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Period-doubling and Neimark–Sacker bifurcations of plant–herbivore models

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Abstract

The interaction between plants and herbivores plays a vital role for understanding community dynamics and ecosystem function given that they are the critical link between primary production and food webs. This paper deals with the qualitative nature of two discrete-time plant–herbivore models. In both discrete-time models, function for plant-limitation is of Ricker type, whereas the effect of herbivore on plant population and herbivore population growth rate are proportional to functional responses of type-II and type-III. Furthermore, we discuss the existence of equilibria and parametric conditions of topological classification for these equilibria. Our analysis shows that positive steady states of both discrete-time plant–herbivore models undergo flip and Hopf bifurcations. Moreover, we implement a hybrid control strategy, based on parameter perturbation and state feedback control, for controlling chaos and bifurcations. Finally, we provide some numerical simulations to illustrate theoretical discussion.

Keywords: Plant–herbivore model; Local stability; Period-doubling bifurcation; Neimark–Sacker bifurcation; Chaos control

1 Introduction

The mathematical framework for plant–herbivore models is identical to interaction between preys and their predators. In other words, such type models are basically modifications of prey–predator systems [1]. The interaction between plants and herbivores has been investigated by many researchers both in differential and difference equations. Kartal [2] investigated the dynamical behavior of a plant–herbivore model including both differential and difference equations. Kang et al. [3] discussed bistability, bifurcation, and chaos control in a discrete-time plant–herbivore model. Liu et al. [4] investigated stability, limit cycle, Hopf bifurcations, and homoclinic bifurcation for a plant–herbivore model with toxin-determined functional response. Li et al. [5] discussed period-doubling and Hopf bifurcations for a plant–herbivore model incorporating plant toxicity in the functional response of plant–herbivore interactions. Similarly, for some other discussions related to qualitative behavior of plant–herbivore models, we refer the interested reader to [6–13] and references therein.

We consider interaction between plants and herbivores by taking into account nonoverlapping generation. For this, the growth rate for herbivore population is assumed to be proportional to a function of nonlinear type dependent upon their feeding rate [14]. Moreover, we suppose that the growth rate for plant population is inversely proportional to population of herbivore, and the feeding rate of herbivore is dependent on plant density [15]. Furthermore, an intraspecific competition is implemented for controlling the plant population density in the absence of herbivore [14]. Then the general discrete-time plant–herbivore interaction is modeled by the following planar system [16]:

$$\begin{aligned} U_{n+1} &= rU_n f(U_n) g(V_n), \\ V_{n+1} &= cV_n h(U_n), \end{aligned} \tag{1.1}$$

where U_n and V_n represent the population densities of plant and herbivore at generation n, respectively. Furthermore, $f(U_n)$ is used for the growth rate of plant, the effect of herbivore population on the plant growth rate is denoted by the function $g(V_n)$, the saturation function for plant density is represented by $h(U_n)$, and the population growth rates for plant and herbivore are denoted by r and c, respectively.

Arguing as in [16], we can choose $f(U_n) = e^{-aU_n}$ as the Ricker growth function [17], where larger values for a give stronger density dependence in the growth rate of plant. Furthermore, we can choose for $g(V_n)$ and $h(U_n)$ functional responses of type II as follows:

$$g(V_n) = \frac{\alpha}{\beta + V_n}, \qquad h(U_n) = \frac{\gamma U_n}{\delta + U_n},$$

where α, β, γ, and δ are positive parameters. Then we rewrite system (1.1) as follows:

$$U_{n+1} = \frac{r\alpha U_n e^{-aU_n}}{\beta + V_n}, \qquad V_{n+1} = \frac{c\gamma V_n U_n}{\delta + U_n}. \tag{1.2}$$

For the dimensionless form of system (1.2), we choose $x_n = aU_n$ and $y_n = V_n/\beta$. Then it follows that

$$x_{n+1} = \frac{kx_n e^{-x_n}}{1 + y_n}, \qquad y_{n+1} = \frac{px_n y_n}{q + x_n}, \tag{1.3}$$

where $k = \frac{r\alpha}{\beta}$, $c\gamma = p$, $a\delta = q$, and x_n and y_n are population densities of plant and herbivore at generation n, respectively.

Similarly, we can implement functional responses of type-III for $g(V_n)$ and $h(U_n)$ as follows:

$$g(V_n) = \frac{\alpha_1}{\beta_1^2 + V_n^2}, \qquad h(U_n) = \frac{\gamma_1 U_n^2}{\delta_1^2 + U_n^2}.$$

Due to implementation of functional responses of type-III, we obtain the following plant–herbivore model:

$$U_{n+1} = \frac{r\alpha_1 U_n e^{-aU_n}}{\beta_1^2 + V_n^2}, \qquad V_{n+1} = \frac{c\gamma_1 V_n U_n^2}{\delta_1^2 + U_n^2}. \tag{1.4}$$

Moreover, implementing the transformations $x_n = aU_n$ and $y_n = V_n/\beta_1$ to system (1.4), we have the following discrete-time plant–herbivore model:

$$x_{n+1} = \frac{\mu x_n e^{-x_n}}{1 + y_n^2}, \qquad y_{n+1} = \frac{\nu x_n^2 y_n}{\eta + x_n^2}, \tag{1.5}$$

where $\mu = r\alpha_1/\beta_1^2$, $c\gamma_1 = \nu$, and $a^2\delta_1^2 = \eta$.

It is worth pointing out the rationality of considering the two types of models, that is, functional response of type-II in system (1.3) and functional response of type-III in system (1.5). For this, note that the functional response of type-III is connected with switching predators, that is, if there is an option of prey, then the predator prefers the more bumper prey type. Consequently, the functional response of type-III has been observed in experiments. On the other hand, in real life studies the functional response of type-II is frequently observed [18]. Therefore system (1.3) is associated with real-life plant–herbivore interaction, and system (1.5) is a good representative for its experimental study.

In further discussion, we study the qualitative behavior for discrete-time plant–herbivore systems (1.3) and (1.5). The main contributions of this paper are as follows:

- Keeping in view standard techniques for stability analysis of discrete-time systems, we analyze the local asymptotic behavior of these plant–herbivore models of discrete nature.
- We investigated that both models undergo flip bifurcation at their positive steady states by implementing bifurcation theory, normal form theory, and center manifold theorem.
- We study Hopf bifurcation for these models at their positive steady states with bifurcation theory of normal forms.
- We introduce a chaos control method based on parameter perturbation and introduce state feedback methodology for controlling chaotic and fluctuating behaviors for these models.
- We present numerical simulations for verification of our theoretical discussions.

In Sects. 2 and 3, we discuss the existence of steady states and their linearized stability for both systems (1.3) and (1.5). In Sect. 4, we investigate period-doubling bifurcations at positive equilibria of systems (1.3) and (1.5). In Sect. 5, we show that the positive steady states of both systems (1.3) and (1.5) undergo Neimark–Sacker bifurcations. Moreover, we implement a hybrid feedback control methodology for controlling chaos and bifurcations for both systems (1.3) and (1.5) in Sect. 6. Finally, in Sect. 7, we present numerical examples to support and illustrate theoretical discussion.

2 Linearized stability of system (1.3)

The equilibria of system (1.3) can be obtained by solving the following algebraic equations:

$$x = \frac{kxe^{-x}}{1+y}, \qquad y = \frac{pxy}{q+x}.$$

We can easily obtain the solutions for aforementioned algebraic system as $(0,0)$, $(\ln(k),0)$, and $(\frac{q}{p-1}, k\exp(-\frac{q}{p-1})-1)$. Thus trivial equilibrium $(0,0)$ always exists, boundary equilibrium $(\ln(k),0)$ exists only for $k>1$, and a unique positive equilibrium $(\frac{q}{p-1}, k\exp(-\frac{q}{p-1})-1)$ for system (1.3) exists if and only if $p>1$, $k>1$, and $0<q<\ln(k)(p-1)$. For simplicity, the unique positive equilibrium point of system (1.3) is $(u^*, ke^{-u^*}-1)$, where $u^* := \frac{q}{p-1}$. Moreover, the Jacobian matrix of system (1.3) evaluated at (x,y) is as follows:

$$J(x,y) = \begin{bmatrix} \frac{ke^{-x}(1-x)}{1+y} & -\frac{ke^{-x}x}{(1+y)^2} \\ \frac{pqy}{(q+x)^2} & \frac{px}{q+x} \end{bmatrix}. \tag{2.1}$$

Taking $(x, y) = (0, 0)$ in (2.1), we obtain the Jacobian matrix $J(0, 0)$ at trivial equilibrium of system (1.3):

$$J(0,0) = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, from $J(0, 0)$ it easy to see that the trivial equilibrium $(0, 0)$ is a sink if and only if $0 < k < 1$, it is saddle point if $k > 1$, and $(0, 0)$ is nonhyperbolic at $k = 1$.

Taking into account the relation between roots and coefficients for a quadratic equation, the following lemma gives locality of roots with respect to the unit disk (see also [19–24]):

Lemma 2.1 *Suppose that* $\mathbb{P}(\kappa) = \kappa^2 - A\kappa + B$ *with* $\mathbb{P}(1) > 0$ *and* κ_1, κ_2 *are roots of* $\mathbb{P}(\kappa) = 0$. *Then, the following results hold*: (i) $|\kappa_1| < 1$ *and* $|\kappa_2| < 1$ *if and only if* $\mathbb{P}(-1) > 0$ *and* $\mathbb{P}(0) < 1$; (ii) $|\kappa_1| < 1$ *and* $|\kappa_2| > 1$, *or* $|\kappa_1| > 1$ *and* $|\kappa_2| < 1$ *if and only if* $\mathbb{P}(-1) < 0$; (iii) $|\kappa_1| > 1$ *and* $|\kappa_2| > 1$ *if and only if* $\mathbb{P}(-1) > 0$ *and* $\mathbb{P}(0) > 1$; (iv) $\kappa_1 = -1$ *or* $\kappa_2 = -1$ *if and only if* $\mathbb{P}(-1) = 0$; *and* (v) κ_1 *and* κ_2 *are conjugate complex numbers with* $|\kappa_1| = 1$ *and* $|\kappa_2| = 1$ *if and only if* $A^2 - 4B < 0$ *and* $\mathbb{P}(0) = 1$.

Furthermore, we suppose that $k > 1$. Then the variational matrix $J(x, y)$ at the boundary equilibrium $(x, y) = (\ln(k), 0)$ is computed as follows:

$$J\big(\ln(k),0\big) = \begin{bmatrix} 1-\ln(k) & -\ln(k) \\ 0 & \frac{p\ln(k)}{q+\ln(k)} \end{bmatrix}. \tag{2.2}$$

Lemma 2.2 *Assume that* $k > 1$. *Then the following statements hold*:

(i) $(\ln(k), 0)$ *is a sink if and only if* $1 < k < e^2$ *and* $p\ln(k) < q + \ln(k)$.

(ii) $(\ln(k), 0)$ *is a saddle point if and only if* $1 < k < e^2$ *and* $p\ln(k) > q + \ln(k)$, *or* $k > e^2$ *and* $p\ln(k) < q + \ln(k)$.

(iii) $(\ln(k), 0)$ *is a source if and only if* $k > e^2$ *and* $p\ln(k) > q + \ln(k)$.

(iv) $(\ln(k), 0)$ *is nonhyperbolic if and only if* $k = e^2$ *or* $p\ln(k) = q + \ln(k)$.

Moreover, for $k \in [1, 20]$, $p \in [0, 50]$, and $q = 35$, the topological classification of $(\ln(k), 0)$ is depicted in Fig. 1.

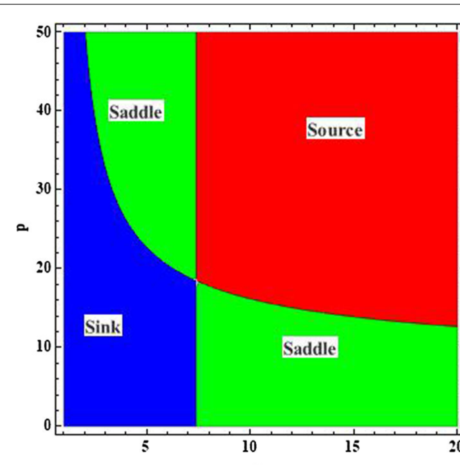


Figure 1 Topological classification of $(\ln(k), 0)$ for $k \in [1, 20]$, $p \in [0, 50]$ and $q = 35$

Next, we assume that $0 < q < (p-1)\ln(k)$, $k > 1$, and $p > 1$. Then the variational matrix (2.1) computed at positive steady state $(u^*, ke^{-u^*} - 1)$ is given as follows:

$$J\left(u^*, ke^{-u^*} - 1\right) = \begin{bmatrix} 1 - u^* & -\frac{u^* e^{u^*}}{k} \\ \frac{(ke^{-u^*}-1)(p-1)}{pu^*} & 1 \end{bmatrix}. \tag{2.3}$$

Moreover, the characteristic polynomial for $J(u^*, ke^{-u^*} - 1)$ is computed as follows:

$$F(\lambda) = \lambda^2 - \left(2 - \frac{q}{p-1}\right)\lambda + 2 - \frac{e^{\frac{q}{p-1}}}{k} - \frac{1}{p} + \frac{e^{\frac{q}{p-1}}}{kp} - \frac{q}{p-1}. \tag{2.4}$$

It must be noted that, due to the conditions for positivity of $(u^*, ke^{-u^*} - 1)$, we have

$$F(1) = \frac{(k - e^{u^*})(p-1)}{kp} > 0.$$

Thus we can implement Lemma 2.1 to prove the following results.

Lemma 2.3 *Let $0 < q < (p-1)\ln(k)$, $k > 1$, and $p > 1$, then the following hold for the topological classification of equilibrium point $(u^*, ke^{-u^*} - 1)$ of system (1.3):*

(i) $(u^*, ke^{-u^*} - 1)$ *is a sink if and only if*

$$\frac{e^{\frac{q}{p-1}}}{k} + \frac{1}{p} + \frac{2q}{p-1} < 5 + \frac{e^{\frac{q}{p-1}}}{kp}$$

and

$$1 + \frac{e^{\frac{q}{p-1}}}{kp} < \frac{e^{\frac{q}{p-1}}}{k} + \frac{1}{p} + \frac{q}{p-1}.$$

(ii) $(u^*, ke^{-u^*} - 1)$ *is a saddle point if and only if*

$$\frac{e^{\frac{q}{p-1}}}{k} + \frac{1}{p} + \frac{2q}{p-1} > 5 + \frac{e^{\frac{q}{p-1}}}{kp}.$$

(iii) $(u^*, ke^{-u^*} - 1)$ *is a source if and only if*

$$\frac{e^{\frac{q}{p-1}}}{k} + \frac{1}{p} + \frac{2q}{p-1} < 5 + \frac{e^{\frac{q}{p-1}}}{kp}$$

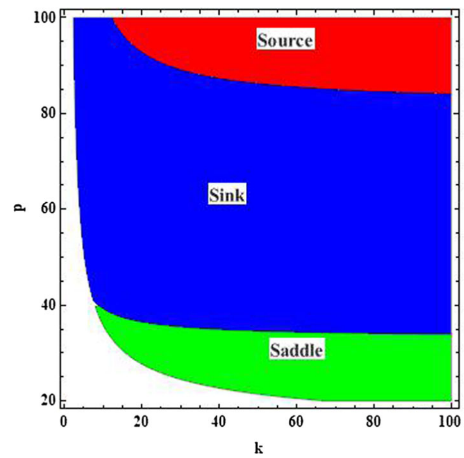
and

$$1 + \frac{e^{\frac{q}{p-1}}}{kp} > \frac{e^{\frac{q}{p-1}}}{k} + \frac{1}{p} + \frac{q}{p-1}.$$

(iv) $(u^*, ke^{-u^*} - 1)$ *is nonhyperbolic with roots $\lambda_1 = -1$ or $\lambda_2 = -1$ of (2.4) if and only if*

$$\frac{e^{\frac{q}{p-1}}}{k} + \frac{1}{p} + \frac{2q}{p-1} = 5 + \frac{e^{\frac{q}{p-1}}}{kp}.$$

Figure 2 Topological classification of $(u^*, ke^{-u^*} - 1)$



(v) $(u^*, ke^{-u^*} - 1)$ *is nonhyperbolic such that complex conjugate roots of* (2.4) *are with modulus one if and only if*

$$k = \frac{e^{\frac{q}{p-1}}(p-1)^2}{1 - 2p + p^2 - pq}$$

and

$$\left(2 - \frac{q}{p-1}\right)^2 < 4\left(2 - \frac{e^{\frac{q}{p-1}}}{k} - \frac{1}{p} + \frac{e^{\frac{q}{p-1}}}{kp} - \frac{q}{p-1}\right).$$

For $k \in [1, 100]$, $p \in [20, 100]$, and $q = 80$, the topological classification of $(u^*, ke^{-u^*} - 1)$ is shown in Fig. 2.

