \frac{1}{|\Omega|}\Bigg\{\sum_{k=0}^{p} \lambda_k {}^\rho I_{t_k^+}^{\alpha_k+\beta_k} |f(\xi_k, x(\xi_k))| \\ &\qquad + \sum_{k=1}^{p}\sum_{i=1}^{k} \frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)}\big[{}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}} |f(t_i, x(t_i))| + |S_i(y(t_i))|\big] \\ &\qquad + \sum_{k=2}^{p}\sum_{i=1}^{k-1} \frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}(t_k^\rho - t_i^\rho)}{\rho^{\beta_k+1}\Gamma(\beta_k+1)}\big[{}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}-1} |f(t_i, x(t_i))| + |S_i^*(y(t_i))|\big] \\ &\qquad + \sum_{k=1}^{p}\sum_{i=1}^{k} \frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k+1}}{\rho^{\beta_k+1}\Gamma(\beta_k+2)}\big[{}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}-1} |f(t_i, x(t_i))| + |S_i^*(y(t_i))|\big] + |\eta|\Bigg\}\Bigg\} \\ &\quad \le \|\phi\|(\Lambda_1 + \Lambda_2) + M_2\Lambda_3 + M_3\Lambda_4 + \frac{|\eta|}{|\Omega|} < r. \end{aligned}$$

Next, for $x, y \in B_r$ and $t \in J_k, k = 1,2,\ldots,p$, we obtain

$$\|\mathcal{P}x + \mathcal{Q}y\|$$

$$
\begin{aligned}
&\leq \sup_{t\in J}\Biggl\{ {}^{\rho}I_{t_k^+}^{\alpha_k}\bigl|f\bigl(t,x(t)\bigr)\bigr| + \sum_{i=1}^{k}\bigl[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}}\bigl|f\bigl(t_i,x(t_i)\bigr)\bigr| + \bigl|S_i\bigl(y(t_i)\bigr)\bigr|\bigr] \\
&\quad + \sum_{i=1}^{k-1}\biggl(\frac{t_k^\rho - t_i^\rho}{\rho}\biggr)\bigl[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1}\bigl|f\bigl(t_i,x(t_i)\bigr)\bigr| + \bigl|S_i^*\bigl(y(t_i)\bigr)\bigr|\bigr] \\
&\quad + \sum_{i=1}^{k}\biggl(\frac{t^\rho - t_k^\rho}{\rho}\biggr)\bigl[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1}\bigl|f\bigl(t_i,x(t_i)\bigr)\bigr| \\
&\quad + \bigl|S_i^*\bigl(y(t_i)\bigr)\bigr|\bigr] + \frac{1}{|\Omega|}\Biggl\{\sum_{k=0}^{p}\lambda_k{}^{\rho}I_{t_k^+}^{\alpha_k+\beta_k}\bigl|f\bigl(\xi_k,x(\xi_k)\bigr)\bigr| \\
&\quad + \sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)}\bigl[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}}\bigl|f\bigl(t_i,x(t_i)\bigr)\bigr| + \bigl|S_i\bigl(y(t_i)\bigr)\bigr|\bigr] \\
&\quad + \sum_{k=2}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}(t_k^\rho - t_i^\rho)}{\rho^{\beta_k+1}\Gamma(\beta_k+1)}\bigl[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1}\bigl|f\bigl(t_i,x(t_i)\bigr)\bigr| + \bigl|S_i^*\bigl(y(t_i)\bigr)\bigr|\bigr] \\
&\quad + \sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k+1}}{\rho^{\beta_k+1}\Gamma(\beta_k+2)}\bigl[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1}\bigl|f\bigl(t_i,x(t_i)\bigr)\bigr| + \bigl|S_i^*\bigl(y(t_i)\bigr)\bigr|\bigr] + |\eta|\Biggr\}\Biggr\} \\
&\leq \|\phi\|(\Lambda_1+\Lambda_2) + M_2\Lambda_3 + M_3\Lambda_4 + \frac{|\eta|}{|\Omega|} < r.
\end{aligned}
$$

Thus, $\mathcal{P}x + \mathcal{Q}y \in B_r$. It follows from the assumptions (H_1) and (3.6) that $\mathcal{Q}$ is a contraction, that is, for $x, y \in B_r$ and $t \in J_0$, we have

$$
\begin{aligned}
&\|\mathcal{Q}y - \mathcal{Q}z\| \\
&\quad\leq \sup_{t\in J}\Biggl\{\frac{1}{|\Omega|}\Biggl\{\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)}\bigl|S_i\bigl(y(t_i)\bigr) - S_i\bigl(z(t_i)\bigr)\bigr| \\
&\qquad + \sum_{k=2}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}(t_k^\rho - t_i^\rho)}{\rho^{\beta_k+1}\Gamma(\beta_k+1)} \\
&\qquad \times \bigl|S_i^*\bigl(y(t_i)\bigr) - S_i^*\bigl(z(t_i)\bigr)\bigr| + \sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k+1}}{\rho^{\beta_k+1}\Gamma(\beta_k+2)}\bigl|S_i^*\bigl(y(t_i)\bigr) - S^*\bigl(z(t_i)\bigr)\bigr|\Biggr\}\Biggr\} \\
&\quad\leq L_2\|y-z\|\frac{1}{|\Omega|}\Biggl\{\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)}\Biggr\} \\
&\qquad + L_3\|y-z\|\frac{1}{|\Omega|}\Biggl\{\sum_{k=2}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}(t_k^\rho - t_i^\rho)}{\rho^{\beta_k+1}\Gamma(\beta_k+1)} \\
&\qquad + \sum_{k=1}^{p}k\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k+1}}{\rho^{\beta_k+1}\Gamma(\beta_k+2)}\Biggr\} \\
&\quad\leq \{L_2\Lambda_3 + L_3\Lambda_4\}\|x-y\|.
\end{aligned}
$$

