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On the oscillation of higher order nonlinear neutral difference equations

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Abstract

In this paper, we shall investigate some oscillation criteria for the solutions of mth-order nonlinear neutral difference equation where $m \geq 1$. The results presented here complement some of the known results reported in the literature. Examples are included to illustrate the importance of the main results.

Keywords: Higher order; Neutral difference equations; Nonlinear; Oscillation

1 Introduction

In this paper, we are concerned with the following higher order neutral difference equation:

$$\Delta^m\bigl[x(n)+p(n)x\bigl(\tau(n)\bigr)\bigr]+q(n)f\bigl(x\bigl(\sigma(n)\bigr)\bigr)=0, \quad n \in N=\{0,1,\ldots\}, \tag{1.1}$$

where $m \geq 1$ and Δ is the forward difference operator defined by

$$\Delta x(n)=x(n+1)-x(n).$$

Throughout this paper, we assume the following conditions to hold:

(H1) $\{q(n)\}$ is a real-valued sequence with $q(n) \geq 0, n \in N$ and $\{q(n)\}$ is not identically zero.

(H2) $\{p(n)\}$ is a real-valued sequence with $0 \leq p(n) < 1, n \in N$.

(H3) $\{\tau(n)\}$ and $\{\sigma(n)\}$ are nondecreasing sequences such that $\tau(n) < n$ with $\lim_{n\to+\infty}\tau(n)=+\infty$ and $\sigma(n) < n$ with $\lim_{n\to+\infty}\sigma(n)=+\infty$.

(H4) $f: R \to R$ is a nondecreasing continuous function such that $xf(x) > 0$ for $x \neq 0$ and

$$-f(-xy) \geq f(xy) \geq f(x)f(y). \tag{1.2}$$

The factorial expression is defined as $(r)^{(s)}=\prod_{i=0}^{s-1}(r-i)$ with $(r)^{(0)}=1$ for all $r \in R = (-\infty,\infty)$ and s, a nonnegative integer.

Let N_0 be a fixed nonnegative integer. By a solution of equation (1.1), we mean a nontrivial real sequence $\{x(n)\}$ which is defined for all $n \geq \min_{i\geq 0}\{\tau(i),\sigma(i)\}$ and satisfies equation (1.1) for $n \geq N_0$. A solution $\{x(n)\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called non-oscillatory. A difference equation is said to be oscillatory if all of its solutions are oscillatory. Otherwise, it is non-oscillatory.

In recent years, the oscillation behavior of neutral difference equations has been studied vigorously, for example, see [1–26] and the references cited therein. This is because of the fact that neutral difference equations find various applications in some variational problems, in natural science and technology.

Agarwal et al. [5] considered the *mth* order neutral difference equation

$$\Delta^m[y_n + p_n y_{n-k}] + q_n f(y_{n-l}) = 0 \tag{1.3}$$

and discussed some oscillation theorems for (1.3), when m is odd, for which every solution of (1.3) either oscillates or tends to zero as $n \to \infty$.

In [4], Agarwal and Grace considered the higher order difference equation

$$\Delta\left(\Delta^{m-1}x(n)\right)^{\alpha} + q(n)x^{\alpha}(n-\tau) = 0 \tag{1.4}$$

and obtained some sufficient conditions for the oscillation of all solutions of (1.4).

Yasar Bolat et al. [9] have taken even order nonlinear neutral difference equation

$$\Delta^m\left[y(k) + p(k)y\left(\tau(k)\right)\right] + q(k)y\left(\sigma(k)\right) = 0, \tag{1.5}$$

and established some criteria for oscillation of bounded solutions only.

Therefore, it is to be noted that, to the best of our knowledge, there is no paper for higher order nonlinear neutral difference equations which ensures that all the solutions are oscillatory when m is odd. Following this notion, our aim in this paper is to provide sufficient conditions which ensure that all solutions of (1.1) are oscillatory.