3 Linearized stability of system (1.5)

Suppose that (x, y) is an arbitrary steady state for system (1.5). Then it solves the following algebraic system:

$$x = \frac{\mu x e^{-x}}{1 + y^2}, \qquad y = \frac{\nu x^2 y}{\eta + x^2}.$$

Simple computation yields the following nonnegative steady states for system (1.5):

$$(0, 0), \qquad \big(\ln(\mu), 0\big), \qquad \big(\nu^*, \sqrt{\mu e^{-\nu^*} - 1}\big),$$

where $\nu^* := \sqrt{\frac{\eta}{\nu-1}}$, $\mu > 1$, $\nu > 1$, and $\sqrt{\eta} < \ln(\mu)\sqrt{\nu - 1}$. Furthermore, the variational matrix for system (1.5) computed at (x, y) is given as follows:

$$V(x, y) = \begin{bmatrix} \frac{\mu e^{-x}(1-x)}{1+y^2} & -\frac{2\mu x y e^{-x}}{(1+y^2)^2} \\ \frac{2xy\eta\nu}{(x^2+\eta)^2} & \frac{\nu x^2}{\eta + x^2} \end{bmatrix}. \qquad (3.1)$$

Then, at $(x,y) = (0,0)$, $V(x,y)$ reduces to

$$V(0,0) = \begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix}.$$

Then from $V(0,0)$ it follows that $(0,0)$ is a sink if and only if $0 < \mu < 1$, saddle if and only if $\mu > 1$, and nonhyperbolic if and only if $\mu = 1$. Moreover, assume that $\mu > 1$. Then the variational matrix $V(x,y)$ computed at semitrivial equilibrium $(x,y) = (\ln(\mu), 0)$ is given as follows:

$$V\bigl(\ln(\mu),0\bigr) = \begin{bmatrix} 1-\ln(\mu) & 0 \\ 0 & \frac{\nu \ln^2(\mu)}{\eta+\ln^2(\mu)} \end{bmatrix}. \tag{3.2}$$

Lemma 3.1 *Assume that* $\mu > 1$. *Then the following statements hold*:

(i) $(\ln(\mu),0)$ *is a sink if and only if* $1 < \mu < e^2$ *and* $\nu(\ln(\mu))^2 < \eta + (\ln(\mu))^2$.

(ii) $(\ln(\mu),0)$ *is a saddle point if and only if* $1 < \mu < e^2$ *and* $\nu(\ln(\mu))^2 > \eta + (\ln(\mu))^2$, *or* $\mu > e^2$ *and* $\nu(\ln(\mu))^2 < \eta + (\ln(\mu))^2$.

(iii) $(\ln(\mu),0)$ *is a source if and only if* $\mu > e^2$ *and* $\nu(\ln(\mu))^2 > \eta + (\ln(\mu))^2$.

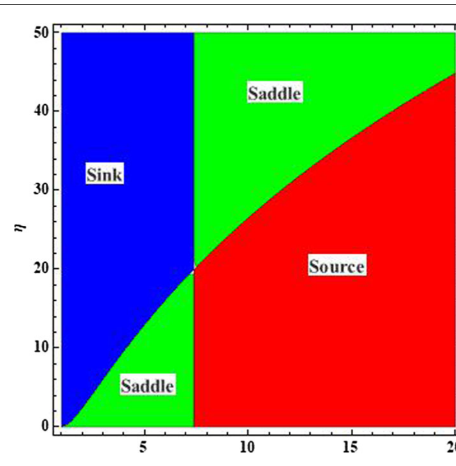
(iv) $(\ln(\mu),0)$ *is nonhyperbolic if and only if* $\mu = e^2$ *or* $\nu(\ln(\mu))^2 = \eta + (\ln(\mu))^2$.

Moreover, for $\mu \in [1,20]$, $\eta \in [0,50]$, and $\nu = 6$, the topological classification for $(\ln(\mu),0)$ is depicted in Fig. 3.

Moreover, we suppose that $\sqrt{\eta} < \ln(\mu)\sqrt{\nu-1}$, $\nu > 1$, and $\mu > 1$. Then the variational matrix computed at positive steady state $(\nu^*, \sqrt{\mu e^{-\nu^*}-1})$ is given as follows:

$$V\Bigl(\nu^*, \sqrt{\mu e^{-\nu^*}-1}\Bigr) = \begin{bmatrix} 1-\frac{\sqrt{\eta}}{\sqrt{\nu-1}} & -\frac{2\sqrt{e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}}\sqrt{\eta}\sqrt{\mu-e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}}}{\mu\sqrt{\nu-1}} \\ \frac{2\sqrt{\mu-e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}}(\nu-1)^{3/2}}{\sqrt{e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}}\sqrt{\eta}\nu} & 1 \end{bmatrix}. \tag{3.3}$$

Figure 3 Topological classification of $(\ln(\mu),0)$



The characteristic polynomial of $V(\nu^*, \sqrt{\mu e^{-\nu^*} - 1})$ is given by

$$F(\lambda) = \lambda^2 - \left(2 - \frac{\sqrt{\eta}}{\sqrt{\nu - 1}}\right)\lambda + 5 - \frac{\sqrt{\eta}}{\sqrt{\nu - 1}} - \frac{4}{\nu} - \frac{4e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}(\nu - 1)}{\mu\nu}. \quad (3.4)$$

Then from (3.4) it follows that

$$F(1) = \frac{4(\mu - e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}})(\nu - 1)}{\mu\nu} > 0.$$

Therefore Lemma 2.1 can be implemented to prove the following results, which give a topological classification for equilibrium $(\nu^*, \sqrt{\mu e^{-\nu^*} - 1})$.

Lemma 3.2 *Let* $\sqrt{\eta} < \ln(\mu)\sqrt{\nu - 1}$, $\nu > 1$, *and* $\mu > 1$. *Then we have the following results for equilibrium* $(\nu^*, \sqrt{\mu e^{-\nu^*} - 1})$ *of system* (1.5):

(i) $(\nu^*, \sqrt{\mu e^{-\nu^*} - 1})$ *is a sink if and only if*

$$\frac{\sqrt{\eta}}{\sqrt{\nu - 1}} + \frac{2}{\nu} + \frac{2e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}(\nu - 1)}{\mu\nu} < 2$$

and

$$4 < \frac{\sqrt{\eta}}{\sqrt{\nu - 1}} + \frac{4}{\nu} + \frac{4e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}(\nu - 1)}{\mu\nu}.$$

(ii) $(\nu^*, \sqrt{\mu e^{-\nu^*} - 1})$ *is a saddle point if and only if*

$$4 > \frac{\sqrt{\eta}}{\sqrt{\nu - 1}} + \frac{4}{\nu} + \frac{4e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}(\nu - 1)}{\mu\nu}.$$

(iii) $(\nu^*, \sqrt{\mu e^{-\nu^*} - 1})$ *is a source if and only if*

$$\frac{\sqrt{\eta}}{\sqrt{\nu - 1}} + \frac{2}{\nu} + \frac{2e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}(\nu - 1)}{\mu\nu} < 2$$

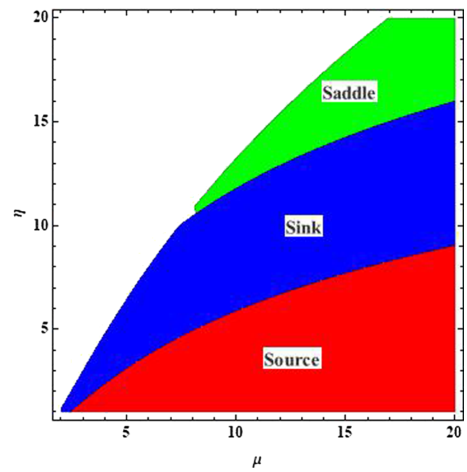
and

$$4 > \frac{\sqrt{\eta}}{\sqrt{\nu - 1}} + \frac{4}{\nu} + \frac{4e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}(\nu - 1)}{\mu\nu}.$$

(iv) $(\nu^*, \sqrt{\mu e^{-\nu^*} - 1})$ *is nonhyperbolic with roots* $\lambda_1 = -1$ *or* $\lambda_2 = -1$ *of* (3.4) *if and only if*

$$\frac{\sqrt{\eta}}{\sqrt{\nu - 1}} + \frac{2}{\nu} + \frac{2e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}(\nu - 1)}{\mu\nu} = 4.$$

Figure 4 Topological classification of $(v^*, \sqrt{\mu e^{-v^*} - 1})$



(v) $(\nu^*, \sqrt{\mu e^{-\nu^*} - 1})$ *is nonhyperbolic such that complex conjugate roots of* (3.4) *are with modulus one if and only if*

$$\frac{\sqrt{\eta}}{\sqrt{\nu - 1}} + \frac{4}{\nu} + \frac{4e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}(\nu - 1)}{\mu\nu} = 4$$

and

$$\left(2 - \frac{\sqrt{\eta}}{\sqrt{\nu - 1}}\right)^2 < 4\left(5 - \frac{\sqrt{\eta}}{\sqrt{\nu - 1}} - \frac{4}{\nu} - \frac{4e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}(\nu - 1)}{\mu\nu}\right).$$

Taking $\mu \in [2, 20]$, $\eta \in [1, 20]$, and $\nu = 3$, the classification for positive steady state $(\nu^*, \sqrt{\mu e^{-\nu^*} - 1})$ is depicted in Fig. 4.

4 Period-doubling bifurcation

In this section, we analyze that positive equilibrium points of systems (1.3) and (1.5) undergo period-doubling bifurcation. For this bifurcation theory, normal forms and center manifold theorem are implemented for the existence and direction of such type of bifurcation. Recently, period-doubling bifurcation related to discrete-time models has been investigated by many authors [19–24].

First, we discuss emergence of flip bifurcation at positive equilibrium point $(u^*, ke^{-u^*} - 1)$ of system (1.3). For this, it follows from part (v) of Lemma 2.3 that $(u^*, ke^{-u^*} - 1)$ is nonhyperbolic with root one of (2.4), say, $\delta_1 = -1$, if the following condition is satisfied:

$$k = \frac{e^{\frac{q}{p-1}}(p - 1)^2}{(5p - 1)(p - 1) - 2pq}, \qquad (5p - 1)(p - 1) > 2pq. \tag{4.1}$$

Then, the second root for (2.4), say, δ_2, satisfies $|\delta_2| \neq 1$ if the following inequalities are satisfied:

$$q \neq 4(p - 1), \qquad q \neq 2(p - 1). \tag{4.2}$$

Moreover, under the assumptions that $p > 1$ and $q > 0$, we consider the following set:

$$\mathbb{S}_1 = \left\{(p,q,k) : k = \frac{e^{\frac{q}{p-1}}(p-1)^2}{(5p-1)(p-1)-2pq}, (5p-1)(p-1) > 2pq, q \neq 2(p-1), 4(p-1)\right\}.$$

Furthermore, let $(p,q,k) \in \mathbb{S}_1$. Then the positive steady-state $(u^*, ke^{-u^*}-1)$ of system (1.3) undergoes flip bifurcation such that k is taken as bifurcation parameter, and it varies in a small neighborhood of k_1 given by

$$k_1 := \frac{e^{\frac{q}{p-1}}(p-1)^2}{(5p-1)(p-1)-2pq}.$$

Moreover, system (1.3) is represented equivalently with the following two-dimensional map:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} \frac{kXe^{-X}}{1+Y} \\ \frac{pXY}{q+X} \end{pmatrix}. \tag{4.3}$$

To discuss period-doubling bifurcation for fixed point $(u^*, ke^{-u^*}-1)$ of (4.3), we suppose that $(p,q,k_1) \in \mathbb{S}_1$. Then it follows that

$$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} \frac{k_1Xe^{-X}}{1+Y} \\ \frac{pXY}{q+X} \end{pmatrix}. \tag{4.4}$$

We take k^* as small perturbation in parameter k_1. Then a perturbed map corresponding to (4.3) is given as follows:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} \frac{(k_1+k^*)Xe^{-X}}{1+Y} \\ \frac{pXY}{q+X} \end{pmatrix}. \tag{4.5}$$

Suppose that $x = X - a^*$ and $y = Y - b^*$ are such that $a^* = \frac{q}{p-1}$ and $b^* = (k_1 + k^*)e^{-a^*} - 1$. Then from (4.5) we obtain the following map with fixed point at $(0,0)$:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_1(x,y,k^*) \\ f_2(x,y,k^*) \end{pmatrix}, \tag{4.6}$$

where

$$\begin{aligned} f_1(x,y,k^*) &= \alpha_{13}x^2 + \alpha_{14}xy + \alpha_{15}y^2 + \alpha_{16}x^3 + \alpha_{17}x^2y + \alpha_{18}xy^2 + \alpha_{19}y^3 + a_1yk^* + a_2xyk^* \\ &\quad + a_3y^2k^* + a_4y(k^*)^2 + O\big((|x| + |y| + k^*)^4\big), \\ f_2(x,y,k^*) &= \alpha_{23}x^2 + \alpha_{24}xy + \alpha_{25}x^3 + \alpha_{26}x^2y + b_1xk^* + b_2x^2k^* + O\big((|x| + |y| + k^*)^4\big), \end{aligned}$$

$$\alpha_{11} = 1 - a^*, \qquad \alpha_{12} = -\frac{a^*e^{a^*}}{k_1}, \qquad \alpha_{13} = \frac{a^*-2}{2}, \qquad \alpha_{14} = \frac{(a^*-1)e^{a^*}}{k_1},$$

$$\alpha_{15} = \frac{a^*e^{2a^*}}{k_1^2}, \qquad \alpha_{16} = \frac{3-a^*}{6}, \qquad \alpha_{17} = \frac{e^{a^*}(2-a^*)}{2k_1}, \qquad \alpha_{18} = \frac{e^{2a^*}(1-a^*)}{k_1^2},$$

$$\alpha_{19} = -\frac{a^* e^{3a^*}}{k_1^3}, \qquad a_1 = \frac{a^* e^{a^*}}{k_1^2}, \qquad a_2 = \frac{e^{a^*}(1-a^*)}{k_1^2}, \qquad a_3 = -\frac{2a^* e^{2a^*}}{k_1^3},$$
$$a_4 = -\frac{a^* e^{a^*}}{k_1^3}, \qquad \alpha_{21} = \frac{pq(k_1 e^{-a^*}-1)}{(q+a^*)^2}, \qquad \alpha_{22} = \frac{pa^*}{q+a^*}, \qquad \alpha_{23} = \frac{pq(1-k_1 e^{-a^*})}{(q+a^*)^3},$$
$$\alpha_{24} = \frac{pq}{(q+a^*)^2}, \qquad \alpha_{25} = \frac{pq(k_1 e^{-a^*}-1)}{(q+a^*)^4}, \qquad \alpha_{26} = -\frac{pq}{(q+a^*)^3}, \qquad b_1 = \frac{pqe^{-a^*}}{(q+a^*)^2},$$
$$b_2 = -\frac{pqe^{-a^*}}{(q+a^*)^3}.$$

Under the assumption that $(p,q,k_1) \in \mathbb{S}_1$, the roots of (2.4) are computed as $\delta_1 = -1$ and

$$\delta_2 := \frac{3(p-1)-q}{p-1}.$$

To obtain a normal form of (4.6), we consider the following similarity transformation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha_{12} & \alpha_{12} \\ -1-\alpha_{11} & \delta_2-\alpha_{11} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{4.7}$$

Then implementing similarity transformation (4.7), we obtain the following normal form:

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & \delta_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_3(u,v,k^*) \\ f_4(u,v,k^*) \end{pmatrix}, \tag{4.8}$$

where

$$\begin{aligned}
f_3(u,v,k^*) &= \left(\frac{(\delta_2-\alpha_{11})\alpha_{16}}{\alpha_{12}(\delta_2+1)} - \frac{\alpha_{25}}{\delta_2+1}\right)x^3 + \left(\frac{(\delta_2-\alpha_{11})\alpha_{17}}{\alpha_{12}(\delta_2+1)} - \frac{\alpha_{26}}{\delta_2+1}\right)x^2y \\
&\quad + \left(\frac{(\delta_2-\alpha_{11})\alpha_{13}}{\alpha_{12}(\delta_2+1)} - \frac{b_2k^*+\alpha_{23}}{\delta_2+1}\right)x^2 + \frac{(\delta_2-\alpha_{11})\alpha_{18}xy^2}{\alpha_{12}(\delta_2+1)} \\
&\quad + \left(\frac{(\delta_2-\alpha_{11})(k^*a_2+\alpha_{14})}{\alpha_{12}(\delta_2+1)} - \frac{\alpha_{24}}{\delta_2+1}\right)xy - \frac{b_1k^*x}{\delta_2+1} + \frac{(\delta_2-\alpha_{11})\alpha_{19}y^3}{\alpha_{12}(\delta_2+1)} \\
&\quad + \frac{(\delta_2-\alpha_{11})(k^*a_3+\alpha_{15})y^2}{\alpha_{12}(\delta_2+1)} + \frac{(\delta_2-\alpha_{11})((k^*)^2a_4+k^*a_1)y}{\alpha_{12}(\delta_2+1)} \\
&\quad + O\bigl(\bigl(|u|+|v|+|k^*|\bigr)^4\bigr), \\
f_4(u,v,k^*) &= \left(\frac{(1+\alpha_{11})\alpha_{16}}{\alpha_{12}(\delta_2+1)} + \frac{\alpha_{25}}{\delta_2+1}\right)x^3 + \left(\frac{(1+\alpha_{11})\alpha_{17}}{\alpha_{12}(\delta_2+1)} + \frac{\alpha_{26}}{\delta_2+1}\right)x^2y \\
&\quad + \left(\frac{(1+\alpha_{11})\alpha_{13}}{\alpha_{12}(\delta_2+1)} + \frac{b_2k^*+\alpha_{23}}{\delta_2+1}\right)x^2 + \frac{(1+\alpha_{11})\alpha_{18}xy^2}{\alpha_{12}(\delta_2+1)} \\
&\quad + \left(\frac{(1+\alpha_{11})(k^*a_2+\alpha_{14})}{\alpha_{12}(\delta_2+1)} + \frac{\alpha_{24}}{\delta_2+1}\right)xy + \frac{b_1k^*x}{\delta_2+1} + \frac{(1+\alpha_{11})\alpha_{19}y^3}{\alpha_{12}(\delta_2+1)} \\
&\quad + \frac{(1+\alpha_{11})(k^*a_3+\alpha_{15})y^2}{\alpha_{12}(\delta_2+1)} + \frac{(1+\alpha_{11})((k^*)^2a_4+k^*a_1)y}{\alpha_{12}(\delta_2+1)} \\
&\quad + O\bigl(\bigl(|u|+|v|+|k^*|\bigr)^4\bigr),
\end{aligned}$$

and

$$x=\alpha_{12}(u+v),\qquad y=-(1+\alpha_{11})u+(\delta_2-\alpha_{11})v.$$

For the implementation of center manifold theorem, we suppose that $W^c(0,0,0)$ represents the center manifold for (4.8) evaluated at $(0,0)$ in a small neighborhood of $k^*=0$. Then $W^c(0,0,0)$ is approximated as follows:

$$W^c(0,0,0)=\left\{\left(u,v,k^*\right)\in\mathbb{R}^3: v=h_1u^2+h_2uk^*+h_3\left(k^*\right)^2+O\left(\left(|u|+\left|k^*\right|\right)^3\right)\right\},$$

where

$$\begin{aligned}h_1&=\left(\frac{(1+\alpha_{11})\alpha_{13}}{\alpha_{12}(1-\delta_2^2)}+\frac{\alpha_{23}}{1-\delta_2^2}\right)\alpha_{12}^2-\left(\frac{(1+\alpha_{11})\alpha_{14}}{\alpha_{12}(1-\delta_2^2)}+\frac{\alpha_{24}}{1-\delta_2^2}\right)\alpha_{12}(1+\alpha_{11})\\&\quad+\frac{\alpha_{15}(1+\alpha_{11})^3}{\alpha_{12}(1-\delta_2^2)},\\h_2&=\frac{b_1\alpha_{12}}{1-\delta_2^2}-\frac{(1+\alpha_{11})^2a_1}{\alpha_{12}(1-\delta_2^2)},h_3=0.\end{aligned}$$

The map restricted to the center manifold $W^c(0,0,0)$ is computed as follows:

$$F:u\to -u+s_1u^2+s_2uk^*+s_3u^2k^*+s_4u\left(k^*\right)^2+s_5u^3+O\left(\left(|u|+\left|k^*\right|\right)^4\right),$$

where

$$\begin{aligned}s_1&=\left(\frac{(\delta_2-\alpha_{11})\alpha_{13}}{\alpha_{12}(1+\delta_2)}-\frac{\alpha_{23}}{1+\delta_2}\right)\alpha_{12}^2-\left(\frac{(\delta_2-\alpha_{11})\alpha_{14}}{\alpha_{12}(1+\delta_2)}-\frac{\alpha_{24}}{1+\delta_2}\right)\alpha_{12}(1+\alpha_{11})\\&\quad+\frac{(\delta_2-\alpha_{11})\alpha_{15}(1+\alpha_{11})^2}{\alpha_{12}(1+\delta_2)},\\s_2&=-\frac{b_1\alpha_{12}}{1+\delta_2}-\frac{(\delta_2-\alpha_{11})a_1(1+\alpha_{11})}{\alpha_{12}(1+\delta_2)},\\s_3&=2\left(\frac{(\delta_2-\alpha_{11})\alpha_{13}}{\alpha_{12}(1+\delta_2)}-\frac{\alpha_{23}}{1+\delta_2}\right)\alpha_{12}^2h_2-\frac{b_2\alpha_{12}^2}{1+\delta_2}\\&\quad+\left(\frac{(\delta_2-\alpha_{11})\alpha_{14}}{\alpha_{12}(1+\delta_2)}-\frac{\alpha_{24}}{1+\delta_2}\right)\alpha_{12}(\delta_2-\alpha_{11})h_2\\&\quad-\left(\left(\frac{(\delta_2-\alpha_{11})\alpha_{14}}{\alpha_{12}(\delta_2+1)}-\frac{\alpha_{24}}{1+\delta_2}\right)\alpha_{12}h_2+\frac{(\delta_2-\alpha_{11})a_2}{1+\delta_2}\right)(1+\alpha_{11})\\&\quad-\frac{b_1\alpha_{12}h_1}{\delta_2+1}-2\frac{(\delta_2-\alpha_{11})^2\alpha_{15}(1+\alpha_{11})h_2}{\alpha_{12}(1+\delta_2)}+\frac{(\delta_2-\alpha_{11})a_3(1+\alpha_{11})^2}{\alpha_{12}(1+\delta_2)}\\&\quad+\frac{(\delta_2-\alpha_{11})^2a_1h_1}{\alpha_{12}(1+\delta_2)},\\s_4&=\frac{(\delta_2-\alpha_{11})^2a_1h_2}{\alpha_{12}(1+\delta_2)}-\frac{b_1\alpha_{12}h_2}{1+\delta_2}-\frac{(\delta_2-\alpha_{11})a_4(1+\alpha_{11})}{\alpha_{12}(1+\delta_2)},\end{aligned}$$

$$
\begin{aligned}
s_5 = {} & \left(\frac{(\delta_2-\alpha_{11})\alpha_{16}}{\alpha_{12}(\delta_2+1)} - \frac{\alpha_{25}}{1+\delta_2}\right)\alpha_{12}^3 - \left(\frac{(\delta_2-\alpha_{11})\alpha_{17}}{\alpha_{12}(1+\delta_2)} - \frac{\alpha_{26}}{\delta_2+1}\right)\alpha_{12}^2(1+\alpha_{11}) \\
& + 2\left(\frac{(\delta_2-\alpha_{11})\alpha_{13}}{\alpha_{12}(1+\delta_2)} - \frac{\alpha_{23}}{1+\delta_2}\right)\alpha_{12}^2 h_1 + \frac{(\delta_2-\alpha_{11})\alpha_{18}(1+\alpha_{11})^2}{1+\delta_2} \\
& + \left(\frac{(\delta_2-\alpha_{11})\alpha_{14}}{\alpha_{12}(1+\delta_2)} - \frac{\alpha_{24}}{\delta_2+1}\right)\alpha_{12}h_1(\delta_2-\alpha_{11}) \\
& - \left(\frac{(\delta_2-\alpha_{11})\alpha_{14}}{\alpha_{12}(1+\delta_2)} - \frac{\alpha_{24}}{\delta_2+1}\right)\alpha_{12}h_1(1+\alpha_{11}) \\
& - \frac{(\delta_2-\alpha_{11})\alpha_{19}(1+\alpha_{11})^3}{\alpha_{12}(\delta_2+1)} - 2\frac{(\delta_2-\alpha_{11})^2\alpha_{15}(1+\alpha_{11})h_1}{\alpha_{12}(\delta_2+1)}.
\end{aligned}
$$

Next, we define the following two nonzero real numbers:

$$
l_1 = \left(\frac{\partial^2 f_1}{\partial u \partial k^*} + \frac{1}{2}\frac{\partial F}{\partial k^*}\frac{\partial^2 F}{\partial u^2}\right)_{(0,0)} = -\frac{b_1\alpha_{12}}{1+\delta_2} - \frac{(\delta_2-\alpha_{11})a_1(1+\alpha_{11})}{\alpha_{12}(1+\delta_2)} \neq 0
$$

and

$$
l_2 = \left(\frac{1}{6}\frac{\partial^3 F}{\partial u^3} + \left(\frac{1}{2}\frac{\partial^2 F}{\partial u^2}\right)^2\right)_{(0,0)} = s_1^2 + s_5 \neq 0.
$$

Due to aforementioned computation, we have the following result about period-doubling bifurcation of system (1.3).

Theorem 4.1 *If $l_1 \neq 0$ and $l_2 \neq 0$, then system* (1.3) *undergoes period-doubling bifurcation at the unique positive equilibrium point when parameter k varies in a small neighborhood of k_1. Furthermore, if $l_2 > 0$, then the period-two orbits that bifurcate from positive equilibrium of* (1.3) *are stable, and if $l_2 < 0$, then these orbits are unstable.*

At the end of this section, we discuss that the positive equilibrium $(\nu^*, \sqrt{\mu e^{-\nu^*} - 1})$ of system (1.5) undergoes flip bifurcation. For this, we first assume that

$$
2\sqrt{\nu-1}(2\nu-1) > \sqrt{\eta}\nu \tag{4.9}
$$

and take

$$
\mu = \frac{2e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}(\nu-1)^{3/2}}{2\sqrt{\nu-1}(2\nu-1) - \sqrt{\eta}\nu}. \tag{4.10}
$$

Suppose that (4.9) and (4.10) hold. Then the characteristic polynomial (3.4) has real roots τ_1 and τ_2 with $\tau_1 = -1$ and $|\tau_2| \neq 1$ if the following inequalities are satisfied:

$$
\frac{\sqrt{\eta}}{\sqrt{\nu-1}} \neq 4, \qquad \frac{\sqrt{\eta}}{\sqrt{\nu-1}} \neq 2. \tag{4.11}
$$

Keeping in view (4.9), (4.10), and (4.11), we define the following set:

$$
\mathbb{S}_2 = \left\{(\mu,\nu,\eta) : \mu = \frac{2e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}(\nu-1)^{\frac{3}{2}}}{2\sqrt{\nu-1}(2\nu-1) - \sqrt{\eta}\nu} \text{ and (4.9), (4.11) hold}\right\}.
$$

Suppose that $(\mu,\nu,\eta)\in\mathbb{S}_2$. Then the positive steady-state $(\nu^*,\sqrt{\mu e^{-\nu^*}-1})$ of system (1.5) undergoes flip bifurcation when μ is taken as bifurcation parameter and varies in a small neighborhood of μ_1 defined as

$$\mu_1 := \frac{2e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}(\nu-1)^{3/2}}{2\sqrt{\nu-1}(2\nu-1)-\sqrt{\eta}\nu}.$$

In the similar fashion as for system (1.3), we construct a perturbed map corresponding to system (1.5) as follows:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} \frac{(\mu_1+\mu^*)Xe^{-X}}{1+Y^2} \\ \frac{\nu X^2 Y}{\eta+X^2} \end{pmatrix}. \tag{4.12}$$

We consider the translations $x = X-\nu^*$ and $y = Y-\sqrt{(\mu_1+\mu^*)e^{-\nu^*}-1}$, where $\nu^* = \sqrt{\frac{\eta}{\nu-1}}$, for the conversion of map (4.12) into the following form having $(0,0)$ as its fixed point:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g_1(x,y,\mu^*) \\ g_2(x,y,\mu^*) \end{pmatrix}, \tag{4.13}$$

where

$$\begin{aligned} g_1(x,y,\mu^*) &= \beta_{13}x^2+\beta_{14}xy+\beta_{15}y^2+\beta_{16}x^3+\beta_{17}x^2y+\beta_{18}xy^2+\beta_{19}y^3+c_1y\mu^* \\ &\quad + c_2xy\mu^*+c_3y^2\mu^*+c_4y(\mu^*)^2+O\big((|x|+|y|+|\mu^*|)^4\big), \\ g_2(x,y,\mu^*) &= \beta_{23}x^2+\beta_{24}xy+\beta_{25}x^3+\beta_{26}x^2y+d_1x\mu^*+d_2x^2\mu^* \\ &\quad + O\big((|x|+|y|+|\mu^*|)^4\big), \end{aligned}$$

$$\beta_{11} = 1-\nu^*, \qquad \beta_{12} = -\frac{2\nu^*e^{\nu^*}\sqrt{\mu_1e^{-\nu^*}-1}}{\mu_1}, \qquad \beta_{13} = \frac{\nu^*}{2}-1,$$

$$\beta_{14} = \frac{2\sqrt{\mu_1e^{-\nu^*}-1}(\nu^*-1)}{e^{-\nu^*}\mu_1}, \qquad \beta_{15} = \frac{\nu^*e^{\nu^*}(3\mu_1-4e^{\nu^*})}{\mu_1^2}, \qquad \beta_{16} = \frac{3-\nu^*}{6},$$

$$\beta_{17} = \frac{(2-\nu^*)\sqrt{\mu_1e^{-\nu^*}-1}}{e^{-\nu^*}\mu_1}, \qquad \beta_{18} = \frac{e^{\nu^*}(1-\nu^*)(3\mu_1-4e^{\nu^*})}{\mu_1^2},$$

$$\beta_{19} = \frac{4\nu^*\sqrt{\mu_1e^{-\nu^*}-1}(2-\mu_1e^{-\nu^*})}{e^{-3\nu^*}\mu_1^3}, \qquad c_1 = \frac{\nu^*e^{\nu^*}(\mu_1e^{-\nu^*}-2)}{\mu_1^2\sqrt{\mu_1e^{-\nu^*}-1}},$$

$$c_2 = \frac{e^{\nu^*}(\mu_1e^{-\nu^*}-2)(1-\nu^*)}{\mu_1^2\sqrt{\mu_1e^{-\nu^*}-1}}, \qquad c_3 = \frac{\nu^*e^{\nu^*}(8e^{\nu^*}-3\mu_1)}{\mu_1^3},$$

$$c_4 = -\frac{\nu^*e^{\nu^*}(3e^{-2\nu^*}\mu_1^2-12\mu_1e^{-\nu^*}+8)}{4\mu_1^3(\mu_1e^{-\nu^*}-1)^{\frac{3}{2}}}, \qquad \beta_{21} = \frac{2\nu^*(\nu-1)^2\sqrt{\mu_1e^{-\nu^*}-1}}{\eta\nu}, \qquad \beta_{22} = 1,$$

$$\beta_{23} = \frac{(\nu-1)^2(\nu-4)\sqrt{\mu_1e^{-\nu^*}-1}}{\eta\nu^2}, \qquad \beta_{24} = \frac{2\nu^*(\nu-1)^2}{\eta\nu},$$

$$\beta_{25} = -\frac{4\nu^*(\nu-1)^3(\nu-2)\sqrt{\mu_1e^{-\nu^*}-1}}{\eta^2\nu^3}, \qquad \beta_{26} = \frac{(\nu-1)^2(\nu-4)}{\eta\nu^2},$$

$$d_1 = \frac{\nu^* e^{-\nu^*}(\nu-1)^2}{\eta\nu\sqrt{\mu_1 e^{-\nu^*}-1}}, \qquad d_2 = \frac{e^{-\nu^*}(\nu-1)^2(\nu-4)}{2\eta\nu^2\sqrt{\mu_1 e^{-\nu^*}-1}}.$$