Similarly, for $x, y \in B_r$ and $t \in J_k$, one can obtain

$$
\|\mathcal{Q}y - \mathcal{Q}z\|
$$

$$\begin{aligned}
&\leq \sup_{t\in J}\Bigg\{\sum_{i=1}^{k}\big|S_i\big(y(t_i)\big)-S_i\big(z(t_i)\big)\big| + \sum_{i=1}^{k-1}\bigg(\frac{t_k^\rho - t_i^\rho}{\rho}\bigg)\big|S_i^*\big(y(t_i)\big)-S_i^*\big(z(t_i)\big)\big| \\
&\quad + \sum_{i=1}^{k}\bigg(\frac{t^\rho - t_k^\rho}{\rho}\bigg)\big|S_i^*\big(y(t_i)\big)-S_i^*\big(z(t_i)\big)\big| \\
&\quad + \frac{1}{|\Omega|}\Bigg\{\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)}\big|S_i\big(y(t_i)\big)-S_i\big(z(t_i)\big)\big| \\
&\quad + \sum_{k=2}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}(t_k^\rho - t_i^\rho)}{\rho^{\beta_k+1}\Gamma(\beta_k+1)}\big|S_i^*\big(y(t_i)\big)-S_i^*\big(z(t_i)\big)\big| \\
&\quad + \sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k+1}}{\rho^{\beta_k+1}\Gamma(\beta_k+2)}\big|S_i^*\big(y(t_i)\big)-S^*\big(z(t_i)\big)\big|\Bigg\}\Bigg\} \\
&\leq L_2\|y-z\|\Bigg\{p+\frac{1}{|\Omega|}\Bigg\{\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)}\Bigg\}\Bigg\} + L_3\|y-z\|\{(2p-1)\frac{T^\rho}{\rho} \\
&\quad + \frac{1}{|\Omega|}\Bigg\{\sum_{k=2}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}(t_k^\rho - t_i^\rho)}{\rho^{\beta_k+1}\Gamma(\beta_k+1)} + \sum_{k=1}^{p}k\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k+1}}{\rho^{\beta_k+1}\Gamma(\beta_k+2)}\Bigg\} \\
&\leq \{L_2\Lambda_3 + L_3\Lambda_4\}\|x-y\|.
\end{aligned}$$

Continuity of f implies that operator $\mathcal{P}$ is continuous. Also, $\mathcal{P}$ is uniformly bounded on B_r as

$$\|\mathcal{P}y\| \leq \|\phi\|(\Lambda_1 + \Lambda_2).$$

In order to prove the compactness of operator $\mathcal{P}$, let $\sup_{(t,y)\in J\times B_r}|f(t,y)| = \bar{f} < \infty$. Then, for $\tau_1,\tau_2 \in J_0$ with $\tau_1 < \tau_2$, we have

$$\begin{aligned}
&\big|(\mathcal{P}y)(\tau_2)-(\mathcal{P}y)(\tau_1)\big| \\
&= \Bigg|\frac{\rho^{1-\alpha_0}}{\Gamma(\alpha_0)}\bigg[\int_0^{\tau_1}s^{\rho-1}\big[\big(\tau_2^\rho - s^\rho\big)^{\alpha_0-1} - \big(\tau_1^\rho - s^\rho\big)^{\alpha_0-1}\big]f\big(s,y(s)\big)\,ds \\
&\quad + \int_{\tau_1}^{\tau_2}s^{\rho-1}\big(\tau_2^\rho - s^\rho\big)^{\alpha_0-1}f\big(s,y(s)\big)\,ds\bigg]\Bigg| \\
&\leq \frac{\bar{f}}{\rho^{\alpha_0}\Gamma(\alpha_0+1)}\big\{2\big(\tau_2^\rho - \tau_1^\rho\big)^{\alpha_0} + \big|\tau_2^{\rho\alpha_0} - \tau_1^{\rho\alpha_0}\big|\big\}.
\end{aligned}$$

Also, for $\tau_1,\tau_2 \in J_k, k = 1,2,\ldots,p$ $(\tau_1 < \tau_2)$, we get

$$\begin{aligned}
&\big|(\mathcal{P}y)(\tau_2)-(\mathcal{P}y)(\tau_1)\big| \\
&= \Bigg|\frac{\rho^{1-\alpha_k}}{\Gamma(\alpha_k)}\bigg[\int_{t_k}^{\tau_1}s^{\rho-1}\big[\big(\tau_2^\rho - s^\rho\big)^{\alpha_k-1} - \big(\tau_1^\rho - s^\rho\big)^{\alpha_k-1}\big]f\big(s,y(s)\big)\,ds \\
&\quad + \int_{\tau_1}^{\tau_2}s^{\rho-1}\big(\tau_2^\rho - s^\rho\big)^{\alpha_k-1}f\big(s,y(s)\big)\,ds\bigg]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{k}\left(\frac{(\tau_2^\rho - t_k^\rho)-(\tau_1^\rho - t_k^\rho)}{\rho}\right)\left(\frac{\rho^{2-\alpha_i}}{\Gamma(\alpha_i-1)}\int_{t_{i-1}}^{t_i} s^{\rho-1}\left(t_i^\rho - s^\rho\right)^{\alpha_i-2} f\left(s,y(s)\right)ds\right)\Bigg| \\
&\le \frac{\bar{f}}{\rho^{\alpha_k}\Gamma(\alpha_k+1)}\left\{2\left(\tau_2^\rho-\tau_1^\rho\right)^{\alpha_k}+\left|\left(\tau_2^\rho-t_k^\rho\right)^{\alpha_k}-\left(\tau_1^\rho-t_k^\rho\right)^{\alpha_k}\right|\right\} \\
&+\bar{f}\sum_{i=1}^{k}\left(\frac{[(\tau_2^\rho-t_k^\rho)-(\tau_1^\rho-t_k^\rho)](t_i^\rho-t_{i-1}^\rho)^{\alpha_i-1}}{\rho^{\alpha_i}\Gamma(\alpha_i)}\right).
\end{aligned}
$$

From the above inequalities, it follows that $|(\mathcal{P}y)(\tau_2) - (\mathcal{P}y)(\tau_1)| \to 0$ as $\tau_2 - \tau_1 \to 0, \forall \tau_1, \tau_2 \in J_k, k = 0,1,\ldots,p$, independent of y. Thus, $\mathcal{P}$ is equicontinuous. So $\mathcal{P}$ is relatively compact on B_r. Hence, by the Arzelá–Ascoli theorem, $\mathcal{P}$ is compact on B_r. Thus all the assumptions of Lemma 3.1 are satisfied. Hence the conclusion of Lemma 3.1 applies, and so the boundary value problem (1.1) has at least one solution on J. □

In the following result, we establish the uniqueness of solutions for problem (1.1) with the aid of the contraction mapping principle.