To obtain our results, we shall need the following lemma.

Lemma 1.1 (see [1]) *Let $x(n)$ be defined for $n \geq n_0 \in N$ and $x(n) > 0$ with $\Delta^n x(n)$ of constant sign for $n \geq n_0$ and not identically zero. Then there exists an integer l, $0 \leq l \leq m$, with $(m+l)$ odd for $\Delta^m x(n) \leq 0$ and $(m+l)$ even for $\Delta^m x(n) \geq 0$ eventually such that*

(i) *$l \leq m-1$ implies $(-1)^{l+k}\Delta^k x(n) > 0$ for all $n \geq n_0$, $l \leq k \leq m-1$.*

(ii) *$l \geq 1$ implies $\Delta^k x(n) > 0$ for all large $n \geq n_0$, $1 \leq k \leq l-1$.*

2 Main results

To obtain the main results, we shall use the following notations.

For all large $n \geq n_0 > 0$, let

$$R_j(n) = f\left(\frac{(\sigma(n)-m+j)^{(j-i)}}{j!}\right)\sum_{r=n}^{\infty}\frac{(r-n+m-j-3)^{(m-j-3)}}{(m-j-3)!}$$
$$\times\left(\sum_{j=r}^{\infty}q(j)\right)f\left(1-p\left(\sigma(r)\right)\right), \quad j \in \{1,2,\ldots,m-3\},$$
$$R_{m-1}(n) = q(n)f\left(1-p\left(\sigma(n)\right)\right)f\left(\frac{(\sigma(n)-1)^{(m-2)}}{(m-1)!}\right),$$
$$R_0(n) = q(n)f\left(1-p\left(\sigma(n)\right)\right)f\left(\sum_{r=\sigma(n)}^{\eta(n)}\frac{(r-\sigma(n)+m-2)^{(m-2)}}{(m-2)!}\right)$$

for some nondecreasing function $\eta(n)$ with $\sigma(n) < \eta(n) \leq n, n \geq n_0$.

Then we shall discuss the following theorems.

Theorem 2.1 *Assume that conditions* (H1)–(H4) *hold and*

$$\sum_{n=n_0}^{\infty}\left(\sum_{j=n}^{\infty}q(j)\right)\left(\sigma(n)\right)^{(m-2)} = \infty. \tag{2.1}$$

Let m be odd. If all the second order equations

$$\Delta^2 y(n) + R_j(n)f\big(y\big(\sigma(n)\big)\big) = 0, \quad j \in \{2,4,\dots,m-1\} \tag{2.2}$$

for $n \geq n_0$ are oscillatory and if there exists a nondecreasing sequence $\{\eta(n)\}$ with $\sigma(n) < \eta(n) \leq n, n \geq n_0$ such that the first order difference equation

$$\Delta v(n) + R_0(n)f\big(v\big(\eta(n)\big)\big) = 0 \tag{2.3}$$

is oscillatory, then every solution of equation (1.1) *oscillates.*

Theorem 2.2 *Assume that conditions* (H1)–(H4) *and* (2.1) *hold. Let m be even. If all the second order equations* (2.2), $j \in \{1,3,\dots,m-3\}$ *and*

$$\Delta^2 u(n) + R_{m-1}(n)f\big(u\big(\sigma(n)\big)\big) = 0, \tag{2.4}$$

for $n \geq n_0$ are oscillatory, then every solution of equation (1.1) *oscillates.*

Proofs of Theorems 2.1 *and* 2.2 Let $\{x(n)\}$ be a non-oscillatory solution of (1.1). Without loss of generality, assume that $x(n) > 0, x(\tau(n)) > 0, x(\sigma(n)) > 0$ for all $n \geq n_0 \geq 0$.

