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Uniqueness of meromorphic solutions of the difference equation $R_1(z)f(z+1)+R_2(z)f(z)=R_3(z)$

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Abstract

This paper mainly concerns the uniqueness of meromorphic solutions of first order linear difference equations of the form

$$R_1(z)f(z+1)+R_2(z)f(z)=R_3(z), \tag{*}$$

where $R_1(z)\not\equiv 0$, $R_2(z)$, $R_3(z)$ are rational functions. Our results indicate that the finite order transcendental meromorphic solution of equation (*) is mainly determined by its zeros and poles except for some special cases. Examples for the sharpness of our results are also given.

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1 Introduction

Throughout the whole paper, for a meromorphic function $f(z)$, we use standard notations of the Nevanlinna theory (see, e.g., [2, 8, 14]) such as $T(r,f)$, $m(r,f)$, and $N(r,f)$ and define respectively the order of growth of $f(z)$, the exponent of convergence of the zeros of $f(z)$, and the exponent of convergence of the poles of $f(z)$ by $\rho(f)$, $\lambda(f)$, $\lambda(1/f)$ as follows:

$$\rho(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r},$$
$$\lambda(f)=\limsup_{r\to\infty}\frac{\log N(r,1/f)}{\log r},$$
$$\lambda(1/f)=\limsup_{r\to\infty}\frac{\log N(r,f)}{\log r}.$$

And we call a meromorphic function $a(z)$ a small function of $f(z)$ if

$$\lim_{r\to\infty, r\notin E}\frac{T(r,a)}{T(r,f)}=0,$$

where E is an exceptional set of finite logarithmic measure. Denote the family of all small functions of $f(z)$ by $S(f)$ and set $\widehat{S}(f)=S(f)\cup\{\infty\}$.

The uniqueness is always one of most essential properties of research objects, such as a function under some given conditions, a solution of a given equation, and so on. The uniqueness theory of meromorphic functions is an important part of Nevanlinna theory. The following is the famous Nevanlinna 5 IM (4 CM) theorem.

Theorem A ([11]) *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions. If $f(z)$ and $g(z)$ share five values IM* (*four values CM*, *respectively*) *in the extended complex plane*, *then* $f(z) \equiv g(z)$ ($f(z) = T(g(z))$, *where T is a Möbius transformation*, *respectively*).

Here and in the following, $f(z)$ and $g(z)$ are said to share the value a CM(IM), provided that $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities (ignoring multiplicities), and $f(z)$ and $g(z)$ are said to share the value ∞ CM(IM), provided that $f(z)$ and $g(z)$ have the same poles with the same multiplicities (ignoring multiplicities).

For about 90 years, lots of researchers have devoted themselves to reducing the number of the shared values, relaxing the CM (IM) shared conditions, or replacing the shared values by sets or small functions in Theorem A (see, e.g., [14]). We recall two relative considerations here. One is to consider the case that $g(z)$ is a derivative, shift, or difference operator of $f(z)$ (see, e.g., [7, 9, 13]). The other is to consider the case that $f(z)$ satisfies some differential equations or difference equations (see, e.g., [1, 4, 5, 10]).

In fact, Heittokangas et al. were the first to consider the case that $f(z)$ shares values and small functions with its shift $f(z+\eta)$ and to prove the following.

Theorem B ([7]) *Let $f(z)$ be a meromorphic function of finite order, and let $\eta \in \mathbb{C}$. If $f(z)$ and $f(z+\eta)$ share three distinct periodic functions $a_1, a_2, a_3 \in \widehat{S}(f)$ with period η CM, then $f(z) = f(z+\eta)$ for all $z \in \mathbb{C}$.*

Cui and Chen considered the uniqueness of meromorphic solutions sharing three values with a meromorphic function to some linear difference equations and proved the following.