Theorem 3.3 *Suppose $f \in C(J \times \mathbb{R}, \mathbb{R})$, assumption (H_1) holds, and the following condition is satisfied*:

(H_3) *there exists a positive constant L_1 such that*

$$
\left|f(t,x) - f(t,y)\right| \le L_1|x-y|, \quad \textit{for } t \in J \textit{ and every } x,y \in \mathbb{R}.
$$

Then there exists a unique solution for problem (1.1) *on J if*

$$
L_1(\Lambda_1 + \Lambda_2) + L_2\Lambda_3 + L_3\Lambda_4 < 1, \tag{3.7}
$$

where Λ_1, Λ_2, Λ_3, and Λ_4 are given by (3.2), (3.3), (3.4), *and* (3.5), *respectively.*

Proof Setting $\sup_{t\in J}|f(t,0)| = M_1$, we consider the set $B_r = \{y \in \mathcal{G} : \|y\| \le r\}$ with

$$
r > \frac{M_1(\Lambda_1+\Lambda_2) + M_2\Lambda_3 + M_3\Lambda_4 + \frac{|\eta|}{|\Omega|}}{1 - L_1(\Lambda_1+\Lambda_2)},
$$

and show that $FB_r \subset B_r$. For $y \in B_r$ and $t \in J_0$, we have

$$
\begin{aligned}
\left|(Fy)(t)\right| &= \left|{}^\rho I_{0^+}^{\alpha_0} f\left(t,y(t)\right) + A\right| \\
&\le {}^\rho I_{0^+}^{\alpha_0}\left[\left|f\left(t,y(t)\right) - f(t,0)\right| + \left|f(t,0)\right|\right] \\
&+ \frac{1}{|\Omega|}\left\{\sum_{k=0}^{p} \lambda_k {}^\rho I_{t_k^+}^{\alpha_k+\beta_k}\left[\left|f\left(\xi_k, y(\xi_k)\right) - f(\xi_k,0)\right| + \left|f(\xi_k,0)\right|\right]\right. \\
&+ \sum_{k=1}^{p}\sum_{i=1}^{k} \frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)} \\
&\times \left[{}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}}\left[\left|f\left(t_i, y(t_i)\right) - f(t_i,0)\right| + \left|f(t_i,0)\right|\right] + \left|S_i\left(y(t_i)\right)\right|\right] \\
&+ \sum_{k=2}^{p}\sum_{i=1}^{k-1} \frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}(t_k^\rho - t_i^\rho)}{\rho^{\beta_k+1}\Gamma(\beta_k+1)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big[{}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}-1}\big[\big|f(t_i,y(t_i))-f(t_i,0)\big|+\big|f(t_i,0)\big|\big]+\big|S_i^*(y(t_i)\big|\Big] \\
&+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho-t_k^\rho)^{\beta_k+1}}{\rho^{\beta_k+1}\Gamma(\beta_k+2)} \\
&\times \Big[{}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}-1}\big[\big|f(t_i,y(t_i))-f(t_i,0)\big|+\big|f(t_i,0)\big|\big]+\big|S_i^*(y(t_i))\big|\Big]+|\eta|\Bigg\} \\
\le{}& \big(L_1\|r\|+M_1\|\big)\Bigg\{\frac{t_1^{\rho\alpha_0}}{\rho^{\alpha_0}\Gamma(\alpha_0+1)}+\frac{1}{|\Omega|}\Bigg\{\sum_{k=0}^{p}\frac{\lambda_k(\xi_k^\rho-t_k^\rho)^{\alpha_k+\beta_k}}{\rho^{\alpha_k+\beta_k}\Gamma(\alpha_k+\beta_k+1)} \\
&+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho-t_k^\rho)^{\beta_k}(t_i^\rho-t_{i-1}^\rho)^{\alpha_{i-1}}}{\rho^{\beta_k+\alpha_{i-1}}\Gamma(\beta_k+1)\Gamma(\alpha_{i-1}+1)} \\
&+\sum_{k=2}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\xi_k^\rho-t_k^\rho)^{\beta_k}(t_k^\rho-t_i^\rho)(t_i^\rho-t_{i-1}^\rho)^{\alpha_{i-1}-1}}{\rho^{\beta_k+\alpha_{i-1}}\Gamma(\beta_k+1)\Gamma(\alpha_{i-1})} \\
&+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho-t_k^\rho)^{\beta_k+1}(t_i^\rho-t_{i-1}^\rho)^{\alpha_{i-1}-1}}{\rho^{\beta_k+\alpha_{i-1}}\Gamma(\beta_k+2)\Gamma(\alpha_{i-1})}\Bigg\}\Bigg\}+\frac{M_2}{|\Omega|}\Bigg\{\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho-t_k^\rho)^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)}\Bigg\} \\
&+\frac{M_3}{|\Omega|}\Bigg\{\sum_{k=2}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\xi_k^\rho-t_k^\rho)^{\beta_k}(t_k^\rho-t_i^\rho)}{\rho^{\beta_k+1}\Gamma(\beta_k+1)}+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho-t_k^\rho)^{\beta_k+1}}{\rho^{\beta_k+1}\Gamma(\beta_k+2)}\Bigg\}+\frac{|\eta|}{|\Omega|} \\
\le{}& \big(L_1\|r\|+M_1\|\big)\{\Lambda_1+\Lambda_2\}+M_2\Lambda_3+M_3\Lambda_4+\frac{|\eta|}{|\Omega|}<r,
\end{aligned}
$$

which, upon taking norm for $t\in J_0$, implies that $\|(Fy)\|<r$. For $y\in B_r$ and $t\in J_k$, we have