Moreover, we take

$$\tau_2 := 3 - \frac{\sqrt{\eta}}{\sqrt{\nu-1}}$$

and construct the following similarity transformation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \beta_{12} & \beta_{12} \\ -1-\beta_{11} & \tau_2-\beta_{11} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{4.14}$$

With the implementation of similarity transformation (4.14), we obtain the following normal for map (4.13):

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & \tau_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} g_3(u,v,\mu^*) \\ g_4(u,v,\mu^*) \end{pmatrix}, \tag{4.15}$$

where

$$\begin{aligned}
g_3(u,v,\mu^*) ={}& \left(\frac{(\tau_2-\beta_{11})\beta_{16}}{\beta_{12}(\tau_2+1)} - \frac{\beta_{25}}{\tau_2+1}\right)x^3 + \left(\frac{(\tau_2-\beta_{11})\beta_{17}}{\beta_{12}(\tau_2+1)} - \frac{\beta_{26}}{\tau_2+1}\right)x^2y \\
&+ \left(\frac{(\tau_2-\beta_{11})\beta_{13}}{\beta_{12}(\tau_2+1)} - \frac{d_2\mu^*+\beta_{23}}{\tau_2+1}\right)x^2 + \frac{(\tau_2-\beta_{11})\beta_{18}xy^2}{\beta_{12}(\tau_2+1)} \\
&+ \left(\frac{(\tau_2-\beta_{11})(\mu^*c_2+\beta_{14})}{\beta_{12}(\tau_2+1)} - \frac{\beta_{24}}{\tau_2+1}\right)xy - \frac{d_1\mu^*x}{\tau_2+1} + \frac{(\tau_2-\beta_{11})\beta_{19}y^3}{\beta_{12}(\tau_2+1)} \\
&+ \frac{(\tau_2-\beta_{11})(\mu^*c_3+\beta_{15})y^2}{\beta_{12}(\tau_2+1)} + \frac{(\tau_2-\beta_{11})((\mu^*)^2c_4+\mu^*c_1)y}{\beta_{12}(\tau_2+1)} \\
&+ O\big((|u|+|v|+|\mu^*|)^4\big), \\
g_4(u,v,\mu^*) ={}& \left(\frac{(1+\beta_{11})\beta_{16}}{\beta_{12}(\tau_2+1)} + \frac{\beta_{25}}{\tau_2+1}\right)x^3 + \left(\frac{(1+\beta_{11})\beta_{17}}{\beta_{12}(\tau_2+1)} + \frac{\beta_{26}}{\tau_2+1}\right)x^2y \\
&+ \left(\frac{(1+\beta_{11})\beta_{13}}{\beta_{12}(\tau_2+1)} + \frac{d_2\mu^*+\beta_{23}}{\tau_2+1}\right)x^2 + \frac{(1+\beta_{11})\beta_{18}xy^2}{\beta_{12}(\tau_2+1)} \\
&+ \left(\frac{(1+\beta_{11})(\mu^*c_2+\beta_{14})}{\beta_{12}(\tau_2+1)} + \frac{\beta_{24}}{\tau_2+1}\right)xy + \frac{d_1\mu^*x}{\tau_2+1} + \frac{(1+\beta_{11})\beta_{19}y^3}{\beta_{12}(\tau_2+1)} \\
&+ \frac{(1+\beta_{11})(\mu^*c_3+\beta_{15})y^2}{\beta_{12}(\tau_2+1)} + \frac{(1+\beta_{11})((\mu^*)^2c_4+\mu^*c_1)y}{\beta_{12}(\tau_2+1)} \\
&+ O\big((|u|+|v|+|\mu^*|)^4\big),
\end{aligned}$$

and

$$x = \beta_{12}(u+v), \qquad y = -(1+\beta_{11})u + (\tau_2-\beta_{11})v.$$

To apply center manifold theorem to map (4.15), we denote by $W^c(0,0,0)$ the center manifold for (4.15) computed at $(0,0)$ in a small neighborhood of $\mu^* = 0$. Then this center manifold for (4.15) is approximated as

$$W^c(0,0,0) = \{(u,v,\mu^*) \in \mathbb{R}^3 : v = p_1u^2 + p_2u\mu^* + p_3(\mu^*)^2 + O((|u| + |\mu^*|)^3)\},$$

where

$$\begin{aligned}
p_1 &= \left(\frac{(1+\beta_{11})\beta_{13}}{\beta_{12}(1-\tau_2^2)} + \frac{\beta_{23}}{1-\tau_2^2}\right)\beta_{12}^2 - \left(\frac{(1+\beta_{11})\beta_{14}}{\beta_{12}(1-\tau_2^2)} + \frac{\beta_{24}}{1-\tau_2^2}\right)\beta_{12}(1+\beta_{11}) \\
&\quad + \frac{\beta_{15}(1+\beta_{11})^3}{\beta_{12}(1-\tau_2^2)}, \\
p_2 &= \frac{d_1\beta_{12}}{1-\tau_2^2} - \frac{(1+\beta_{11})^2c_1}{\beta_{12}(1-\tau_2^2)}, \qquad p_3 = 0.
\end{aligned}$$

Moreover, the map restricted to the center manifold $W^c(0,0,0)$ is computed as follows:

$$G : u \to -u + t_1u^2 + t_2u\mu^* + t_3u^2\mu^* + t_4u(\mu^*)^2 + t_5u^3 + O((|u| + |\mu^*|)^4),$$

where

$$\begin{aligned}
t_1 &= \left(\frac{(\tau_2-\beta_{11})\beta_{13}}{\beta_{12}(1+\tau_2)} - \frac{\beta_{23}}{1+\tau_2}\right)\beta_{12}^2 - \left(\frac{(\tau_2-\beta_{11})\beta_{14}}{\beta_{12}(1+\tau_2)} - \frac{\beta_{24}}{1+\tau_2}\right)\beta_{12}(1+\beta_{11}) \\
&\quad + \frac{(\tau_2-\beta_{11})\beta_{15}(1+\beta_{11})^2}{\beta_{12}(1+\tau_2)}, \\
t_2 &= -\frac{d_1\beta_{12}}{1+\tau_2} - \frac{(\tau_2-\beta_{11})c_1(1+\beta_{11})}{\beta_{12}(1+\tau_2)}, \\
t_3 &= 2\left(\frac{(\tau_2-\beta_{11})\beta_{13}}{\beta_{12}(1+\tau_2)} - \frac{\beta_{23}}{1+\tau_2}\right)\beta_{12}^2p_2 - \frac{d_2\beta_{12}^2}{1+\tau_2} \\
&\quad + \left(\frac{(\tau_2-\beta_{11})\beta_{14}}{\beta_{12}(1+\tau_2)} - \frac{\beta_{24}}{1+\tau_2}\right)\beta_{12}(\tau_2-\beta_{11})p_2 \\
&\quad - \left(\left(\frac{(\tau_2-\beta_{11})\beta_{14}}{\beta_{12}(\tau_2+1)} - \frac{\beta_{24}}{1+\tau_2}\right)\beta_{12}p_2 + \frac{(\tau_2-\beta_{11})c_2}{1+\tau_2}\right)(1+\beta_{11}) \\
&\quad - \frac{d_1\beta_{12}p_1}{\tau_2+1} - 2\frac{(\tau_2-\beta_{11})^2\beta_{15}(1+\beta_{11})p_2}{\beta_{12}(1+\tau_2)} + \frac{(\tau_2-\beta_{11})c_3(1+\beta_{11})^2}{\beta_{12}(1+\tau_2)} \\
&\quad + \frac{(\tau_2-\beta_{11})^2c_1p_1}{\beta_{12}(1+\tau_2)}, \\
t_4 &= \frac{(\tau_2-\beta_{11})^2c_1p_2}{\beta_{12}(1+\tau_2)} - \frac{d_1\beta_{12}p_2}{1+\tau_2} - \frac{(\tau_2-\beta_{11})c_4(1+\beta_{11})}{\beta_{12}(1+\tau_2)}, \\
t_5 &= \left(\frac{(\tau_2-\beta_{11})\beta_{16}}{\beta_{12}(\tau_2+1)} - \frac{\beta_{25}}{1+\tau_2}\right)\beta_{12}^3 - \left(\frac{(\tau_2-\beta_{11})\beta_{17}}{\beta_{12}(1+\tau_2)} - \frac{\beta_{26}}{\tau_2+1}\right)\beta_{12}^2(1+\beta_{11}) \\
&\quad + 2\left(\frac{(\tau_2-\beta_{11})\beta_{13}}{\beta_{12}(1+\tau_2)} - \frac{\beta_{23}}{1+\tau_2}\right)\beta_{12}^2p_1 + \frac{(\tau_2-\beta_{11})\beta_{18}(1+\beta_{11})^2}{1+\tau_2} \\
&\quad + \left(\frac{(\tau_2-\beta_{11})\beta_{14}}{\beta_{12}(1+\tau_2)} - \frac{\beta_{24}}{\tau_2+1}\right)\beta_{12}p_1(\tau_2-\beta_{11}) \\
&\quad - \left(\frac{(\tau_2-\beta_{11})\beta_{14}}{\beta_{12}(1+\tau_2)} - \frac{\beta_{24}}{\tau_2+1}\right)\beta_{12}p_1(1+\beta_{11}) \\
&\quad - \frac{(\tau_2-\beta_{11})\beta_{19}(1+\beta_{11})^3}{\beta_{12}(\tau_2+1)} - 2\frac{(\tau_2-\beta_{11})^2\beta_{15}(1+\beta_{11})p_1}{\beta_{12}(\tau_2+1)}.
\end{aligned}$$

Furthermore, we define the following nonzero real numbers:

$$m_1 = \left(\frac{\partial^2 g_1}{\partial u \partial \mu^*} + \frac{1}{2}\frac{\partial G}{\partial \mu^*}\frac{\partial^2 G}{\partial u^2}\right)_{(0,0)} = -\frac{d_1\beta_{12}}{1+\tau_2} - \frac{(\tau_2-\beta_{11})c_1(1+\beta_{11})}{\beta_{12}(1+\tau_2)} \neq 0$$

and

$$m_2 = \left(\frac{1}{6}\frac{\partial^3 G}{\partial u^3} + \left(\frac{1}{2}\frac{\partial^2 G}{\partial u^2}\right)^2\right)_{(0,0)} = t_1^2 + t_5 \neq 0.$$

Due to aforementioned computation, we have the following result about period-doubling bifurcation of system (1.5).

Theorem 4.2 *If $m_1 \neq 0$ and $m_2 \neq 0$, then system (1.5) undergoes period-doubling bifurcation at the unique positive equilibrium point when the parameter μ varies in a small neighborhood of μ_1. Furthermore, if $m_2 > 0$, then the period-two orbits that bifurcate from positive equilibrium of (1.5) are stable, and if $m_2 < 0$, then these orbits are unstable.*

5 Neimark–Sacker bifurcation

In this section, we investigate when positive steady-states of systems (1.3) and (1.5) undergo Neimark–Sacker bifurcation. For this bifurcation, theory of normal forms is implemented for the existence and direction of such a type of bifurcation. Recently, Neimark–Sacker bifurcation related to discrete-time models has been investigated by many authors [19–27]. Furthermore, in the case of continuous systems, we refer to [28–32] for some recent discussions related to Hopf bifurcation.

First, we show that the positive equilibrium $(u^*, ke^{-u^*} - 1)$ of system (1.3) undergoes Hopf bifurcation such that k is selected as a bifurcation parameter. For this, we see that the characteristic polynomial (2.4) has complex roots if the following inequality holds:

$$\left(2 - \frac{q}{p-1}\right)^2 < 4\left(2 - \frac{e^{\frac{q}{p-1}}}{k} - \frac{1}{p} + \frac{e^{\frac{q}{p-1}}}{kp} - \frac{q}{p-1}\right). \tag{5.1}$$

Furthermore, we suppose that η_1 and η_2 are complex roots of (2.4). Then these roots satisfy $|\eta_1| = |\eta_2| = 1$ whenever the following conditions hold:

$$k = \frac{e^{\frac{q}{p-1}}(p-1)^2}{(p-1)^2 - pq}, \qquad (p-1)^2 > pq. \tag{5.2}$$

Keeping in view (5.1) and (5.2), we consider the following set:

$$\mathbb{S}_3 = \left\{(p,q,k) \in \mathbb{R}_+^3 : p > 1, q > 0, k = \frac{e^{\frac{q}{p-1}}(p-1)^2}{(p-1)^2 - pq}, pq < (p-1)^2\right\}.$$

Suppose that $(p,q,k) \in \mathbb{S}_3$. Then the positive equilibrium $(u^*, ke^{-u^*} - 1)$ of system (1.3) undergoes Hopf bifurcation whenever k varies in a small neighborhood of k_2 given as

$$k_2 := \frac{e^{\frac{q}{p-1}}(p-1)^2}{(p-1)^2 - pq}.$$

Assume that $(p,q,k) \in \mathbb{S}_3$. Then plant–herbivore model (1.3) is represented equivalently by the following two-dimensional map:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} \frac{k_2 X e^{-X}}{1+Y} \\ \frac{pXY}{q+X} \end{pmatrix}. \tag{5.3}$$

Assume that $\tilde{k}$ represents a small perturbation in k_2. Then the corresponding perturbed map for (5.3) is expressed as follows:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} \frac{(k_2+\tilde{k}) X e^{-X}}{1+Y} \\ \frac{pXY}{q+X} \end{pmatrix}. \tag{5.4}$$

To translate the positive fixed point of (5.4) at $(0,0)$, we implement the translations $x = X - u^*$ and $y = Y - ((k_2+\tilde{k})e^{-u^*} - 1)$ with $u^* = \frac{q}{p-1}$. Then from (5.4) it follows that

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h_1(x,y) \\ h_2(x,y) \end{pmatrix}, \tag{5.5}$$

where

$$\begin{aligned}
&h_1(x,y) = m_{13}x^2 + m_{14}xy + m_{15}y^2 + m_{16}x^3 + m_{17}x^2y + m_{18}xy^2 + m_{19}y^3 + O\big((|x|+|y|)^4\big),\\
&h_2(x,y) = m_{23}x^2 + m_{24}xy + m_{25}x^3 + m_{26}x^2y + O\big((|x|+|y|)^4\big),\\
&m_{11} = 1-u^*, \qquad m_{12} = -\frac{u^* e^{u^*}}{(k_2+\tilde{k})}, \qquad m_{13} = \frac{u^*-2}{2}, \qquad m_{14} = \frac{(u^*-1)e^{u^*}}{(k_2+\tilde{k})},\\
&m_{15} = \frac{u^* e^{2u^*}}{(k_2+\tilde{k})^2}, \qquad m_{16} = \frac{3-u^*}{6}, \qquad m_{17} = \frac{e^{u^*}(2-u^*)}{2(k_2+\tilde{k})}, \qquad m_{18} = \frac{e^{2u^*}(1-u^*)}{(k_2+\tilde{k})^2},\\
&m_{19} = -\frac{u^* e^{3u^*}}{(k_2+\tilde{k})^3}, \qquad m_{21} = \frac{pq((k_2+\tilde{k})e^{-u^*}-1)}{(q+u^*)^2}, \qquad m_{22} = \frac{pu^*}{q+u^*},\\
&m_{23} = \frac{pq(1-(k_2+\tilde{k})e^{-u^*})}{(q+u^*)^3}, \qquad m_{24} = \frac{pq}{(q+u^*)^2}, \qquad m_{25} = \frac{pq((k_2+\tilde{k})e^{-u^*}-1)}{(q+u^*)^4},\\
&m_{26} = -\frac{pq}{(q+u^*)^3}.
\end{aligned}$$