Theorem C ([4]) *Let $f(z)$ be a finite order transcendental meromorphic solution of the equation*

$$A_1(z)f(z+1) + A_2(z)f(z) = 0, \tag{1.1}$$

where $A_1(z)$, $A_2(z)$ are nonzero polynomials such that $A_1(z) + A_2(z) \not\equiv 0$. If a meromorphic function $g(z)$ shares $0, 1, \infty$ *CM with $f(z)$, then either $f(z) \equiv g(z)$ or $f(z)g(z) \equiv 1$.*

Theorem D ([5]) *Let $f(z)$ be a finite order transcendental meromorphic solution of the equation*

$$A_1(z)f(z+1) + A_2(z)f(z) = A_3(z),$$

where $A_1(z)$, $A_2(z)$, $A_3(z)$ are nonzero polynomials such that $A_1(z) + A_2(z) \not\equiv 0$. If a meromorphic function $g(z)$ shares $0, 1, \infty$ *CM with $f(z)$, then one of the following cases holds*:

(i) $f(z) \equiv g(z)$;

(ii) $f(z) + g(z) = f(z)g(z)$;

(iii) *there exist a polynomial* $\beta(z) = az + b_0$ *and a constant* a_0 *satisfying* $e^{a_0} \neq e^{b_0}$ *such that*

$$f(z) = \frac{1 - e^{\beta(z)}}{e^{\beta(z)}(e^{a_0 - b_0} - 1)}, \qquad g(z) = \frac{1 - e^{\beta(z)}}{1 - e^{b_0 - a_0}},$$

where $a_0 \neq 0$, b_0 *are constants.*

Remark 1 Obviously, if (1.1) admits a meromorphic solution $f(z)$, then for each periodic entire function $h(z)$ with period 1 (chosen by the method of Ozawa in [12]), $f(z)h(z)$ is also a meromorphic solution of (1.1). This means that (1.1) may admit infinitely many solutions.

Examples are provided in [4] and [5] to show that all cases of Theorem C and Theorem D can happen, and the number of shared values cannot be reduced. When looking at Theorem C and Theorem D and considering Remark 1, instead of trying to improve them directly, we are interested in the following natural question:

Question *What can we say about the uniqueness of finite order transcendental meromorphic solution of the equation*

$$R_1(z)f(z+1) + R_2(z)f(z) = R_3(z), \tag{1.2}$$

where $R_1(z) \not\equiv 0, R_2(z), R_3(z)$ *are rational functions? That is, how can we guarantee the uniqueness of such solution by its zeros and poles?*

For the question above, we discuss two cases $R_3(z) \equiv 0$ and $R_3(z) \not\equiv 0$ separately since they are quite different and prove the following two results.

Theorem 1.1 *Let* $f(z)$ *and* $g(z)$ *be two finite order transcendental meromorphic solutions of equation* (1.2), *where* $R_3(z) \equiv 0$. *Suppose that* $f(z)$ *and* $g(z)$ *share* $0, \infty$ *CM. Then*

$$f(z) \equiv e^{2k_0\pi iz + a_0} g(z)$$

for some integer k_0 *and constant* a_0. *What is more,* $f(z) \equiv g(z)$ *provided that one of the following cases holds*:

(i) *there exist two points* z_1, z_2 *such that* $f(z_j) = g(z_j) \neq 0$ $(j = 1, 2)$ *and* $z_1 - z_2 \notin \mathbb{Q}$;

(ii) $f(z) - g(z)$ *has a zero* z_3 *of multiplicity* ≥ 2 *such that* $f(z_3) = g(z_3) \neq 0$.