$$
\begin{aligned}
&\big|(Fy)(t)\big| \\
&=\Bigg|{}^\rho I_{t_k^+}^{\alpha_k}f(t,y(t))+\sum_{i=1}^{k}\Big[{}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}}f(t_i,y(t_i))+S_i(y(t_i))\Big] \\
&\quad+\sum_{i=1}^{k-1}\left(\frac{t_k^\rho-t_i^\rho}{\rho}\right)\Big[{}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}-1}f(t_i,y(t_i))+S_i^*(y(t_i))\Big] \\
&\quad+\sum_{i=1}^{k}\left(\frac{t^\rho-t_k^\rho}{\rho}\right)\Big[{}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}-1}f(t_i,y(t_i))+S_i^*(y(t_i))\Big]+A\Bigg| \\
&\le {}^\rho I_{t_k^+}^{\alpha_k}\big[\big|f(t,y(t))-f(t,0)\big|+\big|f(t,0)\big|\big] \\
&\quad+\sum_{i=1}^{k}\Big[{}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}}\big[\big|f(t_i,y(t_i))-f(t_i,0)\big|+\big|f(t_i,0)\big|\big]+\big|S_i(y(t_i))\big|\Big] \\
&\quad+\sum_{i=1}^{k-1}\left(\frac{t_k^\rho-t_i^\rho}{\rho}\right) \\
&\quad\times\Big[{}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}-1}\big[\big|f(t_i,y(t_i))-f(t_i,0)\big|+\big|f(t_i,0)\big|\big]+\big|S_i^*(y(t_i))\big|\Big] \\
&\quad+\sum_{i=1}^{k}\left(\frac{t^\rho-t_k^\rho}{\rho}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times \bigl[{}^{\rho}I_{t_{i-1}^{+}}^{\alpha_{i-1}-1}\bigl[\bigl|f\bigl(t_i,y(t_i)\bigr)-f(t_i,0)\bigr|+\bigl|f(t_i,0)\bigr|\bigr]+\bigl|S_i^{*}\bigl(y(t_i)\bigr)\bigr|\bigr]\\
&+\frac{1}{|\Omega|}\Biggl\{\sum_{k=0}^{p}\lambda_k{}^{\rho}I_{t_k^{+}}^{\alpha_k+\beta_k}\bigl[\bigl|f\bigl(\xi_k,y(\xi_k)\bigr)-f(\xi_k,0)\bigr|+\bigl|f(\xi_k,0)\bigr|\bigr]\\
&+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^{\rho}-t_k^{\rho})^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)}\bigl[{}^{\rho}I_{t_{i-1}^{+}}^{\alpha_{i-1}}\bigl[\bigl|f\bigl(t_i,y(t_i)\bigr)-f(t_i,0)\bigr|+\bigl|f(t_i,0)\bigr|\bigr]+\bigl|S_i(y(t_i)\bigr|\bigr]\\
&+\sum_{k=2}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\xi_k^{\rho}-t_k^{\rho})^{\beta_k}(t_k^{\rho}-t_i^{\rho})}{\rho^{\beta_k+1}\Gamma(\beta_k+1)}\\
&\times \bigl[{}^{\rho}I_{t_{i-1}^{+}}^{\alpha_{i-1}-1}\bigl[\bigl|f\bigl(t_i,y(t_i)\bigr)-f(t_i,0)\bigr|+\bigl|f(t_i,0)\bigr|\bigr]+\bigl|S_i^{*}(y(t_i)\bigr|\bigr]\\
&+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^{\rho}-t_k^{\rho})^{\beta_k+1}}{\rho^{\beta_k+1}\Gamma(\beta_k+2)}\\
&\times \bigl[{}^{\rho}I_{t_{i-1}^{+}}^{\alpha_{i-1}-1}\bigl[\bigl|f\bigl(t_i,y(t_i)\bigr)-f(t_i,0)\bigr|+\bigl|f(t_i,0)\bigr|\bigr]+\bigl|S_i^{*}\bigl(y(t_i)\bigr)\bigr|\bigr]+|\eta|\Biggr\}\\
\le{}& \bigl(L_1\|r\|+M_1\|\bigr)\Biggl\{\frac{(t_{k+1}^{\rho}-t_k^{\rho})^{\alpha_k}}{\rho^{\alpha_k}\Gamma(\alpha_k+1)}+\sum_{i=1}^{k}\frac{(t_i^{\rho}-t_{i-1}^{\rho})^{\alpha_{i-1}}}{\rho^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}\\
&+\sum_{i=1}^{k-1}\biggl(\frac{t_k^{\rho}-t_i^{\rho}}{\rho}\biggr)\biggl(\frac{(t_i^{\rho}-t_{i-1}^{\rho})^{\alpha_{i-1}-1}}{\rho^{\alpha_{i-1}-1}\Gamma(\alpha_{i-1})}\biggr)\\
&+\sum_{i=1}^{k}\biggl(\frac{t_{k+1}^{\rho}-t_k^{\rho}}{\rho}\biggr)\biggl(\frac{(t_i^{\rho}-t_{i-1}^{\rho})^{\alpha_{i-1}-1}}{\rho^{\alpha_{i-1}-1}\Gamma(\alpha_{i-1})}\biggr)+\frac{1}{|\Omega|}\Biggl\{\sum_{k=0}^{p}\frac{\lambda_k(\xi_k^{\rho}-t_k^{\rho})^{\alpha_k+\beta_k}}{\rho^{\alpha_k+\beta_k}\Gamma(\alpha_k+\beta_k+1)}\\
&+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^{\rho}-t_k^{\rho})^{\beta_k}(t_i^{\rho}-t_{i-1}^{\rho})^{\alpha_{i-1}}}{\rho^{\beta_k+\alpha_{i-1}}\Gamma(\beta_k+1)\Gamma(\alpha_{i-1}+1)}\\
&+\sum_{k=2}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\xi_k^{\rho}-t_k^{\rho})^{\beta_k}(t_k^{\rho}-t_i^{\rho})(t_i^{\rho}-t_{i-1}^{\rho})^{\alpha_{i-1}-1}}{\rho^{\beta_k+\alpha_{i-1}}\Gamma(\beta_k+1)\Gamma(\alpha_{i-1})}\\
&+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^{\rho}-t_k^{\rho})^{\beta_k+1}(t_i^{\rho}-t_{i-1}^{\rho})^{\alpha_{i-1}-1}}{\rho^{\beta_k+\alpha_{i-1}}\Gamma(\beta_k+2)\Gamma(\alpha_{i-1})}\Biggr\}\Biggr\}\\
&+M_2\Biggl\{k+\frac{1}{|\Omega|}\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^{\rho}-t_k^{\rho})^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)}\Biggr\}\\
&+M_3\Biggl[\sum_{i=1}^{k-1}\biggl(\frac{t_k^{\rho}-t_i^{\rho}}{\rho}\biggr)+\sum_{i=1}^{k}\biggl(\frac{t_{k+1}^{\rho}-t_k^{\rho}}{\rho}\biggr)+\frac{1}{|\Omega|}\Biggl\{\sum_{k=2}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\xi_k^{\rho}-t_k^{\rho})^{\beta_k}(t_k^{\rho}-t_i^{\rho})}{\rho^{\beta_k+1}\Gamma(\beta_k+1)}\\
&+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^{\rho}-t_k^{\rho})^{\beta_k+1}}{\rho^{\beta_k+1}\Gamma(\beta_k+2)}\Biggr\}\Biggr]+\frac{|\eta|}{|\Omega|}\\
\le{}& \bigl(L_1\|r\|+M_1\|\bigr)\{\Lambda_1+\Lambda_2\}+M_2\Lambda_3+M_3\Lambda_4+\frac{|\eta|}{|\Omega|}<r.
\end{aligned}
$$

Consequently, we get $\|Fy\|<r$ for $t\in J_k, k=0,1,\dots,p$. Thus $FB_r\subset B_r$.