Moreover, the characteristic equation for the variational matrix of system (5.5) computed at $(0,0)$ is given as follows:

$$\eta^2 - P(\tilde{k})\eta + Q(\tilde{k}) = 0, \tag{5.6}$$

where

$$P(\tilde{k}) = 2 - \frac{q}{p-1},$$

and

$$Q(\tilde{k}) = 2 - \frac{e^{\frac{q}{p-1}}}{k_2 + \tilde{k}} - \frac{1}{p} + \frac{e^{\frac{q}{p-1}}}{p(k_2 + \tilde{k})} - \frac{q}{p-1}.$$

Assume that $(p, q, k) \in \mathbb{S}_3$. Then the complex roots for (5.6) are computed as follows:

$$\eta_1 = \frac{P(\tilde{k}) - i\sqrt{4Q(\tilde{k}) - (P(\tilde{k}))^2}}{2}$$

and

$$\eta_2 = \frac{P(\tilde{k}) + i\sqrt{4Q(\tilde{k}) - (P(\tilde{k}))^2}}{2}.$$

Then it easily follows that

$$|\eta_1| = |\eta_2| = \sqrt{2 - \frac{e^{\frac{q}{p-1}}}{k_2 + \tilde{k}} - \frac{1}{p} + \frac{e^{\frac{q}{p-1}}}{p(k_2 + \tilde{k})} - \frac{q}{p-1}}.$$

Also, we have that

$$\left(\frac{d|\eta_2|}{d\tilde{k}}\right)_{\tilde{k}=0} = \left(\frac{d|\eta_1|}{d\tilde{k}}\right)_{\tilde{k}=0} = \frac{e^{-\frac{q}{p-1}}(1 + p^2 - p(2+q))^2}{2(p-1)^3 p} \neq 0.$$

Since $-2 < P(0) < 2$ as $(p, q, k) \in \mathbb{S}_3$. Moreover, a simple computation yields that $P(0) = 2 - \frac{q}{p-1}$, and we suppose that $P(0) \neq 0$ and $P(0) \neq -1$, that is,

$$q \neq 2(p-1), \qquad q \neq 3(p-1). \tag{5.7}$$

Suppose that (5.7) holds and $(p, q, k) \in \mathbb{S}_3$. Then it follows that $P(0) \neq \pm 2, 0, -1$, that is, $\eta_1^m, \eta_2^m \neq 1$ for all $m = 1, 2, 3, 4$ at $\tilde{k} = 0$. Therefore both roots of (5.6) do not lie in the intersection of the unit circle with the coordinate axes when $\tilde{k} = 0$.

Furthermore, we suppose that $\kappa = \frac{P(0)}{2}$ and $\omega = \frac{\sqrt{4Q(0) - (P(0))^2}}{2}$. Then to convert (5.5) into normal form, we take into account the following similarity transformation:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} m_{12} & 0 \\ \kappa - m_{11} & -\omega \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{5.8}$$

Due to implementation of similarity transformation (5.8), we can obtain the following normal form for (5.5):

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} \kappa & -\omega \\ \omega & \kappa \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \tilde{f}(u,v) \\ \tilde{g}(u,v) \end{pmatrix}, \tag{5.9}$$

where

$$\tilde{f}(u,v) = \frac{m_{13}}{m_{12}}x^2 + \frac{m_{14}}{m_{12}}xy + \frac{m_{15}}{m_{12}}y^2 + \frac{m_{16}}{m_{12}}x^3 + \frac{m_{17}}{m_{12}}x^2y + \frac{m_{18}}{m_{12}}xy^2 + \frac{m_{19}}{m_{12}}y^3 + O\big((|u| + |v|)^4\big),$$

$$\begin{aligned}\tilde{g}(u,v) &= \left(\frac{(\kappa - m_{11})m_{13}}{m_{12}\omega} - \frac{m_{23}}{\omega}\right)x^2 + \left(\frac{(\kappa - m_{11})m_{14}}{m_{12}\omega} - \frac{m_{24}}{\omega}\right)xy \\ &+ \left(\frac{(\kappa - m_{11})m_{15}}{m_{12}\omega}\right)y^2 + \left(\frac{(\kappa - m_{11})m_{16}}{m_{12}\omega} - \frac{m_{25}}{\omega}\right)x^3 \\ &+ \left(\frac{(\kappa - m_{11})m_{17}}{m_{12}\omega} - \frac{m_{26}}{\omega}\right)x^2y + \left(\frac{(\kappa - m_{11})m_{18}}{m_{12}\omega}\right)xy^2 \\ &+ \left(\frac{(\kappa - m_{11})m_{19}}{m_{12}\omega}\right)y^3 + O\big((|u| + |v|)^4\big),\end{aligned}$$

$x = m_{12}u$, and $y = (\kappa - m_{11})u - \omega v$. To discuss the direction for Hopf bifurcation, we consider the following first Lyapunov exponent computed as

$$L = \left(\left[-\operatorname{Re}\left(\frac{(1-2\eta_1)\eta_2^2}{1-\eta_1}\zeta_{20}\zeta_{11}\right) - \frac{1}{2}|\zeta_{11}|^2 - |\zeta_{02}|^2 + \operatorname{Re}(\eta_2\zeta_{21})\right]\right)_{\tilde{k}=0},$$

where

$$\begin{aligned}\zeta_{20} &= \frac{1}{8}\big[\tilde{f}_{uu} - \tilde{f}_{vv} + 2\tilde{g}_{uv} + i(\tilde{g}_{uu} - \tilde{g}_{vv} - 2\tilde{f}_{uv})\big], \\ \zeta_{11} &= \frac{1}{4}\big[\tilde{f}_{uu} + \tilde{f}_{vv} + i(\tilde{g}_{uu} + \tilde{g}_{vv})\big], \\ \zeta_{02} &= \frac{1}{8}\big[\tilde{f}_{uu} - \tilde{f}_{vv} - 2\tilde{g}_{uv} + i(\tilde{g}_{uu} - \tilde{g}_{vv} + 2\tilde{f}_{uv})\big], \\ \zeta_{21} &= \frac{1}{16}\big[\tilde{f}_{uuu} + \tilde{f}_{uvv} + \tilde{g}_{uuv} + \tilde{g}_{vvv} + i(\tilde{g}_{uuu} + \tilde{g}_{uvv} - \tilde{f}_{uuv} - \tilde{f}_{vvv})\big].\end{aligned}$$

Arguing as in [33–37], we present the following result on the existence and direction of Neimark–Sacker bifurcation at positive steady state of system (1.3).

Theorem 5.1 *Suppose that* (5.7) *holds and* $L \neq 0$. *Then the positive equilibrium* $(u^*, ke^{-u^*} - 1)$ *of system* (1.3) *undergoes Hopf bifurcation as the bifurcation parameter* k *varies in a small neighborhood of* $k_2 = \frac{e^{\frac{q}{p-1}}(p-1)^2}{(p-1)^2 - pq}$. *Furthermore, if* $L < 0$, *then an attracting invariant closed curve bifurcates from the equilibrium point for* $k > k_2$, *and if* $L > 0$, *then a repelling invariant closed curve bifurcates from the equilibrium point for* $k < k_2$.

Finally, in this section, we discuss when the positive steady state $(v^*, \sqrt{\mu e^{-v^*} - 1})$ of system (1.5) undergoes Hopf bifurcation when μ is taken as a bifurcation parameter.

Suppose that the following parametric conditions hold:

$$\eta < 4\sqrt{\eta}\sqrt{v-1}, \qquad 4(v-1)^{3/2} - \sqrt{\eta}v > 0. \tag{5.10}$$

Furthermore, we assume that

$$\mu = \frac{4e^{\frac{\sqrt{\eta}}{\sqrt{v-1}}}(v-1)^{3/2}}{4(v-1)^{3/2} - \sqrt{\eta}v}. \tag{5.11}$$

Then, keeping in view conditions (5.10) and (5.11), we consider the set

$$\mathbb{S}_4 = \Bigg\{(\mu,\nu,\eta) : \nu > 1, \mu,\eta > 0, \mu = \frac{4e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}(\nu-1)^{3/2}}{4(\nu-1)^{3/2} - \sqrt{\eta}\nu},$$
$$\eta < 4\sqrt{\eta}\sqrt{\nu-1}, 4(\nu-1)^{3/2} > \sqrt{\eta}\nu\Bigg\}.$$

Moreover, assume that $(\mu,\nu,\eta) \in \mathbb{S}_4$. Then the positive steady state $(\nu^*, \sqrt{\mu e^{-\nu^*} - 1})$ of system (1.5) undergoes Hopf bifurcation as μ varies in a small neighborhood of μ_2 given as

$$\mu_2 := \frac{4e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}(\nu-1)^{3/2}}{4(\nu-1)^{3/2} - \sqrt{\eta}\nu}.$$

Next, we assume that $(\mu,\nu,\eta) \in \mathbb{S}_4$. Then system (1.5) is represented by the following two-dimensional map:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} \frac{\mu_2 X e^{-X}}{1+Y^2} \\ \frac{\nu X^2 Y}{\eta + X^2} \end{pmatrix}. \tag{5.12}$$

Taking a small perturbation $\tilde{\mu}$ in μ_2 corresponding to map (5.12), we have the following perturbed mapping:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} \frac{(\mu_2+\tilde{\mu}) X e^{-X}}{1+Y^2} \\ \frac{\nu X^2 Y}{\eta + X^2} \end{pmatrix}. \tag{5.13}$$

Furthermore, taking into account the translations $x = X - \nu^*$ and $y = Y - \sqrt{(\mu_1 + \mu^*)e^{-\nu^*} - 1}$, where $\nu^* = \sqrt{\frac{\eta}{\nu-1}}$, (5.13) is converted into following map:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} k_1(x,y) \\ k_2(x,y) \end{pmatrix}, \tag{5.14}$$

where

$$k_1(x,y) = b_{13}x^2 + b_{14}xy + b_{15}y^2 + b_{16}x^3 + b_{17}x^2y + b_{18}xy^2 + b_{19}y^3 + O\big((|x|+|y|)^4\big),$$
$$k_2(x,y) = b_{23}x^2 + b_{24}xy + b_{25}x^3 + b_{26}x^2y + O\big((|x|+|y|)^4\big), \qquad b_{11} = 1 - \nu^*,$$
$$b_{12} = -\frac{2\nu^* e^{\nu^*}\sqrt{(\mu_2+\tilde{\mu})e^{-\nu^*} - 1}}{\mu_2 + \tilde{\mu}}, \qquad b_{13} = \frac{\nu^*}{2} - 1,$$
$$b_{14} = \frac{2\sqrt{(\mu_2+\tilde{\mu})e^{-\nu^*} - 1}(\nu^* - 1)}{e^{-\nu^*}(\mu_2+\tilde{\mu})}, \qquad b_{15} = \frac{\nu^* e^{\nu^*}(3(\mu_2+\tilde{\mu}) - 4e^{\nu^*})}{(\mu_2+\tilde{\mu})^2},$$
$$b_{16} = \frac{3-\nu^*}{6}, \qquad b_{17} = \frac{(2-\nu^*)\sqrt{(\mu_2+\tilde{\mu})e^{-\nu^*} - 1}}{e^{-\nu^*}(\mu_2+\tilde{\mu})},$$
$$b_{18} = \frac{e^{\nu^*}(1-\nu^*)(3(\mu_2+\tilde{\mu}) - 4e^{\nu^*})}{(\mu_2+\tilde{\mu})^2}, \qquad b_{19} = \frac{4\nu^*\sqrt{(\mu_2+\tilde{\mu})e^{-\nu^*} - 1}(2 - (\mu_2+\tilde{\mu})e^{-\nu^*})}{e^{-3\nu^*}(\mu_2+\tilde{\mu})^3},$$

$$b_{21}=\frac{2\nu^*(\nu-1)^2\sqrt{(\mu_2+\tilde{\mu})e^{-\nu^*}-1}}{\eta\nu},\qquad b_{22}=1,$$
$$b_{23}=\frac{(\nu-1)^2(\nu-4)\sqrt{(\mu_2+\tilde{\mu})e^{-\nu^*}-1}}{\eta\nu^2},\qquad b_{24}=\frac{2\nu^*(\nu-1)^2}{\eta\nu},$$
$$b_{25}=-\frac{4\nu^*(\nu-1)^3(\nu-2)\sqrt{(\mu_2+\tilde{\mu})e^{-\nu^*}-1}}{\eta^2\nu^3},\qquad b_{26}=\frac{(\nu-1)^2(\nu-4)}{\eta\nu^2}.$$

Then the characteristic equation for the variational matrix of (5.14) computed at its equilibrium $(0,0)$ is given as follows:

$$\eta^2-R(\tilde{\mu})\eta+S(\tilde{\mu})=0, \tag{5.15}$$

where

$$R(\tilde{\mu})=2-\frac{\sqrt{\eta}}{\sqrt{\nu-1}}$$

and

$$S(\tilde{\mu})=5-\frac{\sqrt{\eta}}{\sqrt{\nu-1}}-\frac{4}{\nu}-\frac{4e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}(\nu-1)}{(\mu_2+\tilde{\mu})\nu}.$$

Assume that $(\mu,\nu,\eta)\in\mathbb{S}_4$. Then the complex roots for (5.17) are computed as follows:

$$\eta_1=\frac{R(\tilde{\mu})-i\sqrt{4S(\tilde{\mu})-(R(\tilde{\mu}))^2}}{2}$$

and

$$\eta_2=\frac{R(\tilde{\mu})+i\sqrt{4S(\tilde{\mu})-(R(\tilde{\mu}))^2}}{2}.$$

Moreover, it follows that

$$|\eta_1|=|\eta_2|=\sqrt{5-\frac{\sqrt{\eta}}{\sqrt{\nu-1}}-\frac{4}{\nu}-\frac{4e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}(\nu-1)}{(\mu_2+\tilde{\mu})\nu}}.$$

Similarly, a simple computation yields that

$$\left(\frac{d|\eta_2|}{d\tilde{\mu}}\right)_{\tilde{\mu}=0}=\left(\frac{d|\eta_1|}{d\tilde{\mu}}\right)_{\tilde{\mu}=0}=\frac{e^{-\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}(\sqrt{\eta}\nu-4(\nu-1)^{3/2})^2}{8(\nu-1)^2\nu}\neq 0.$$

Furthermore, according to the existence condition for bifurcation, we have that $-2<R(0)<2$. Next, it follows that $R(0)=2-\frac{\sqrt{\eta}}{\sqrt{\nu-1}}$. Moreover, we suppose that $R(0)\neq 0$ and $R(0)\neq -1$, that is,

$$\sqrt{\eta}\neq 2\sqrt{\nu-1},\qquad \sqrt{\eta}\neq 3\sqrt{\nu-1}. \tag{5.16}$$

Assume that $(\mu,\nu,\eta)\in\mathbb{S}_4$ and (5.16) holds. Then it follows that $R(0)\neq\pm2,0,-1$, and due to these restrictions, we have that $\eta_1^m,\eta_2^m\neq1$ for all $m=1,2,3,4$ at $\tilde{\mu}=0$. Therefore, at $\tilde{\mu}=0$, the roots of (5.14) do not lie in the intersection of the unit circle with the coordinate axes.