Theorem 1.2 *Let* $f(z)$ *and* $g(z)$ *be two finite order transcendental meromorphic solutions of equation* (1.2), *where* $R_3(z) \not\equiv 0$. *Suppose that* $f(z)$ *and* $g(z)$ *share* $0, \infty$ *CM. Then either* $f(z) \equiv g(z)$ *or*

$$f(z) = \frac{R_3(z)}{2R_2(z)}\left(e^{a_1 z + a_0} + 1\right)$$

and

$$g(z) = \frac{R_3(z)}{2R_2(z)}\left(e^{-a_1 z - a_0} + 1\right),$$

where a_1, a_0 are constants such that $e^{-a_1} = e^{a_1} = -1$, and the coefficients of (1.2) *satisfy*

$$R_1(z)R_3(z+1) \equiv R_3(z)R_2(z+1).$$

Remark 2 From the proof of Theorem 1.1, we see that it still holds, even if $R_1(z)$ or $R_2(z)$ is a transcendental meromorphic function. Unfortunately, we still wonder what happens if $R_3(z) \not\equiv 0$ and one of coefficients is a transcendental meromorphic function.

From Theorem 1.2, we get the following corollaries.

Corollary 1.1 *Let $f(z)$ and $g(z)$ be two finite order transcendental meromorphic solutions of equation* (1.2), *where $R_3(z) \not\equiv 0$ such that $R_1(z)R_3(z+1) \not\equiv R_3(z)R_2(z+1)$. If $f(z)$ and $g(z)$ share* $0, \infty$ *CM, then $f(z) \equiv g(z)$.*

Corollary 1.2 *Let $f(z)$ and $g(z)$ be two finite order transcendental meromorphic solutions of equation* (1.2), *where*

$$R_1(z) + R_2(z) \not\equiv R_3(z), \qquad R_1(z)\big[R_3(z+1) - R_1(z+1)\big] \not\equiv \big[R_3(z) - R_2(z)\big]R_2(z+1).$$

If $f(z)$ and $g(z)$ share $1, \infty$ *CM, then $f(z) \equiv g(z)$.*

Corollary 1.3 *Let $f(z)$ and $g(z)$ be two finite order transcendental meromorphic solutions of equation* (1.2), *where $R_3(z) \equiv 0$ and $R_1(z) \not\equiv -R_2(z)$. Suppose that $f(z)$ and $g(z)$ share* $1, \infty$ *CM. Then either $f(z) \equiv g(z)$ or $f(z)g(z) \equiv 1$ such that*

$$f(z) = e^{a_1 z + a_0} \quad \textit{and} \quad g(z) = e^{-a_1 z - a_0},$$

where a_1, a_0 are constants such that $e^{-a_1} = e^{a_1} = -1$, and the coefficients of (1.2) *satisfy $R_1(z) \equiv R_2(z)$.*

Remark 3 Corollary 1.1 and Corollary 1.2 follow from Theorem 1.2 immediately. And their proofs are thus omitted.

Remark 4 From Corollary 1.3, one can find that equation (1.2), where $R_3(z) \equiv 0$ and $R_1(z) \not\equiv -R_2(z)$, is equivalent to the equation

$$f(z+1) + f(z) = 0,$$

provided that it admits two distinct finite order transcendental meromorphic solutions sharing $1, \infty$ CM.

We should give some examples in which $f(z) \not\equiv g(z)$ for our results before the proofs of them. These examples show that the conditions in these results cannot be omitted.

Example 1

(1) The entire functions $f_1(z) = z3^z e^{2\pi iz}$ and $g_1(z) = z3^z$, and the meromorphic functions $f_2(z) = z3^z e^{2\pi iz}/\cos(2\pi z)$ and $g_2(z) = z3^z/\cos(2\pi z)$ satisfy the equation

$$\frac{z}{3(z+1)}f(z+1) - f(z) = 0.$$

Here $f_j(z)$ and $g_j(z)$ share $0, \infty$ CM, $f_j(z) = e^{2\pi iz}g_j(z)$, $f_j(z)$ and $g_j(z)$ have only one zero $z_0 = 0$, and all zeros of $f_j(z) - g_j(z)$ such that $f_j(z) = g_j(z) \neq 0$ are simple ($j = 1, 2$).