Now, for $y, z \in \mathcal{G}$ and $t \in J_0$, we have

$$\begin{aligned}
&\bigl|(Fy)(t) - (Fz)(t)\bigr| \\
&\quad \le {}^{\rho}I_{0^+}^{\alpha_0}\bigl|f\bigl(t, y(t)\bigr) - f\bigl(t, z(t)\bigr)\bigr| + \frac{1}{|\Omega|}\Biggl\{\sum_{k=0}^{p}\lambda_k {}^{\rho}I_{t_k^+}^{\alpha_k+\beta_k}\bigl|f\bigl(\xi_k, y(\xi_k)\bigr) - f\bigl(\xi_k, z(\xi_i)\bigr)\bigr| \\
&\qquad + \sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)}\bigl[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}}\bigl|f\bigl(t_i, y(t_i)\bigr) - f\bigl(t_i, z(\xi_i)\bigr)\bigr| + \bigl|S_i\bigl(y(t_i)\bigr) - S_i\bigl(z(t_i)\bigr)\bigr|\bigr] \\
&\qquad + \sum_{k=2}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}(t_k^\rho - t_i^\rho)}{\rho^{\beta_k+1}\Gamma(\beta_k+1)} \\
&\qquad \times \bigl[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1}\bigl|f\bigl(t_i, y(t_i)\bigr) - f\bigl(t_i, z(t_i)\bigr)\bigr| + \bigl|S_i^*\bigl(y(t_i)\bigr) - S_i^*\bigl(z(t_i)\bigr)\bigr|\bigr] \\
&\qquad + \sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k+1}}{\rho^{\beta_k+1}\Gamma(\beta_k+2)} \\
&\qquad \times \bigl[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1}\bigl|f\bigl(t_i, y(t_i)\bigr) - f\bigl(t_i, z(t_i)\bigr)\bigr| + \bigl|S_i^*\bigl(y(t_i)\bigr) - S^*\bigl(z(t_i)\bigr)\bigr|\bigr]\Biggr\} \\
&\quad \le \bigl\{L_1(\Lambda_1 + \Lambda_2) + L_2\Lambda_3 + L_3\Lambda_4\bigr\}\|x - y\|.
\end{aligned}$$

In a similar way for $t \in J_k$, we obtain

$$\begin{aligned}
&\bigl|(Fy)(t) - (Fz)(t)\bigr| \\
&\quad \le {}^{\rho}I_{t_k^+}^{\alpha_k}\bigl|f\bigl(t, y(t)\bigr) - f\bigl(t, z(t)\bigr)\bigr| \\
&\qquad + \sum_{i=1}^{k}\bigl[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}}\bigl|f\bigl(t_i, y(t_i)\bigr) - f\bigl(t_i, z(t_i)\bigr)\bigr| + \bigl|S_i\bigl(y(t_i)\bigr) - S_i\bigl(z(t_i)\bigr)\bigr|\bigr] \\
&\qquad + \sum_{i=1}^{k-1}\biggl(\frac{t_k^\rho - t_i^\rho}{\rho}\biggr)\bigl[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1}\bigl|f\bigl(t_i, y(t_i)\bigr) - f\bigl(t_i, z(t_i)\bigr)\bigr| + \bigl|S_i^*\bigl(y(t_i)\bigr) - S_i^*\bigl(z(t_i)\bigr)\bigr|\bigr] \\
&\qquad + \sum_{i=1}^{k}\biggl(\frac{t^\rho - t_k^\rho}{\rho}\biggr)\bigl[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1}\bigl|f\bigl(t_i, y(t_i)\bigr) - f\bigl(t_i, z(t_i)\bigr)\bigr| + \bigl|S_i^*\bigl(y(t_i)\bigr) - S_i^*\bigl(z(t_i)\bigr)\bigr|\bigr] \\
&\qquad + \frac{1}{|\Omega|}\Biggl\{\sum_{k=0}^{p}\lambda_k {}^{\rho}I_{t_k^+}^{\alpha_k+\beta_k}\bigl|f\bigl(\xi_k, y(\xi_k)\bigr) - f\bigl(\xi_k, z(\xi_i)\bigr)\bigr| \\
&\qquad + \sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)}\bigl[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}}\bigl|f\bigl(t_i, y(t_i)\bigr) - f\bigl(t_i, z(\xi_i)\bigr)\bigr| + \bigl|S_i\bigl(y(t_i)\bigr) - S_i\bigl(z(t_i)\bigr)\bigr|\bigr] \\
&\qquad + \sum_{k=2}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}(t_k^\rho - t_i^\rho)}{\rho^{\beta_k+1}\Gamma(\beta_k+1)} \\
&\qquad \times \bigl[{}^{\rho}I_{t_{i-1}^+}^{\alpha_{i-1}-1}\bigl|f\bigl(t_i, y(t_i)\bigr) - f\bigl(t_i, z(t_i)\bigr)\bigr| + \bigl|S_i^*\bigl(y(t_i)\bigr) - S_i^*\bigl(z(t_i)\bigr)\bigr|\bigr] \\
&\qquad + \sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k+1}}{\rho^{\beta_k+1}\Gamma(\beta_k+2)}
\end{aligned}$$

$$\times \Bigl[{}^{\rho}I^{\alpha_{i-1}-1}_{t^+_{i-1}}\bigl|f\bigl(t_i,y(t_i)\bigr)-f\bigl(t_i,z(t_i)\bigr)\bigr| + \bigl|S^*_i\bigl(y(t_i)\bigr)-S^*\bigl(z(t_i)\bigr)\bigr|\Bigr]\Biggr\}$$

$$\le \bigl\{L_1(\Lambda_1+\Lambda_2)+L_2\Lambda_3+L_3\Lambda_4\bigr\}\|y-z\|.$$

Consequently, we obtain

$$\|Fy-Fz\| \le \bigl\{L_1(\Lambda_1+\Lambda_2)+L_2\Lambda_3+L_3\Lambda_4\bigr\}\|y-z\|, \quad t\in J_k, k=0,1,2,\dots,p,$$

which, in view of (3.7), implies that F is a contraction. Thus the conclusion of the theorem follows by the contraction mapping principle. □