Let

$$z(n) = x(n) + p(n)x\big(\tau(n)\big) \geq x(n) > 0.$$

Then (1.1) becomes

$$\Delta^m z(n) = -q(n)f\big(x\big(\sigma(n)\big)\big) \leq 0 \quad \text{for } n \geq n_1 \geq n_0. \tag{2.5}$$

From Lemma 1.1, it is easy to check

$$\Delta^{m-1} z(n) > 0 \quad \text{for } n \geq n_1. \tag{2.6}$$

Also, from (2.5), we have $\Delta^m z(n) \leq 0$ eventually.

So, $z(n)$ satisfies Lemma 1.1 for some $l \in \{1,2,\dots,m-3\}$ and $(l+m)$ odd. Also, by Lemma 1.1, $\Delta z(n) > 0$. Since $z(n)$ is increasing, we have

$$\begin{aligned}\big(1-p(n)\big)z(n) &\leq z(n) - p(n)z\big(\tau(n)\big)\\ &= x(n) - p(n)p\big(\tau(n)\big)x\big(\tau\big(\tau(n)\big)\big)\\ &\leq x(n) \quad \text{for } n \geq n_1.\end{aligned}$$

That is,

$$\bigl(1-p(n)\bigr)z(n) \le x(n) \quad \text{for } n \ge n_1. \tag{2.7}$$

Now the following three cases are considered: $l \in \{1,2,\dots,m-3\}$, $l = m-1$, $l = 0$.

Case (i): $l \in \{1,2,\dots,m-3\}$. From discrete Taylor's formula, we have

$$\begin{aligned} -\Delta^{l+1}z(n) &= \sum_{j=l+1}^{m-2} \frac{(s-n+j-l-2)^{(j-l-1)}}{(j-l-1)!}(-1)^{j-l}\Delta^j z(s) \\ &\quad + (-1)^{m-l-3}\sum_{r=n}^{s-1}\frac{(r-n+m-l-3)^{(m-l-3)}}{(m-l-3)!}\Delta^{m-1}z(r) \end{aligned} \tag{2.8}$$

for $s \ge n \ge n_1$. Using Lemma 1.1 in (2.8), we obtain

$$-\Delta^{l+1}z(n) \ge \sum_{r=n}^{\infty}\frac{(r-n+m-l-3)^{(m-l-3)}}{(m-l-3)!}\Delta^{m-1}z(r). \tag{2.9}$$

Summing up equation (1.1) from r to $u-1$ and letting $u \to \infty$, we have

$$\Delta^{m-1}z(r) \ge \sum_{j=r}^{\infty} q(j) f\bigl(x(\sigma(r))\bigr) \quad \text{for } n \ge n_2 \ge n_1. \tag{2.10}$$

Substituting (2.10) in (2.9), we have

$$-\Delta^{l+1}z(n) \ge \sum_{r=n}^{\infty}\frac{(r-n+m-l-3)^{(m-l-3)}}{(m-l-3)!}\left(\sum_{j=r}^{\infty}q(j)\right)f\bigl(x(\sigma(r))\bigr). \tag{2.11}$$

Using (2.7) and (1.2) in (2.11), we get

$$\begin{aligned} -\Delta^{l+1}z(n) &\ge \sum_{r=n}^{\infty}\frac{(r-n+m-l-3)^{(m-l-3)}}{(m-l-3)!} \\ &\quad \times \left(\sum_{j=r}^{\infty}q(j)\right)f\bigl(1-p(\sigma(r))\bigr)f\bigl(z(\sigma(r))\bigr). \end{aligned} \tag{2.12}$$

From (2.10), we can see that

$$\begin{aligned} (n)^{(m-l-1)}\Delta^{m-1}z(n) &\ge (n)^{(m-l-1)}\left(\sum_{j=n}^{\infty}q(j)\right)x\bigl(\sigma(n)\bigr) \\ &\ge (n)^{(m-l-1)}\left(\sum_{j=n}^{\infty}q(j)\right)\bigl(\sigma(n)\bigr)^{(l-1)} \\ &\ge \sum_{j=n}^{\infty}q(j)\bigl(\sigma(n)\bigr)^{(m-l-2)}. \end{aligned}$$