Furthermore, we take $\alpha=\frac{R(0)}{2}$ and $\beta=\frac{\sqrt{4S(0)-(R(0))^2}}{2}$. Then, to obtain the normal form for (5.14) at $\tilde{\mu}=0$, we consider the following similarity transformation:

$$\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}b_{12}&0\\\alpha-b_{11}&-\beta\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}. \tag{5.17}$$

Due to implementation of similarity transformation (5.17), we obtain the following normal form for (5.14):

$$\begin{pmatrix}u\\v\end{pmatrix}\to\begin{pmatrix}\alpha&-\beta\\\beta&\alpha\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}+\begin{pmatrix}\tilde{h}(u,v)\\\tilde{k}(u,v)\end{pmatrix}, \tag{5.18}$$

where

$$\begin{aligned}\tilde{h}(u,v)&=\frac{b_{13}}{b_{12}}x^2+\frac{b_{14}}{b_{12}}xy+\frac{b_{15}}{b_{12}}y^2+\frac{b_{16}}{b_{12}}x^3+\frac{b_{17}}{b_{12}}x^2y+\frac{b_{18}}{b_{12}}xy^2+\frac{b_{19}}{b_{12}}y^3+O\bigl((|u|+|v|)^4\bigr),\\
\tilde{k}(u,v)&=\left(\frac{(\alpha-b_{11})b_{13}}{b_{12}\beta}-\frac{b_{23}}{\beta}\right)x^2+\left(\frac{(\alpha-b_{11})b_{14}}{b_{12}\beta}-\frac{b_{24}}{\beta}\right)xy\\
&\quad+\left(\frac{(\alpha-b_{11})b_{15}}{b_{12}\beta}\right)y^2+\left(\frac{(\alpha-b_{11})b_{16}}{b_{12}\beta}-\frac{b_{25}}{\beta}\right)x^3\\
&\quad+\left(\frac{(\alpha-b_{11})b_{17}}{b_{12}\beta}-\frac{b_{26}}{\beta}\right)x^2y+\left(\frac{(\alpha-b_{11})b_{18}}{b_{12}\beta}\right)xy^2\\
&\quad+\left(\frac{(\alpha-b_{11})b_{19}}{b_{12}\beta}\right)y^3+O\bigl((|u|+|v|)^4\bigr),\end{aligned}$$

$x=b_{12}u$, and $y=(\alpha-b_{11})u-\beta v$. To discuss the direction for Hopf bifurcation, we consider the following first Lyapunov exponent computed as

$$\Upsilon=\left(\left[-\operatorname{Re}\left(\frac{(1-2\eta_1)\eta_2^2}{1-\eta_1}\zeta_{20}\zeta_{11}\right)-\frac{1}{2}|\zeta_{11}|^2-|\zeta_{02}|^2+\operatorname{Re}(\eta_2\zeta_{21})\right]\right)_{\tilde{\mu}=0},$$

where

$$\begin{aligned}\zeta_{20}&=\frac{1}{8}\bigl[\tilde{h}_{uu}-\tilde{h}_{vv}+2\tilde{k}_{uv}+i(\tilde{k}_{uu}-\tilde{k}_{vv}-2\tilde{h}_{uv})\bigr],\\
\zeta_{11}&=\frac{1}{4}\bigl[\tilde{h}_{uu}+\tilde{h}_{vv}+i(\tilde{k}_{uu}+\tilde{k}_{vv})\bigr],\\
\zeta_{02}&=\frac{1}{8}\bigl[\tilde{h}_{uu}-\tilde{h}_{vv}-2\tilde{k}_{uv}+i(\tilde{k}_{uu}-\tilde{k}_{vv}+2\tilde{h}_{uv})\bigr],\\
\zeta_{21}&=\frac{1}{16}\bigl[\tilde{h}_{uuu}+\tilde{h}_{uvv}+\tilde{k}_{uuv}+\tilde{k}_{vvv}+i(\tilde{k}_{uuu}+\tilde{k}_{uvv}-\tilde{h}_{uuv}-\tilde{h}_{vvv})\bigr].\end{aligned}$$

Next, due to aforementioned computation, we state the following theorem, which gives conditions for the existence and direction of Hopf bifurcation for positive equilibrium of system (1.5).

Theorem 5.2 *Suppose that* (5.16) *holds and* $\Upsilon \neq 0$. *Then the positive equilibrium* $(v^*, \sqrt{\mu e^{-v^*} - 1})$ *of system* (1.5) *undergoes Hopf bifurcation as the bifurcation parameter* μ *varies in a small neighborhood of* $\mu_2 = \frac{4e^{\frac{\sqrt{\eta}}{\sqrt{v-1}}}(v-1)^{3/2}}{4(v-1)^{3/2}-\sqrt{\eta}v}$. *Furthermore, if* $\Upsilon < 0$, *then an attracting invariant closed curve bifurcates from the equilibrium point for* $\mu > \mu_2$, *and if* $\Upsilon > 0$, *then a repelling invariant closed curve bifurcates from the equilibrium point for* $\mu < \mu_2$.

6 Chaos control

For controlling random irregular and fluctuating behavior in a biological system, chaos control is considered to be a practical tool for avoiding this chaotic and complex behavior. For further details related to biological meanings of chaos control and its practical use in the real world, we refer to [38].

In this section we implement a simple chaos control technique for both systems (1.3) and (1.5). Moreover, there are various chaos control methods for discrete-time dynamical systems. For further details related to these techniques, we refer to [39–61].

Here we implement a hybrid control method (also see [24, 47]). The hybrid control technique is based on parameter perturbation and a state feedback control method. First, we implement this methodology to system (1.3) as follows:

$$\begin{aligned} x_{n+1} &= \alpha\left(\frac{kx_n e^{-x_n}}{1+y_n}\right) + (1-\alpha)x_n, \\ y_{n+1} &= \alpha\left(\frac{px_n y_n}{q+x_n}\right) + (1-\alpha)y_n, \end{aligned} \tag{6.1}$$

where $0 < \alpha < 1$ is a control parameter. Similarly, implementation of a hybrid control strategy to system (1.5) yields:

$$\begin{aligned} x_{n+1} &= \beta\left(\frac{\mu x_n e^{-x_n}}{1+y_n^2}\right) + (1-\beta)x_n, \\ y_{n+1} &= \beta\left(\frac{v x_n^2 y_n}{\eta+x_n^2}\right) + (1-\beta)y_n, \end{aligned} \tag{6.2}$$

where $0 < \beta < 1$ is a control parameter for a hybrid control strategy. Next, system (6.1) is controllable as long as its steady state $(u^*, ke^{-u^*} - 1)$ is locally asymptotically stable. Then for particular choice of control parameter α, we can obtain the desired interval for controlling chaos and bifurcation. Furthermore, the variational matrix for the controlled system (6.1) at its positive equilibrium $(u^*, ke^{-u^*} - 1)$ is computed as follows:

$$\begin{bmatrix} 1-\alpha u^* & -\frac{\alpha u^* e^{u^*}}{k} \\ \frac{\alpha(ke^{-u^*}-1)(p-1)}{pu^*} & 1 \end{bmatrix}.$$

Then the characteristic polynomial for the aforementioned variational matrix is given by

$$F(\eta) = \eta^2 - \left(2 - \frac{\alpha q}{p-1}\right)\eta + 1 - \frac{q\alpha}{p-1} + \alpha^2 - \frac{e^{\frac{q}{p-1}}\alpha^2}{k} - \frac{\alpha^2}{p} + \frac{e^{\frac{q}{p-1}}\alpha^2}{kp}. \tag{6.3}$$

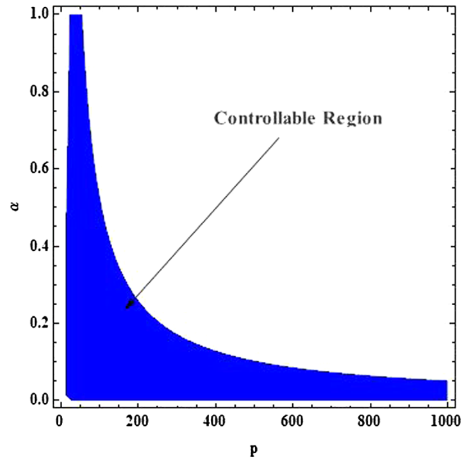


Figure 5 Controllable region for system (6.1)

Lemma 6.1 *Assume that* $0 < q < (p-1)\ln(k)$, $k > 1$, *and* $p > 1$. *Then the positive steady state* $(u^*, ke^{-u^*} - 1)$ *of the controlled system* (6.1) *is a sink if and only if*

$$\left|2 - \frac{\alpha q}{p-1}\right| < 2 - \frac{q\alpha}{p-1} + \alpha^2 - \frac{e^{\frac{q}{p-1}}\alpha^2}{k} - \frac{\alpha^2}{p} + \frac{e^{\frac{q}{p-1}}\alpha^2}{kp} < 2.$$

For $0 < \alpha < 1$, $p \in [1, 1000]$, $k = 60$, and $q = 50$, the controllable region for system (6.1) is depicted in Fig. 5 in the $p\alpha$-plane.

Furthermore, we suppose that $\sqrt{\eta} < \ln(\mu)\sqrt{\nu - 1}$, $\mu > 1$, and $\nu > 1$. Then controlled system (6.2) has a unique positive steady-state $(\nu^*, \sqrt{\mu e^{-\nu^*} - 1})$, which is similar to the positive equilibrium point of (1.5). Moreover, the variational matrix for controlled system (6.2) is computed as follows:

$$\begin{bmatrix} 1 - \frac{\beta\sqrt{\eta}}{\sqrt{\nu-1}} & -\frac{2\beta\sqrt{e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}}\sqrt{\eta}\sqrt{\mu - e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}}}{\mu\sqrt{\nu-1}} \\ \frac{2\beta\sqrt{\mu - e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}}(\nu-1)^{3/2}}{\sqrt{e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}}}\sqrt{\eta}\nu} & 1 \end{bmatrix}.$$

Furthermore, the characteristic polynomial for the aforementioned variational matrix is given by

$$P(\eta) = \eta^2 - \left(2 - \frac{\beta\sqrt{\eta}}{\sqrt{\nu-1}}\right)\eta + 1 - \frac{\beta\sqrt{\eta}}{\sqrt{\nu-1}} + \frac{4\beta^2(\mu - e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}})(\nu-1)}{\mu\nu}. \tag{6.4}$$

Lemma 6.2 *Suppose that* $\sqrt{\eta} < \ln(\mu)\sqrt{\nu - 1}$, $\mu > 1$, *and* $\nu > 1$. *Then the positive steady state* $(\nu^*, \sqrt{\mu e^{-\nu^*} - 1})$ *of system* (6.2) *is locally asymptotically stable if the following condition is satisfied*:

$$\left|2 - \frac{\beta\sqrt{\eta}}{\sqrt{\nu-1}}\right| < 2 - \frac{\beta\sqrt{\eta}}{\sqrt{\nu-1}} + \frac{4\beta^2(\mu - e^{\frac{\sqrt{\eta}}{\sqrt{\nu-1}}})(\nu-1)}{\mu\nu} < 2.$$

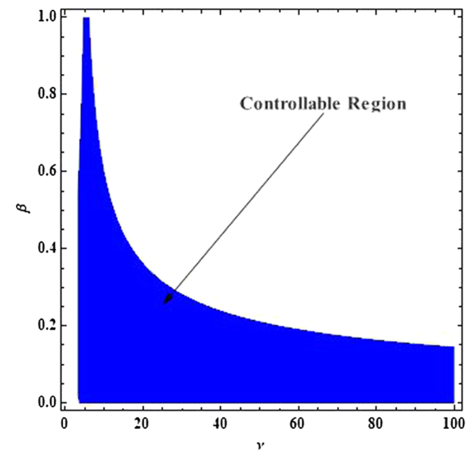
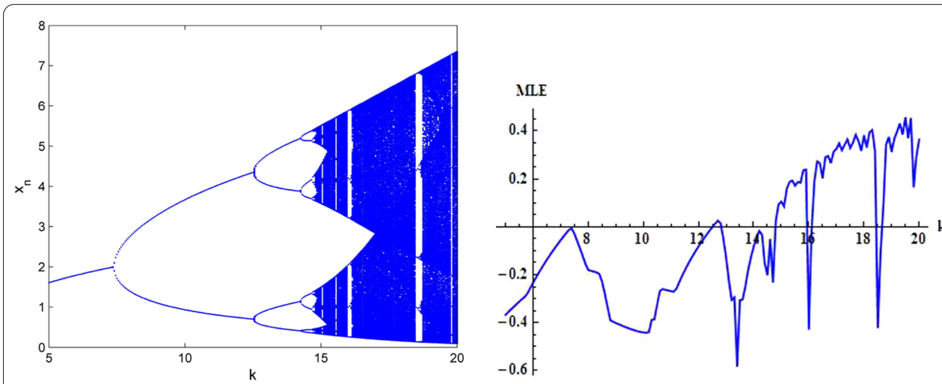


Figure 6 Controllable region for system (6.2)



(a) Bifurcation diagram for x_n (b) Maximum Lyapunov exponents

Figure 7 Bifurcation diagram and MLE for system (1.3) with $p = 20.1$, $q = 45.1$, $k \in [5, 20]$, and $(x_0, y_0) = (2.36, 3.17)$

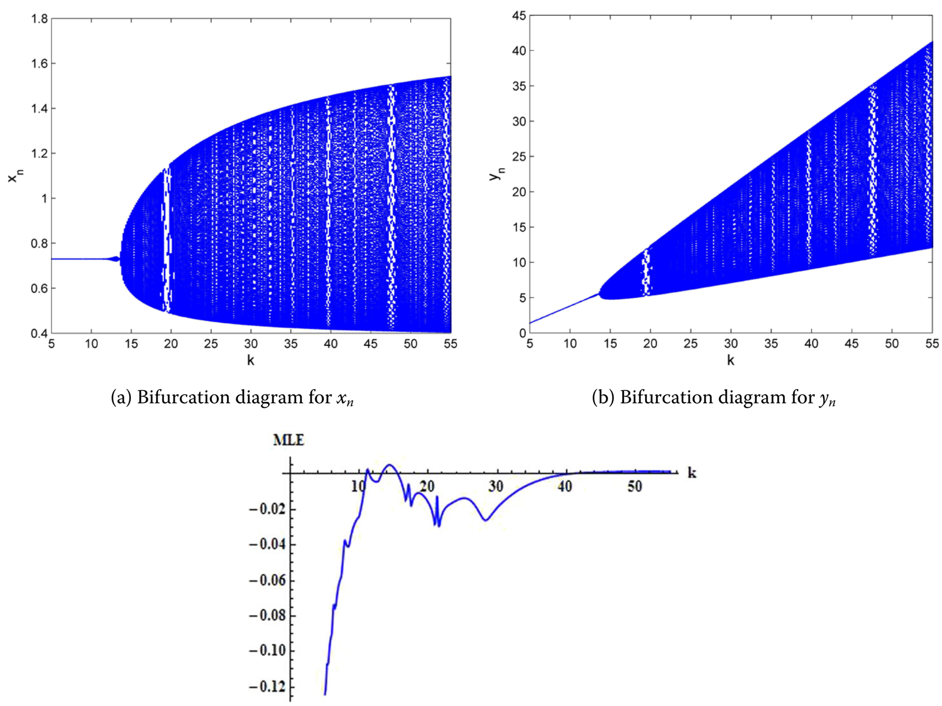
For $0 < \beta < 1$, $\nu \in [1, 100]$, $\eta = 30$, and $\mu = 40$, the controllable region for system (6.2) is depicted in Fig. 6 in the $\nu\beta$-plane.

7 Numerical simulation and discussion

Example 7.1 First, we choose $p = 20.1$, $q = 45.1$, and $k \in [5, 20]$. Then system (1.3) undergoes flip bifurcation as bifurcation parameter k varies in the interval $[5, 20]$. Moreover, the bifurcation diagram for plant population density x_n is depicted in Fig. 7a, and the corresponding maximum Lyapunov exponents (MLE) are depicted in Fig. 7b.