Example 3.4 With $\rho = 1/2, \alpha_0 = 5/4, \alpha_1 = 7/4, \beta_0 = 1/2, \beta_1 = 3/2, \lambda_0 = 1/3, \lambda_1 = 1/4, \xi_0 = 1/2, \xi_1 = 3/2, t_1 = 3/4$, we consider the problem

$$\begin{cases} {}^{\rho}_{c}D^{\alpha_k}_{t^+_k}y(t) = \frac{1}{(t+9)^2}\bigl(\frac{|y(t)|+2}{|y(t)|+1}+\cos t\bigr), & t\in[0,2], t\neq 3/4, k=0,1,\\ \triangle y(3/4) = \frac{|y(3/4)|}{12+|y(3/4)|}, \qquad \Delta\delta y(3/4) = \frac{|y(3/4)|}{9+|y(3/4)|}, & k=1,2,\dots,p,\\ u(0) = \sum_{k=0}^{1}\lambda_k{}^{\rho}I^{\beta_k}_{t^+_k}y(\xi_k)-1, \qquad \delta y(0)=0. \end{cases} \tag{3.8}$$

Using the given data, we find that $|\Omega| \approx 0.438425$, $\Lambda_1 \approx 12.512411$, $\Lambda_2 \approx 1.442181$, $\Lambda_3 \approx 1.260667$, $\Lambda_4 \approx 2.903232$, where Ω, Λ_1, Λ_2, Λ_3 and Λ_4 are given by (2.10), (3.2), (3.3), (3.4), and (3.5), respectively. Clearly, all the assumptions of Theorem 3.2 hold with $L_2 = 1/12, L_3 = 1/9, M_2 = M_3 = 1, \phi(t) = \frac{2+\cos t}{(t+9)^2}$, and $p = 1$. Also $L_2\Lambda_3 + L_3\Lambda_4 \approx 0.4276369325 < 1$. Therefore, by Theorem 3.2, we deduce that the impulsive integro-initial value problem (3.8) has at least one solution on $[0,2]$. Furthermore, the hypothesis of Theorem 3.3 is satisfied with $L_1 = 1/81, L_2 = 1/12, L_3 = 1/9, M_2 = M_3 = 1$. Moreover, $L_1(\Lambda_1 + \Lambda_2) + L_2\Lambda_3 + L_3\Lambda_4 \approx 0.599915849 < 1$. So, Theorem 3.3 implies that the impulsive integro-initial value problem (3.8) has a unique solution on $[0,2]$.

4 Extremal solutions

Here we discuss the existence of extremal solutions for problem (1.1). Before presenting the main result, we define lower and upper solutions for the problem at hand and prove a new comparison result.

Definition 4.1 Function $y(t)$ is said to be a lower solution of problem (1.1) if

$$\begin{cases} {}^{\rho}_{c}D^{\alpha_k}_{t^+_k}y(t) \le f(t,y(t)), & 1<\alpha_k\le 2, k=0,1,2,\dots,p, t\in J',\\ \triangle y(t_k) \le S_k(y(t_k)), \qquad \Delta\delta y(t_k) \le S^*_k(y(t_k)), & k=1,2,\dots,p,\\ y(0) \le \sum_{k=0}^{p}\lambda_k{}^{\rho}I^{\beta_k}_{t^+_k}y(\xi_k)+\eta, \qquad \delta y(0)=0, & t_k<\xi_k<t_{k+1}. \end{cases} \tag{4.1}$$

By reversing the inequalities in the above definition, we obtain the corresponding definition of an upper solution of (1.1).

Lemma 4.2 (Comparison result) *If* $\sum_{k=0}^{p} \frac{\lambda_k (\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k} \Gamma(\beta_k+1)} < 1$ *and* $y \in \mathcal{E} = PC_\delta^1(J,\mathbb{R}) \cap AC_\delta^2(J_k,\mathbb{R})$ *satisfies*

$$\begin{cases} {}^\rho_c D_{t_k^+}^{\alpha_k} y(t) \geq 0, \quad 0 < \alpha_k \leq 2, k = 0,1,2,\ldots,p, t \in J', \\ \triangle y(t_k) \geq 0, \quad \triangle \delta y(t_k) \geq 0, \quad k = 1,2,\ldots,p, \\ y(0) \geq \sum_{k=0}^{p} \lambda_k {}^\rho I_{t_k^+}^{\beta_k} y(\xi_k), \quad \delta y(0) = 0, \quad t_k < \xi_k < t_{k+1}, \end{cases} \tag{4.2}$$

then $y(t) \geq 0, \forall t \in J$.

Proof Consider a modified form of problem (2.11) given by

$$\begin{cases} {}^\rho_c D_{t_k^+}^{\alpha_k} y(t) = g(t), \quad 0 < \alpha_k \leq 2, k = 0,1,2,\ldots,p, t \in J', \\ \triangle y(t_k) = S_k, \quad \triangle \delta y(t_k) = S_k^*, \quad k = 1,2,\ldots,p, \\ y(0) = \sum_{k=0}^{p} \lambda_k {}^\rho I_{t_k^+}^{\beta_k} y(\xi_k) + \eta, \quad \delta y(0) = 0, \quad t_k < \xi_k < t_{k+1}, \end{cases} \tag{4.3}$$

where $g(t) \in C(J,\mathbb{R}^+)$ and S_k, S_k^* $(k = 1,2,\ldots,p), \eta$ are nonnegative constants.

Then the solution of problem (4.3) is

$$y(t) = \begin{cases} {}^\rho I_{0^+}^{\alpha_0} g(t) + \sigma, \quad t \in J_0, \\ {}^\rho I_{t_k^+}^{\alpha_k} g(t) + \sum_{i=1}^{k} [{}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}} g(t_i) + S_i] + \sum_{i=1}^{k-1} (\frac{t_k^\rho - t_i^\rho}{\rho}) [{}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}-1} g(t_i) + S_i^*] \\ \quad + \sum_{i=1}^{k} (\frac{t^\rho - t_k^\rho}{\rho}) [{}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}-1} g(t_i) + S_i^*] + \sigma, \\ \quad t \in J_k, k = 1,2,\ldots,p, \end{cases} \tag{4.4}$$

where

$$\begin{aligned} \sigma = \frac{1}{\Omega} \Biggl\{ & \sum_{k=0}^{p} \lambda_k {}^\rho I_{t_k^+}^{\alpha_k+\beta_k} g(\xi_k) + \sum_{k=1}^{p} \sum_{i=1}^{k} \frac{\lambda_k (\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k} \Gamma(\beta_k+1)} \bigl[{}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}} g(t_i) + S_i\bigr] \\ & + \sum_{k=2}^{p} \sum_{i=1}^{k-1} \frac{\lambda_k (\xi_k^\rho - t_k^\rho)^{\beta_k} (t_k^\rho - t_i^\rho)}{\rho^{\beta_k+1} \Gamma(\beta_k+1)} \bigl[{}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}-1} g(t_i) + S_i^*\bigr] \\ & + \sum_{k=1}^{p} \sum_{i=1}^{k} \frac{\lambda_k (\xi_k^\rho - t_k^\rho)^{\beta_k+1}}{\rho^{\beta_k+1} \Gamma(\beta_k+2)} \bigl[{}^\rho I_{t_{i-1}^+}^{\alpha_{i-1}-1} g(t_i) + S_i^*\bigr] + \eta \Biggr\}. \end{aligned} \tag{4.5}$$

In view of the nonnegative nature of the function $g(t)$ and constants S_k, S_k^*, η, the conclusion of Lemma 4.2 follows from (4.4). □

Our next result, dealing with the extremal solutions of (1.1), relies on the following fixed point theorem [34].