Hence from (2.1), we get

$$\sum^{\infty}(s)^{(m-l-1)}\Delta^{m-1}z(s)=\infty. \tag{2.13}$$

Consider the equality

$$\begin{aligned}
&\sum_{j=l-1}^{m-2}(-1)^{(j+l+1)}\frac{(n-m+j+1)^{(j-l+1)}}{(j-l+1)!}\Delta^{j}z(n)\\
&\quad=\sum_{j=l-1}^{m-2}(-1)^{(j+l+1)}\frac{(n_2)^{(j-l+1)}}{(j-l+1)!}\Delta^{j}z(n_1+m-j-2)\\
&\qquad+(-1)^{m+l-1}\sum_{s=n_2}^{m-2}(s)^{(m-l-1)}\Delta^{m-1}z(s)
\end{aligned}$$

with $l\in\{1,2,\dots,m-1\}$ and $(l+m)$ is odd.

Now from the above, there exists an integer $n\geq n_3\geq n_2$ such that

$$\Delta^{l-1}z(n)\geq(n-m+l+1)\Delta^{l}z(n) \tag{2.14}$$

and

$$z(n)\geq\frac{(n-m+l)^{(l-1)}}{l!}\Delta^{l-1}z(n). \tag{2.15}$$

Then we can find an integer $N\geq n_3$ such that

$$z(\sigma(n))\geq\frac{(\sigma(n)-m+l)^{(l-1)}}{l!}\Delta^{l-1}z(\sigma(n))\quad\text{for } n\geq N. \tag{2.16}$$

Using (2.16) in (2.12), we have

$$\begin{aligned}
-\Delta^{l+1}z(n)\geq{}&\sum_{r=n}^{\infty}\frac{(r-n+m-l-3)^{(m-l-3)}}{(m-l-3)!}\left(\sum_{j=r}^{\infty}q(j)\right)f\bigl(1-p(\sigma(r))\bigr)\\
&\times f\left(\frac{(\sigma(n)-m+l)^{(l-1)}}{l!}\right)f\bigl(\Delta^{l-1}z(\sigma(n))\bigr).
\end{aligned}$$

That is,

$$-\Delta^{l+1}z(n)\geq R_l(n)f\bigl(\Delta^{l-1}z(\sigma(n))\bigr).$$

Let $y(n)=\Delta^{l-1}z(n)$.

Then $y(n)>0$ for $n\geq N$ and the above inequality becomes

$$\Delta^2y(n)+R_l(n)f\bigl(y(\sigma(n))\bigr)\leq 0\quad\text{for } n\geq N.$$

Thus the last inequality has an eventually positive solution. By a well-known result in [14, p. 186, Corollary 7.6.1], we can see that the equation

$$\Delta^2 y(n) + R_l(n) f\big(y\big(\sigma(n)\big)\big) = 0 \quad \text{for } n \geq N$$

also has an eventually positive solution, which contradicts our assumption.

Case (ii): $l = m - 1$. Substituting $l = m - 1$ in inequality (2.16), we get

$$z\big(\sigma(n)\big) \geq \frac{(\sigma(n) - 1)^{(m-2)}}{(m-1)!} \Delta^{m-2} z\big(\sigma(n)\big).$$

From (1.1), (1.2), (2.7), and the above inequality, we have

$$\begin{aligned} -\Delta\big(\Delta^{m-1} z(n)\big) &= q(n) f\big(x\big(\sigma(n)\big)\big) \\ &\geq q(n) f\big(1 - p\big(\sigma(n)\big)\big) f\big(z\big(\sigma(n)\big)\big) \\ &\geq q(n) f\big(1 - p\big(\sigma(n)\big)\big) f\left(\frac{(\sigma(n) - 1)^{(m-2)}}{(m-1)!}\right) f\big(\Delta^{m-2} z\big(\sigma(n)\big)\big) \\ &= R_{m-1}(n) f\big(\Delta^{m-2} z\big(\sigma(n)\big)\big). \end{aligned}$$

Set $u(n) = \Delta^{m-2} z(n) > 0$ for $n \geq n_1$.