Example 7.2 Next, we choose $p = 7.3$, $q = 4.6$, $k \in [5, 55]$, and initial values $(x_0, y_0) = (0.73, 5.49)$. Then system (1.3) undergoes Hopf bifurcation as the bifurcation parameter k varies in a small neighborhood of $k = 13.48167321833215$. If we choose $(p, q, k) = (7.3, 4.6, 13.482)$, then the positive steady state for system (1.3) is given by (0.730159, 5.49591). Furthermore, the characteristic equation for the variational matrix is given by

$$\eta^2 - 1.26984126984127\eta + 1 = 0. \tag{7.1}$$



(a) Bifurcation diagram for x_n

(b) Bifurcation diagram for y_n

(c) Maximum Lyapunov exponents

Figure 8 Bifurcation diagrams and MLE for system (1.3) with $p = 7.3$, $q = 4.6$, $k \in [5, 55]$, and $(x_0, y_0) = (0.73, 5.49)$

The complex conjugate roots for (7.1) are $\eta_1 = 0.634921 + 0.772577i$ and $\eta_2 = 0.634921 - 0.772577i$ with $|\eta_1| = |\eta_2| = 1$. Thus, we have $(p, q, k) = (7.3, 4.6, 13.48167321833215) \in \mathbb{S}_3$. Moreover, bifurcation diagrams and MLE are depicted in Fig. 8. Taking $q = 4.6$, $p = 7.3$, and $k = 13.46, 13.48167, 13.6, 14, 16, 19.5$, phase portraits for system (1.3) are depicted in Fig. 9. To apply a hybrid control strategy, we choose $(p, q, k) = (7.3, 4.6, 19.5)$. Then controlled system (6.1) takes the following form:

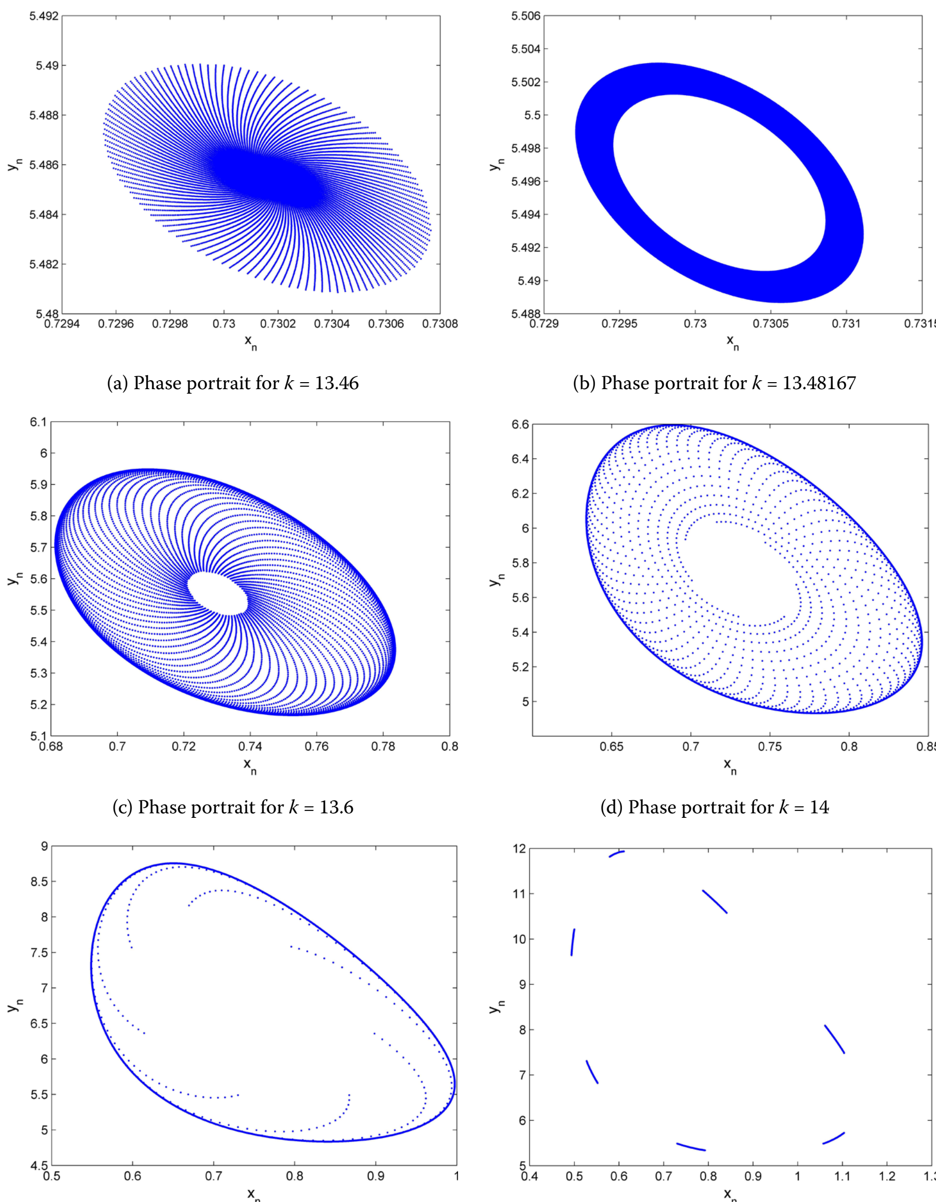
$$\begin{aligned} x_{n+1} &= \alpha\left(\frac{19.5x_n e^{-x_n}}{1 + y_n}\right) + (1 - \alpha)x_n, \\ y_{n+1} &= \alpha\left(\frac{7.3x_n y_n}{4.6 + x_n}\right) + (1 - \alpha)y_n. \end{aligned} \tag{7.2}$$

Then system (7.2) has a positive fixed point $(0.730159, 8.39573)$. Furthermore, the characteristic equation for the variational matrix of (7.2) is computed as follows:

$$\eta^2 - (2 - 0.730159\alpha)\eta + 1 - 0.730159\alpha + 0.771162\alpha^2 = 0. \tag{7.3}$$

Due to the Jury condition, the equilibrium point $(0.730159, 8.39573)$ is a sink if and only if $0 < \alpha < 0.946829$. Choosing $\alpha = 0.945$, plots for the controlled system (7.2) are shown in Fig. 10.

Example 7.3 Now we consider system (1.5) for the numerical verification of flip bifurcation. For this, we choose $\mu \in [2, 22]$, $\nu = 8.5$, $\eta = 55.8$, and the initial conditions $(x_0, y_0) =$



(a) Phase portrait for $k = 13.46$

(b) Phase portrait for $k = 13.48167$

(c) Phase portrait for $k = 13.6$

(d) Phase portrait for $k = 14$

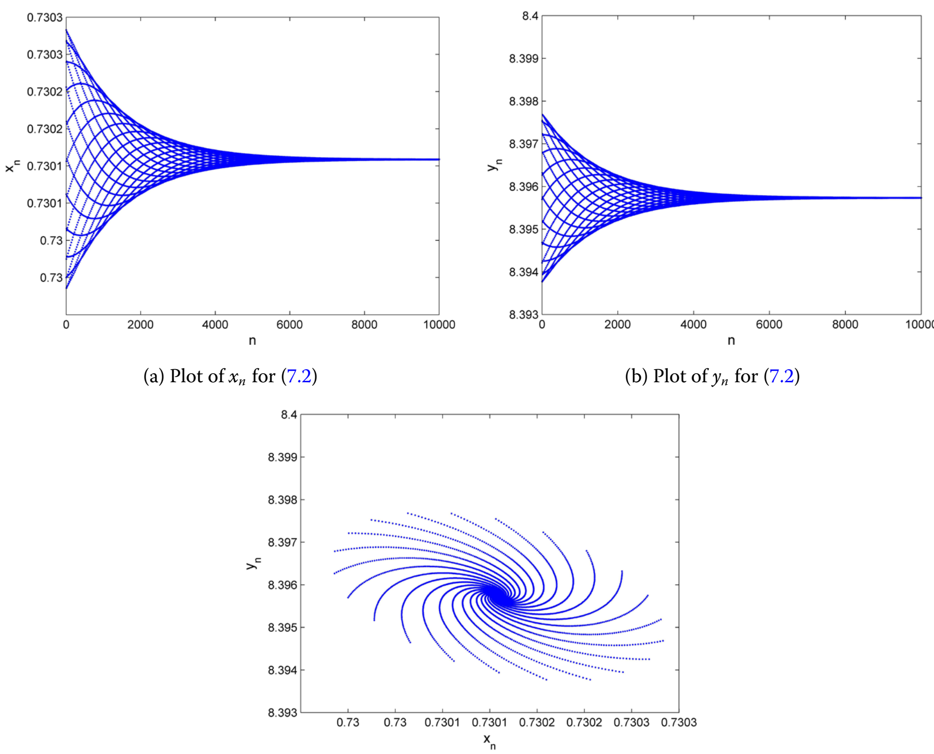
(e) Phase portrait for $k = 16$

(f) Phase portrait for $k = 19.5$

Figure 9 Phase portraits of system (1.3) for $p = 7.3$, $q = 4.6$, $x_0 = 0.73$, $y_0 = 5.49$ with different values of k

$(2.72, 0.83)$. Then the population density of plants x_n undergoes flip bifurcation. We can see the bifurcation diagram for x_n and corresponding MLE in Fig. 11.

Example 7.4 At the end of this section, we verify the existence of Hopf bifurcation for system (1.5) by taking into account some particular parametric values. For such verification, we choose $\nu = 4.1$, $\eta = 1.5$, $\mu \in [2, 8]$, and $(x_0, y_0) = (0.6956, 0.5465)$. Then system (1.5) undergoes Hopf bifurcation as the parameter μ varies in a small neighborhood of a particular value $\mu = 2.603801522628164$. Furthermore, if we choose the parametric values $\mu = 2.603801522628164$, $\nu = 4.1$, and $\eta = 1.5$, then the unique positive fixed point for system (1.5) is $(0.695608, 0.546535)$. At $(0.695608, 0.546535)$ the characteristic equation

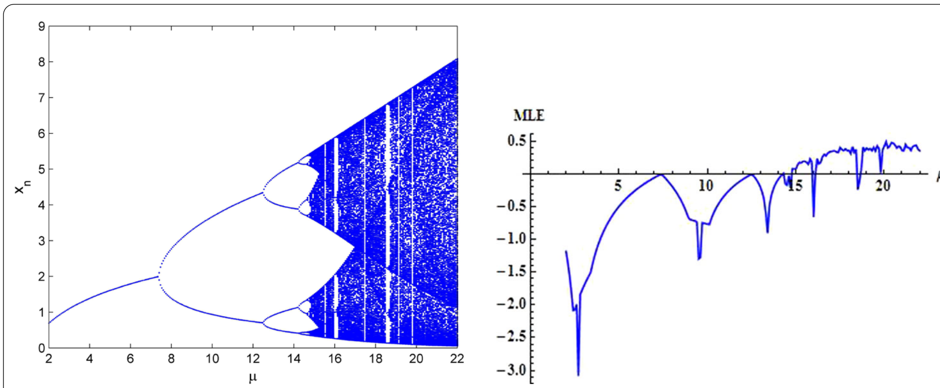


(a) Plot of x_n for (7.2)

(b) Plot of y_n for (7.2)

(c) Phase portrait for (7.2)

Figure 10 Plots of the controlled system (7.2) with $\alpha = 0.945$ and $(x_0, y_0) = (0.73, 8.3957)$



(a) Bifurcation diagram for x_n

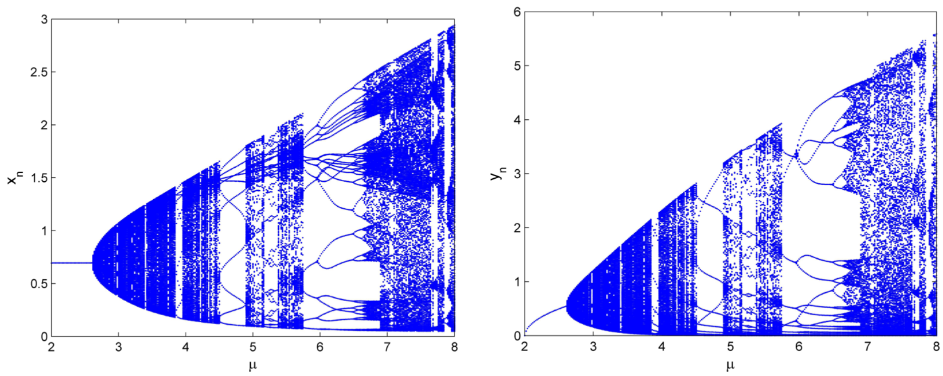
(b) Maximum Lyapunov exponents

Figure 11 Bifurcation diagram and MLE for system (1.5) with $\nu = 8.5$, $\eta = 55.8$, $\mu \in [2, 22]$, and $(x_0, y_0) = (2.72, 0.83)$

for system (1.5) is

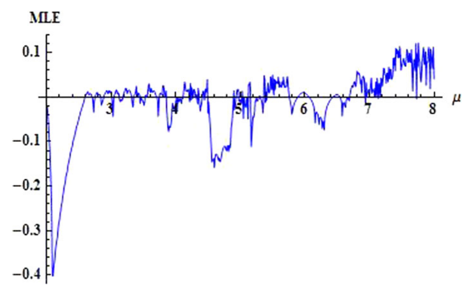
$$\eta^2 - 1.3043916563597475\eta + 1 = 0. \tag{7.4}$$

Furthermore, $\eta_1 = 0.652196 + 0.758051i$ and $\eta_2 = 0.652196 - 0.758051i$ are the roots of (7.4) with modulus $|\eta_1| = |\eta_2| = 1$. Therefore it follows that $(\mu, \nu, \eta) = (2.603801522628164, 4.1, 1.5) \in \mathbb{S}_4$, and bifurcation diagrams and MLE are depicted in Fig. 12. Moreover, for



(a) Bifurcation diagram for x_n

(b) Bifurcation diagram for y_n



(c) Maximum Lyapunov exponents

Figure 12 Bifurcation diagrams and MLE for system (1.5) with $\nu = 4.1$, $\eta = 1.5$, $\mu \in [2, 8]$, and $(x_0, y_0) = (0.6956, 0.5465)$

various values of μ, the phase portraits for system (1.5) are shown in Fig. 13. Lastly, we check the effectiveness of hybrid control strategy for system (1.5). For this purpose, we choose $(\mu, \nu, \eta) = (8, 4.1, 1.5)$. Then due to this choice, controlled system (6.2) is given by

$$\begin{aligned} x_{n+1} &= \beta\left(\frac{8x_n e^{-x_n}}{1+y_n^2}\right) + (1-\beta)x_n, \\ y_{n+1} &= \beta\left(\frac{4.1x_n^2 y_n}{1.5+x_n^2}\right) + (1-\beta)y_n. \end{aligned} \tag{7.5}$$

Then system (7.5) has a unique positive equilibrium point $(0.695608, 1.72921)$, and the characteristic equation of the Jacobian matrix of (7.5) evaluated at $(0.695608, 1.72921)$ is

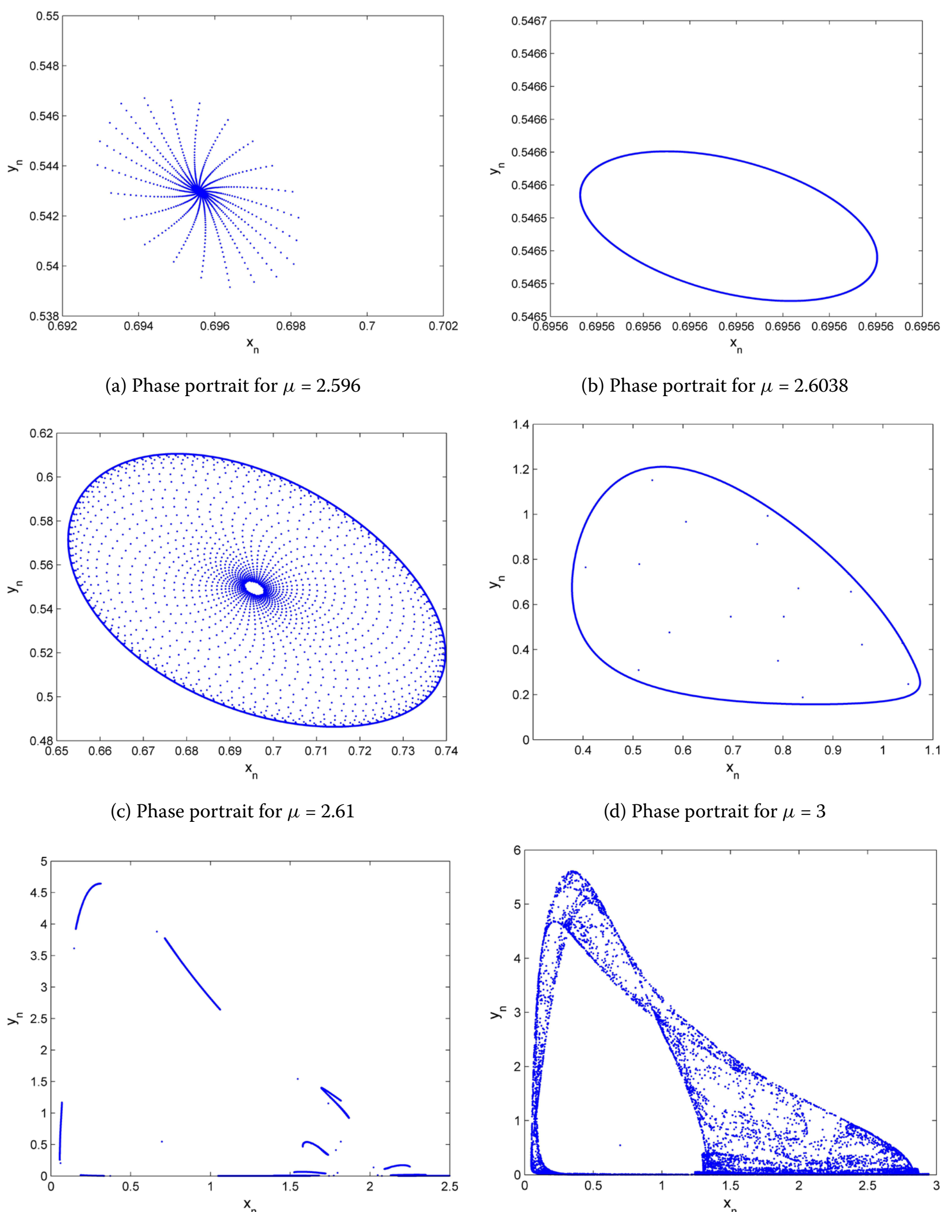
$$\eta^2 - (2 - 0.695608\beta)\eta + 1 - 0.695608\beta + 2.26643\beta^2 = 0. \tag{7.6}$$