(2) $f(z) = (e^{\pi iz} + 1)/2$ and $g(z) = (e^{-\pi iz} + 1)/2$ satisfy the equation

$$f(z+1) + f(z) = 1.$$

Here $f(z)$ and $g(z)$ share $0, \infty$ CM and $f(z) = e^{\pi iz}g(z)$, $R_3(z) \equiv 1 \neq 0$ and $R_1(z)R_3(z+1) \equiv R_3(z)R_2(z+1) \equiv 1$.

(3) $f(z) = e^{\pi iz}$ and $g(z) = e^{-\pi iz}$ satisfy the equation

$$f(z+1) + f(z) = 0.$$

Here $f(z)$ and $g(z)$ share $1, \infty$ CM, $e^{-\pi i} = e^{\pi i} = -1$, and the coefficients of (1.2) satisfy $R_1(z) \equiv R_2(z) \equiv 1$; $R_1(z) + R_2(z) \not\equiv -R_3(z)$, but

$$R_1(z)\big[R_3(z+1) - R_1(z+1)\big] \equiv -1 \equiv \big[R_3(z) - R_2(z)\big]R_2(z+1).$$

This shows that the condition $R_1(z)[R_3(z+1) - R_1(z+1)] \equiv -1 \equiv [R_3(z) - R_2(z)]R_2(z+1)$ in Corollary 1.2 cannot deleted.

(4) $f(z) = z3^z e^{2\pi iz}/\cos^2(2\pi z)$ and $g(z) = z3^z/\cos(2\pi z)$ share 0 CM and ∞ IM, and they satisfy the equation

$$\frac{z}{3(z+1)}f(z+1) - f(z) = 0,$$

but $f(z) \not\equiv g(z)$. This indicates that the shared condition "CM" cannot be replaced by "IM" and the number of CM shared values cannot be reduced in Theorem 1.1.

Remark 5 We still wonder what happens if the shared condition "CM" is replaced by "IM" or the number of CM shared values is reduced in Theorem 1.2.

2 Proof of Theorem 1.1

To prove Theorem 1.1, we need the following lemma, which is a very important result in studying the difference analogues of Nevanlinna theory and difference equations, proved by Chiang and Feng [3] and by Halburd and Korhonen [6] independently.

Lemma 2.1 ([3, 6]) *Let $f(z)$ be a meromorphic function of finite order $\rho(f) = \rho$, ε be a positive constant, η_1 and η_2 be two distinct complex constants. Then*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O\big(r^{\rho-1+\varepsilon}\big) = o\big(T(r,f)\big).$$

Proof of Theorem 1.1 Since $f(z)$ and $g(z)$ are finite order transcendental meromorphic functions and share $0, \infty$ CM, we have

$$\frac{f(z)}{g(z)} = e^{P(z)}, \tag{2.1}$$

where $P(z)$ is a polynomial such that $\deg P(z) \le \max\{\rho(f), \rho(g)\}$.

We can get from (1.2) and (2.1) that

$$\frac{g(z+1)e^{P(z+1)}}{g(z)e^{P(z)}} = \frac{f(z+1)}{f(z)} = -\frac{R_2(z)}{R_1(z)} = \frac{g(z+1)}{g(z)}.$$

Thus $e^{P(z+1)-P(z)} \equiv 1$ and hence $P(z+1) - P(z)$ must be a constant. More precisely, $P(z+1) - P(z) = 2k_0\pi i$ for some integer k_0. Then we obtain easily that $P(z) = 2k_0\pi iz + a_0$ and hence

$$f(z) \equiv e^{2k_0\pi iz+a_0}g(z), \tag{2.2}$$

where a_0 is a constant. The first conclusion is thus proved.

Next, we discuss two cases for the second conclusion.