Then the above inequality becomes

$$-\Delta\big(\Delta u(n)\big) \geq R_{m-1}(n) f\big(u\big(\sigma(n)\big)\big).$$

That is,

$$\Delta^2 u(n) + R_{m-1}(n) f\big(u\big(\sigma(n)\big)\big) \leq 0,$$

which has an eventually positive solution. Thus we get a contradiction as in Case (i).

Case (iii): $l = 0$. In this case, m is odd. From discrete Taylor's formula, we have

$$\begin{aligned} z(n) = {} & \sum_{j=0}^{m-2} \frac{(s - n + j - 1)^{(j)}}{j!} (-1)^j \Delta^j z(s) \\ & + \sum_{r=n}^{s-1} \frac{(r - n + m - 2)^{(m-2)}}{(m-2)!} \Delta^{m-1} z(r) \quad \text{for } s \geq n. \end{aligned}$$

Considering Lemma 1.1 with $l = 0$ and using this in the above equation, we get

$$z(n) \geq \sum_{r=n}^{s-1} \frac{(r - n + m - 2)^{(m-2)}}{(m-2)!} \Delta^{m-1} z(r) \quad \text{for } n \geq n_1 \geq n_0.$$

Then we can find an integer $n_2 \geq n_1$ and a nondecreasing function $\eta(n)$ with $\sigma(n) < \eta(n) \leq n$ such that

$$z\big(\sigma(n)\big) \geq \sum_{r=\sigma(n)}^{\eta(n)} \frac{(r - \sigma(n) + m - 2)^{(m-2)}}{(m-2)!} \Delta^{m-1} z\big(\eta(n)\big) \quad \text{for } n \geq n_2 \geq n_1. \tag{2.17}$$

From (1.1), (1.2), (2.7), and (2.17), we have

$$\begin{aligned}-\Delta\big(\Delta^{m-1}z(n)\big) &= q(n)f\big(x\big(\sigma(n)\big)\big)\\ &\geq q(n)f\big(1-p\big(\sigma(n)\big)\big)f\big(z\big(\sigma(n)\big)\big)\\ &\geq q(n)f\big(1-p\big(\sigma(n)\big)\big)f\left(\sum_{r=\sigma(n)}^{\eta(n)}\frac{(r-\sigma(n)+m-2)^{(m-2)}}{(m-2)!}\right)\\ &\quad\times f\big(\Delta^{m-1}z\big(\eta(n)\big)\big)\\ &= R_0(n)f\big(\Delta^{m-1}z\big(\eta(n)\big)\big).\end{aligned}$$

Let $v(n) = \Delta^{m-1}z(\eta(n))$. Then $v(n) > 0$ for $n \geq n_2$, and the above inequality becomes

$$\Delta v(n) + R_0(n)f\big(v\big(\sigma(n)\big)\big) \leq 0,$$

for which an eventually positive solution exists. By a well-known result in [14, p. 186, Corollary 7.6.1], we have equation (2.3) also has an eventually positive solution, which contradicts our assumption. This completes the proof. □

Example 2.3 Consider the third order difference equation

$$\Delta^3\left[x(n)+\frac{1}{2}x(n-1)\right]+4x(n-2)=0. \tag{E1}$$

Here, $0 \leq p(n) = \frac{1}{2} < 1$, $q(n) = 4n$, $\tau(n) = n-1 < n$, $\sigma(n) = n-2 < n$, and $f(u) = \frac{u}{n}$.