Lemma 4.3 *Let* $[a,b]$ *be a nonempty order interval of a subset* Y *of an ordered Banach space* X *and let* $P : [a,b] \to [a,b]$ *be a nondecreasing mapping. If each sequence* $\{Py_n\} \subset P([a,b])$ *converges whenever* $\{y_n\}$ *is a monotone sequence in* $[a,b]$*, then the sequence of* P*-iterates of* a *converges to the least fixed point* y_* *of* P *and the sequence of* P*-iterates of* b

converges to the greatest fixed point y^* *of* P. *Moreover,*

$$y_* = \min\{x \in [a,b] : x \geq Px\}, \qquad y^* = \max\{x \in [a,b] : x \leq Px\}.$$

Theorem 4.4 *Assume that*

(A_1) *the functions* $f(t,y)$, $S_k(y)$, $S_k^*(y), k = 1,\dots,p$, *are continuous and nondecreasing in* y;

(A_2) *there exist lower and upper solutions* y_0 *and* $z_0 \in \mathcal{E}$ *for problem* (1.1), *respectively, such that* $y_0 \leq z_0$;

(A_3) $\sum_{k=0}^{p} \frac{\lambda_k(\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k}\Gamma(\beta_k+1)} < 1$.

Then problem (1.1) *has extremal solutions in the sector* $[y_0, z_0]$.

Proof Consider problem (2.11) with $h(t) = f(t,v(t)), S_k = S_k(v(t_k))$ and $S_k^* = S_k^*(v(t_k)), k = 1,2,\dots,p$. Let us consider the operator F defined by (3.1) from $[y_0,z_0]$ to $\mathcal{E}$ such that $y(t) = Fv(t)$. First, it will be shown that F maps $[y_0,z_0]$ into $[y_0,z_0]$.

Let $y_1 = Fy_0, z_1 = Fz_0$. Then y_1, z_1 are well defined and respectively satisfy the problems

$$\begin{cases} {}^\rho_c D^{\alpha_k}_{t_k^+} y_1(t) = f(t, y_0(t)), \quad 1 < \alpha_k \leq 2, k = 0,1,2,\dots,p, t \in J', \\ \triangle y_1(t_k) = S_k(y_0(t_k)), \qquad \triangle \delta y_1(t_k) = S_k^*(y_0(t_k)), \quad k = 1,2,\dots,p, \\ y_1(0) = \sum_{k=0}^{p} \lambda_k {}^\rho I^{\beta_k}_{t_k^+} y_1(\xi_k) + \eta, \qquad \delta y_1(0) = 0, \quad t_k < \xi_k < t_{k+1} \end{cases} \tag{4.6}$$

and

$$\begin{cases} {}^\rho_c D^{\alpha_k}_{t_k^+} z_1(t) = f(t, z_0(t)), \quad 1 < \alpha_k \leq 2, k = 0,1,2,\dots,p, t \in J', \\ \triangle z_1(t_k) = S_k(z_0(t_k)), \qquad \triangle \delta z_1(t_k) = S_k^*(z_0(t_k)), \quad k = 1,2,\dots,p, \\ z_1(0) = \sum_{k=0}^{p} \lambda_k {}^\rho I^{\beta_k}_{t_k^+} z_1(\xi_k) + \eta, \qquad \delta z_1(0) = 0, \quad t_k < \xi_k < t_{k+1}. \end{cases} \tag{4.7}$$

Setting $u = y_1 - y_0$ and using the definition of a lower solution, we get

$$\begin{cases} {}^\rho_c D^{\alpha_k}_{t_k^+} u(t) \geq 0, \quad 1 < \alpha_k \leq 2, k = 0,1,2,\dots,p, t \in J', \\ \triangle u(t_k) \geq 0, \qquad \triangle \delta u(t_k) \geq 0, \quad k = 1,2,\dots,p, \\ u(0) \geq \sum_{k=0}^{p} \lambda_k {}^\rho I^{\beta_k}_{t_k^+} u(\xi_k), \qquad \delta u(0) = 0, \quad t_k < \xi_k < t_{k+1}, \end{cases} \tag{4.8}$$

which, by Lemma 4.2, implies that $u(t) \geq 0, \forall t \in J$. Thus $Fy_0 \geq y_0$. Similarly, using the definition of an upper solution, one can show that $Fz_0 \leq z_0$.

Now, we define $\omega = z_1 - y_1$ and use (4.6) and (4.7) together with assumption A_1 to obtain

$$\begin{cases} {}^\rho_c D^{\alpha_k}_{t_k^+} \omega(t) = f(t, z_0(t)) - f(t, y_0(t)) \geq 0, \quad 1 < \alpha_k \leq 2, k = 0,1,2,\dots,p, t \in J', \\ \triangle \omega(t_k) = S_k(z_0(t_k)) - S_k(y_0(t_k)) \geq 0, \quad k = 1,2,\dots,p, \\ \triangle \delta \omega(t_k) = S_k^*(z_0(t_k)) - S_k^*(y_0(t_k)) \geq 0, \quad k = 1,2,\dots,p, \\ \omega(0) = \sum_{k=0}^{p} \lambda_k {}^\rho I^{\beta_k}_{t_k^+} \omega(\xi_k), \qquad \delta \omega(0) = 0, \quad t_k < \xi_k < t_{k+1}. \end{cases} \tag{4.9}$$

Applying Lemma 4.2, we deduce that $\omega(t) \geq 0$, that is, $Fz_0 \geq Fy_0$. Thus F is nondecreasing and $y_0 \leq Fy \leq z_0$ for any $y \in [y_0,z_0]$. In consequence, $F[y_0,z_0] \subset [y_0,z_0]$ and $\|Fy\| \leq \max\{\|y_0\|, \|z_0\|\} := \Delta$.