Case (i): There exist two points z_1, z_2 such that $f(z_j) = g(z_j) \neq 0$ and $z_1 - z_2 \notin \mathbb{Q}$, then from (2.1) and (2.2), we have

$$e^{2k_0\pi iz_j+a_0}g(z_j) = f(z_j) = g(z_j) \neq 0 \quad (j = 1, 2), \tag{2.3}$$

which gives

$$e^{2k_0\pi iz_1+a_0} = 1 = e^{2k_0\pi iz_2+a_0}.$$

This indicates that $k_0(z_1 - z_2)$ is an integer. Suppose that $k_0 \neq 0$, then $z_1 - z_2$ must be a rational number. This contradicts our assumption $z_1 - z_2 \notin \mathbb{Q}$. Thus $k_0 = 0$. From (2.3) and $f(z_1) = g(z_1) \neq 0$, we get $e^{a_0} = 1$ and prove that $f(z) \equiv g(z)$.

Case (ii): $f(z) - g(z)$ has a zero z_3 of multiplicity ≥ 2 such that $f(z_3) = g(z_3) \neq 0$. From (2.2), we see that $e^{2k_0\pi iz_3+a_0} = 1$.

Differentiating both sides of (2.2), we get

$$f'(z) - e^{2k_0\pi iz+a_0}g'(z) = 2k_0\pi ie^{2k_0\pi iz+a_0}g(z). \tag{2.4}$$

Suppose that $k_0 \neq 0$. By the assumption that z_3 is a zero of $f(z) - g(z)$ with multiplicity ≥ 2, $e^{2k_0\pi iz_3+a_0} = 1$ and (2.4), we can deduce the following contradiction:

$$\begin{aligned} 0 &= f'(z_3) - g'(z_3) = f'(z_3) - e^{2k_0\pi iz_3+a_0}g'(z_3) \\ &= 2k_0\pi ie^{2k_0\pi iz_3+a_0}g(z_3) = 2k_0\pi ig(z_3) \neq 0. \end{aligned}$$

Thus $k_0 = 0$. From (2.2) and $f(z_3) = g(z_3) \neq 0$, we can also get $e^{a_0} = 1$ and prove that $f(z) \equiv g(z)$. □

3 Proof of Theorem 1.2

Since $f(z)$ and $g(z)$ are finite order transcendental meromorphic functions and share $0, \infty$ CM, equation (2.1) still holds. Keep in mind that $R_1(z)R_2(z) \not\equiv 0$. Otherwise, (1.2) cannot admit any transcendental meromorphic solution.

We can get from (1.2) and (2.1) that

$$R_1(z)g(z+1) + R_2(z)g(z) = R_3(z) \tag{3.1}$$

and

$$R_1(z)e^{P(z+1)}g(z+1) + R_2(z)e^{P(z)}g(z) = R_3(z). \tag{3.2}$$

By (3.1) and (3.2), we obtain

$$R_2(z)\bigl[e^{P(z)-P(z+1)} - 1\bigr]g(z) = R_3(z)\bigl[e^{-P(z+1)} - 1\bigr]. \tag{3.3}$$

If $e^{P(z)-P(z+1)} - 1 \equiv 0$, then from (3.3), $e^{-P(z+1)} - 1 \equiv 0$. This means that $f(z) \equiv g(z)$.

If $e^{P(z)-P(z+1)} - 1 \not\equiv 0$, we can solve out $g(z)$ from (3.3) as the form

$$g(z) = \frac{R_3(z)[e^{-P(z+1)} - 1]}{R_2(z)[e^{P(z)-P(z+1)} - 1]}. \tag{3.4}$$

Combining (3.1) with (3.4), we get

$$\frac{R_1(z)R_3(z+1)[e^{-P(z+2)} - 1]}{R_2(z+1)[e^{P(z+1)-P(z+2)} - 1]} + \frac{R_3(z)[e^{-P(z+1)} - 1]}{e^{P(z)-P(z+1)} - 1} = R_3(z).$$

Equally,

$$R_1(z)h(z+1)\bigl[e^{-P(z+2)} - 1\bigr] + R_2(z)h(z)\bigl[e^{-P(z+1)} - 1\bigr] = R_3(z), \tag{3.5}$$

where

$$h(z) = \frac{R_3(z)}{R_2(z)[e^{P(z)-P(z+1)} - 1]}.$$

Set

$$P(z) = a_n z^n + \cdots + a_1 z + a_0, \tag{3.6}$$

where $a_n \neq 0, \ldots, a_1, a_0$ are constants and n is an integer.