Also,

$$\sum_{n=n_0}^{\infty}\left(\sum_{j=n}^{\infty}q(j)\right)\big(\sigma(n)\big)^{(m-2)} = \sum_{n=n_0}^{\infty}\left(\sum_{j=n}^{\infty}4j\right)(n-2)=\infty.$$

We can easily see that all the conditions of Theorem 2.1 are satisfied, and hence all the solutions of equation (E1) are oscillatory.

One of such solutions is $x(n) = (-1)^n$.

Example 2.4 Consider the second order difference equation

$$\Delta^2\left[x(n)+\frac{1}{4}x(n-2)\right]+\frac{5n+3}{n-1}x(n-1)=0. \tag{E2}$$

Here, $0 \leq p(n) = \frac{1}{4} < 1$, $q(n) = \frac{5n+3}{n-1}$, $\tau(n) = n-2 < n$, $\sigma(n) = n-1 < n$, and $f(u) = u$.

Also,

$$\sum_{n=n_0}^{\infty}\left(\sum_{j=n}^{\infty}q(j)\right)\big(\sigma(n)\big)^{(m-2)} = \sum_{n=n_0}^{\infty}\left(\sum_{j=n}^{\infty}\frac{5j+3}{j-1}\right)=\infty.$$

We check that all the conditions of Theorem 2.2 are satisfied. In fact, $x(n) = n(-1)^n$ is an oscillatory solution of (E2).

Next we shall present the following theorems.

Theorem 2.5 *Assume that conditions* (H1)–(H4) *and* (2.1) *hold. Let m be odd. If*

$$f\left(\frac{(\sigma(n)-m+j)^{(j-i)}}{j!}\right)\sum_{r=n_0}^{\infty}\frac{(r-n+m-j-2)^{(m-j-2)}}{(m-j-2)!}$$
$$\times\left(\sum_{j=r}^{\infty}q(j)\right)f\bigl(1-p\bigl(\sigma(r)\bigr)\bigr)=\infty \tag{2.18}$$

for $j\in\{2,4,\ldots,m-1\}$ *and if there exists a nondecreasing sequence* $\{\eta(n)\}$ *with* $\sigma(n)<\eta(n)\leq n, n\geq n_0$ *such that equation* (2.3) *is oscillatory, then every solution of equation* (1.1) *oscillates.*

Theorem 2.6 *Assume that conditions* (H1)–(H4) *and* (2.1) *hold. Let m be even. If condition* (2.18), $j\in\{1,3,\ldots,m-3\}$ *holds for all large n and if*

$$f\left(\frac{(\sigma(n))^{(m-2)}}{(m-1)!}\right)f\bigl(1-p\bigl(\sigma(n)\bigr)\bigr)\left(\sum_{r=n_0}^{\infty}q(r)\right)=\infty, \tag{2.19}$$

then all the solutions of (1.1) *are oscillatory.*

Proofs of Theorems 2.5 *and* 2.6 Assume that $\{x(n)\}$ is a non-oscillatory solution of (1.1). Without loss of generality, assume that $x(n)>0, x(\tau(n))>0, x(\sigma(n))>0$ for all $n\geq n_0\geq 0$.

Let

$$z(n)=x(n)+p(n)x\bigl(\tau(n)\bigr)\geq x(n)>0.$$

Proceeding as in the proof of Theorems 2.1 and 2.2, we get the following three cases: $l\in\{1,2,\ldots,m-3\}, l=m-1, l=0$.