Now, according to the Jury condition, the roots of (7.6) lie inside the open unit disk if and only if

$$|2 - 0.695608\beta| < 2 - 0.695608\beta + 2.26643\beta^2 < 2.$$

Or, equivalently,

$$4 - 1.39122\beta + 2.26643\beta^2 > 0, \qquad 2.26643\beta < 0.695608.$$



(a) Phase portrait for $\mu = 2.596$

(b) Phase portrait for $\mu = 2.6038$

(c) Phase portrait for $\mu = 2.61$

(d) Phase portrait for $\mu = 3$

(e) Phase portrait for $\mu = 6.8$

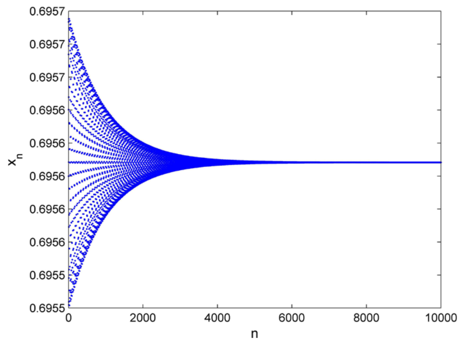
(f) Phase portrait for $\mu = 8$

Figure 13 Phase portraits of system (1.5) for $\nu = 4.1$, $\eta = 1.5$, $x_0 = 0.6956$, $y_0 = 0.5465$ with different values of μ

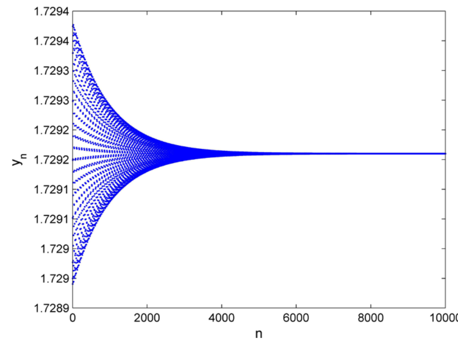
From aforementioned inequalities it follows that $0 < \beta < 0.306918$. Thus the unique positive equilibrium point $(0.695608, 1.72921)$ of the controlled system (7.5) is locally asymptotically stable if and only if $0 < \beta < 0.306918$. The plots of the controlled system (7.5) are shown in Fig. 14 for $\beta = 0.304$.

8 Concluding remarks

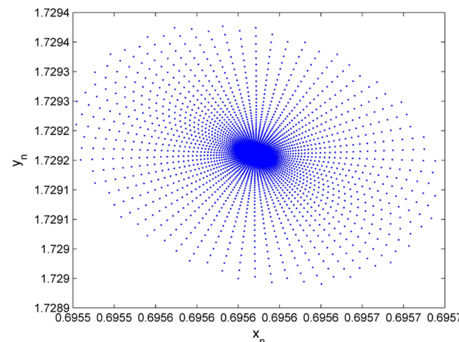
This paper is concerned with qualitative behavior of two discrete-time plant–herbivore models in exponential forms. The models are proposed by taking into account that the function for plant limitation is of Ricker type, whereas the effect of herbivore on plant



(a) Plot of x_n for (7.5)



(b) Plot of y_n for (7.5)



(c) Phase portrait for (7.5)

Figure 14 Plots for the controlled system (7.5) with $\beta = 0.304$ and $(x_0, y_0) = (0.6956, 1.729)$

population and herbivore population growth rate are proportional to functional responses of type-II and type-III, respectively. The parametric conditions for local asymptotic stability of equilibria of both systems are investigated. Due to implementation of bifurcation theory and center manifold theorem, we obtained that both models undergo Neimark–Sacker bifurcation and period-doubling bifurcation at their positive steady states. Our results show that parameters related to growth rates of plants have strong stability effects or vice versa. To control chaotic behaviors of the systems, a hybrid control strategy is implemented. The effectiveness of this control strategy is illustrated through numerical simulations. Moreover, complex dynamics for both models is exhibited through periodic orbits, quasi-periodic orbits, and chaotic sets and windows. Furthermore, in present discussion of qualitative analysis for plant–herbivore interaction, Holling type-II and III functional responses are implemented with Ricker-type function for plant self-limitation. It is interesting to implement some other choice of plant self-limitation function. In future, we will apply a Beverton–Holt-type function for plant self-limitation.

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None of the authors has any competing interests in this manuscript.

Authors' contributions

Both authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

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References

1. Buckley, Y.M., Rees, M., Sheppard, A.W., Smyth, M.J.: Stable coexistence of an invasive plant and biocontrol agent: a parameterized coupled plant–herbivore model. J. Appl. Ecol. **42**, 70–79 (2005)
2. Kartal, S.: Dynamics of a plant–herbivore model with differential–difference equations. Cogent Math. **3**(1), 1136198 (2016)
3. Kang, Y., Armbruster, D., Kuang, Y.: Dynamics of a plant–herbivore model. J. Biol. Dyn. **2**(2), 89–101 (2008)
4. Liu, R., Feng, Z., Zhu, H., DeAngelis, D.L.: Bifurcation analysis of a plant–herbivore model with toxin-determined functional response. J. Differ. Equ. **245**, 442–467 (2008)
5. Li, Y., Feng, Z., Swihart, R., Bryant, J., Huntley, H.: Modeling plant toxicity on plant–herbivore dynamics. J. Dyn. Differ. Equ. **18**(4), 1021–1024 (2006)
6. Feng, Z., Qiu, Z., Liu, R., DeAngelis, D.L.: Dynamics of a plant–herbivore–predator system with plant-toxicity. Math. Biosci. **229**(2), 190–204 (2011)
7. Kuang, Y., Huisman, J., Elser, J.J.: Stoichiometric plant–herbivore models and their interpretation. Math. Biosci. Eng. **1**(2), 215–222 (2004)
8. Sui, G., Fan, M., Loladze, I., Kuang, Y.: The dynamics of a stoichiometric plant–herbivore model and its discrete analog. Math. Biosci. Eng. **4**(1), 29–46 (2007)
9. Din, Q.: Global behavior of a plant–herbivore model. Adv. Differ. Equ. **2015**, 119 (2015)
10. Saha, T., Bandyopadhyay, M.: Dynamical analysis of a plant–herbivore model bifurcation and global stability. J. Appl. Math. Comput. **19**(1–2), 327–344 (2005)
11. Castillo-Chavez, C., Feng, Z., Huang, W.: Global dynamics of a plant–herbivore model with toxin-determined functional response. SIAM J. Appl. Math. **72**(4), 1002–1020 (2012)
12. Jothi, S.S., Gunasekaran, M.: Chaos and bifurcation analysis of plant–herbivore system with intra-specific competitions. Int. J. Adv. Res. **3**(8), 1359–1363 (2015)
13. Zhao, Y., Feng, Z., Zheng, Y., Cen, X.: Existence of limit cycles and homoclinic bifurcation in a plant–herbivore model with toxin-determined functional response. J. Differ. Equ. **258**(8), 2847–2872 (2015)
14. Harper, J.L.: Population Biology of Plants. Academic Press, New York (1977)
15. Crawley, M.J.: Herbivory: The Dynamics of Animal–Plant Interactions. University of California Press, Los Angeles (1983)
16. Abbott, K.C., Dwyer, G.: Food limitation and insect outbreaks: complex dynamics in plant–herbivore models. J. Anim. Ecol. **76**, 1004–1014 (2007)
17. Ricker, W.E.: Stock and recruitment. J. Fish. Res. Board Can. **11**, 559–623 (1954)
18. Leeuwen, E.V., Jansen, V.A.A., Bright, P.W.: How population dynamics shape the functional response in a one-predator–two-prey system. Ecology **88**(6), 1571–1581 (2007)
19. He, Z., Lai, X.: Bifurcation and chaotic behavior of a discrete-time predator–prey system. Nonlinear Anal., Real World Appl. **12**, 403–417 (2011)
20. Liu, X., Xiao, D.: Complex dynamic behaviors of a discrete-time predator–prey system. Chaos Solitons Fractals **32**, 80–94 (2007)
21. Jing, Z., Yang, J.: Bifurcation and chaos in discrete-time predator–prey system. Chaos Solitons Fractals **27**, 259–277 (2006)
22. Agiza, H.N., ELabbasy, E.M., EL-Metwally, H., Elsadany, A.A.: Chaotic dynamics of a discrete prey–predator model with Holling type II. Nonlinear Anal., Real World Appl. **10**, 116–129 (2009)
23. Li, B., He, Z.: Bifurcations and chaos in a two-dimensional discrete Hindmarsh–Rose model. Nonlinear Dyn. **76**(1), 697–715 (2014)
24. Yuan, L.-G., Yang, Q.-G.: Bifurcation, invariant curve and hybrid control in a discrete-time predator–prey system. Appl. Math. Model. **39**(8), 2345–2362 (2015)
25. Din, Q.: Neimark–Sacker bifurcation and chaos control in Hassell–Varley model. J. Differ. Equ. Appl. **23**(4), 741–762 (2017)
26. Din, Q., Gümüş, Ö.A., Khalil, H.: Neimark–Sacker bifurcation and chaotic behaviour of a modified host-parasitoid model. Z. Naturforsch. A **72**(1), 25–37 (2017)
27. Din, Q., Elsadany, A.A., Khalil, H.: Neimark–Sacker bifurcation and chaos control in a fractional-order plant–herbivore model. Discrete Dyn. Nat. Soc. **2017**, Article ID 6312964 (2017)
28. Wang, F., Yang, Y.: Quasi-synchronization for fractional-order delayed dynamical networks with heterogeneous nodes. Appl. Math. Comput. **339**, 1–14 (2018)
29. Han, M., Sheng, L., Zhang, X.: Bifurcation theory for finitely smooth planar autonomous differential systems. J. Differ. Equ. **264**(5), 3596–3618 (2018)
30. Tian, H., Han, M.: Bifurcation of periodic orbits by perturbing high-dimensional piecewise smooth integrable systems. J. Differ. Equ. **263**(11), 7448–7474 (2017)
31. Wang, Z., Wang, X., Li, Y., Huang, X.: Stability and Hopf bifurcation of fractional-order complex-valued single neuron model with time delay. Int. J. Bifurc. Chaos **27**(13), 1750209 (2017)

32. Wang, Z., Xie, Y., Lu, J., Li, Y.: Stability and bifurcation of a delayed generalized fractional-order prey–predator model with interspecific competition. Appl. Math. Comput. **347**, 360–369 (2019)
33. Guckenheimer, J., Holmes, P.: Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer, New York (1983)
34. Robinson, C.: Dynamical Systems: Stability, Symbolic Dynamics and Chaos Boca Raton, New York (1999)
35. Wiggins, S.: Introduction to Applied Nonlinear Dynamical Systems and Chaos. Springer, New York (2003)
36. Wan, Y.H.: Computation of the stability condition for the Hopf bifurcation of diffeomorphism on R^2. SIAM J. Appl. Math. **34**, 167–175 (1978)
37. Kuznetsov, Y.A.: Elements of Applied Bifurcation Theory. Springer, New York (1997)
38. Weiss, J.N., Garfinkel, A., Spano, M.L., Ditto, W.L.: Chaos and chaos control in biology. J. Clin. Invest. **93**(4), 1355–1360 (1994)
39. Ott, E., Grebogi, C., Yorke, J.A.: Controlling chaos. Phys. Rev. Lett. **64**(11), 1196–1199 (1990)
40. Romeiras, F.J., Grebogi, C., Ott, E., Dayawansa, W.P.: Controlling chaotic dynamical systems. Physica D **58**, 165–192 (1992)
41. Ogata, K.: Modern Control Engineering, 2nd edn. Prentice Hall, Englewood (1997)
42. Chen, G., Dong, X.: From Chaos to Order: Perspectives, Methodologies, and Applications. World Scientific, Singapore (1998)
43. Elaydi, S.N.: An Introduction to Difference Equations, 3rd edn. Springer, New York (2005)
44. Lynch, S.: Dynamical Systems with Applications Using Mathematica. Birkhäuser, Boston (2007)
45. Ran, J., Liu, Y., He, J., Li, X.: Controlling Neimark–Sacker bifurcation in delayed species model using feedback controller. Adv. Math. Phys. **2016**, Article ID 2028037 (2016)
46. Fradkov, A.L., Evans, R.J.: Control of chaos: methods and applications in engineering. Annu. Rev. Control **29**, 33–56 (2005)
47. Luo, X.S., Chen, G.R., Wang, B.H., et al.: Hybrid control of period-doubling bifurcation and chaos in discrete nonlinear dynamical systems. Chaos Solitons Fractals **18**, 775–783 (2004)
48. Zhang, X., Zhang, Q.L., Sreeram, V.: Bifurcation analysis and control of a discrete harvested prey–predator system with Beddington–DeAngelis functional response. J. Franklin Inst. **347**, 1076–1096 (2010)
49. Chen, G.R., Moiola, J.L., Wang, H.O.: Bifurcation control: theories, methods and applications. Int. J. Bifurc. Chaos **10**, 511–548 (2000)
50. Yu, P.: Bifurcation Dynamics in Control Systems. Springer, Berlin (2003)
51. Scholl, E., Schuster, H.G.: Handbook of Chaos Control. Wiley-VCH, Weinheim (2008)
52. Nguyen, L.H., Hong, K.-S.: Hopf bifurcation control via a dynamic state-feedback control. Phys. Lett. A **376**(4), 442–446 (2012)
53. Chen, Z., Yu, P.: Controlling and anti-controlling Hopf bifurcations in discrete maps using polynomial functions. Chaos Solitons Fractals **26**, 1231–1248 (2005)
54. ELabbasy, E.M., Agiza, H.N., EL-Metwally, H., Elsadany, A.A.: Bifurcation analysis, chaos and control in the Burgers mapping. Int. J. Nonlinear Sci. **4**, 171–185 (2007)
55. Chen, G.R., Fang, J.Q., Hong, Y.G., Qin, H.S.: Controlling Hopf bifurcations: discrete-time systems. Discrete Dyn. Nat. Soc. **5**, 29–33 (2000)
56. Din, Q.: Complexity and chaos control in a discrete-time prey–predator model. Commun. Nonlinear Sci. Numer. Simul. **49**, 113–134 (2017)
57. Din, Q.: Bifurcation analysis and chaos control in discrete-time glycolysis models. J. Math. Chem. **56**(3), 904–931 (2018)
58. Din, Q., Donchev, T., Kolev, D.: Stability, bifurcation analysis and chaos control in chlorine dioxide–iodine–malonic acid reaction. MATCH Commun. Math. Comput. Chem. **79**(3), 577–606 (2018)
59. Din, Q.: A novel chaos control strategy for discrete-time Brusselator models. J. Math. Chem. **56**(10), 3045–3075 (2018)
60. Din, Q.: Stability, bifurcation analysis and chaos control for a predator–prey system. J. Vib. Control (2018). https://doi.org/10.1177/1077546318790871
61. Din, Q., Iqbal, M.A.: Bifurcation analysis and chaos control for a discrete-time enzyme model. Z. Naturforsch. A (2018). https://doi.org/10.1515/zna-2018-0254