Notice that $g(z)$ is transcendental. From (3.4), we see that $\deg P(z) \geq 1$. We claim that $\deg P(z) = 1$. Otherwise, $n = \deg P(z) \geq 2$.

It is clear that

$$\deg\bigl[P(z+2) - P(z+1)\bigr] = \deg\bigl[P(z+1) - P(z)\bigr] = n - 1. \tag{3.7}$$

Therefore, $\rho(e^{P(z+2)-P(z+1)}) = n-1$ and

$$\begin{aligned} T(r,h) &= T\left(r,\frac{1}{h}\right) + O(1) = T\left(r,\frac{R_2(z)}{R_3(z)}\left[e^{P(z)-P(z+1)}-1\right]\right) + O(1) \\ &= T\left(r,e^{P(z)-P(z+1)}\right) + O(\log r), \end{aligned}$$

which means $\rho(h) = n-1$. By Lemma 2.1, for each $\varepsilon \in (0,1)$,

$$m\left(r,\frac{h(z+1)}{h(z)}\right) = O\left(r^{\rho(h)-1+\varepsilon}\right) = O\left(r^{n-2+\varepsilon}\right) = o\left(r^{n-1}\right). \tag{3.8}$$

Rewrite (3.5) as the form

$$\begin{aligned} &R_1(z)h(z+1) + R_2(z)h(z)e^{P(z+2)-P(z+1)} \\ &\quad = \left[R_3(z) + R_1(z)h(z+1) + R_2(z)h(z)\right]e^{P(z+2)}. \end{aligned} \tag{3.9}$$

Suppose that $R_3(z) + R_1(z)h(z+1) + R_2(z)h(z) \not\equiv 0$. Then from (3.7), (3.9) and the fact $\rho(h) = n-1$, we can deduce the following contradiction:

$$\begin{aligned} n &= \rho\left(\left[R_3(z) + R_1(z)h(z+1) + R_2(z)h(z)\right]e^{P(z+2)}\right) \\ &= \rho\left(R_1(z)h(z+1) + R_2(z)h(z)e^{P(z+2)-P(z+1)}\right) \le n-1. \end{aligned}$$

Thus, $R_3(z) + R_1(z)h(z+1) + R_2(z)h(z) \equiv 0$ and hence we get from (3.9) that

$$R_1(z)h(z+1) + R_2(z)h(z)e^{P(z+2)-P(z+1)} = 0. \tag{3.10}$$

By (3.8) and (3.10), we get

$$\begin{aligned} T\left(r,e^{P(z+2)-P(z+1)}\right) &= m\left(r,e^{P(z+2)-P(z+1)}\right) \\ &= m\left(r,-\frac{R_1(z)h(z+1)}{R_2(z)h(z)}\right) \le o\left(r^{n-1}\right) + O(\log r), \end{aligned}$$

which contradicts $\rho(e^{P(z+2)-P(z+1)}) = n-1 \ge 1$. Thus, we prove that $\deg P(z) = 1$ and get from (3.6) that $P(z) = a_1 z + a_0$, where $a_1 \neq 0$.

Now, submitting $P(z) = a_1 z + a_0$ into (3.4), we obtain

$$g(z) = \frac{cR_3(z)}{R_2(z)}\left(e^{-a_1 z - a_1 - a_0} - 1\right), \tag{3.11}$$

where $c = (e^{-a_1} - 1)^{-1} \neq 0$.