Case (i): $l\in\{1,2,\ldots,m-3\}$. From discrete Taylor's formula, we have

$$\Delta^l z(n)=\sum_{j=l}^{m-2}\frac{(s-n+j-l-2)^{(j-l)}}{(j-l)!}(-1)^{j-l}\Delta^j z(s)$$
$$+(-1)^{m-l-1}\sum_{r=n}^{s-1}\frac{(r-n+m-l-2)^{(m-l-2)}}{(m-l-2)!}\Delta^{m-1}z(r) \tag{2.20}$$

for $s\geq n$. Using (1.2), (2.7), (2.10), and Lemma 1.1 in (2.20), we obtain

$$\Delta^l z(n)\geq\sum_{r=n}^{\infty}\frac{(r-n+m-l-2)^{(m-l-2)}}{(m-l-2)!}\left(\sum_{j=r}^{\infty}q(j)\right)$$
$$\times f\bigl(1-p\bigl(\sigma(n)\bigr)\bigr)f\bigl(z\bigl(\sigma(n)\bigr)\bigr). \tag{2.21}$$

From (2.15), there exist $n_2\geq n_1$ and a positive constant $c>0$ such that

$$z\bigl(\sigma(n)\bigr)\geq\frac{(\sigma(n)-m+l)^{(l-1)}}{l!}\Delta^{l-1}z\bigl(\sigma(n)\bigr)\quad\text{for } n\geq n_2 \tag{2.22}$$

and

$$\Delta^{l-1} z\big(\sigma(n)\big) \geq c \quad \text{for } n \geq n_2. \tag{2.23}$$

Using (2.22) and (2.23) in (2.21), we get

$$\begin{aligned} \infty > \Delta^{l} z(n_2) \geq \sum_{r=n_2}^{\infty} \frac{(r-n+m-l-2)^{(m-l-2)}}{(m-l-2)!} \\ \times \left(\sum_{j=r}^{\infty} q(j)\right) f\big(1-p\big(\sigma(n)\big)\big) f(c) f\left(\frac{(\sigma(n)-m+l)^{(l-1)}}{l!}\right), \end{aligned}$$

which contradicts (2.18).

Case (ii): $l = m-1$. Summing up equation (1.1) from r to $u-1$ and letting $u \to \infty$, we have

$$\Delta^{m-1} z(r) \geq \sum_{j=r}^{\infty} q(j) f\big(1-p\big(\sigma(n)\big)\big) f\big(z\big(\sigma(n)\big)\big) \quad \text{for } n \geq n_1 \geq n_0. \tag{2.24}$$

Using (2.22) and (2.23) in (2.24), we get a contradiction to (2.19).

Case (iii): $l = 0$. The proof for this case is similar to the proof of Case (iii) in Theorems 2.1 and 2.2 and is hence omitted. This completes the proof. □

Example 2.7 Consider the first order difference equation

$$\Delta\left[x(n) + \frac{3}{4} x(n-1)\right] + \frac{1}{2} x(n-2) = 0. \tag{E3}$$

Here, $0 \leq p(n) = \frac{3}{4} < 1$, $q(n) = \frac{1}{2}$, $\tau(n) = n-1 < n$, $\sigma(n) = n-2 < n$, and $f(u) = u$.

We can find that all the hypotheses of Theorem 2.5 are fulfilled. Also, equation (E3) has an oscillatory solution given by $x(n) = \frac{(-1)^n}{2}$.

Example 2.8 Consider the fourth order difference equation

$$\Delta^4\left[x(n) + \frac{1}{2} x(n-1)\right] + 8(n+2) x(n-1) = 0. \tag{E4}$$

Here, $0 \leq p(n) = \frac{1}{2} < 1$, $q(n) = 8(n+2)$, $\tau(n) = n-1 < n$, $\sigma(n) = n-1 < n$, and $f(u) = \frac{u}{n+2}$.

It is noted that all the conditions of Theorem 2.6 are satisfied. Also, we can find an oscillatory solution given by $x(n) = (-1)^n$ for equation (E4).

3 Conclusion

In this paper, by using discrete Taylor's formula, the summing averaging technique, and the comparison method, the oscillatory behavior of every solution of equation (1.1) is discussed in Theorems 2.1 and 2.2, Theorems 2.5 and 2.6. Here, some sufficient conditions are proved. These sufficient conditions, which are new, extend and complement some of the known results in the literature. Also, the examples reveal the illustration of the proved results.

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