Let $\{y_n\}$ be a monotone sequence in $[y_0, z_0]$. Then $y_0 \leq Fy_n \leq z_0$ and $\|Fy_n\| \leq \Delta$. Next we show that the sequence $\{Fy_n\}$ is equicontinuous. For any $(t,y) \in J \times [-\Delta, \Delta]$, there exist positive constants K_1, K_2 such that $|f(t,y)| \leq K_1, |S_k^*(y)| \leq K_2$. Then, for any $\tau_1, \tau_2 \in J_k$ with $\tau_1 \leq \tau_2, k = 1, 2, \ldots, p$, we obtain

$$\begin{aligned}
&\bigl\|F(y)(\tau_2) - F(y)(\tau_1)\bigr\| \\
&= \Biggl\| \frac{\rho^{1-\alpha_k}}{\Gamma(\alpha_k)} \biggl[\int_{t_k}^{\tau_1} s^{\rho-1} \bigl[\bigl(\tau_2^\rho - s^\rho\bigr)^{\alpha_k - 1} - \bigl(\tau_1^\rho - s^\rho\bigr)^{\alpha_k - 1} \bigr] f\bigl(s, y(s)\bigr)\, ds \\
&\quad + \int_{\tau_1}^{\tau_2} s^{\rho-1} \bigl(\tau_2^\rho - s^\rho\bigr)^{\alpha_k - 1} f\bigl(s, y(s)\bigr)\, ds \biggr] \\
&\quad + \sum_{i=1}^{k} \biggl(\frac{(\tau_2^\rho - t_k^\rho) - (\tau_1^\rho - t_k^\rho)}{\rho} \biggr) \\
&\quad \times \biggl(\frac{\rho^{2-\alpha_i}}{\Gamma(\alpha_i - 1)} \int_{t_{i-1}}^{t_i} s^{\rho-1} \bigl(t_i^\rho - s^\rho\bigr)^{\alpha_i - 2} f\bigl(s, y(s)\bigr)\, ds + S_i^*\bigl(y(t_i)\bigr) \biggr) \Biggr\| \\
&\leq \frac{K_1}{\rho^{\alpha_k} \Gamma(\alpha_k + 1)} \bigl\{ 2\bigl(\tau_2^\rho - \tau_1^\rho\bigr)^{\alpha_k} + \bigl| \bigl(\tau_2^\rho - t_k^\rho\bigr)^{\alpha_k} - \bigl(\tau_1^\rho - t_k^\rho\bigr)^{\alpha_k} \bigr| \bigr\} \\
&\quad + K_1 \sum_{i=1}^{k} \biggl(\frac{[(\tau_2^\rho - t_k^\rho) - (\tau_1^\rho - t_k^\rho)](t_i^\rho - t_{i-1}^\rho)^{\alpha_i - 1}}{\rho^{\alpha_i} \Gamma(\alpha_i)} \biggr) + K_2 \sum_{i=1}^{k} \biggl(\frac{(\tau_2^\rho - t_k^\rho) - (\tau_1^\rho - t_k^\rho)}{\rho} \biggr),
\end{aligned}$$

which tends to zero as $\tau_2 - \tau_1 \to 0$ independent of y. A similar conclusion follows for $\tau_1, \tau_2 \in J_0$. Thus, $\{Fy_n\}$ is equicontinuous on all $J_k, 0 \leq k \leq p$. So F is relatively compact on $[y_0, z_0]$. Hence, by the Arzelá–Ascoli theorem, F is compact on $[y_0, z_0]$, and consequently $\{Fy_n\}$ converges in $F([y_0, z_0])$. Thus all the hypotheses of Lemma 4.3 hold, and the conclusion of Lemma 4.3 implies that F has the least and greatest fixed points in $[y_0, z_0]$. This shows that problem (1.1) has extremal solutions on $[y_0, z_0]$. □

Example 4.5 Consider the problem

$$\begin{cases}
{}^\rho_c D^{\alpha_k}_{t_k^+} y(t) = \frac{t(t^{1/3} - (1/2)^{1/3})^2}{1100} (1 + (y(t))^3), \quad t \in [0,1], t \neq 1/2, k = 0, 1, \\
\Delta y(t_1) = \frac{1}{4} \tan^{-1} y(t_1), \qquad \Delta \delta y(t_1) = \frac{y(t_1)}{5}, \\
y(0) = \sum_{k=0}^{1} \lambda_k {}^\rho I^{\beta_k}_{t_k^+} y(\xi_k) + 1/4, \qquad \delta y(0) = 0,
\end{cases} \tag{4.10}$$

where $\rho = 1/3, \alpha_0 = 5/4, \alpha_1 = 3/2, \beta_0 = 1/2, \beta_1 = 3/2, \lambda_0 = 1/10, \lambda_1 = 1/7, \xi_0 = 1/4, \xi_1 = 3/4, t_1 = 1/2, f(t,y) = \frac{t(t^{1/3} - (1/2)^{1/3})^2}{1100}(1 + y^3)$, $S_1(y) = \frac{1}{4} \tan^{-1} y$, and $S_1^*(y) = \frac{y}{5}$.

We take $y_0(t) = 0$ as the lower solution and

$$z_0(t) = \begin{cases}
1 + \frac{t^{2\rho}}{2\rho^2}, & 0 \leq t \leq \frac{1}{2}, \\
1 + \frac{t^{2\rho}}{\rho^2}, & \frac{1}{2} < t \leq 1,
\end{cases}$$

as the upper solution of problem (4.10). With the given data, it is found that

$$\sum_{k=0}^{1} \frac{\lambda_k (\xi_k^\rho - t_k^\rho)^{\beta_k}}{\rho^{\beta_k} \Gamma(\beta_k + 1)} = 0.1768586259 < 1.$$

Also, assumption (A_1) is clearly satisfied. Thus, by Theorem 4.4, problem (4.10) has extremal solutions on $[y_0, z_0]$.

5 Conclusions

We have developed an existence theory for impulsive multi-order nonlinear Caputo-type generalized fractional differential equations equipped with nonlocal conditions involving Katugampola type generalized fractional integrals. The work presented in this paper is new and significantly contributes to the existing literature on the topic. By fixing the parameters involved in the problem, we can obtain some new results as special cases of those derived in this paper. For instance, our results correspond to those for nonlinear single order Caputo-type generalized fractional differential equations with generalized fractional integro-initial conditions if we set $\alpha_k = \alpha$. The results obtained in [26] appear as a special case of those established in Sect. 4 for $\rho = 1$.

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