By (3.1) and (3.11), we have

$$\left(\frac{cR_1(z)R_3(z+1)}{R_2(z+1)}e^{-a_1} + cR_3(z)\right)e^{-a_1 z - a_1 - a_0} = (1+c)R_3(z) + \frac{cR_1(z)R_3(z+1)}{R_2(z+1)}.$$

Comparing the orders of both sides of the equation above, we can deduce that

$$\frac{R_1(z)R_3(z+1)}{R_2(z+1)}e^{-a_1} + R_3(z) \equiv 0, \tag{3.12}$$

and hence

$$(1+c)R_3(z)+\frac{cR_1(z)R_3(z+1)}{R_2(z+1)}\equiv 0. \tag{3.13}$$

From (3.12) and (3.13), we get

$$e^{-a_1}=-\frac{R_3(z)R_2(z+1)}{R_1(z)R_3(z+1)}=\frac{c}{1+c}=e^{a_1},$$

which yields that $e^{-a_1}=e^{a_1}=-1$, since $c=(e^{-a_1}-1)^{-1}\neq 0$.

Finally, we obtain from (2.1) and (3.11) that

$$f(z)=\frac{R_3(z)}{2R_2(z)}\bigl(e^{a_1z+a_0}+1\bigr)$$

and

$$g(z)=\frac{R_3(z)}{2R_2(z)}\bigl(e^{-a_1z-a_0}+1\bigr),$$

where $e^{-a_1}=e^{a_1}=-1$. What is more, from (3.12) or (3.13), we see that

$$R_1(z)R_3(z+1)\equiv R_3(z)R_2(z+1)$$

holds for this case.

4 Proof of Corollary 1.3

Set $F(z)=f(z)-1$ and $G(z)=g(z)-1$. Then $F(z)$ and $G(z)$ share $0,\infty$ CM, since $f(z)$ and $g(z)$ share $1,\infty$ CM.

Submitting $f(z)=F(z)+1$ and $g(z)=G(z)+1$ into (1.2), we see that both $F(z)$ and $G(z)$ satisfy the equation of the form

$$R_1(z)f(z+1)+R_2(z)f(z)=R_3^*(z), \tag{4.1}$$

where

$$R_3^*(z)=-R_1(z)-R_2(z)\not\equiv 0, \tag{4.2}$$

by the assumption $R_1(z)\not\equiv -R_2(z)$. Thus, by Theorem 1.2, either $F(z)\equiv G(z)$ and hence $f(z)\equiv g(z)$, or

$$F(z)=\frac{R_3^*(z)}{2R_2(z)}\bigl(e^{a_1z+b_0}+1\bigr) \tag{4.3}$$

and

$$G(z)=\frac{R_3^*(z)}{2R_2(z)}\bigl(e^{-a_1z-b_0}+1\bigr), \tag{4.4}$$

where a_1, b_0 are constants such that $e^{-a_1} = e^{a_1} = -1$, and the coefficients of (4.1) satisfy

$$R_1(z)R_3^*(z+1) = R_3^*(z)R_2(z+1). \tag{4.5}$$

From (4.2) and (4.5), we get

$$R_1(z)R_1(z+1) = R_2(z)R_2(z+1),$$

which indicates that $R_1(z) \equiv R_2(z)$.

Now, $R_3^*(z) = -R_1(z) - R_2(z) = -2R_2(z)$. By this fact and (4.3)–(4.4), we see that

$$F(z) = -\bigl(e^{a_1 z + b_0} + 1\bigr)$$

and

$$G(z) = -\bigl(e^{-a_1 z - b_0} + 1\bigr).$$

Finally, we can finish our proof by denoting $a_0 = b_0 + \pi i$ and using $f(z) = F(z) + 1$ and $g(z) = G(z) + 1$.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors drafted the manuscript, read and approved the final manuscript.

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